On a class of stochastic partial differential equations

Jian Song

Abstract

This paper concerns the stochastic partial differential equation with multiplicative noise \( \frac{\partial u}{\partial t} = \mathcal{L}u + u \dot{W} \), where \( \mathcal{L} \) is the generator of a symmetric Lévy process \( X \), \( \dot{W} \) is a Gaussian noise and \( u \dot{W} \) is understood both in the senses of Stratonovich and Skorohod. The Feynman-Kac type of representations for the solutions and the moments of the solutions are obtained, and the Hölder continuity of the solutions is also studied. As a byproduct, when \( \gamma(x) \) is a nonnegative and nonnegative-definite function, a sufficient and necessary condition for \( \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds \) to be exponentially integrable is obtained.

1 Introduction

In [39], Walsh developed the theory of stochastic integrals with respect to martingale measures and used it to study the stochastic partial differential equations (SPDEs) driven by space-time Gaussian white noise. Dalang in his seminal paper [17] extended the definition of Walsh’s stochastic integral and applied it to solve SPDEs with Gaussian noise white in time and homogeneously colored in space (white-colored noise). Recently, the theories on SPDEs with white-colored noise have been extensively developed, and one can refer to, for instance, [13, 15, 16, 32, 37] and the references therein. For the SPDEs with white-colored noise, the methods used in the above-mentioned literature relies on the martingale structure of the noise, and hence cannot be applied to the case when the noise is colored in time. On the other hand, SPDEs driven by a Gaussian noise which is colored in time and (possibly) colored in space have attracted more and more attention.

In the present article, we consider the following SPDE in \( \mathbb{R}^d \),

\[
\begin{cases}
\frac{\partial u}{\partial t} = \mathcal{L}u + u \dot{W}, & t \geq 0, x \in \mathbb{R}^d \\
u(0, x) = u_0(x), & x \in \mathbb{R}^d.
\end{cases}
\]

(1.1)

In the above equation, \( \mathcal{L} \) is the generator of a Lévy process \( \{X_t, t \geq 0\} \), \( u_0(x) \) is a continuous and bounded function, and the noise \( \dot{W} \) is a (generalized) Gaussian random field independent of \( X \) with the covariance function given by

\[\mathbb{E}[\dot{W}(t, x)\dot{W}(s, y)] = |t - s|^{-\beta_0} \gamma(x - y),\]

(1.2)
where $\beta_0 \in (0, 1)$ and $\gamma$ is a nonnegative and nonnegative-definite (generalized) function. The product $u \dot{W}$ in (1.1) is understood either in the Stratonovich sense or in the Skorohod sense. Throughout the paper, we assume that $X$ is a symmetric Lévy process with characteristic exponent $\Psi(\xi)$, i.e., $\mathbb{E}\exp(i\xi X_t) = \exp(-t\Psi(\xi))$. Note that the symmetry implies that $\Psi(\xi)$ is a real-valued nonnegative function. Furthermore, we assume that $X$ has transition functions denoted by $q_t(x)$, which also entails that $\lim_{|\xi| \to \infty} \Psi(\xi) = \infty$ by Riemann-Lebesgue lemma.

When $\mathcal{L} = \frac{1}{2}\Delta$ where $\Delta$ is the Laplacian operator, and $\dot{W}$ is colored in time and white in space, Hu and Nualart [28] investigated the conditions to obtain a unique mild solution for (1.1) in the Skorohod sense, and obtained the Feynman-Kac formula for the moments of the solution. When $\mathcal{L} = \frac{1}{2}\Delta$, and $\dot{W}$ is a fractional white noise with Hurst parameters $H_0 \in (\frac{1}{2}, 1)$ in time and $(H_1, \ldots, H_d) \in (\frac{1}{2}, 1)^d$ in space, i.e., $\beta_0 = 2 - 2H_0$ and $\gamma(x) = \prod_{i=1}^d |x_i|^{2H_i - 2}$, Hu et al. [30] obtained a Feynman-Kac formula for a weak solution under the condition $2H_0 + \sum_{i=1}^d H_i > d + 1$ for the SPDE in the Stratonovich sense. This result was extended to the case $\mathcal{L} = -(-\Delta)^{\alpha/2}$ in Chen et al. [11]. A recent paper [27] by Hu et al. studied (1.1) in both senses when $\mathcal{L} = \frac{1}{2}\Delta$ and $\dot{W}$ is a general Gaussian noise using the techniques of Malliavin calculus and Fourier analysis, obtained the Feynman-Kac formulas for the solutions and the moments of the solutions, and investigated Hölder continuity of the Feynman-Kac functional and the intermittency of the solutions.

There has been fruitful literature on (1.1) in the sense of Skorohod, especially when $\dot{W}$ is white in time. For instance, when $\mathcal{L} = \frac{1}{2}\Delta$, (1.1) is the well-known parabolic Anderson model ([1]) and has been extensively investigated in, for example, [6, 7, 8, 10, 35]. Foondun and Khoshnevisan [20, 21] studied the general nonlinear SPDEs. When $\dot{W}$ is colored both in time and in space, $\mathcal{L}$ is a fractional Laplacian, the intermittency property of (1.1) was investigated in Balan and Conus [4, 5].

The main purpose of the current paper is to study (1.1) in both senses of Stratonovich and Skorohod under the assumptions Hypothesis (I) in Section 3 and Hypothesis (II) in Section 5.1 respectively. Under Hypothesis (I), we will obtain Feynman-Kac type of representations for a mild solution to (1.1) in the Stratonovich sense and for the moments of the solution (Theorem 4.6 and Theorem 4.7). Under Hypothesis (II), we will show that the mild solution to (1.1) in the Skorohod sense exists uniquely, and obtain the Feynman-Kac formula for the moments of the solution (Theorem 5.3 and Theorem 5.5). Furthermore, under stronger conditions, we can get Hölder continuity of the solutions in both senses (Theorem 4.11 and Theorem 5.9). As a byproduct, we show that Hypothesis (I) is a sufficient and necessary condition for the Hamiltonian $\int_0^t \int_0^t |r-s|^{-\beta_0 \gamma} (X_r - X_s) dr ds$ to be exponentially integrable (Proposition 3.2 and Theorem 3.3).

There are two key ingredients to prove the main result Theorem 4.6 for the Stratonovich case. One is to obtain the exponential integrability of $\int_0^t \int_0^t |r-s|^{-\beta_0 \gamma} (X_r - X_s) dr ds$. When $X$ is a Brownian motion, Le Gall’s moment method ([34]) was applied in [30] to get the exponential integrability, and when $X$ is a symmetric $\alpha$-stable process, the techniques from large deviation were employed in [11, 12]. However, in the current paper, we cannot apply directly either of the two approaches due to the lacks of the self-similarity of the Lévy
process $X$ and the homogeneity of the spatial kernel function $\gamma(x)$. Instead, to get the desired exponential integrability, we estimate the moments of $\int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds$ directly using Fourier analysis inspired by [27] and the techniques for the computation of moments used in [29]. The other key ingredient is to justify that the Feynman-Kac representation (4.10) is a mild solution to (1.1) in the sense of Definition 4.5. To this goal, we will apply the Malliavin calculus and follow the “standard” approach used in [28, 30, 11, 27].

We get the existence of the solution to (1.1) in the Stratonovich sense by finding its Feynman-Kac representation directly, while in this article we do not address its uniqueness which will be our future work. A possible “probabilistic” treatment that was used in [3] is to express the Duhamel solution as a sum of multiple Stratonovich integrals, and then investigate its relationship (the Hu-Meyer formula [31]) with the Wiener chaos expansion. Another approach is to consider (1.1) pathwisely as a “deterministic” equation. Hu et al. [27] obtained the existence and uniqueness of (1.1) in the Stratonovich sense when $\mathcal{L} = \frac{1}{2} \Delta$ and $\dot{W}$ is a general Gaussian noise, by linking it to a general pathwise equation for which the authors obtained the existence and uniqueness in the framework of weighted Besov spaces. For general SPDEs, one can refer to [9, 19, 23, 24] for the rough path treatment. Recently, Deya [18] applied Hairer’s regularity structures theory ([25]) to investigate a nonlinear heat equation driven by a space-time fractional noise.

For (1.1) in the Skorohod sense, we obtain the existence and uniqueness result by studying the chaos expansion of the solution as has been done in [28, 5, 27]. We apply the approximation method initiated in [28] to get the Feynman-Kac type of representation for the moments of the solution. One possibly can also obtain the representation by directly computing the expectations of the products of Wiener chaoses as in [14].

The rest of the paper is organized as follows. In Section 2, we recall some preliminaries on the Gaussian noise and Malliavin calculus. In Section 3, we provide a sufficient and necessary condition for the Hamiltonian $\int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds$ to be exponentially integrable. In Section 4, the Feynman-Kac formula for a mild solution to (1.1) in the Stratonovich sense is obtained, the Feynman-Kac formula for the moments of the solution is provided, and the Hölder continuity of the solution is studied. Finally, in Section 5, we obtain the existence and uniqueness of the mild solution in the Skorohod sense under some condition, find the Feynman-Kac formula for the moments, and investigate the Hölder continuity of the solution.

2 Preliminaries

In this section, we introduce the stochastic integral with respect to the noise $\dot{W}$ and recall some material from Malliavin calculus which will be used.

Let $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ be the space of smooth functions on $\mathbb{R}_+ \times \mathbb{R}^d$ with compact supports, and the Hilbert space $\mathcal{H}$ be the completion of $C_0^\infty(\mathbb{R}_+ \times \mathbb{R}^d)$ endowed with the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}_+^2} \int_{\mathbb{R}^{2d}} \varphi(s, x) \psi(t, y) |t-s|^{-\beta_0} \gamma(x-y) ds dt dx dy,$$

(2.1)
where $\beta_0 \in (0, 1)$ and $\gamma$ is a nonnegative and nonnegative-definite function. In a complete probability space $(\Omega, \mathcal{F}, P)$, we define an isonormal Gaussian process (see, e.g., [36, Definition 1.1.1]) \( W = \{W(h), h \in \mathcal{H}\} \) with the covariance function given by

\[ \mathbb{E}[W(\varphi)W(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}}. \tag{2.2} \]

In this paper, we will also use the following stochastic integral to denote \( W(\varphi), \)

\[ W(\varphi) := \int_0^\infty \int_{\mathbb{R}^d} \varphi(s, x)W(ds, dx). \]

Denote \( \mathcal{S}(\mathbb{R}^d) \) the Schwartz space of rapidly decreasing functions and let \( \mathcal{S}'(\mathbb{R}^d) \) denote its dual space of tempered distributions. Let \( \hat{\varphi} \) or \( \mathcal{F}\varphi \) be the Fourier transform of \( \varphi \in \mathcal{S}'(\mathbb{R}^d): \)

\[ \hat{\varphi}(\xi) = \mathcal{F}\varphi(\xi) := \int_{\mathbb{R}^d} e^{-i\xi \cdot x} \varphi(x)dx. \]

By the Bochner-Schwartz theorem [22, Theorem 3], the spectral measure \( \mu \) of the process \( W \) defined by

\[ \int_{\mathbb{R}^d} \gamma(x)\varphi(x)dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\varphi}(\xi)\mu(d\xi), \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^d) \tag{2.3} \]

exists and is tempered (meaning that there exists \( p \geq 1 \) such that \( \int_{\mathbb{R}^d}(1 + |\xi|^2)^{-p}\mu(d\xi) < \infty \)).

The inner product in (2.1) now can be represented by:

\[ \langle \varphi, \psi \rangle_{\mathcal{H}} = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \hat{\varphi}(s, \xi)\overline{\hat{\psi}(t, \xi)}|t - s|^{-\beta_0}\mu(d\xi)dsdt, \quad \forall \varphi, \psi \in \mathcal{S}(\mathbb{R}^d), \tag{2.4} \]

where the Fourier transform is with respect to the space variable only, and \( \overline{z} \) is the complex conjugate of \( z \).

Throughout the paper, we assume that the covariance function \( \gamma(x) \) possesses the following properties .

1. \( \gamma(x) \) is locally integrable.
2. The Fourier transform \( \hat{\gamma}(\xi) \) is a nonnegative measurable function, and hence \( \mu(d\xi) = \hat{\gamma}(\xi)d\xi \) is absolutely continuous with respect to the Lebesgue measure.
3. \( \gamma(x) : \mathbb{R}^d \to [0, \infty] \) is a continuous function, where \([0, \infty]\) is the usual one-point compactification of \([0, \infty)\).
4. \( \gamma(x) < \infty \) if and only if \( x \neq 0 \) OR \( \hat{\gamma} \in L^\infty(\mathbb{R}^d) \) and \( \gamma(x) < \infty \) when \( x \neq 0 \).

The function \( \gamma(x) \) with the above four properties covers a number of kernels such as the Riesz kernel \( |x|^{-\beta} \) with \( \beta \in (0, d) \), the Cauchy Kernel \( \prod_{j=1}^d(x_j^2 + c)^{-1} \), the Poisson kernel
\[ (|x|^2 + c)^{-(d+1)/2}, \text{ and the Ornstein-Uhlenbeck kernel } e^{-c|x|^\alpha} \text{ with } \alpha \in (0, 2), \text{ for some constant } c \in (0, \infty). \]

Properties (1) and (2) make the spatial kernel \( \gamma(x) \) a function of positive type ([33, Definition 5.1]). Therefore by [33, Lemma 5.6], for any two Borel probability measures \( \nu_1(dx) \) and \( \nu_2(dx) \), the following identity holds,

\[
\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \gamma(x - y)\nu_1(dx)\nu_2(dy) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\gamma}(\xi)\mathcal{F}\nu_1(\xi)\mathcal{F}\nu_2(\xi)d\xi, \quad (2.5)
\]

where \( \mathcal{F}\nu_i(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x}\nu_i(dx) \) is the Fourier transform of \( \nu_i \) for \( i = 1, 2 \). The above formula, as in [5, Appendix] for instance, can be generalized to \( \nu_i(dx) = f_i(x)dx \) with \( f_i \) belonging to the space \( L^1_c(\mathbb{R}^d) \) of integrable complex-valued functions for \( i = 1, 2 \), with \( \nu_2(dy) \) on the left-hand side being replaced by its complex conjugate.

If we let \( \nu_2(dx) \) be the Dirac delta measure \( \delta_0(x)dx \), then we actually have

\[
\int_{\mathbb{R}^d} \gamma(x)\nu_1(dx) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \hat{\gamma}(\xi)\mathcal{F}\nu_1(\xi)d\xi = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}\nu_1(\xi)\mu(d\xi), \quad (2.6)
\]

where \( \nu_1(dx) \) is any Borel probability measure or the measure of the form \( \nu_1(dx) = f_1(x)dx \) with \( f_1 \in L^1_c(\mathbb{R}^d) \). This allows us to have the following lemma.

**Lemma 2.1.** For a \( d \)-dimensional random variable \( Y \), we have

\[
\mathbb{E}[\gamma(Y)] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbb{E}[e^{-i\xi \cdot Y}] \mu(d\xi).
\]

Especially, for any \( a \in \mathbb{R}^d \), we have

\[
\mathbb{E}[\gamma(X_t + a)] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathbb{E}[e^{-i\xi \cdot (X_t + a)}] \mu(d\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot a} e^{-t\Psi(\xi)} \mu(d\xi).
\]

Now we briefly recall some useful knowledge in Malliavin calculus. The reader is referred to [36] for more details. Let \( D \) be the Malliavin derivative, which is an operator mapping from the Sobolev space \( \mathbb{D}^{1,2} \subset L^2(\Omega) \) endowed with the norm \( \|F\|_{1,2} = \sqrt{\mathbb{E}[F^2] + \mathbb{E}[\|DF\|^2]} \) to \( L^2(\Omega; \mathcal{H}) \). The divergence operator \( \delta \) is defined as the the dual operator of \( D \) by the duality \( \mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u \rangle_{\mathcal{H}}] \) for all \( F \in \mathbb{D}^{1,2} \) and \( u \in L^2(\Omega; \mathcal{H}) \) in the domain of \( \delta \). Note that when \( u \in \mathcal{H}, \delta(u) = W(u) \), and that the operator \( \delta \) is also called the Skorohod integral since it coincides with the Skorohod integral in the case of Brownian motion. When \( F \in \mathbb{D}^{1,2} \) and \( h \in \mathcal{H}, \) we have

\[
\delta(Fh) = F \circ \delta(h), \quad (2.7)
\]

where \( \circ \) means the wick product. For \( u \) in the domain of \( \delta \), we also denote \( \delta(u) \) by \( \int_0^\infty \int_{\mathbb{R}^d} u(s, y)W^\circ(ds, dy) \) in this article. The following two formulas will be used in the proofs.

\[
FW(h) = \delta(Fh) + \langle DF, h \rangle_{\mathcal{H}}, \quad (2.8)
\]
for all $F \in \mathbb{D}^{1,2}$ and $h \in \mathcal{H}$.

$$
\mathbb{E}[FW(h)W(g)] = \mathbb{E}[\langle D^2F, h \otimes g \rangle_{\mathcal{H}^{\otimes 2}}] + \mathbb{E}[F]\langle h, g \rangle_{\mathcal{H}},
$$

(2.9)

for all $F \in \mathbb{D}^{2,2}, h \in \mathcal{H}, g \in \mathcal{H}$.

The Wiener chaos expansion has been used in, e.g., [28, 4], to deal with (1.1) in the Skorohod sense. Here we recall some basic facts. Let $F$ be a square integrable random variable measurable with respect to the $\sigma$-algebra generated by $W$. Then $F$ has the chaos expansion

$$
F = \mathbb{E}[F] + \sum_{n=1}^{\infty} F_n,
$$

where $F_n$ belongs to the $n$-th Wiener chaos space $\mathbb{H}_n$. Moreover, $F_n = I_n(f_n)$ for some $f_n \in \mathcal{H}^{\otimes n}$, and the expansion is unique if we require that all $f_n$'s are symmetric in its $n$ variables. Here $I_n : \mathcal{H}^{\otimes n} \to \mathbb{H}_n$ is the multiple Wiener integral. We have the following isometry

$$
\mathbb{E}[|I_n(f_n)|^2] = n!\|\tilde{f}_n\|_{\mathcal{H}^{\otimes n}}^2,
$$

(2.10)

where $\tilde{f}_n$ is the symmetrization of $f_n$.

### 3 On the exponential integrability

In this section, we will show that Hypothesis (I) below is a sufficient and necessary condition such that for all $\lambda, t > 0$

$$
\mathbb{E}\left[\exp \left(\lambda \int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X_r - X_s)drds\right)\right] < \infty.
$$

**Hypothesis (I).** The spectral measure $\mu$ satisfies

$$
\int_{\mathbb{R}^d} \frac{1}{1 + (\Psi(\xi))^{1-\beta_0}} \mu(d\xi) < \infty.
$$

**Remark 3.1.** When $\mathcal{L} = -(-\Delta)^{\alpha/2}$ for $\alpha \in (0, 2]$ and $\gamma(x)$ is of one of the forms $\prod_{j=1}^d |x_j|^{\beta_j}$, $|x|^{-\beta}$ and $\delta_0(x)$, Hypothesis (I) then coincides with the conditions in [29, 11].

First, we prove that Hypothesis (I) is a necessary condition.

**Proposition 3.2.**

$$
\mathbb{E}\int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X_r - X_s)drds < \infty, \text{ for all } t > 0
$$

if and only if $\mu$ satisfies Hypothesis (I).
Proof. By Lemma 2.6,
\[
\int_0^t \int_0^t |r - s|^{-\beta_0} \mathbb{E} \left[ \gamma(X_r - X_s) \right] dr ds = \frac{1}{(2\pi)^d} \int_0^t \int_0^t |r - s|^{-\beta_0} \int_{\mathbb{R}^d} e^{-|r-s|^2 / 2} \mu(d\xi) dr ds,
\]
and the result follows from Fubini’s theorem and Lemma 3.7. 

The following theorem is the main result in this section.

**Theorem 3.3.** Let the measure \( \mu \) satisfy Hypothesis (I), then for all \( t, \lambda > 0 \),
\[
\mathbb{E} \left[ \exp \left( \lambda \int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X_r - X_s) ds dr \right) \right] < \infty.
\]

**Remark 3.4.** The above theorem, together with Proposition 3.2, actually declares the equivalence between the integrability and the exponential integrability of \( \int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X_r - X_s) ds dr \). This surprising result is mainly a consequence of the Markovian property of the Lévy process \( X \). A result in the same flavor for \( X = \mathcal{B} \) where \( \mathcal{B} \) is a standard Brownian motion and \( f \) is a positive measurable function has been discovered by Khas’minskii [26] (see, e.g., [38, Lemma 2.1]).

Proof. Note that \( \int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X_r - X_s) ds dr = 2 \int_0^t \int_0^r |r - s|^{-\beta_0} \gamma(X_r - X_s) ds dr \), and equivalently we will study the exponential integrability of \( \int_0^t \int_0^r |r - s|^{-\beta_0} \gamma(X_r - X_s) ds dr \).

Inspired by the method in the proof of [29, Theorem 1], we estimate the \( n \)-th moments as follows.
\[
\mathbb{E} \left( \int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X_r - X_s) ds dr \right)^n = \int_{[0 < s < r < t]^n} \mathbb{E} \left( \prod_{j=1}^n |r_j - s_j|^{-\beta_0} \gamma(X_{r_j} - X_{s_j}) \right) ds dr
\]
\[
= n! \int_{[0 < s < r < t]^n \cap [0 < r_1 < r_2 \ldots < r_n < t]} \mathbb{E} \left( \prod_{j=1}^n |r_j - s_j|^{-\beta_0} \gamma(X_{r_j} - X_{s_j}) \right) ds dr
\]
\[
\leq n! \int_{[0 < s < r < t]^n \cap [0 < r_1 < r_2 \ldots < r_n < t]} \prod_{j=1}^n |r_j - \eta_j|^{-\beta_0} \mathbb{E} \left[ \gamma(X_{r_j} - X_{\eta_j}) \right] ds dr.
\]

The last inequality, where \( \eta_j \) is the point in the set \( \{ r_{j-1}, s_j, s_{j+1}, \ldots, s_n \} \) which is closest to \( r_j \) from the left, holds since \( \mathbb{E} \left[ \gamma(X_{r_j} - X_{s_j}) \right] = \mathbb{E} \left[ \gamma(X_{r_j} - X_{\eta_j} + X_{\eta_j} - X_{s_j}) \right] \leq \mathbb{E} \left[ \gamma(X_{r_j} - X_{\eta_j}) \right] \) by the independent increment property of \( X \) and Lemma 3.9. Note that \( ds dr \) actually means \( ds_1 \ldots ds_n dr_1 \ldots dr_n \) in the above last three integrals. Throughout the article, we will take this kind of abuse of the notation for simpler exposition.

Fix the points \( r_1 < \cdots < r_n \), we can decompose the set \( [0 < s < r < t]^n \cap [0 < r_1 < r_2 \cdots < r_n < t] \) into \( (2n - 1)!! \) disjoint subsets depending on which interval the \( s_i \)'s are placed in. More precisely, \( s_1 \) must be in \( (0, r_1) \), while \( s_2 \) could be in \( (0, s_1) \), \( (s_1, r_1) \) or \( (r_1, r_2) \). Similarly, there are \( (2j - 1) \) choices to place \( s_j \). Over each subset, we denote the integral by
where \( \sigma(1) < \cdots < \sigma(n) \) are \( n \) distinct elements in the set \( \{2, 3, \ldots, 2n\} \). Hence

\[
\mathbb{E}\left[\left(\int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X_r - X_s) ds dr \right)^n\right] \leq n! \times \text{sum of the } (2n-1)!! \text{ terms of } I'_\sigma.
\]

(3.1)

Next, for fixed \( n \), we will provide a uniform upper bound for all \( I'_\sigma \)'s. Noting that \( X_{\sigma(j)} - X_{\sigma(j)-1} \) and letting \( y_j = z_j - z_{j-1} \), we have

\[
I'_\sigma := \int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X_r - X_s) ds dr
\]

where \( \sigma(i) \leq \cdots \leq \sigma(n) \) are \( n \) distinct elements in the set \( \{2, 3, \ldots, 2n\} \). Hence

\[
\mathbb{E}\left[\left(\int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X_r - X_s) ds dr \right)^n\right] \leq n! \times \text{sum of the } (2n-1)!! \text{ terms of } I'_\sigma.
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\[
I'_\sigma = \int_{[0<z_1<\cdots<z_n]} \prod_{j=1}^n |y_{\sigma(j)}|^{-\beta_0} \mathbb{E}[\gamma(X_{y_{\sigma(j)}})] dy
\]

\[
\leq \frac{t^n}{n!} \int_{[0<z_1<\cdots<z_n]} \prod_{j=1}^n |z_j - z_{j-1}|^{-\beta_0} \mathbb{E}[\gamma(X_{z_j - z_{j-1}})] dz
\]

\[
= \frac{t^n}{n!} \int_{[0<z_1<\cdots<z_n]} \prod_{j=1}^n |z_j - z_{j-1}|^{-\beta_0} e^{-(z_j - z_{j-1})\Psi(\xi_j)} \mu(d\xi) dz.
\]

(3.2)

Note that

\[
\int_{[0<z_1<\cdots<z_n]} \prod_{j=1}^n |z_j - z_{j-1}|^{-\beta_0} e^{-(z_j - z_{j-1})\Psi(\xi_j)} \mu(d\xi) ds,
\]

(3.3)

where

\[
\Omega^n_t = \left\{ (s_1, \ldots, s_n) \in [0, \infty)^n : \sum_{j=1}^n s_j \leq t \right\}.
\]

(3.4)

For fixed large \( N \), denote

\[
\varepsilon_N = \int_{[|\xi| \leq N]} \frac{1}{(\Psi(\xi))^{1-\beta_0}} \mu(d\xi), \quad \text{and} \quad m_N = \mu([|\xi| \leq N]).
\]

(3.5)

Thus, by (3.1), (3.2), (3.3) and Proposition 3.5, we have

\[
\mathbb{E}\left[\left(\lambda \int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X_r - X_s) ds dr \right)^n\right] \leq (2n-1)!! \lambda^n t^n \sum_{k=0}^n \binom{n}{k} \left(\frac{\Gamma(1-\beta_0) t^{1-\beta_0}}{\Gamma(k(1-\beta_0) + 1)}\right)^k m_N^{k} [A_0 \varepsilon_N]^{n-k}.
\]

(3.6)

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Now, for fixed $t$ and $\lambda$, we can choose $N$ sufficiently large such that $4A_0\lambda t\varepsilon_N < 1$. Consequently,

$$
\begin{align*}
    \mathbb{E} \left[ \exp \left( \lambda \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) ds dr \right) \right] & \leq \sum_{n=0}^{\infty} \lambda^n t^n \frac{(2n-1)!!}{n!} \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(1-\beta_0) t^{1-\beta_0}}{\Gamma(k(1-\beta_0) + 1)} m_N^k [A_0 \varepsilon_N]^{n-k} \\
    &= \sum_{k=0}^{\infty} \frac{\Gamma(1-\beta_0) t^{1-\beta_0}}{\Gamma(k(1-\beta_0) + 1)} m_N^k [A_0 \varepsilon_N]^{n-k} \sum_{n=0}^{\infty} \lambda^n t^n \frac{(2n-1)!!}{n!} \binom{n}{k} [A_0 \varepsilon_N]^n \\
    &\leq \sum_{k=0}^{\infty} \frac{\Gamma(1-\beta_0) t^{1-\beta_0}}{\Gamma(k(1-\beta_0) + 1)} m_N^k [A_0 \varepsilon_N]^{n-k} \sum_{n=0}^{\infty} [4A_0 \lambda t \varepsilon_N]^n \\
    &= \frac{1}{1 - 4A_0 \lambda t \varepsilon_N} \sum_{k=0}^{\infty} \frac{\Gamma(1-\beta_0) t^{1-\beta_0}}{\Gamma(k(1-\beta_0) + 1)} (4A_0 \lambda t)^k < \infty,
\end{align*}
$$

where in the second inequality we used the estimate $\frac{(2n-1)!!}{n!} \binom{n}{k} \leq 2^n \cdot 2^n = 4^n$. The proof is concluded.

The following proposition, which plays a key role in this article, is a generalized version of Lemma 3.3 in [27].

**Proposition 3.5.** For $\beta_0 \in [0, 1)$, assume

$$
\int_{\mathbb{R}^d} \frac{1}{1 + (\Psi(\xi))^{1-\beta_0}} \mu(d\xi) < \infty.
$$

Then there exists a positive constant $A_0$ depending on $\beta_0$ only such that for all $N > 0$,

$$
\int_{\Omega^n} \int_{\mathbb{R}^d} \prod_{j=1}^{n} r_j^{-\beta_0} e^{-r_j \Psi(\xi)} \mu(d\xi) dr \leq \sum_{k=0}^{n} \binom{n}{k} \frac{\Gamma(1-\beta_0) t^{1-\beta_0}}{\Gamma(k(1-\beta_0) + 1)} m_N^k [A_0 \varepsilon_N]^{n-k},
$$

where $\varepsilon_N$ and $m_N$ are given by (3.5), and $\Omega^n$ is given by (3.4).

**Proof.** The proof essentially follows the approach used in the proof of [27, Lemma 3.3].

First note that the assumption implies that $\lim_{N \to \infty} \varepsilon_N = 0$, and since $\mu(d\xi)$ is a tempered measure, then $m_N < \infty$ for all $N > 0$. For a subset $S$ of $\{1, 2, \ldots, n\}$, we denote its
complement by $S^c$, i.e., $S^c := \{1, 2, \ldots, n\} \setminus S$.

$$
\int_{\mathbb{R}^n} \int_{\Omega^R} \prod_{j=1}^n r_j^{-\beta_0} e^{-r_j \Psi(\xi_j)} dr \mu(d\xi)
= \int_{\mathbb{R}^n} \int_{\Omega^R} \prod_{j=1}^n r_j^{-\beta_0} e^{-r_j \Psi(\xi_j)} [I_{\|\xi_j\| \leq N} + I_{\|\xi_j\| > N}] dr \mu(d\xi)
= \sum_{S \subseteq \{1, 2, \ldots, n\}} \int_{\mathbb{R}^n} \int_{\Omega^R} \prod_{l \in S} r_l^{-\beta_0} e^{-r_l \Psi(\xi_l)} I_{\|\xi_l\| \leq N} \prod_{j \in S^c} r_j^{-\beta_0} e^{-r_j \Psi(\xi_j)} dr \mu(d\xi)
\leq \sum_{S \subseteq \{1, 2, \ldots, n\}} \int_{\mathbb{R}^n} \int_{\Omega^R} \prod_{l \in S} r_l^{-\beta_0} I_{\|\xi_l\| \leq N} \prod_{j \in S^c} r_j^{-\beta_0} e^{-r_j \Psi(\xi_j)} I_{\|\xi_j\| > N} dr \mu(d\xi).
$$

Note that $\Omega_i^R \subseteq \Omega_i^S \times \Omega_i^{S^c}$, where $\Omega_i^I = \{(r_i, i \in I) : r_i \geq 0, \sum_{j \in I} r_i \leq t\}$ for any $I \subseteq \{1, 2, \ldots, n\}$. Therefore,

$$
\int_{\mathbb{R}^n} \int_{\Omega_i^R} \prod_{j=1}^n r_j^{-\beta_0} e^{-r_j \Psi(\xi_j)} dr \mu(d\xi)
\leq \sum_{S \subseteq \{1, 2, \ldots, n\}} \int_{\mathbb{R}^n} \int_{\Omega_i^S \times \Omega_i^{S^c}} \prod_{l \in S} r_l^{-\beta_0} I_{\|\xi_l\| \leq N} \prod_{j \in S^c} r_j^{-\beta_0} e^{-r_j \Psi(\xi_j)} I_{\|\xi_j\| > N} dr \mu(d\xi).
$$

By Lemma 3.11, we have

$$
\int_{\Omega_i^S} \prod_{l \in S} r_l^{-\beta_0} dr = \frac{(\Gamma(1 - \beta_0) t^{1-\beta_0})^{|S|}}{\Gamma(|S| (1 - \beta_0) + 1)}.
$$

On the other hand, there exists $A_0 > 0$ depending on $\beta_0$ only such that

$$
\int_{\Omega_i^S} \prod_{j \in S^c} r_j^{-\beta_0} e^{-r_j \Psi(\xi_j)} dr \leq \int_{[0, t]^{|S^c|}} \prod_{j \in S^c} r_j^{-\beta_0} e^{-r_j \Psi(\xi_j)} dr
\leq \prod_{j \in S^c} \int_0^t r_j^{-\beta_0} e^{-r_j \Psi(\xi_j)} dr \leq \prod_{j \in S^c} A_0(\Psi(\xi_j))^{-1+\beta_0},
$$

where the last equality holds since $\int_0^t r_j^{-\beta_0} e^{-r_j \Psi(\xi_j)} dr = a^{-1+\beta_0} \int_0^t s^{-\beta_0} e^{-s} ds \leq a^{-1+\beta_0} \int_0^\infty s^{-\beta_0} e^{-s} ds$. 
Therefore,
\[
\int_{\mathbb{R}^d} \int_{\Omega^c} \prod_{j=1}^n r_j^{-\beta_0} e^{-r_j \Psi(\xi_j)} \, dr \, d\mu(d\xi)
\]
\[
\leq \sum_{S \subset \{1, 2, \ldots, n\}} \int_{\mathbb{R}^d} \frac{(\Gamma(1 - \beta_0) t^{1-\beta_0})^{\vert S \vert}}{\Gamma(\vert S \vert) (1 - \beta_0) + 1} \prod_{i \in S} I_{[\{1, \ldots, n\} \setminus S]}(\xi_i) \prod_{j \in S^c} A_0(\Psi(\xi_j))^{-1+\beta_0} I_{[\{1, \ldots, n\} \setminus S]}(\xi_j) \mu(d\xi)
\]
\[
= \sum_{S \subset \{1, 2, \ldots, n\}} \frac{(\Gamma(1 - \beta_0) t^{1-\beta_0})^{\vert S \vert}}{\Gamma(\vert S \vert) (1 - \beta_0) + 1} A_0^{\vert S \vert} m_N^{\vert S \vert} C_N^{\vert S \vert}
\]
\[
= \sum_{k=0}^n \binom{n}{k} \frac{(\Gamma(1 - \beta_0) t^{1-\beta_0})^k}{\Gamma(k(1 - \beta_0) + 1)} A_0^{-k} m_N^k C_N^{k},
\]
and the proof is concluded.

\[\blacksquare\]

**Remark 3.6.** If we assume the following stronger condition,
\[
\int_{\mathbb{R}^d} \frac{1}{1 + (\Psi(\xi))^{1-\beta_0 - \varepsilon_0}} \mu(d\xi) < \infty
\]
for some \(\varepsilon_0 \in (0, 1 - \beta_0)\), we may prove that for all \(\lambda, t > 0\)
\[
E \left[ \lambda \exp \left( \int_0^t \int_0^t |r - s|^{-\beta_0 - 2\varepsilon_0} (X_r - X_s) ds dr \right) \right] < \infty, \text{ when } p < \frac{1}{1 - \varepsilon_0}, \tag{3.7}
\]
without involving Proposition 3.5.

Now we estimate the integral over \(\mathbb{R}^d\) in the last term of (3.2) first. By (3.2) and Lemma 3.10, there exists \(C > 0\) depending only on \(1 - \beta_0 - \varepsilon_0\) and \(\mu(d\xi)\), such that
\[
I_\sigma \leq C^n \int_{[0 < z_1 < z_2 \cdots < z_n < t]} \prod_{j=1}^n |z_j - z_{j-1}|^{-\beta_0} \prod_{j=1}^n (1 + (z_j - z_{j-1})^{-1+\beta_0 + \varepsilon_0}) dz.
\]

Denote \(\tau = (\tau_1, \ldots, \tau_n)\) and \(|\tau| = \sum_{j=1}^n \tau_j\). Then
\[
\prod_{j=1}^n [1 + (z_j - z_{j-1})^{-1+\beta_0 + \varepsilon_0}] = \sum_{\tau \in \{0, 1\}^n} \prod_{j=1}^n (z_j - z_{j-1})^{\tau_j (-1+\beta_0 + \varepsilon_0)} = \sum_{\tau \in \{0, 1\}^n} J_\tau = \sum_{m=0}^n \sum_{|\tau|=m} J_\tau.
\]

When \(|\tau| = m\) and \(t \geq 1\), by Lemma 3.11, we have
\[
\int_{[0 < z_1 < z_2 \cdots < z_n < t]} \prod_{j=1}^n |z_j - z_{j-1}|^{-\beta_0} J_\tau dz \leq \frac{C^m m \varepsilon_0 + (n-m)(1-\beta_0)}{\Gamma(m \varepsilon_0 + (n-m)(1-\beta_0))} \leq \frac{C^m n(1-\beta_0)}{\Gamma(n \varepsilon_0 + 1)},
\]
noting that \(\varepsilon_0 < 1 - \beta_0\).
Note that there are \( \binom{n}{m} \) J\_\( x \)'s for \( |x| = m \), and hence
\[
I_x \leq C^n \frac{t^{n(2-\beta_0)}}{n!} \sum_{m=0}^{n} \binom{n}{m} \frac{1}{\Gamma(n\varepsilon_0 + 1)} \leq C^n \frac{t^{n(2-\beta_0)}}{n!} (n+1)^2 \frac{1}{(n\varepsilon_0/3)^{n\varepsilon_0}}, \tag{3.8}
\]
where in the last step we use the properties \( \binom{n}{m} \leq 2^n \) and \( \Gamma(x + 1) \geq (x/3)^x \).

Combining (3.1) and (3.8), we have, for all \( \lambda > 0 \) and \( t > 0 \),
\[
\mathbb{E} \left[ \left( \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s)dsdr \right)^n \right] \leq (Ct^{2-\beta_0})^n (n!)^{1-\varepsilon_0},
\]
where \( C > 0 \) depends on \( \beta_0, \varepsilon_0 \) and \( \mu(d\xi) \), and then (3.7) follows.

**Lemma 3.7.** There exist positive constants \( C_1 \) and \( C_2 \) depending on \( \beta_0 \) only such that
\[
\frac{1}{1 + x^{1-\beta_0}} \int_0^t \int_0^t s^{-\beta_0} e^{-s} ds \leq \int_0^t \int_0^t s^{-\beta_0} e^{-s} ds \leq \frac{1}{1 + x^{1-\beta_0}} (C_1 + C_2 t^{1-\beta_0}), \forall x > 0.
\]
Similarly, there exist positive constants \( D_1 \) and \( D_2 \) depending on \( \beta_0 \) only such that
\[
\frac{2}{1 + x^{1-\beta_0}} \int_0^t \int_0^s r^{-\beta_0} e^{-r} drds \leq \int_0^t \int_0^t |r-s|^{-\beta_0} e^{-|r-s|} dsdr \leq \frac{2}{1 + x^{1-\beta_0}} (D_1 t + D_2 t^{2-\beta_0}), \forall x > 0.
\]

**Proof.** An change of variable implies that
\[
\int_0^t \int_0^t s^{-\beta_0} e^{-s} ds = x^{\beta_0-1} \int_0^x r^{-\beta_0} e^{-r} dr.
\]
The first inequality is a consequence of the following observation. When \( x \geq 1 \),
\[
x^{\beta_0-1} \int_0^t r^{-\beta_0} e^{-r} dr \leq x^{\beta_0-1} \int_0^x r^{-\beta_0} e^{-r} dr \leq x^{\beta_0-1} \int_0^\infty r^{-\beta_0} e^{-r} dr,
\]
and when \( 0 < x < 1 \),
\[
\int_0^t s^{-\beta_0} e^{-s} ds \leq \int_0^t s^{-\beta_0} e^{-s} ds \leq \int_0^t s^{-\beta_0} ds.
\]
The second estimate follows from the first one and the following equality
\[
\int_0^t \int_0^t |r-s|^{-\beta_0} e^{-|r-s|} dsdr = 2 \int_0^t \int_0^t (r-s)^{-\beta_0} e^{-x(r-s)} dsdr = 2 \int_0^t \int_0^r s^{-\beta_0} e^{-xs} dsdr.
\]

**Remark 3.8.** Using similar approach in the above proof, we can show that the two inequalities hold for
\[
\sup_{a \in \mathbb{R}} \int_0^t s^{-\beta_0} e^{-|s+a|} ds \quad \text{and} \quad \sup_{a \in \mathbb{R}} \int_0^t \int_0^t |r-s|^{-\beta_0} e^{-|r-s+a|} drds
\]
as well. It suffices to show that the upper bounds hold. We prove the first one as an illustration. When $0 < x < 1$, 
\[ \int_0^t s^{-\beta_0} e^{-|s+a|^x} ds \leq \int_0^t s^{-\beta_0} ds; \]
when $x \geq 1$, 
\[ \int_0^t s^{-\beta_0} e^{-|s+a|^x} ds \leq x^{\beta_0-1} \int_0^\infty s^{-\beta_0} e^{-|s+ax|^x} ds \leq C x^{\beta_0-1} \]
where 
\[ C = \sup_{a \in \mathbb{R}} \int_0^\infty s^{-\beta_0} e^{-|s+ax|^x} ds \leq \int_0^1 s^{-\beta_0} ds + \sup_{a \in \mathbb{R}} \int_1^\infty e^{-|s+ax|^x} ds \leq \int_0^1 s^{-\beta_0} ds + \int_0^\infty e^{-|s|} ds < \infty, \]
and the upper bound is obtained.

**Lemma 3.9.** $\mathbb{E}[\gamma(X_t + a)] \leq \mathbb{E}[\gamma(X_t)]$, for all $a \in \mathbb{R}^d$.

**Proof.** By Lemma 2.1,
\[ \mathbb{E}[\gamma(X_t + a)] = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi \cdot a} e^{-t\Psi(\xi)} \mu(d\xi) \leq \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t\Psi(\xi)} \mu(d\xi) = \mathbb{E}[\gamma(X_t)]. \]

**Lemma 3.10.** Suppose 
\[ \int_{\mathbb{R}^d} \frac{1}{1 + (\Psi(\xi))^\alpha} \mu(d\xi) < \infty, \]
for some $\alpha > 0$, then there exists a constant $C > 0$ depending on $\mu(d\xi)$ and $\alpha$ only, such that 
\[ \int_{\mathbb{R}^d} e^{-x\Psi(\xi)} \mu(d\xi) \leq C (1 + x^{-\alpha}), \forall x > 0. \]

**Proof.** Since $\lim_{|\xi| \to \infty} \Psi(\xi) = \infty$, we can choose $M > 0$ such that $\Psi(\xi) > 1$ when $|\xi| > M$. Clearly
\[ \int_{\mathbb{R}^d} e^{-x\Psi(\xi)} \mu(d\xi) = \int_{|\xi| \leq M} e^{-x\Psi(\xi)} \mu(d\xi) + \int_{|\xi| > M} e^{-x\Psi(\xi)} \mu(d\xi). \]
The first integral on the right-hand side is bounded by $\mu([|\xi| \leq M])$ which is finite. For the second integral, note that $y^\alpha e^{-y}$ is uniformly bounded for all $y \geq 0$, and hence there exists a constant $C$ depending on $\alpha$ only such that
\[ \int_{|\xi| > M} e^{-x\Psi(\xi)} \mu(d\xi) \leq C \int_{|\xi| > M} x^{-\alpha} (\Psi(\xi))^{-\alpha} \mu(d\xi) \leq x^{-\alpha} \int_{|\xi| > M} \frac{2C}{1 + (\Psi(\xi))^{\alpha}} \mu(d\xi). \]

**Lemma 3.11.** Suppose $\alpha_i \in (-1, 1)$, $i = 1, \ldots, n$ and let $\alpha = \alpha_1 + \cdots + \alpha_n$. Then
\[ \prod_{0 < r_1 < \cdots < r_n < t} \prod_{i=1}^n (r_i - r_{i-1})^{\alpha_i} dr_1 \cdots dr_n = \frac{\prod_{i=1}^n \Gamma(\alpha_i + 1)}{\Gamma(\alpha + n + 1)} t^{\alpha+n}, \]
where $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt$ is the Gamma function.
Proof. The result follows from a direct computation of the iterated integral with respect to \( r_n, r_{n-1}, \ldots, r_1 \) orderly. The properties \( \Gamma(x+1) = x\Gamma(x) \) and \( B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \) are used in the computation, where \( B(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}dt \) for \( x, y > 0 \) is the beta function. \( \blacksquare \)

4 Stratonovich equation

In this section, we will use the approximation method ([28, 30, 11, 27]) to study (1.1) in the Stratonovich sense.

4.1 Definition of \( \int_0^t \int_{\mathbb{R}^d} \delta_0(X_{t-r}^x - y)W(dr, dy) \)

Denote \( g_\varepsilon(t) := \frac{1}{t}I_{[0,\delta]}(t) \) for \( t \geq 0 \) and \( p_\varepsilon(x) = \frac{1}{\varepsilon^d}p(\frac{x}{\varepsilon}) \) for \( x \in \mathbb{R}^d \), where \( p(x) \in C_0^\infty(\mathbb{R}^d) \) is a symmetric probability density function and its Fourier transform \( \hat{p}(\xi) \geq 0 \) for all \( \xi \in \mathbb{R}^d \). We also have that for all \( \varepsilon \in \mathbb{R}^d \), \( \lim_{\varepsilon \to 0} \hat{p}_\varepsilon(\xi) = 1 \).

Let

\[
\Phi_{t,x}^{\varepsilon,\delta}(r, y) := \int_0^t g_\varepsilon(t-s-r)p_\varepsilon(X_s^x - y)ds \cdot I_{[0,\delta]}(r). \quad (4.1)
\]

Formal computations suggest that

\[
\lim_{\varepsilon,\delta \downarrow 0} \int_0^t \int_{\mathbb{R}^d} \Phi_{t,x}^{\varepsilon,\delta}(r, y)W(dr, dy) = \int_0^t \int_{\mathbb{R}^d} \delta_0(X_{t-r}^x - y)W(dr, dy),
\]

where \( \delta_0(x) \) is the Dirac delta function. This formal derivation is validated by the following theorem.

**Theorem 4.1.** Let the measure \( \mu \) satisfy Hypothesis (I), then \( W(\Phi_{t,x}^{\varepsilon,\delta}) \) is well-defined a.s. and forms a Cauchy sequence in \( L^2 \) when \( (\varepsilon, \delta) \to 0 \) with the limit denoted by

\[
W(\delta_0(X_{t-r}^x - \cdot)I_{[0,\delta]}(\cdot)) = \int_0^t \int_{\mathbb{R}^d} \delta_0(X_{t-r}^x - y)W(dr, dy).
\]

Furthermore, \( W(\delta_0(X_{t-r}^x - \cdot)I_{[0,\delta]}(\cdot)) \) is Gaussian distributed conditional on \( X \) with variance

\[
\text{Var} \left[ W(\delta_0(X_{t-r}^x - \cdot)I_{[0,\delta]}(\cdot)) | X \right] = \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s)dsdr. \quad (4.2)
\]

**Proof.** Let \( \varepsilon_1, \delta_1, i = 1, 2 \) be positive numbers, then by (2.1)

\[
\langle \Phi_{t,x}^{\varepsilon_1,\delta_1}, \Phi_{t,x}^{\varepsilon_2,\delta_2} \rangle_{\mathcal{H}} = \int_{[0,\delta_1]} \int_{\mathbb{R}^{2d}} p_{\varepsilon_1}(X_{s_1}^x - y_1)p_{\varepsilon_2}(X_{s_2}^x - y_2)\gamma(y_1 - y_2)g_{\varepsilon_1}(t-s_1-r_1)g_{\varepsilon_2}(t-s_2-r_2)|r_1 - r_2|^{-\beta_0}dy_1dy_2dr_1dr_2ds_1ds_2. \quad (4.3)
\]
Hence
\[ \langle \Phi_{t,x}^{\varepsilon_1,\delta_1}, \Phi_{t,x}^{\varepsilon_2,\delta_2} \rangle_{\mathcal{H}} \geq 0. \]

By [30, Lemma A.3], there exists a positive constant \( C \) depending on \( \beta_0 \) only, such that
\[
\int_{[0,t]^2} g_{\delta_1}(t - s_1 - r_1)g_{\delta_2}(t - s_2 - r_2)|r_1 - r_2|^{-\beta_0} dr_1 dr_2 \leq C|s_1 - s_2|^{-\beta_0}. \tag{4.4}
\]

Therefore,
\[
\langle \Phi_{t,x}^{\varepsilon_1,\delta_1}, \Phi_{t,x}^{\varepsilon_2,\delta_2} \rangle_{\mathcal{H}} \leq C \int_{[0,t]^2} \int_{\mathbb{R}^d} p_{\varepsilon_1}(X_{s_1}^x - y_1)p_{\varepsilon_2}(X_{s_2}^x - y_2)|s_1 - s_2|^{-\beta_0} dy_1 dy_2 ds_1 ds_2
\]
\[
= \frac{C}{(2\pi)^d} \int_{[0,t]^2} \int_{\mathbb{R}^d} \mathcal{F}(p_{\varepsilon_1}(X_{s_1}^x - \cdot))(\xi) \mathcal{F}(p_{\varepsilon_2}(X_{s_2}^x - \cdot))(\xi)|s_1 - s_2|^{-\beta_0} d\mu(\xi)ds_1 ds_2
\]
\[
= \frac{C}{(2\pi)^d} \int_{[0,t]^2} \int_{\mathbb{R}^d} \widehat{p}_{\varepsilon_1}(\xi)\widehat{p}_{\varepsilon_2}(\xi) \exp(-i\xi \cdot (X_{s_1} - X_{s_2}))|s_1 - s_2|^{-\beta_0} d\mu(\xi)ds_1 ds_2
\]
\[
\leq C(\varepsilon_1, \varepsilon_2) \int_{[0,t]^2} |s_1 - s_2|^{-\beta_0} ds_1 ds_2 < \infty. \tag{4.5}
\]

The second equality above holds because \( \mathcal{F}(\phi(\cdot - a))(\xi) = \exp(-ia \cdot \xi)\widehat{\phi}(\xi) \) and that we can apply the Parseval’s identity since \( \int_{\mathbb{R}^d} \widehat{p}_{\varepsilon_1}(\xi)\widehat{p}_{\varepsilon_2}(\xi)d\mu(\xi) \leq \int_{\mathbb{R}^d} \widehat{p}_{\varepsilon_1}(\xi)d\mu(\xi) = \int_{\mathbb{R}^d} p_{\varepsilon_1}(x)\gamma(x)dx < \infty \). Hence, for \( \varepsilon, \delta > 0, \Phi_{t,x}^{\varepsilon,\delta} \in \mathcal{H} \) a.s. and \( W(\Phi_{t,x}^{\varepsilon,\delta}) \) is well-defined a.s.

Now we show that \( W(\Phi_{t,x}^{\varepsilon,\delta}) \) forms a Cauchy sequence in \( L^2 \) when \( (\varepsilon, \delta) \to 0 \), for which it suffices to show that \( \mathbb{E}[\langle \Phi_{t,x}^{\varepsilon_1,\delta_1}, \Phi_{t,x}^{\varepsilon_2,\delta_2} \rangle_{\mathcal{H}}] \) converges as \( (\varepsilon_1, \delta_1) \) and \( (\varepsilon_2, \delta_2) \) tend to zero. By the formula (2.4) for the inner product using Fourier transforms,
\[
\langle \Phi_{t,x}^{\varepsilon_1,\delta_1}, \Phi_{t,x}^{\varepsilon_2,\delta_2} \rangle_{\mathcal{H}} = \frac{1}{(2\pi)^d} \int_{[0,t]^2} \int_{\mathbb{R}^d} \mathcal{F}(p_{\varepsilon_1}(X_{s_1}^x - \cdot))(\xi) \mathcal{F}(p_{\varepsilon_2}(X_{s_2}^x - \cdot))(\xi)
\]
\[
g_{\delta_1}(t - s_1 - r_1)g_{\delta_2}(t - s_2 - r_2)|r_1 - r_2|^{-\beta_0} d\mu(\xi)dr_1 dr_2 ds_1 ds_2
\]
\[
= \frac{1}{(2\pi)^d} \int_{[0,t]^2} \int_{\mathbb{R}^d} \widehat{p}_{\varepsilon_1}(\xi)\widehat{p}_{\varepsilon_2}(\xi) \exp(-i\xi \cdot (X_{s_1} - X_{s_2}))
\]
\[
g_{\delta_1}(t - s_1 - r_1)g_{\delta_2}(t - s_2 - r_2)|r_1 - r_2|^{-\beta_0} d\mu(\xi)dr_1 dr_2 ds_1 ds_2.
\]

By Fubini’s theorem and thanks to (4.4) and Proposition 3.2, we can apply the dominated convergence theorem and get that
\[
\mathbb{E}[\langle \Phi_{t,x}^{\varepsilon_1,\delta_1}, \Phi_{t,x}^{\varepsilon_2,\delta_2} \rangle_{\mathcal{H}}] \rightarrow \frac{1}{(2\pi)^d} \int_{[0,t]^2} \int_{\mathbb{R}^d} \mathbb{E} \exp(-i\xi \cdot (X_{s_1} - X_{s_2}))|s_1 - s_2|^{-\beta_0} d\mu(\xi)ds_1 ds_2
\]
\[
= \int_{[0,t]^2} |s_1 - s_2|^{-\beta_0} \mathbb{E} \gamma(X_{s_2} - X_{s_1}) ds_1 ds_2 \tag{4.6}
\]
as \( (\varepsilon_1, \delta_1) \) and \( (\varepsilon_2, \delta_2) \) go to zero.
Finally, conditional on $X$, $W(\Phi^{d,\delta})$ is Gaussian and hence the limit (in probability) $W(\delta_0(X_{t-}^{x} - \cdot))$ is also Gaussian. To show the formula (4.2) for conditional variance, it suffices to show that
\[
\langle \Phi^{d,\delta}, \Phi^{d,\delta} \rangle_{L^2} \rightarrow \int_{[0,t]^2} |s_1 - s_2|^{-\beta_0 \gamma(X_{s_2} - X_{s_1})} ds_1 ds_2
\]
in $L^1(\Omega)$ as $(\varepsilon, \delta) \rightarrow 0$. Noting that, by Lemma 4.2, the inside integral in (4.3)
\[
\int_{[0,t]^2} \int_{\mathbb{R}^d} p_{\varepsilon}(X_{s_1}^{x} - y_1)p_{\varepsilon}(X_{s_2}^{x} - y_2) \gamma(y_1 - y_2)
\]
converges to $|s_1 - s_2|^{-\beta_0 \gamma(X_{s_1} - X_{s_2})}$ a.s. as $(\varepsilon, \delta)$ goes to zero, because of (4.6) we can apply Scheffé’s lemma to get that the convergence is also in $L^1(\Omega \times [0, t]^2, P \times m)$ where $m$ is the Lebesgue measure on $[0, t]^2$. Consequently it follows that the convergence (4.7) holds in $L^1(\Omega)$.

**Lemma 4.2.** When $a - b \neq 0$,
\[
\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}^d} p_{\varepsilon}(a - y_1)p_{\varepsilon}(b - y_2) \gamma(y_1 - y_2) dy_1 dy_2 = \gamma(a - b).
\]

**Proof.** The change of variables $x_1 = y_1 - y_2$, $x_2 = y_2$ implies that $\int_{\mathbb{R}^d} p_{\varepsilon}(a - y_1)p_{\varepsilon}(b - y_2) \gamma(y_1 - y_2) dy_1 dy_2 = \int_{\mathbb{R}^d} p_{\varepsilon}(a - x_1 - x_2)p_{\varepsilon}(b - x_2) \gamma(x_1) dx_1 dx_2 = \int_{\mathbb{R}^d} (p_{\varepsilon} * p_{\varepsilon})(a - b - x_1) \gamma(x_1) dx_1 = \int_{\mathbb{R}^d} \frac{1}{\varepsilon} (p * \frac{a - b - x}{\varepsilon}) \gamma(x) dx$. Since the convolution $p * p$ is also a smooth probability density function with compact support, it suffices to prove the following result.

**Lemma 4.3.** Let $f_{\varepsilon}(x) = \frac{1}{\varepsilon^d} f(\frac{x}{\varepsilon})$, where $f \in C_0^\infty(\mathbb{R}^d)$ is a symmetric probability density function. Then we have
\[
\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} f_{\varepsilon}(a - x) \gamma(x) dx = \gamma(a), \ \forall a \neq 0.
\]

**Proof.** Suppose that the support of the function $f$ is inside $[-M, M]$. Let the positive number $\varepsilon$ be sufficiently small such that $\gamma(x)$ is continuous on $[a - M\varepsilon, a + M\varepsilon]$. By the mean value theorem, we have
\[
\int_{\mathbb{R}} f_{\varepsilon}(a - x) \gamma(x) dx = \int_{[a - M\varepsilon, a + M\varepsilon]} f_{\varepsilon}(a - x) \gamma(x) dx
\]
\[
= \gamma(a_{\varepsilon}) \int_{[a - M\varepsilon, a + M\varepsilon]} f_{\varepsilon}(a - x) dx = \gamma(a_{\varepsilon}),
\]
where $a_{\varepsilon} \in [a - M\varepsilon, a + M\varepsilon]$. The result follows if we let $\varepsilon$ go to zero.
4.2 Feynman-Kac formula

For positive numbers $\varepsilon$ and $\delta$, define

$$\dot{W}^{\varepsilon, \delta}(t, x) := \int_0^t \int_{\mathbb{R}^d} g_\delta(t-s)p_\varepsilon(x-y)W(ds, dy) = W(\phi^{\varepsilon, \delta}_{t, x}),$$

(4.8)

where

$$\phi^{\varepsilon, \delta}_{t, x}(s, y) = g_\delta(t-s)p_\varepsilon(x-y) \cdot I_{[0, t]}(s).$$

Then $\dot{W}^{\varepsilon, \delta}(t, x)$ exists in the classical sense and it is an approximation of $\dot{W}(t, x)$. Taking advantage of $\dot{W}^{\varepsilon, \delta}(t, x)$, we can define the integral $\int_0^T \int_{\mathbb{R}^d} v(t, x)W(dt, dx)$ in the Stratonovich sense as follows.

Definition 4.4. Suppose that $v = \{v(t, x), t \geq 0, x \in \mathbb{R}^d\}$ is a random field satisfying

$$\int_0^T \int_{\mathbb{R}^d} |v(t, x)| dx dt < \infty, \text{ a.s.,}$$

and that the limit in probability $\lim_{\varepsilon, \delta \downarrow 0} \int_0^T \int_{\mathbb{R}^d} v(t, x)\dot{W}^{\varepsilon, \delta}(t, x)dx dt$ exists. The we denote the limit by

$$\int_0^T \int_{\mathbb{R}^d} v(t, x)W(dt, dx) := \lim_{\varepsilon, \delta \downarrow 0} \int_0^T \int_{\mathbb{R}^d} v(t, x)\dot{W}^{\varepsilon, \delta}(t, x)dx dt.$$

and call it Stratonovich integral.

Let $\mathcal{F}_t$ be the $\sigma$-algebra generated by $\{W(s, x), 0 \leq s \leq t, x \in \mathbb{R}^d\}$, and we say that a random field $\{F(t, x), t \geq 0, x \in \mathbb{R}^d\}$ is adapted if $\{F(t, x), t \geq 0\}$ is adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ for all $x \in \mathbb{R}^d$. Denote the convolution between the function $q_t$ and $f$ by $Q_t f$, i.e.,

$$Q_t f(x) := \int_{\mathbb{R}^d} q_t(x-y)f(y)dy.$$

A mild solution to (1.1) in the Stratonovich sense is defined as follows.

Definition 4.5. An adapted random field $u = \{u(t, x), t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution to (1.1) with initial condition $u_0 \in C_b(\mathbb{R}^d)$, if for all $t \geq 0$ and $x \in \mathbb{R}^d$ the following integral equation holds

$$u(t, x) = Q_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y)u(s, y)W(ds, dy),$$

(4.9)

where the stochastic integral is in the Stratonovich sense of Definition 4.4.

The following theorem is the main result in this section.
Theorem 4.6. Let the measure $\mu$ satisfy Hypothesis (I). Then

$$u(t, x) = \mathbb{E}^X \left[ u_0(X_t^x) \exp \left( \int_0^t \int_{\mathbb{R}^d} \delta_0(X_{t-r}^x - y) W(dr, dy) \right) \right]$$

(4.10)

is well-defined and it is a mild solution to (1.1) in the Stratonovich sense.

**Proof.** Consider the following approximation of (1.1)

$$\begin{cases}
  u^{\varepsilon, \delta}(t, x) = \mathcal{L}u^{\varepsilon, \delta}(t, x) + u^{\varepsilon, \delta}(t, x) \dot{W}^{\varepsilon, \delta}(t, x), \\
  u^{\varepsilon, \delta}(0, x) = u_0(x).
\end{cases}$$

(4.11)

By the classical Feynman-Kac formula,

$$u^{\varepsilon, \delta}(t, x) = \mathbb{E}^X \left[ u_0(X_t^x) \exp \left( \int_0^t \dot{W}^{\varepsilon, \delta}(r, X_{t-r}^x) dr \right) \right] = \mathbb{E}^X \left[ u_0(X_t^x) \exp \left( W(\Phi_{t,x}^{\varepsilon, \delta}) \right) \right]$$

where $\Phi_{t,x}^{\varepsilon, \delta}$ is defined in (4.1) and the last equality follows from the stochastic Fubini’s theorem, is a mild solution to (4.11), i.e.,

$$u^{\varepsilon, \delta}(t, x) = Q_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y) u^{\varepsilon, \delta}(s, y) \dot{W}^{\varepsilon, \delta}(s, y) ds dy.$$  

(4.12)

To prove the result, it suffices to show that as $(\varepsilon, \delta)$ tends to zero, both sides of (4.12) converge respectively in probability to those of (4.9) with $u(t, x)$ given in (4.10). We split the proof in two steps for easier interpretation.

**Step 1.** First, we show that $u^{\varepsilon, \delta}(t, x) \to u(t, x)$ in $L^p$ for all $p > 1$. By Theorem 4.1, as $(\varepsilon, \delta) \to 0$, $W(\Phi_{t,x}^{\varepsilon, \delta})$ converges to $W(\delta_0(X_{t-} - \cdot)I_{[0,t]}(\cdot))$ in probability, and hence it suffices to show that

$$\sup_{\varepsilon, \delta > 0} \sup_{t \in [0,T], x \in \mathbb{R}^d} \mathbb{E}[|u^{\varepsilon, \delta}(t, x)|^p] < \infty.$$ 

Note that $W(\Phi_{t,x}^{\varepsilon, \delta})$ is Gaussian conditional on $X$, and hence

$$\mathbb{E} \left[ \exp \left( pW(\Phi_{t,x}^{\varepsilon, \delta}) \right) \right] = \mathbb{E} \left[ \exp \left( \frac{p^2}{2} \|\Phi_{t,x}^{\varepsilon, \delta}\|_H^2 \right) \right].$$

By (2.4) and (4.4), in a similar way of proving (4.5), we can show that there exists a positive constant $C$ depending on $\beta_0$ only such that

$$\|\Phi_{t,x}^{\varepsilon, \delta}\|_H^2 \leq C \int_{[0,t]^2} \int_{\mathbb{R}^d} (\hat{\rho}_{\varepsilon}(\xi))^2 \exp(-i\xi \cdot (X_r - X_s)) |r-s|^{-\beta_0} \mu(d\xi) dr ds.$$
Therefore,

\[ E[\|\Phi_{t, x}^{\varepsilon, \delta}\|_{H}^{2n}] \leq C^n \int_{[0, t]^{2n}} \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} (\tilde{p}_{\varepsilon}(\xi_j))^2 E \exp(-i \sum_{j=1}^{n} \xi_j \cdot (X_{r_j} - X_{s_j})) \]

\[ \prod_{j=1}^{n} |r_j - s_j|^{-\beta_0} \prod_{j=1}^{n} \mu(d\xi_j)dr_jds_j \]

\[ \leq C^n \int_{[0, t]^{2n}} \int_{\mathbb{R}^{nd}} E \exp(-i \sum_{j=1}^{n} \xi_j \cdot (X_{r_j} - X_{s_j})) \prod_{j=1}^{n} |r_j - s_j|^{-\beta_0} \prod_{j=1}^{n} \mu(d\xi_j)dr_jds_j \]

\[ = E \left[ \left( C \int_{0}^{t} \int_{0}^{t} |r - s|^{-\beta_0} \gamma(X_r - X_s)drds \right)^{n} \right]. \]

The second inequality above holds because \( \sup_{\xi \in \mathbb{R}^d} \tilde{p}_{\varepsilon}(\xi) \leq 1 \) and \( E \exp(-i \sum_{j=1}^{n} \xi_j \cdot (X_{r_j} - X_{s_j})) \) is a positive real number. Thus there is constant \( C > 0 \) depending on \( \beta_0 \) only such that

\[ \sup_{\varepsilon, \delta > 0} \sup_{t \in [0, T], x \in \mathbb{R}^d} E \left[ \exp\left( \frac{p^2}{2} \|\Phi_{t, x}^{\varepsilon, \delta}\|_{H}^{2} \right) \right] \leq E \left[ \exp\left( C \frac{p^2}{2} \int_{0}^{t} \int_{0}^{t} |r - s|^{-\beta_0} \gamma(X_r - X_s)drds \right) \right], \]

where the term on the right-hand side is finite by Theorem 3.3.

**Step 2.** Now by Definition 4.4, it suffices to show that

\[ I^{\varepsilon, \delta} := \int_{0}^{t} \int_{\mathbb{R}^d} q_{t-s}(x-y)(u^{\varepsilon, \delta}(s, y) - u(s, y))\hat{W}^{\varepsilon, \delta}(s, y)dsdy \]

converges in \( L^2 \) to zero. Denoting \( v_{s, y}^{\varepsilon, \delta} = u^{\varepsilon, \delta}(s, y) - u(s, y) \) and noting that \( \hat{W}^{\varepsilon, \delta}(s, y) = W(\phi_{s, y}^{\varepsilon, \delta}) \) we have

\[ E[(I^{\varepsilon, \delta})^2] = \int_{[0, t]^{2}} \int_{\mathbb{R}^{2d}} q_{t-s_1}(x-y_1)q_{t-s_2}(x-y_2) E\left[v_{s_1, y_1}^{\varepsilon, \delta} v_{s_2, y_2}^{\varepsilon, \delta} W(\phi_{s_1, y_1}^{\varepsilon, \delta}) W(\phi_{s_2, y_2}^{\varepsilon, \delta})\right] dy_1dy_2ds_1ds_2. \]

Use the following notations \( V_{t, x}^{\varepsilon, \delta}(X) = \int_{0}^{t} \hat{W}^{\varepsilon, \delta}(r, X_{t-r})dr = W(\Phi_{t, x}^{\varepsilon, \delta}(X)) \), \( V_{t, x}(X) = \int_{0}^{t} \int_{\mathbb{R}^d} \delta_0(X_{t-r} - y)W(dr, dy) = W(\delta_0(X_{t-r} - \cdot)|_{[0, t]}(\cdot)) \), and

\[ A^{\varepsilon, \delta}(s_1, y_1, s_2, y_2) = \prod_{j=1}^{2} u_0(X_{s_j}^j + y_j) \left[ \exp\left(V_{s_j, y_j}^{\varepsilon, \delta}(X_j^j)\right) - \exp\left(V_{s_j, y_j}^{\varepsilon, \delta}(X_j^j)\right) \right]. \]

where \( X^1 \) and \( X^2 \) are two independent copies of \( X \). Then

\[ E\left[v_{s_1, y_1}^{\varepsilon, \delta} v_{s_2, y_2}^{\varepsilon, \delta} W(\phi_{s_1, y_1}^{\varepsilon, \delta}) W(\phi_{s_2, y_2}^{\varepsilon, \delta})\right] = E[X^1, X^2] E^W\left[A^{\varepsilon, \delta}(s_1, y_1, s_2, y_2) W(\phi_{s_1, y_1}^{\varepsilon, \delta}) W(\phi_{s_2, y_2}^{\varepsilon, \delta})\right]. \]

By the integration by parts formula (2.9),

\[ E^W\left[A^{\varepsilon, \delta}(s_1, y_1, s_2, y_2) W(\phi_{s_1, y_1}^{\varepsilon, \delta}) W(\phi_{s_2, y_2}^{\varepsilon, \delta})\right] = E^W\left[D^{2} A^{\varepsilon, \delta}(s_1, y_1, s_2, y_2), \phi_{s_1, y_1}^{\varepsilon, \delta} \otimes \phi_{s_2, y_2}^{\varepsilon, \delta}\right]_{\mathcal{H}^\otimes 2} + E^W\left[A^{\varepsilon, \delta}(s_1, y_1, s_2, y_2), \phi_{s_1, y_1}^{\varepsilon, \delta} \otimes \phi_{s_2, y_2}^{\varepsilon, \delta}\right]_{\mathcal{H}^\otimes 2}, \]
and hence we have

$$\mathbb{E} \left[ v_{s_1,y_1}^\varepsilon v_{s_2,y_2}^\varepsilon W(\phi_{s_1,y_1}^\varepsilon, \phi_{s_2,y_2}^\varepsilon) W(\phi_{s_3,y_3}^\varepsilon) \right] = \mathbb{E} [ A^\varepsilon(s_1,y_1,s_2,y_2) B^\varepsilon(s_1,y_1,s_2,y_2)] + \mathbb{E} [ v_{s_1,y_1}^\varepsilon v_{s_2,y_2}^\varepsilon \phi_{s_1,y_1}^\varepsilon \phi_{s_2,y_2}^\varepsilon \phi_{s_3,y_3}^\varepsilon ] \mathcal{H},$$

where

$$B^\varepsilon(s_1,y_1,s_2,y_2) = \sum_{j,k=1}^2 \langle \phi_{s_1,y_1}^\varepsilon, \phi_{s_2,y_2}^\varepsilon \rangle (X^j) - \delta(X_{s_j}^j + y_j - \cdot) I_{[0,s_j]}(\cdot) \langle \phi_{s_2,y_2}^\varepsilon, \phi_{s_k,y_k}^\varepsilon \rangle (X^k) - \delta(X_{s_k}^k + y_k - \cdot) I_{[0,s_k]}(\cdot) \mathcal{H}.$$

Therefore,

$$\mathbb{E} [(J^\varepsilon)^2] = J_1^\varepsilon + J_2^\varepsilon,$$

with the notations

$$J_1^\varepsilon = \int [0,T]^2 \int_{\mathbb{R}^2} q_{t_1} (x-y_1) q_{t_2} (x-y_2) \mathbb{E} [ A^\varepsilon(s_1,y_1,s_2,y_2) B^\varepsilon(s_1,y_1,s_2,y_2) ] dy_1 dy_2 ds_1 ds_2$$

and

$$J_2^\varepsilon = \int [0,T]^2 \int_{\mathbb{R}^2} q_{t_1} (x-y_1) q_{t_2} (x-y_2) \mathbb{E} [ v_{s_1,y_1}^\varepsilon v_{s_2,y_2}^\varepsilon \phi_{s_1,y_1}^\varepsilon \phi_{s_2,y_2}^\varepsilon ] \mathcal{H} dy_1 dy_2 ds_1 ds_2.$$

Now the problem is reduced to show that both $J_1^\varepsilon$ and $J_2^\varepsilon$ converge to zero as $(\varepsilon, \delta) \to 0$.

By the result in **Step 1**, we have

$$\lim_{\varepsilon,\delta \to 0} \mathbb{E} [(v_{s,y}^\varepsilon)^2] = 0,$$

and similar arguments imply that

$$\lim_{\varepsilon,\delta \to 0} \mathbb{E} [(A_{s,y}^\varepsilon)^2] = 0,$$

for all $(s,y) \in [0,T] \times \mathbb{R}^d$. Also note that both sup $\sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \mathbb{E} [(v_{s,y}^\varepsilon)^2]$ and sup $\sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \mathbb{E} [(A_{s,y}^\varepsilon)^2]$ are finite.

First we can prove $\lim_{\varepsilon,\delta \to 0} J_1^\varepsilon = 0$ by the dominated convergence theorem, noting that Lemma 4.8 implies

$$\sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \sup_{(s_1,y_1) \in [0,T] \times \mathbb{R}^d} \sup_{(s_2,y_2) \in [0,T] \times \mathbb{R}^d} \mathbb{E} [(B^\varepsilon(s_1,y_1,s_2,y_2))^2] < \infty.$$
Now we show $\lim_{\varepsilon, \delta \downarrow 0} J_{2}^{\varepsilon, \delta} = 0$. By (2.4) and (4.4), we have

$$\langle \phi_{s, y_{1}}, \phi_{s, y_{2}} \rangle_{\mathcal{H}} \leq C|s_{1} - s_{2}|^{-\beta_0} \int_{\mathbb{R}^{d}} \exp \left( -i\xi \cdot (y_{1} - y_{2}) \right) (\hat{p}_{\varepsilon}(\xi))^{2} \mu(d\xi),$$

therefore,

$$J_{2}^{\varepsilon, \delta} \leq C \int_{0}^{t} \int_{0}^{t} \int_{\mathbb{R}^{2d}} q_{t-s_{1}}(x-y_{1}) q_{t-s_{2}}(x-y_{2}) K^{\varepsilon, \delta}(s_{1}, y_{1}, s_{2}, y_{2}) |s_{1} - s_{2}|^{-\beta_0} dy_{1} dy_{2} ds_{1} ds_{2}$$

where

$$K^{\varepsilon, \delta}(s_{1}, y_{1}, s_{2}, y_{2}) := \left( \mathbb{E}(v_{s_{1}, y_{1}}^{\varepsilon, \delta}) \right)^{1/2} \left( \mathbb{E}(v_{s_{2}, y_{2}}^{\varepsilon, \delta}) \right)^{1/2} \int_{\mathbb{R}^{d} p_{\varepsilon}(\xi)} \exp \left( -i\xi \cdot (y_{1} - y_{2}) \right) (\hat{p}_{\varepsilon}(\xi))^{2} \mu(d\xi).$$

Denote

$$L_{s_{1}, s_{2}}^{\varepsilon} := \int_{\mathbb{R}^{d}} \exp \left( -i\xi \cdot (y_{1} - y_{2}) \right) (\hat{p}_{\varepsilon}(\xi))^{2} \mu(d\xi).$$

Hence

$$K^{\varepsilon, \delta}(s_{1}, y_{1}, s_{2}, y_{2}) \leq C L_{s_{1}, s_{2}}^{\varepsilon}.$$

For the integral of $L_{s_{1}, s_{2}}^{\varepsilon}$, we have

$$\int_{0}^{t} \int_{0}^{t} \int_{\mathbb{R}^{2d}} q_{t-s_{1}}(x-y_{1}) q_{t-s_{2}}(x-y_{2}) L_{s_{1}, s_{2}}^{\varepsilon} |s_{1} - s_{2}|^{-\beta_0} dy_{1} dy_{2} ds_{1} ds_{2}$$

$$= \int_{0}^{t} \int_{0}^{t} \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^{d}} q_{t-s_{1}}(x-y_{1}) q_{t-s_{2}}(x-y_{2}) \exp(-i\xi \cdot (y_{1} - y_{2}))$$

$$\left( \hat{p}_{\varepsilon}(\xi) \right)^{2} |s_{1} - s_{2}|^{-\beta_0} \mu(d\xi) dy_{1} dy_{2} ds_{1} ds_{2}$$

$$= \int_{0}^{t} \int_{0}^{t} \int_{\mathbb{R}^{d}} \exp \left( -(t-s_{1}) \Psi(\xi) \right) \exp \left( -(t-s_{2}) \Psi(\xi) \right) (\hat{p}_{\varepsilon}(\xi))^{2} |s_{1} - s_{2}|^{-\beta_0} \mu(d\xi) ds_{1} ds_{2}$$

$$\xrightarrow{\varepsilon \rightarrow 0} \int_{0}^{t} \int_{0}^{t} \int_{\mathbb{R}^{d}} \exp \left( -(t-s_{1}) \Psi(\xi) \right) \exp \left( -(t-s_{2}) \Psi(\xi) \right) |s_{1} - s_{2}|^{-\beta_0} \mu(d\xi) ds_{1} ds_{2}$$

$$= \int_{0}^{t} \int_{0}^{t} \int_{\mathbb{R}^{2d}} q_{t-s_{1}}(x-y_{1}) q_{t-s_{2}}(x-y_{2}) \gamma(y_{1} - y_{2}) |s_{1} - s_{2}|^{-\beta_0} dy_{1} dy_{2} ds_{1} ds_{2} ds_{2} w, \quad (4.13)$$

where the convergence follows from the dominated convergence theorem, the last equality follows from the formula (2.5), and the last term is finite by Lemma 3.7.

We have shown that $K^{\varepsilon, \delta}(s_{1}, y_{1}, s_{2}, y_{2})$ which converges to zero almost everywhere, is bounded by the sequence $L_{s_{1}, s_{2}}^{\varepsilon}$ which converges to $\gamma(y_{1} - y_{2})$, and thanks to (4.13), we can apply the generalized dominated convergence theorem to get that $\lim_{\varepsilon, \delta \downarrow 0} J_{2}^{\varepsilon, \delta} = 0$. 

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Using Theorem 4.6, by direct computation we can get the following Feynman-Kac type of representation for the moments of the solution to (1.1).

**Theorem 4.7.** Let \( \mu \) satisfy Hypothesis (I), then the solution given by (4.10) has finite moments of all orders. Furthermore, for any positive integer \( p \),

\[
\mathbb{E}[u(t, x)^p] = \mathbb{E} \left[ \prod_{j=1}^{p} u_0(X_t^j + x) \exp \left( \frac{1}{2} \sum_{j,k=1}^{p} \int_{0}^{t} |r - s|^{-\beta_0} \gamma(X_t^j - X_s^k) dr ds \right) \right],
\]

where \( X_1, \ldots, X_p \) are \( p \) independent copies of \( X \).

**Lemma 4.8.** Let the measure \( \mu \) satisfy Hypothesis (I). Then, for any \( n \in \mathbb{N} \),

\[
\sup_{\varepsilon, \delta > 0} \sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \sup_{(r,z) \in [0,T] \times \mathbb{R}^d} \mathbb{E} \left[ \left( \langle \Phi_{s,y}^{\varepsilon,\delta}, \Phi_{r,z}^{\varepsilon,\delta'} (X) \rangle_{\mathcal{H}} \right)^n \right] < \infty,
\]

and

\[
\sup_{\varepsilon, \delta > 0} \sup_{(s,y) \in [0,T] \times \mathbb{R}^d} \sup_{(r,z) \in [0,T] \times \mathbb{R}^d} \mathbb{E} \left[ \left( \langle \Phi_{s,y}^{\varepsilon,\delta}, \delta(X_{r-\cdot} - \cdot) I_{[0,r]}(\cdot) \rangle_{\mathcal{H}} \right)^n \right] < \infty.
\]

**Proof.** First of all, \( \langle \Phi_{s,y}^{\varepsilon,\delta}, \Phi_{r,z}^{\varepsilon,\delta'} (X) \rangle_{\mathcal{H}} \) is a nonnegative real number by (2.1), and by (2.4)

\[
\langle \Phi_{s,y}^{\varepsilon,\delta}, \Phi_{r,z}^{\varepsilon,\delta'} (X) \rangle_{\mathcal{H}} = \int_{0}^{r} \int_{0}^{s} \int_{\mathbb{R}^d} \hat{p}_\varepsilon(\xi) \hat{p}_{\varepsilon'}(\xi) \exp(-i \xi \cdot (X_{r}^z - y)) g_{\delta}(r - \mu) d\mu d\tau d\xi d\nu.
\]

Therefore, denoting \( D = [0, r] \times [0, s] \times [0, r] \), as in the first step of the proof for Theorem 4.6, we have

\[
\mathbb{E} \left[ \left( \langle \Phi_{s,y}^{\varepsilon,\delta}, \Phi_{r,z}^{\varepsilon,\delta'} (X) \rangle_{\mathcal{H}} \right)^n \right] = \int_{D^n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^{n} g_{\delta}(r - \mu_j - \tau_j) g_{\delta}(s - \nu_j) |\mu_j - \nu_j|^{-\beta_0} \prod_{j=1}^{n} \hat{p}_\varepsilon(\xi_j) \hat{p}_{\varepsilon'}(\xi_j) \mathbb{E} \left[ \exp \left( -i \sum_{j=1}^{n} \xi_j \cdot (X_{r_j}^z - y) \right) \right] \mu(\xi) d\sigma d\mu d\nu
\]

\[
\leq C^n \int_{[0,r]^n} \int_{[0,s]^n} \prod_{j=1}^{n} |r - s - \tau_j|^{-\beta_0} \exp \left( -\varepsilon + \varepsilon' \sum_{j=1}^{n} |\xi_j|^2 \right) \mathbb{E} \left[ \exp \left( -i \sum_{j=1}^{n} \xi_j \cdot (X_{r_j}^z - y) \right) \right] \mu(\xi) d\tau
\]

\[
\leq C^n \int_{[0,r]^n} \int_{[0,s]^n} \prod_{j=1}^{n} |r - s - \tau_j|^{-\beta_0} \mathbb{E} \left[ \exp \left( -i \sum_{j=1}^{n} \xi_j \cdot X_{r_j} \right) \right] \mu(\xi) d\tau.
\]
Thus, we have, denoting $\eta_j = \xi_j + \xi_{j+1} + \cdots + \xi_n$,

\[
\mathbb{E} \left[ \left( \tilde{\phi}^{\xi,\delta}_{k,y}, \Phi_{r,z}^{\epsilon,\delta'} (X) \right)_n \right] ^n \\
\leq C^n \int_{[0<\tau_1<\cdots<\tau_n<r]} \prod_{j=1}^n |r - \tau_j|^{-\beta_0} \mathbb{E} \left[ \exp \left( -i \sum_{j=1}^n \xi_j \cdot X_{\tau_j} \right) \right] \mu(d\xi) d\tau \\
= C^n \int_{[0<\tau_1<\cdots<\tau_n<r]} \prod_{j=1}^n |r - \tau_j|^{-\beta_0} \mathbb{E} \left[ \exp \left( -i \sum_{j=1}^n \eta_j \cdot (X_{\tau_j} - X_{\tau_{j-1}}) \right) \right] \mu(d\xi) d\tau \text{ (let } \tau_0 = 0) \\
\leq C^n \int_{[0<\tau_1<\cdots<\tau_n<r]} \prod_{j=1}^n |\tau_j + (s - r)|^{-\beta_0} \exp \left( -\sum_{j=1}^n (\tau_j - \tau_{j-1}) \Psi(\xi_j) \right) \mu(d\xi) d\tau \text{ (by Lemma 4.9)} \\
=: C^n! U_n(r, s)
\]

When $s - r \geq 0$, for all $0 < r \leq s < T$,

\[
U_n(r, s) \leq \int_{[0<\tau_1<\cdots<\tau_n<r]} \prod_{j=1}^n |\tau_j|^{-\beta_0} \exp \left( -\sum_{j=1}^n (\tau_j - \tau_{j-1}) \Psi(\xi_j) \right) \mu(d\xi) d\tau \\
\leq \int_{[0<\tau_1<\cdots<\tau_n<T]} \prod_{j=1}^n |\tau_j - \tau_{j-1}|^{-\beta_0} \exp \left( -\sum_{j=1}^n (\tau_j - \tau_{j-1}) \Psi(\xi_j) \right) \mu(d\xi) d\tau.
\]

By Proposition 3.5, $U_n(r, s)$ is uniformly bounded by a finite number depending on $(T, n, \beta_0)$ and the measure $\mu$ only.

When $r - s > 0$, the set $[0 < \tau_1 < \cdots < \tau_n < r]$ is the union of $A_k^r$ for $k = 0, 1, 2, \ldots, n$
where \( A_k = [0 = \tau_0 < \tau_1 < \cdots < \tau_k < r - s < \tau_{k+1} < \cdots < \tau_n < r] \). On each \( A_k \), we have

\[
\int_{A_k} \int_{\mathbb{R}^d} \prod_{j=1}^n |r - s - \tau_j|^{-\beta_0} \exp \left( -\sum_{j=1}^n (\tau_j - \tau_{j-1})\Psi(\xi_j) \right) \mu(d\xi)d\tau
\]

\[
= \int_{A_k} \int_{\mathbb{R}^d} \prod_{j=1}^k |r - s - \tau_j|^{-\beta_0} \exp \left( -\sum_{j=1}^k (\tau_j - \tau_{j-1})\Psi(\xi_j) \right)
\]

\[
(\tau_{k+1} - (r - s))^{-\beta_0} \exp \left( -(\tau_{k+1} - (r - s) + (r - s) - \tau_k)\Psi(\xi_j) \right)
\]

\[
\prod_{j=k+2}^n (\tau_j - (r - s))^{-\beta_0} \exp \left( -\sum_{j=k+2}^n (\tau_j - \tau_{j-1})\Psi(\xi_j) \right) \mu(d\xi)d\tau
\]

\[
\leq \int_{A_k} \int_{\mathbb{R}^d} \prod_{j=1}^k |r - s - \tau_j|^{-\beta_0} \exp \left( -\sum_{j=1}^k (\tau_j - \tau_{j-1})\Psi(\xi_j) \right)
\]

\[
(\tau_{k+1} - (r - s))^{-\beta_0} \exp \left( -(\tau_{k+1} - (r - s))\Psi(\xi_{k+1}) \right)
\]

\[
\prod_{j=k+2}^n (\tau_j - (r - s))^{-\beta_0} \exp \left( -\sum_{j=k+2}^n (\tau_j - \tau_{j-1})\Psi(\xi_j) \right) \mu(d\xi)d\tau
\]

\[
= M_1(s, r) \times M_2(s, r).
\]

By Lemma 3.10, we have

\[
\sup_{0 < s < r < T} M_1(s, r) \leq C^k \sup_{0 < s < r < T} \int_{0 < \tau_1 < \cdots < \tau_k < r - s} \prod_{j=1}^k (r - s - \tau_j)^{-\beta_0} (1 + (\tau_j - \tau_{j-1})^{-1+\beta_0})d\tau
\]

\[
< \infty.
\]

For \( M_2(s, r) \), let \( \theta_j = \tau_j - (r - s), j = k + 1, \ldots, n \), and assume \( \theta_k = 0 \), then for all \( 0 < s < r < T \),

\[
M_2(s, r) = \int_{0 < \theta_{k+1} < \cdots < \theta_n < s} \int_{\mathbb{R}^{(n-k)d}} \prod_{j=k+1}^{n} \theta_j^{-\beta_0} \exp \left( -\sum_{j=k+1}^{n} (\theta_j - \theta_{j-1})\Psi(\xi_j) \right) \mu(d\xi)d\theta
\]

\[
\leq \int_{0 < \theta_{k+1} < \cdots < \theta_n < T} \int_{\mathbb{R}^{(n-k)d}} \prod_{j=k+1}^{n} (\theta_j - \theta_{j-1})^{-\beta_0} \exp \left( -\sum_{j=k+1}^{n} (\theta_j - \theta_{j-1})\Psi(\xi_j) \right) \mu(d\xi)d\theta,
\]
and the last integral is bounded by a finite number depending on \((n - k, T, \beta_0)\) and \(\mu\) by Proposition 3.5.

Thus we have shown that when \(r - s > 0, \sup_{0 < s < r < T} U_n(r, s) < \infty\), and the first inequality is obtained. Finally, since \(\langle \delta_{s, t}^{\varepsilon', \delta}, \Phi_{s, t}^{\varepsilon, \delta}(X) \rangle_{\mathcal{H}}\) converges to \(\langle \delta_{s, t}^{\varepsilon', \delta}, \Phi_{s, t}^{\varepsilon, \delta}(X) \rangle_{\mathcal{H}}\) in probability as \((\varepsilon', \delta') \to 0\), the second the inequality follows from the first one and Fatou’s lemma.

**Lemma 4.9.** For any \(t > 0\) and \(a \in \mathbb{R}^d\),
\[
\int_{\mathbb{R}^d} \exp(-t\psi(\xi + a))\mu(d\xi) \leq \int_{\mathbb{R}^d} \exp(-t\psi(\xi))\mu(d\xi).
\]

**Proof.** By the formula (2.6), we have
\[
\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(-t\psi(\xi + a))\mu(d\xi) = \int_{\mathbb{R}^d} q_t(x)e^{-ia \cdot x}\gamma(x)dx
\]
\[
\leq \int_{\mathbb{R}^d} q_t(x)\gamma(x)dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \exp(-t\psi(\xi))\mu(d\xi).
\]

4.3 Hölder continuity

**Hypothesis (S1).** The spectral measure \(\mu\) satisfies that for all \(z \in \mathbb{R}^d\), there exist \(\alpha_1 \in (0, 1)\) and \(C > 0\) such that
\[
\int_0^T \int_0^T \int_{\mathbb{R}^d} |r - s|^{-\beta_0} e^{-|r - s|\psi(\xi)} \left(1 - e^{-i\xi \cdot z}\right) \mu(d\xi)drds \leq C|z|^{2\alpha_1}.
\]

**Hypothesis (T1).** The spectral measure \(\mu\) satisfies that for all \(a\) in a bounded subset of \(\mathbb{R}\), there exist \(\alpha_2 \in (0, 1)\) and \(C > 0\) such that
\[
\int_0^T \int_0^T \int_{\mathbb{R}^d} |r - s|^{-\beta_0} \left|\exp\left(-|r - s|\psi(\xi)\right) - \exp\left(-|r - s + a|\psi(\xi)\right)\right| \mu(d\xi)drds \leq C|a|^{\alpha_2}.
\]

**Remark 4.10.** A sufficient condition for Hypothesis (S1) to hold is the following
\[
\int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha_1}}{1 + (\psi(\xi))^{1 - \beta_0}}\mu(d\xi) < \infty \tag{4.15}
\]
due to Lemma 3.7 and the fact that \(1 - \cos x \leq |x|^{2\alpha_1}\). Note that \(\alpha_1 < 1 - \beta_0\) is a necessary condition for (4.15) to hold. This is because \(\mu(A) < \infty\) for any bounded set \(A \subset \mathbb{R}^d\), \(\mu(\mathbb{R}^d) = \gamma(0) = \infty\), \(\lim_{\xi \to \infty} \psi(\xi) = \infty\) and \(\limsup_{\|\xi\| \to \infty} \frac{\psi(\xi)}{\|\xi\|^2} < \infty\).

Similarly, a sufficient condition for Hypothesis (T1) to be true is that
\[
\int_{\mathbb{R}^d} \frac{(\psi(\xi))^{\alpha_2}}{1 + (\psi(\xi))^{1 - \beta_0}}\mu(d\xi) < \infty \tag{4.16}
\]
because of Remark 3.8 and the fact that \(|e^{-x} - e^{-y}| \leq (e^{-x} + e^{-y})|x - y|^\alpha\) for \(x, y \geq 0\) and \(\alpha \in (0, 1]\). Indeed for \(a > 0\), \(e^a - 1 \leq (e^a + 1)(a \wedge 1)\), and hence \(e^a - 1 \leq (e^a + 1)a^\alpha\) for \(\alpha \in (0, 1]\). One necessary condition for (4.16) to hold is \(\alpha_2 < 1 - \beta_0\).
Theorem 4.11. Let \( u_0(x) \equiv 1 \). If the measure \( \mu \) satisfies Hypothesis (S1), then the solution \( u(t, x) \) given by the Feynman-Kac formula (4.10) has a version that is \( \theta_1 \)-Hölder continuous in \( x \) on any compact set of \([0, \infty) \times \mathbb{R}^d\), with \( \theta_1 < \alpha_1 \); Similarly, if \( \mu \) satisfies Hypothesis (T1), the solution \( u(t, x) \) has a version that is \( \theta_2 \)-Hölder continuous in \( t \) on any compact set of \([0, \infty) \times \mathbb{R}^d\), with \( \theta_2 < \alpha_2/2 \).

Proof. Recall that \( V_{t,x} = \int_{\mathbb{R}^d} \delta(X^x_{t,s} - y)W(ds, dy) \). Noting that \(|e^a - e^b| \leq (e^a + e^b)|a - b|\), we have for any \( p > 0 \)

\[
\mathbb{E}^W[|\mathbb{E}^X[\exp(V_{t,x}) - \exp(V_{s,y})]|^p] \leq C\mathbb{E}^W \left[ \left( \mathbb{E}^X[\exp(2V_{t,x}) + \exp(2V_{s,y})] \right)^{p/2} \right]^{1/2}.
\]

By Theorem 3.3, \( \mathbb{E}[\exp(pV_{t,x}) + \exp(pV_{s,y})] \) is bounded on any set of \([0, \infty) \times \mathbb{R}^d\). On the other hand,

\[
(\mathbb{E}^W\left[ (\mathbb{E}^X[(V_{t,x} - V_{s,y})^2])^p \right])^{1/2} \leq \left( \mathbb{E}^X\left[ \mathbb{E}^W[(V_{t,x} - V_{s,y})^2] \right] \right)^{1/2} \leq C_p \mathbb{E}^W[|V_{t,x} - V_{s,y}|^2]^{p/2},
\]

where the first inequality follows from Minkowski’s inequality and the second one holds because of the equivalence between the \( L^p \)-norm and \( L^2 \)-norm of Gaussian random variables. For the spatial estimate, by Hypothesis (S1),

\[
\mathbb{E}^{X} \mathbb{E}^{W}[|V_{t,x} - V_{s,y}|^2] = 2 \int_0^t \int_{\mathbb{R}^d} |r - s|^{-\beta_0} \mathbb{E}^{X} [\gamma(X_r - X_s) - \gamma(X_r - X_s + x - y))] drds
\]

\[
= 2 \int_0^t \int_{\mathbb{R}^d} |r - s|^{-\beta_0} e^{-|r-s|} \mathbb{E}^{W} [\Psi(r)] \mu(dx)drds \leq C|x - y|^{2\alpha_1}.
\]

Therefore

\[
\mathbb{E}^{W}[|\mathbb{E}^{X}[\exp(V_{t,x}) - \exp(V_{s,y})]|^p] \leq C_p |x - y|^{\alpha_1 p},
\]

and the Hölder continuity of \( u(t, x) \) in space follows from Komogorov’s continuity criterion.

Now assume that \( 0 \leq s < t \leq T \), then

\[
\mathbb{E}[|V_{t,x} - V_{s,x}|^2] = \mathbb{E}
\]

\[
\left( \int_0^s \int_{\mathbb{R}^d} \delta_0(X^x_{t-r} - z) - \delta_0(X^x_{s-r} - z) \right) W(dr, dz) + \int_s^t \int_{\mathbb{R}^d} \delta_0(X^x_{t-r} - z) W(dr, dz)
\]

\[
\leq 2(A + B),
\]

where

\[
A = \mathbb{E}\left[ \left( \int_0^s \int_{\mathbb{R}^d} \delta_0(X^x_{t-r} - z) - \delta_0(X^x_{s-r} - z) \right) W(dr, dz) \right]^2,
\]

and

\[
B = \mathbb{E}\left[ \left( \int_s^t \int_{\mathbb{R}^d} \delta_0(X^x_{t-r} - z) W(dr, dz) \right)^2 \right].
\]
For the first term $A$, by Hypothesis (T1), we have

$$A = \mathbb{E} \left[ \int_0^s \int_0^s |s_1 - s_2|^{-\beta_0} \left[ \gamma(X_{t-s_1} - X_{s-s_2}) + \gamma(X_{s-s_1} - X_{s-s_2}) - 2\gamma(X_{t-s_1} - X_{s-s_2}) \right] ds_1 ds_2 \right]$$

$$\leq 2 \int_0^s \int_0^s \int_{\mathbb{R}^d} |s_1 - s_2|^{-\beta_0} \exp \left( -|s_1 - s_2|\Psi(\xi) \right) - \exp \left( -|t - s - s_1 + s_2|\Psi(\xi) \right) \mu(d\xi) ds_1 ds_2$$

$$\leq C |t - s|^{\alpha_2}.$$

For the term $B$, we have

$$B = \int_s^t \int_s^t |s_1 - s_2|^{-\beta_0} \mathbb{E} \gamma(X_{s_1} - X_{s_2}) ds_1 ds_2 = \int_0^{t-s} \int_0^{t-s} |s_1 - s_2|^{-\beta_0} \mathbb{E} \gamma(X_{s_1} - X_{s_2}) ds_1 ds_2$$

$$= \int_{\mathbb{R}^d} \int_0^{t-s} \int_0^{t-s} |s_1 - s_2|^{-\beta_0} \exp \left( -|s_1 - s_2|\Psi(\xi) \right) ds_1 ds_2 \mu(d\xi).$$

By Lemma 3.7, we have that for $(t - s)$ in a bounded domain, there exists a constant $C$ such that

$$\int_0^{t-s} \int_0^{t-s} |s_1 - s_2|^{-\beta_0} \exp \left( -|s_1 - s_2|\Psi(\xi) \right) ds_1 ds_2 \leq C(t - s) \frac{1}{1 + (\Psi(\xi))^{1-\beta_0}}.$$

Hence $B \leq C(t - s)$, and

$$\mathbb{E}^W \left[ \mathbb{E}^X \left[ \exp(V_{t,x}) - \exp(V_{s,x}) \right]^p \right] \leq C \left( \mathbb{E}[(V_{t,x} - V_{s,x})^2] \right)^{p/2} \leq C(A + B)^{p/2} \leq C(t - s)^{p\alpha_2/2}.$$

The Hölder continuity in time is obtained by Kolmogorov’s criterion.

5 Skorohod equation

In this section, we consider (1.1) in the Skorohod sense, i.e., we consider the following SPDE,

$$\begin{cases}
\frac{\partial u}{\partial t} = Lu + u \circ \dot{W}, & t \geq 0, x \in \mathbb{R}^d \\
u(0, x) = u_0(x), & x \in \mathbb{R}^d,
\end{cases}$$

(5.1)

where the symbol $\circ$ means the wick product.

5.1 Existence and uniqueness of the mild solution

In this subsection, we will obtain the existence and uniqueness of the mild solution to (5.1) under the following assumption.
Hypothesis (II). The spectral measure $\mu$ satisfies
\[
\int_{\mathbb{R}^d} \frac{1}{1 + \Psi(\xi)} \mu(d\xi) < \infty.
\]

Remark 5.1. When $\mathcal{L} = -(-\Delta)^{\alpha/2}$ for $\alpha \in (0, 2]$ and $\gamma(x)$ is of one of the forms $\prod_{j=1}^d |x_j|^{\beta_j}$, $|x|^{-\beta}$ and $\delta_0(x)$, Hypothesis (II) is equivalent to $\beta < \alpha$, where $\beta = \sum_{j=1}^d \beta_j$ for the first case and $\beta = 1$ for the third one. It is also a necessary condition for (5.1) to have a unique mild solution ([5]).

Definition 5.2. An adapted random field $u = \{u(t, x), t \geq 0, x \in \mathbb{R}^d\}$ is a mild solution to (5.1) with initial condition $u_0 \in C_1^r(\mathbb{R}^d)$, if for all $t \geq 0$ and $x \in \mathbb{R}^d$, $\mathbb{E}[u^2(t, x)] < \infty$, and the following integral equation holds
\[
u(t, x) = Q_t u_0(x) + \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x - y)u(s, y)W^c(ds, dy),
\]  
(5.2)

where the stochastic integral is in the Skorohod sense.

Suppose that $u = \{u(t, x), t \geq 0, x \in \mathbb{R}^d\}$ is a solution to (5.2), then for fixed $(t, x)$, the square integrable random variable $u(t, x)$ can be expressed uniquely as the Wiener chaos expansion,
\[
u(t, x) = \sum_{n=0}^{\infty} I_n(f_n(\cdot, t, x)),
\]  
(5.3)

where $f_n(\cdot, t, x)$ is symmetric in $\mathcal{H}^{\otimes n}$. On the other hand, if we apply (5.2) repeatedly, as in [28, 29], we can find an explicit representations for $f_n$ with $n \geq 1$
\[
u_n(s_1, x_1, \ldots, s_n, x_n, t, x) = \frac{1}{n!} q_{t-s_{\sigma(n)}}(x - x_{\sigma(n)}) \cdots q_{s_{\sigma(2)} - s_{\sigma(1)}}(x_{\sigma(2)} - x_{\sigma(1)}) Q_{s_{\sigma(1)}} u_0(x_{\sigma(1)}).
\]

Here $\sigma$ denotes the permutation of $\{1, 2, \ldots, n\}$ such that $0 < s_{\sigma(1)} < \cdots < s_{\sigma(n)} < t$. Note that $f_0(t, x) = Q_t u_0(x)$.

Therefore, to obtain the existence and uniqueness of the solution to (5.2), it suffices to prove
\[
\sum_{n=0}^{\infty} n! \|f_n(\cdot, t, x)\|_H^{\otimes n}^2 < \infty, \quad \forall (t, x) \in [0, T] \times \mathbb{R}^d.
\]  
(5.4)

Theorem 5.3. Let the measure $\mu$ satisfy Hypothesis (II). Then (5.4) holds, and consequently, $u(t, x)$ given by (5.3) is the unique mild solution to (5.1).

Proof. Without loss of generality, we assume that $u_0(x) \equiv 1$. Now we have
\[
= n! \int_{[0,t]^{2n}} \int_{\mathbb{R}^{2nd}} h_n(s, y, t, x)h_n(r, z, t, x) \prod_{j=1}^{n} |s_j - r_j|^{-\beta_0} \prod_{j=1}^{n} |y_j - z_j| dydzdr,
\]

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where
\[ h_n(s_1, \ldots, s_n, x_1, \ldots, x_n, t, x) = \frac{1}{n!} q_{t-s_{\sigma(n)}}(x-x_{\sigma(n)}) \cdots q_{s_{\sigma(2)}-s_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)}). \] (5.5)

Then by (2.1) and (2.5),
\[ n!\| f_n(\cdot, t, x) \|^2_{\mathcal{H}^{\otimes n}} \]
\[ \leq n! \int_{[0,t]^n} \int_{\mathbb{R}^d} \mathcal{F} h_n(s, \cdot, t, x)(\xi) \overline{\mathcal{F} h_n(s, \cdot, t, x)(\xi)} \mu(d\xi) \prod_{j=1}^n |s_j - r_j|^{-\beta_0} ds dr, \]
\[ \leq n! \int_{[0,t]^n} A_{t,x}(s) A_{t,x}(r) \prod_{j=1}^n |s_j - r_j|^{-\beta_0} ds dr \]
\[ \leq n! \int_{[0,t]^n} A^2_{t,x}(s) \prod_{j=1}^n |s_j - r_j|^{-\beta_0} ds dr, \] (using \(2ab \leq a^2 + b^2\) and the symmetry of the integral)

where
\[ A_{t,x}(s) = \left( \int_{\mathbb{R}^d} |\mathcal{F} h_n(s, \cdot, t, x)(\xi)|^2 \mu(d\xi) \right)^{1/2}, \] (5.6)

with
\[ \mathcal{F} h_n(s, \cdot, t, x)(\xi) = \frac{1}{n!} e^{-i\xi \cdot (\xi_{\sigma(1)} + \cdots + \xi_{\sigma(n)})} \prod_{j=1}^n \exp \left[ -[s_{\sigma(j+1)} - s_{\sigma(j)}] \Psi(\xi_{\sigma(1)} + \cdots + \xi_{\sigma(j)}) \right], \] (5.7)

where we use the convention \(s_{\sigma(n+1)} = t\).

Note that \( \int_0^t \int_0^t |s - r|^{-\beta_0} ds dr \leq 2 \int_0^t r^{-\beta_0} dr \int_0^t |f(s)| ds \) and let \( D_t = 2 \int_0^t r^{-\beta_0} dr \). Therefore,
\[ n!\| f_n(\cdot, t, x) \|^2_{\mathcal{H}^{\otimes n}} \leq D_t^n n! \int_{[0,t]^n} A^2_{t,x}(s) ds \]
\[ = D_t^n n! \int_{[0,t]^n} \int_{\mathbb{R}^d} |\mathcal{F} h_n(s, \cdot, t, x)(\xi)|^2 \mu(d\xi) ds \]
\[ = D_t^n \frac{1}{n!} \int_{[0,t]^n} \int_{\mathbb{R}^d} \prod_{j=1}^n \exp \left[ -2[s_{\sigma(j+1)} - s_{\sigma(j)}] \Psi(\xi_{\sigma(1)} + \cdots + \xi_{\sigma(j)}) \right] \mu(d\xi) ds \]
\[ \leq D_t^n \frac{1}{n!} \int_{[0,t]^n} \int_{\mathbb{R}^d} \prod_{j=1}^n \exp \left[ -2[s_{\sigma(j+1)} - s_{\sigma(j)}] \Psi(\xi_{\sigma(j)}) \right] \mu(d\xi) ds \] (by Lemma 4.9)
\[ = D_t^n \int_{[0<s_1<\cdots<s_n<t]} \int_{\mathbb{R}^d} \prod_{j=1}^n \exp \left[ -2|s_{j+1} - s_j| \Psi(\xi_j) \right] \mu(d\xi) ds. \]

Similar as in the proof of Theorem 3.3, we can apply Proposition 3.5 with \( \beta_0 = 0 \) for the last integral and then get the following estimate
\[ n!\| f_n(\cdot, t, x) \|^2_{\mathcal{H}^{\otimes n}} \leq D_t^n \sum_{k=0}^n \binom{n}{k} \frac{t^k}{k!} n^k \varepsilon N^n - k, \]
where $\varepsilon_N$ and $m_N$ are given in (3.5) with $\beta_0 = 0$. Hence, if we choose $N$ sufficiently large such that $2D_tA_0\varepsilon_N < 1$, then we have

$$\sum_{n=0}^{\infty} n!\|f_n(\cdot, t, x)\|_{H^n}^2 \leq \sum_{n=0}^{\infty} D_t^n \sum_{k=0}^{n} \binom{n}{k} \frac{t^k}{k!} m_N^n [A_0\varepsilon_N]^{n-k} \leq \sum_{k=0}^{\infty} \frac{t^k}{k!} m_N^n \sum_{n=k}^{\infty} \frac{t^k}{k!} m_N^n D_t^{2n} \frac{m_N^n}{2^n} < \infty.$$  

**Remark 5.4.** Let $\eta(x)$ be a locally integrable function, then as in [27], the result of the above theorem still holds if the temporal kernel $|r - s|^{-\beta_0}$ is replaced by $\eta(r - s)$.

The following theorem provides the Feynman-Kac type of representations for the solution and the moments of the solution when the spectral measure $\mu$ satisfies the stronger condition Hypothesis (I). The proof is similar to the one in [30] and we omit it here.

**Theorem 5.5.** If we assume that $\mu$ satisfy Hypothesis (I), then

$$u(t, x) = \mathbb{E}^{X} \left[ u_0(X_t^x) \exp \left( \int_0^t \int \delta_0(X^x_{r-s}) W(dr, dy) - \frac{1}{2} \int_0^t \int \frac{|r-s|^{-\beta_0}\gamma(X^x_{r}) - X^x_{s}) drds \right) \right], \quad (5.8)$$

is the unique mild solution to (5.1) in the Skorohod sense. Consequently, for any positive integer $p$, we have

$$\mathbb{E}[u(t,x)^p] = \mathbb{E} \left[ \prod_{j=1}^{p} u_0(X_t^x + x) \exp \left( \sum_{1 \leq j < k \leq p} \int_0^t \int \frac{|r-s|^{-\beta_0}\gamma(X^j_r - X^k_s) drds \right) \right], \quad (5.9)$$

where $X_1, \ldots, X_p$ are $p$ independent copies of $X$.

### 5.2 Feynman-Kac formula for the moments of the solution

When the measure $\mu$ satisfies Hypothesis (II) but not Hypothesis (I), the representation (5.8) may be invalid since $\int_0^t \int |r-s|^{-\beta_0}\gamma(X^x_{r} - X^x_{s}) drds$ might be infinite a.s. (see [30] for the case that $X$ is a $d$-dimensional Brownian motion and $\gamma(x) = \prod_{j=1}^{d} |x_j|^{-\beta_j}$, $\beta_j \in (0, 1), j = 1, \ldots, d$). However, the Feynman-Kac formula (5.9) for the moments still holds as stated in the following theorem.

**Theorem 5.6.** Let the measure $\mu$ satisfy Hypothesis (II), then the Feynman-Kac formula (5.9) for the moments of the mild solution to (5.1) holds.

**Proof.** We will adopt the approximation method used in [28, Section 5] to prove the result. The proof is split into three steps for easier reading.
Step 1. Consider the approximation of (5.1),

\[
\begin{align*}
    u^{\varepsilon,\delta}(t, x) &= \mathcal{L}u^{\varepsilon,\delta}(t, x) + u^{\varepsilon,\delta}(t, x) \circ \dot{W}^{\varepsilon,\delta}(t, x), \\
    u^{\varepsilon,\delta}(0, x) &= u_0(x).
\end{align*}
\] (5.10)

Recall that \(\dot{W}^{\varepsilon,\delta}(t, x)\) is defined in (4.8). If \(u^{\varepsilon,\delta}(t, x) \in \mathbb{D}^{1,2}\), then by (2.7)

\[
u^{\varepsilon,\delta}(t, x) \circ \dot{W}^{\varepsilon,\delta}(t, x) = \int_0^t \int_{\mathbb{R}^d} g_\delta(t - s)p_\varepsilon(x - y)u^{\varepsilon,\delta}(t, x)W^\omega(ds, dy).
\]

Therefore, the mild solution to (5.10) is, as defined in [28], an adapted random field \(\{u^{\varepsilon,\delta}(t, x), t \geq 0, x \in \mathbb{R}^d\}\) which is square integrable for all fixed \((t, x)\) and satisfies the following integral equation,

\[
u^{\varepsilon,\delta}(t, x) = Q_tu_0(x) + \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y)u^{\varepsilon,\delta}(s, y) \circ \dot{W}^{\varepsilon,\delta}(s, y)dsdy
\]

\[
= Q_tu_0(x) + \int_0^t \int_{\mathbb{R}^d} \left( \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y)g_\delta(s-r)p_\varepsilon(y-z)u^{\varepsilon,\delta}(s, y)dsdy \right) W^\omega(dr, dz).
\]

Denote

\[
Z_{t,x}^{\varepsilon,\delta}(r, z) = \int_0^t \int_{\mathbb{R}^d} q_{t-s}(x-y)g_\delta(s-r)p_\varepsilon(y-z)u^{\varepsilon,\delta}(s, y)dsdy.
\]

Thus to show that an adapted and square integrable process \(\{u^{\varepsilon,\delta}(t, x), t \geq 0, x \in \mathbb{R}^d\}\) is a mild solution to (5.10), it is equivalent to show \(u^{\varepsilon,\delta}(t, x) = Q_tu_0(x) + \delta(Z_{t,x}^{\varepsilon,\delta})\). Therefore by the definition of the divergence operator \(\delta\), it is equivalent to show that for any \(F \in \mathbb{D}^{1,2}\) with mean zero,

\[
\mathbb{E}[Fu^{\varepsilon,\delta}(t, x)] = \mathbb{E}[(Z_{t,x}^{\varepsilon,\delta}, DF)_H].
\] (5.11)

Let

\[
u^{\varepsilon,\delta}(t, x) = \mathbb{E}^X \left[ u_0(X_t^x) \exp \left( W(\Phi_{t,x}^{\varepsilon,\delta}) - \frac{1}{2}\|\Phi_{t,x}^{\varepsilon,\delta}\|_H^2 \right) \right],
\] (5.12)

where \(\Phi_{t,x}^{\varepsilon,\delta}\) is given by (4.1). Using a similar argument based on the technique of S-transform as in the proof of [28, Proposition 5.2], we can show that \(u^{\varepsilon,\delta}(t, x)\) given by (5.12) satisfies (5.11), and hence it is a mild solution to (5.10).

Step 2. In this step, we will show that

\[
\lim_{\varepsilon,\delta \downarrow 0} \mathbb{E} \left( \left( u^{\varepsilon,\delta}(t, x) \right)^p \right) = \mathbb{E} \left[ \prod_{j=1}^p u_0(X_t^j + x) \exp \left( \sum_{1 \leq j < k \leq p} I_{[0, t]}(r) \left| r - s \right|^{\beta_0} \gamma(X_t^j - X_t^k)drds \right) \right].
\] (5.13)

Without loss of the generality, we assume \(u_0(x) \equiv 1\) from now on. Denote

\[
\Phi_{t,x}^{\varepsilon,\delta}(r, y) := \int_0^t g_\delta(t - s - r)p_\varepsilon(X_t^j + x - y)ds \cdot I_{[0, t]}(r), \quad j = 1, \ldots, p.
\]
The $p$-moment of $u^{\varepsilon, \delta}(t, x)$ is
\[
\mathbb{E}\left[\left(u^{\varepsilon, \delta}(t, x)\right)^p\right] = \mathbb{E}^W \mathbb{E}^X \prod_{j=1}^p \exp\left(W(\Phi_{t,x}^{\varepsilon,\delta, j}) - \frac{1}{2} \|\Phi_{t,x}^{\varepsilon,\delta,j}\|_H^2\right)
\]
\[
= \mathbb{E}^X \exp\left(\frac{1}{2} \sum_{j=1}^p \|\Phi_{t,x}^{\varepsilon,\delta,j}\|_H^2 - \frac{1}{2} \sum_{j=1}^p \|\Phi_{t,x}^{\varepsilon,\delta,j}\|_H^2\right) = \mathbb{E}^X \exp\left(\sum_{1 \leq i < j \leq p} \langle \Phi_{t,x}^{\varepsilon,\delta,j}, \Phi_{t,x}^{\varepsilon,\delta,k}\rangle_H\right).
\]
As in the proof of Theorem 4.1, we can show that
\[
\sup_{\varepsilon > 0} \mathbb{E}\left[\exp\left(\lambda \langle \Phi_{t,x}^{\varepsilon,\delta,j}, \Phi_{t,x}^{\varepsilon,\delta,k}\rangle_H\right)\right] < \infty.
\]
By (2.4) and (4.4), there exists a positive constant $C$ depending on $\beta_0$ only such that
\[
\langle \Phi_{t,x}^{\varepsilon,\delta,j}, \Phi_{t,x}^{\varepsilon,\delta,k}\rangle_H \leq C \int_{[0,t]^2} \int_{\mathbb{R}^d} (\tilde{\rho}_\varepsilon(\xi))^2 \exp(-i\xi \cdot (X_r^j - X_s^k)) |r-s|^{-\beta_0} \mu(d\xi) dr ds.
\]
Hence to obtain (5.14), it is sufficient to prove that for any $\lambda > 0$,
\[
\sup_{\varepsilon > 0} \mathbb{E}\left[\exp\left(\lambda \int_{[0,t]^2} \int_{\mathbb{R}^d} (\tilde{\rho}_\varepsilon(\xi))^2 \exp(-i\xi \cdot (X_r - \tilde{X}_s)) |r-s|^{-\beta_0} \mu(d\xi) dr ds\right)\right] < \infty.
\]
Note that
\[ 
\mathbb{E} \left[ \left( \lambda \int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X_r - \tilde{X}_s)drds \right)^n \right] = \lambda^n \int_{[0,t]^2n} \prod_{j=1}^n |r_j - s_j|^{-\beta_0} \mathbb{E} \left[ \prod_{j=1}^n \gamma(X_{r_j} - \tilde{X}_{s_j}) \right] drds 
\]
\[ = \lambda^n (n!)^2 \int_{[0,t]^2n} h_n(s, y, t, 0) \int_{\mathbb{R}^{2nd}} h_n(r, z, t, 0) \prod_{j=1}^n |s_j - r_j|^{-\beta_0} \prod_{j=1}^n \gamma(y_j - z_j) dydzdrds, \]
where \( h_n \) is given by (5.5), and the last equality is obtained by using the independent increment property of \( X \). Then (5.16) can be obtained as in the proof of Theorem 5.3.

**Step 3.** As in the proof of Theorem 4.6, we can show that \( \sup_{\varepsilon, \delta > 0} \sup_{t \in [0,T], x \in \mathbb{R}^d} \mathbb{E}[|u^{\varepsilon, \delta}(t, x)|^p] < \infty \), \( u^{\varepsilon, \delta}(t, x) \) converges to a limit denoted by \( u(t, x) \) in \( L^p \) for any \( p > 0 \) as \((\varepsilon, \delta)\) goes to zero, and moreover, \( u(t, x) \) satisfies the formula (5.9). Therefore, by the uniqueness of the mild solution to (5.1), to conclude the proof, we only need to show that \( u(t, x) \) is a mild solution to (5.1), i.e.,
\[ \mathbb{E}[Fu(t, x)] = \mathbb{E}[(Z_{t,x}, DF)_\mathcal{H}], \tag{5.17} \]
for any \( F \in \mathbb{D}^{1,2} \) with \( \mathbb{E}[F] = 0 \), where \( Z_{t,x}(r, z) = q_{t-r}(x - z)u(r, z) \).

In a way similar to the proof of Theorem 4.1, we can prove that \( \lim_{\varepsilon, \delta \downarrow 0} \mathbb{E}[\|Z^{\varepsilon, \delta}_{t,x} - Z_{t,x}\|_\mathcal{H}^2] = 0 \). Then we can show the equality (5.17) by letting \((\varepsilon, \delta)\) in (5.11) go to zero, noting that \( F \in \mathbb{D}^{1,2} \) and \( \lim_{\varepsilon, \delta \downarrow 0} u^{\varepsilon, \delta}(t, x) = u(t, x) \) in \( L^2 \).

**Remark 5.7.** In the second step of the proof, actually we proved that under Hypothesis (II), (5.16) holds, i.e., for any \( \lambda > 0 \)
\[ \mathbb{E} \left[ \exp \left( \lambda \int_0^t \int_0^t |r - s|^{-\beta_0} \gamma(X_r - \tilde{X}_s)drds \right) \right] < \infty. \]

### 5.3 Hölder continuity

**Hypothesis (S2).** The spectral measure \( \mu \) satisfies that for all \( a \in \mathbb{R}^d \), there exist \( \alpha_1 \in (0, 1] \) and \( C > 0 \) such that
\[ \sup_{z \in \mathbb{R}^d} \int_T^t \int_{\mathbb{R}^d} e^{-s \Psi(z \xi + z)} (1 - e^{-i(z \xi + z)a}) \mu(d\xi)ds \leq C|a|^{2\alpha_1}. \]

**Hypothesis (T2).** The spectral measure \( \mu \) satisfies, for some \( \alpha_2 \in (0, 1) \),
\[ \int_{\mathbb{R}^d} \left( \frac{\Psi(\xi)}{1 + \Psi(\xi)} \right)^{\alpha_2} \mu(d\xi) < \infty. \]

**Remark 5.8.** Similar to the Stratonovich case, we have the following sufficient condition for Hypothesis (S2) to hold:
\[ \sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|\xi + z|^{2\alpha_1}}{1 + \Psi(\xi + z)} \mu(d\xi) < \infty. \]
Furthermore, if \( \eta(\xi) := \Psi(\xi)/|\xi|^{2\alpha_1} \) is a Lévy characteristic exponent (which is equivalent to say that \(-\eta(\xi)\) is continuous, conditionally positive definite and \(\eta(0) = 0\), see, e.g., [2, Theorem 1.2.17]; a special case in which \(\eta(\xi)\) is the characteristic exponent of a symmetric\) stable process is that \(\Psi(\xi) = |\xi|^\alpha\) with \(\alpha > 2\alpha_1\), and the condition (5.20) below holds, then condition (5.18) is equivalent to

\[
\int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha_1}}{1 + \Psi(\xi)} \mu(d\xi) < \infty. \tag{5.19}
\]

Clearly (5.18) implies (5.19). Now we show that the inverse is true, if we assume that \(\eta(\xi)\) is the characteristic exponent of a certain Lévy process \(\{Y_t, t \geq 0\}\) and that the following condition holds

\[
\sup_{z \in \mathbb{R}^d} \mu([|\xi + z| \leq M]) < \infty, \tag{5.20}
\]

where \(M\) is a positive number such that \(\eta(\xi) \geq 1\) for all \(|\xi| \geq M\).

\[
\int_{\mathbb{R}^d} \frac{|\xi + z|^{2\alpha_1}}{1 + \Psi(\xi + z)} \mu(d\xi) = \int_{[|\xi + z| \leq M]} \frac{|\xi + z|^{2\alpha_1}}{1 + \Psi(\xi + z)} \mu(d\xi) + \int_{[|\xi + z| > M]} \frac{|\xi + z|^{2\alpha_1}}{1 + \Psi(\xi + z)} \mu(d\xi)
\leq M^{2\alpha_1} \sup_{z \in \mathbb{R}^d} \mu([|\xi + z| \leq M]) + 2 \int_{\mathbb{R}^d} \frac{1}{1 + \eta(\xi + z)} \mu(d\xi)
= C + 2 \int_0^\infty e^{-t} \mathbb{E}\left[e^{i(\xi + z)\cdot Y_t}\right] dt \mu(d\xi) = C + 2 \mathbb{E}\left[\int_0^\infty e^{-t} e^{i\xi \cdot Y_t} e^{-i\xi \cdot Y_t} dt \mu(d\xi)\right]
= C + 2 \mathbb{E}\left[\int_0^\infty \gamma(Y_t) e^{-t} e^{i\xi \cdot Y_t} dt\right] \leq C + 2 \mathbb{E}\left[\int_0^\infty \gamma(Y_t) e^{-t} dt\right] = C + 2 \int_{\mathbb{R}^d} \frac{1}{1 + \eta(\xi)} \mu(d\xi) \leq D + 2 \int_{\mathbb{R}^d} \frac{|\xi|^{2\alpha_1}}{1 + \Psi(\xi)} \mu(d\xi),
\]

where \(D\) is another constant that may be different from \(C\).

On the other hand, Hypothesis (T2) actually implies and hence is equivalent to the condition

\[
\sup_{z \in \mathbb{R}^d} \int_{\mathbb{R}^d} \frac{(\Psi(\xi + z))^{\alpha_2}}{1 + \Psi(\xi + z)} \mu(d\xi) < \infty. \tag{5.21}
\]

Note that for all \(z \in \mathbb{R}^d\),

\[
\int_{\mathbb{R}^d} \frac{(\Psi(\xi + z))^{\alpha_2}}{1 + \Psi(\xi + z)} \mu(d\xi) \leq \int_{\mathbb{R}^d} \left(\frac{1}{1 + \Psi(\xi + z)}\right)^{1-\alpha_2} \mu(d\xi)
= \int_{\mathbb{R}^d} \frac{1}{\Gamma(1-\alpha_2)} \int_0^\infty t^{-\alpha_2} e^{-[1+\Psi(\xi+z)]t} dt \mu(d\xi)
= \int_{\mathbb{R}^d} \frac{1}{\Gamma(1-\alpha_2)} \int_0^\infty t^{-\alpha_2} e^{-t} \mathbb{E}[e^{i(\xi + z)\cdot Y_t}] dt \mu(d\xi)
= \frac{1}{\Gamma(1-\alpha_2)} \mathbb{E}\left[\int_0^\infty t^{-\alpha_2} e^{-t} \int_{\mathbb{R}^d} e^{i\xi \cdot Y_t} \mu(d\xi) dt\right]
\leq \frac{1}{\Gamma(1-\alpha_2)} \mathbb{E}\left[\int_0^\infty t^{-\alpha_2} e^{-t} \int_{\mathbb{R}^d} e^{i\xi \cdot Y_t} \mu(d\xi) dt\right] = \int_{\mathbb{R}^d} \left(\frac{1}{1 + \Psi(\xi)}\right)^{1-\alpha_2} \mu(d\xi),
\]

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where the first equality follows from the formula \( c^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-ct} dt \) for \( c > 0 \) and \( \alpha \in (0, 1) \). Finally Hypothesis \((T2)\) implies (5.21) because of the following equivalence

\[
\int_{\mathbb{R}^d} \frac{\langle \Psi(\xi) \rangle^2}{1 + \langle \Psi(\xi) \rangle} \mu(d\xi) \leq \infty \iff \int_{\mathbb{R}^d} \left( \frac{1}{1 + \langle \Psi(\xi) \rangle} \right)^{1-\alpha} \mu(d\xi) \leq \infty
\]

which is due to the facts \( \lim_{|\xi| \to \infty} \Psi(\xi) = \infty \) and \( \mu(A) < \infty \) for bounded \( A \in \mathcal{B}(\mathbb{R}^d) \).

**Theorem 5.9.** Let \( u_0(x) \equiv 1 \) and \( u(t, x) \) be the unique mild solution to (5.1). If \( \mu \) satisfies Hypothesis \((S2)\), then \( u(t, x) \) has a version that is \( \theta_1 \)-Hölder continuous in \( x \) with \( \theta_1 < \alpha_1 \) on any compact set of \([0, \infty) \times \mathbb{R}^d\); Similarly, if \( \mu \) satisfies Hypothesis \((T2)\), the solution \( u(t, x) \) has a version that is \( \theta_2 \)-Hölder continuous in \( t \) with \( \theta_2 < [\alpha_2 \wedge (1 - \beta_0)]/2 \) on any compact set of \([0, \infty) \times \mathbb{R}^d\).

**Proof.** Let \( u(t, x) = 1 + \sum_{n=1}^{\infty} I_n(h_n(\cdot, t, x)) \) and \( u(s, y) = 1 + \sum_{n=1}^{\infty} I_n(h_n(\cdot, s, y)) \), where \( h_n \) is given by (5.5). Then for \( p > 2 \),

\[
\|u(t, x) - u(s, y)\|_{L^p} \leq \sum_{n=1}^{\infty} \|I_n(h_n(\cdot, t, x)) - I_n(h_n(\cdot, s, y))\|_{L^p}
\]

\[
\leq \sum_{n=1}^{\infty} (p-1)^{n/2} \|I_n(h_n(\cdot, t, x)) - I_n(h_n(\cdot, s, y))\|_{L^2}
\]

\[
= \sum_{n=1}^{\infty} (p-1)^{n/2} \sqrt{n!} \|h_n(\cdot, t, x) - h_n(\cdot, s, y)\|_{\mathcal{H} \otimes \mathcal{N}}, \tag{5.22}
\]

where the last inequality holds due to the equivalence of \( L^p \) norms for \( p > 1 \) on any Wiener chaos space \( \mathbb{H}_n \) ([36, Theorem 1.4.1]), and the last equality follows from (2.10).

**Step 1.** First, we study the spatial continuity. Suppose that \( s = t \), similar as in the proof of Theorem 5.3, we have

\[
n! \|h_n(\cdot, t, x) - h_n(\cdot, t, y)\|_{\mathcal{H} \otimes \mathcal{N}}^2
\]

\[
= n! \left( \|h_n(\cdot, t, x)\|_{\mathcal{H} \otimes \mathcal{N}}^2 + \|h_n(\cdot, t, y)\|_{\mathcal{H} \otimes \mathcal{N}}^2 - 2 \langle h_n(\cdot, t, x), h_n(\cdot, t, y) \rangle_{\mathcal{H} \otimes \mathcal{N}} \right)
\]

\[
= \frac{2}{n!} \int_{[0, t]^{2n}} \int_{\mathbb{R}^d} \left[ 1 - e^{-i(x-y) \cdot (\xi_1 + \cdots + \xi_n)} \right] \prod_{j=1}^{n} \exp \left[ -[r_{\sigma(j+1)} - r_{\sigma(j)}] \Psi(\xi_{\sigma(1)} + \cdots + \xi_{\sigma(j)}) \right]
\]

\[
\prod_{j=1}^{n} \exp \left[ -[s_{\eta(j+1)} - s_{\eta(j)}] \Psi(\xi_{\eta(1)} + \cdots + \xi_{\eta(j)}) \right] \mu(d\xi) \prod_{j=1}^{n} |r_j - s_j|^{-\beta_0} dr ds
\]

where \( \sigma \) and \( \eta \) are permutations of the set \( \{1, 2, \ldots, n\} \) such that \( r_{\sigma(1)} < r_{\sigma(2)} < \cdots < r_{\sigma(n)} \) and \( s_{\eta(1)} < s_{\eta(2)} < \cdots < s_{\eta(n)} \). Denote

\[
A^2(r) = \int_{\mathbb{R}^d} \left[ 1 - e^{-i(x-y) \cdot (\xi_1 + \cdots + \xi_n)} \right] \prod_{j=1}^{n} \exp \left[ -2[r_{\sigma(j+1)} - r_{\sigma(j)}] \Psi(\xi_{\sigma(1)} + \cdots + \xi_{\sigma(j)}) \right] \mu(d\xi).
\]

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Recall the notations $D_t = 2\int_0^t s^{-\beta_0} ds$ and $\Omega_t^n = \{(s_1, \ldots, s_n) \in [0, \infty)^n : \sum_{j=1}^n s_j \leq t\}$. We have

$$n!\|h_n(\cdot, t, x) - h_n(\cdot, t, y)\|^2_{H^n} \leq \frac{2}{n!} \int_{[0, t]^n} A^2(r) \prod_{j=1}^n |s_j - r_j|^{-\beta_0} ds dr \leq \frac{2}{n!} D_t^n \int_{[0, t]^n} A^2(r) dr$$

$$= 2D_t^n \int_{[0<r_1<r_2<\ldots<r_n<t]} \int_{\mathbb{R}^d} [1 - e^{-i(x-y)\cdot(\xi_1+\cdots+\xi_n)}] \prod_{j=1}^n \exp \left[ - 2|r_{j+1} - r_j| \Psi(\xi_1 + \cdots + \xi_j) \right] \mu(d\xi) dr$$

$$\leq 2D_t^n \sup_{z \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} [1 - e^{-i(x-y)\cdot(z+\xi_n)}] \exp \left[ - 2s_j \Psi(z + \xi_1 + \cdots + \xi_j) \right] \mu(d\xi_n) ds_n$$

$$\times \int_{\Omega_t^n-1} \int_{\mathbb{R}^{n-1}d} \prod_{j=1}^{n-1} \exp \left[ - 2s_j \Psi(\xi_1 + \cdots + \xi_j) \right] \mu(d\xi_1) \cdots \mu(d\xi_{n-1}) ds_1 \cdots ds_{n-1}$$

$$\leq CD_t^n |x - y|^{2\alpha_1} \int_{\Omega_t^n-1} \int_{\mathbb{R}^{n-1}d} \prod_{j=1}^{n-1} \exp \left[ - 2s_j \Psi(\xi_j) \right] \mu(d\xi) ds. \quad \text{(By Hypothesis (S2))}$$

Applying Lemma (3.5), we have

$$\sqrt{n!}\|h_n(\cdot, t, x) - h_n(\cdot, t, y)\|_{H^n} \leq |x - y|^{\alpha_1} C D_t^{n/2} \sum_{k=0}^{n-1} \sqrt{\binom{n-1}{k} \frac{t^k}{k!} m_N^k [A_0 \epsilon_N]^{n-k}}.$$

As in the proof of Theorem 5.3, we can choose $N$ large enough, such that

$$\sum_{n=1}^\infty D_t^{n/2} \sum_{k=0}^n \sqrt{\binom{n}{k} \frac{t^k}{k!} m_N^k [A_0 \epsilon_N]^{n-k}} < \infty,$$

and hence there exists a constant $C$ such that

$$\|u(t, x) - u(t, y)\|_{L^p} \leq C |x - y|^{\alpha_1},$$

which implies the spatial Hölder continuity of $u(t, x)$.

**Step 2.** Now we consider the Hölder continuity in time, assuming that $0 \leq s < t \leq T$
and \( x = y \). Then for the estimation on the \( n \)-th chaos space, we have

\[
\text{\( n! \| h_n(\cdot, t, x) - h_n(\cdot, s, x) \|_{\mathcal{H}^\otimes n}^2 \)} = n! \left( \| h_n(\cdot, t, x) \|_{\mathcal{H}^\otimes n}^2 + \| h_n(\cdot, s, x) \|_{\mathcal{H}^\otimes n}^2 - 2 \langle h_n(\cdot, t, x), h_n(\cdot, s, x) \rangle_{\mathcal{H}^\otimes n} \right)
\]

\[
= n! \left[ \int_{[0,t]^{2n}} \int_{\mathbb{R}^{nd}} \mathcal{F} h_n(u, \cdot, t, x)(\xi) \mathcal{F} h_n(v, \cdot, t, x)(\xi) \mu(d\xi) \prod_{j=1}^n |u_j - v_j|^{-\beta_0} dv du \\
+ \int_{[0,t]^{2n}} \int_{\mathbb{R}^{nd}} \mathcal{F} h_n(u, \cdot, s, x)(\xi) \mathcal{F} h_n(v, \cdot, s, x)(\xi) \mu(d\xi) \prod_{j=1}^n |u_j - v_j|^{-\beta_0} dv du \\
- 2 \int_{[0,t]^{n} \times [0,s]^{n}} \int_{\mathbb{R}^{nd}} \mathcal{F} h_n(u, \cdot, t, x)(\xi) \mathcal{F} h_n(v, \cdot, s, x)(\xi) \mu(d\xi) \prod_{j=1}^n |u_j - v_j|^{-\beta_0} dv du \right].
\]

Therefore

\[
\text{\( n! \| h_n(\cdot, t, x) - h_n(\cdot, s, x) \|_{\mathcal{H}^\otimes n}^2 \)} \leq n!(D_n + D'_n), \tag{5.23}
\]

where

\[
D_n = \int_{[0,t]^{2n}} \int_{\mathbb{R}^{nd}} \mathcal{F} h_n(u, \cdot, t, x)(\xi) \mathcal{F} h_n(v, \cdot, t, x)(\xi) \mu(d\xi) \prod_{j=1}^n |u_j - v_j|^{-\beta_0} dv du \\
- \int_{[0,t]^{n} \times [0,s]^{n}} \int_{\mathbb{R}^{nd}} \mathcal{F} h_n(u, \cdot, t, x)(\xi) \mathcal{F} h_n(v, \cdot, s, x)(\xi) \mu(d\xi) \prod_{j=1}^n |u_j - v_j|^{-\beta_0} dv du,
\]

and

\[
D'_n = \int_{[0,t]^{n} \times [0,s]^{n}} \int_{\mathbb{R}^{nd}} \mathcal{F} h_n(u, \cdot, t, x)(\xi) \mathcal{F} h_n(v, \cdot, t, x)(\xi) \mu(d\xi) \prod_{j=1}^n |u_j - v_j|^{-\beta_0} dv du \\
- \int_{[0,s]^{2n}} \int_{\mathbb{R}^{nd}} \mathcal{F} h_n(u, \cdot, t, x)(\xi) \mathcal{F} h_n(v, \cdot, s, x)(\xi) \mu(d\xi) \prod_{j=1}^n |u_j - v_j|^{-\beta_0} dv du.
\]

We will just estimate \( D_n \), and \( D'_n \) will share the same upper bound of \( D_n \).

Clearly, \( D_n = A_n + B_n \) where

\[
A_n = \int_{[0,t]^{n} \times ([0,t]^{n} \setminus [0,s]^{n})} \int_{\mathbb{R}^{nd}} \mathcal{F} h_n(u, \cdot, t, x)(\xi) \mathcal{F} h_n(v, \cdot, t, x)(\xi) \mu(d\xi) \prod_{j=1}^n |u_j - v_j|^{-\beta_0} dv du \tag{5.24}
\]

and

\[
B_n = \int_{[0,t]^{n} \times [0,s]^{n}} \int_{\mathbb{R}^{nd}} \left( \mathcal{F} h_n(v, \cdot, t, x)(\xi) - \mathcal{F} h_n(v, \cdot, s, x)(\xi) \right) \mathcal{F} h_n(u, \cdot, t, x)(\xi) \mu(d\xi) \prod_{j=1}^n |u_j - v_j|^{-\beta_0} dv du.	ag{5.25}
\]
To get an estimation for the right-hand side of (5.23), we will separate the rest of the proof into three parts for easier reading.

**Step 2(a).** In this part, we will estimate $A_n$ given in (5.24). Note that $[0, t]^n = \bigcup_{k_j \in \{0,1\}} I_{k_1} \times I_{k_2} \times \cdots \times I_{k_n}$ with $I_1 = [0, s]$ and $I_2 = [s, t]$. Hence $[0, t]^n \setminus [0, s]^n$ is the union of $2^n - 1$ disjoint interval products, each of which contains at least one $[s, t]$. Denote $E_{n,j}$ the product of $n$ intervals, all of which are $[0, t]$ except that the $j$-th interval is $[s, t]$. Therefore, for the term $A_n$, we have

$$A_n \leq 2^n \sup_{j=1,\ldots,n} \int_{[0,t]^n \times E_{n,j}} \int_{\mathbb{R}^d} \mathcal{F} h_n(u, t, x) \mathcal{F} h_n(v, t, x)(\xi) \mu(d\xi) \prod_{j=1}^n |u_j - v_j|^{-\beta_0} dvdu$$

$$\leq 2^n \sup_{j=1,\ldots,n} \int_{[0,t]^n \times E_{n,j}} \left( A_{t,x}^2(u) + A_{t,x}^2(v) \right) \prod_{j=1}^n |u_j - v_j|^{-\beta_0} dvdu \quad (5.26)$$

with $A_{t,x}(u)$ given in (5.6). Denoting $D_t = 2 \int_0^t |s|^{-\beta_0} ds$, for positive function $f$, we have the following estimates

$$\int_0^t \int_0^t f(u)|u - v|^{-\beta_0} dvdu \leq D_t \int_0^t f(u)du,$n

$$\int_0^t \int_s^t f(u)|u - v|^{-\beta_0} dvdu \leq \frac{2\beta_0}{1 - \beta_0} (t - s)^{1-\beta_0} \int_0^t f(u)du,$n

and

$$\int_0^t \int_s^t f(v)|u - v|^{-\beta_0} dvdu \leq D_t \int_s^t f(v)dv.$n

Applying those estimates, we get

$$\int_{[0,t]^n \times E_{n,j}} \left( A_{t,x}^2(u) + A_{t,x}^2(v) \right) \prod_{j=1}^n |u_j - v_j|^{-\beta_0} dvdu$$

$$\leq \frac{2\beta_0}{1 - \beta_0} (t - s)^{1-\beta_0} D_t^{n-1} \int_{[0,t]^n} A_{t,x}^2(u)du + D_t^n \int_{E_{n,j}} A_{t,x}^2(v)dv. \quad (5.27)$$

Note that Hypothesis (T2) implies

$$\int_{\mathbb{R}^d} \frac{1}{1 + (\Psi(\xi))^{1-\alpha_2}} \mu(d\xi) < \infty, \quad (5.28)$$

and hence there exists $C > 0$ depending on the measure $\mu$ and $\alpha_2$ such that for all $x > 0$

$$\int_{\mathbb{R}^d} e^{-x\Psi(\xi)} \mu(d\xi) \leq C(1 + x^{\alpha_2 - 1})$$

by Lemma 3.10. On the other hand, by Lemma 3.11, we have

$$\int_{[0<v_1<v_2<\cdots<v_n<t]} \int_{\mathbb{R}^d} \prod_{j=1}^n (1 + (v_{j+1} - v_j)^{\alpha_2 - 1}) \mu(d\xi)dv$$

$$\leq C^n \sum_{\tau \in \{0,1\}^n} \prod_{j=1}^n \Gamma(\tau_j(\alpha_2 - 1) + 1) \left( \sum_{j=1}^n (\tau_j(\alpha_2 - 1) + n + 1) \right) \sum_{j=1}^n t^{\tau_j(\alpha_2 - 1) + n}. \quad (5.29)$$
Combining (5.28) and (5.29) and using the approach in Remark 3.6, we have for \( t \in [0, T] \) with \( T \geq 1 \),
\[
\int_{0,t} A^2_{t,x}(u) du \leq \frac{C^n}{n!} \sum_{m=0}^{n} \binom{n}{m} t^{m(\alpha_2 - 1) + n} \frac{1}{\Gamma(m(\alpha_2 - 1) + n + 1)} \leq \frac{(2C)^n}{n!} \frac{T^n}{\Gamma(n\alpha_2 + 1)}. \tag{5.30}
\]
Similarly, for all \( j \in \{1, 2, \ldots, n\} \) and \( 0 \leq s < t \leq T \) with \( T \geq 1 \), we have
\[
\int_{E_{n,j}} A^2_{t,x}(v) dv \leq \frac{1}{(n!)^2} \int_{0,t} [n-1 \times [s,t]] \int_{\mathbb{R}^d} \prod_{j=1}^{n} \exp \left( -2(v_{\sigma(j+1)} - v_{\sigma(j)})\psi(\xi_{\sigma(j)}) \right) \mu(d\xi) dv
\leq \frac{C^n}{(n!)^2} \int_{0,t} [n-1 \times [s,t]] \prod_{j=1}^{n} \left( 1 + (v_{\sigma(j+1)} - v_{\sigma(j)})^{\alpha_2 - 1} \right) dv
\leq \frac{C^n}{(n!)^2} \left( \int_{0,t} - \int_{0,[s]} \right) \prod_{j=1}^{n} \left( 1 + (v_{\sigma(j+1)} - v_{\sigma(j)})^{\alpha_2 - 1} \right) dv
= \frac{C^n}{n!} \left( \int_{0,v_{1} \cdots v_{n}} - \int_{0,v_{1} \cdots v_{n}} \right) \prod_{j=1}^{n} \left( 1 + (v_{j+1} - v_{j})^{\alpha_2 - 1} \right) dv
= \frac{C^n}{n!} \sum_{\tau \in \{0,1\}^n} \frac{\prod_{j=1}^{n} \Gamma(\tau_j(\alpha_2 - 1) + 1)}{\Gamma(\sum_{j=1}^{n} \tau_j(\alpha_2 - 1) + n + 1)} (t\sum_{j=1}^{n} \tau_j(\alpha_2 - 1) + n - s\sum_{j=1}^{n} \tau_j(\alpha_2 - 1) + n)
\leq \frac{1}{n!} \frac{C^n}{\Gamma(n\alpha_2 + 1)} nT^n (t - s)^{\alpha_2}. \tag{5.31}
\]
The last inequality holds because \( t\sum_{j=1}^{n} \tau_j(\alpha_2 - 1) + n - s\sum_{j=1}^{n} \tau_j(\alpha_2 - 1) + n \leq nT^n (t - s)^{\alpha_2} \) for all \( n \) and \( \tau \). Combining the above (5.30) and (5.31) with (5.26) and (5.27), we have
\[
A_n \leq \frac{1}{n!} \frac{C^n}{\Gamma(n\alpha_2 + 1)} \left( (t - s)^{1-\beta_0} + (t - s)^{\alpha_2} \right), \tag{5.32}
\]
where \( C \) depends on the measure \( \mu \), \( T \), \( \beta_0 \) and \( \alpha_2 \).

**Step 2(b).** The term \( B_n \) given in (5.25) will be estimated in this part.

\[
B_n \leq \frac{1}{(n!)^2} \int_{0,t} [n \times [s,t]] \int_{\mathbb{R}^d} \left| e^{-(t-v_\sigma(n))\Psi(\xi_1 + \cdots + \xi_n)} - e^{-(s-v_\sigma(n))\Psi(\xi_1 + \cdots + \xi_n)} \right|
\prod_{j=1}^{n-1} e^{-(v_{\sigma(j+1)} - v_{\sigma(j)})\psi(\xi_{\sigma(1)} + \cdots + \xi_{\sigma(j)})} \mathcal{F} h_n(u, \ldots, t, x)(\xi) \mu(d\xi) \prod_{j=1}^{n} |u_j - v_j|^{-\beta_0} dv du
\leq 2(t-s)^{\alpha_2} \frac{1}{(n!)^2} \int_{0,t} [n \times [s,t]] \int_{\mathbb{R}^d} \left( \Psi(\xi_1 + \cdots + \xi_n) \right)^{\alpha_2} \prod_{j=1}^{n} e^{-(v_{\sigma(j+1)} - v_{\sigma(j)})\psi(\xi_{\sigma(1)} + \cdots + \xi_{\sigma(j)})}
\prod_{j=1}^{n} e^{-(u_{\sigma(j+1)} - u_{\sigma(j)})\psi(\xi_{\sigma(1)} + \cdots + \xi_{\sigma(j)})} \mu(d\xi) \prod_{j=1}^{n} |u_j - v_j|^{-\beta_0} dv du,
\]
where \( v_{n+1} = s, u_{n+1} = t \) and \( \sigma \) and \( \eta \) are permutations such that \( 0 < v_{\sigma(1)} < \cdots < v_{\sigma(n)} < t \) and \( 0 < u_{\eta(1)} < \cdots < u_{\eta(n)} < t \), and in the last step we used the inequality \(|e^{-x} - e^{-y}| \leq |e^{-x} + e^{-y}| |x - y|^\alpha \leq 2|x - y|^\alpha \) for \( x, y > 0 \) and \( \alpha \in (0, 1] \).

Let

\[
A_t^2(u) = \int_{\mathbb{R}^d} (\Psi(\xi_1 + \cdots + \xi_n))^{\alpha_2} \prod_{j=1}^n e^{-2(u_{\eta(j)+1} - u_{\eta(j)})}\Psi(\xi_{\eta(j)+1} + \cdots + \xi_{\eta(j)}) \mu(d\xi)
\]

and

\[
A_s^2(v) = \int_{\mathbb{R}^d} (\Psi(\xi_1 + \cdots + \xi_n))^{\alpha_2} \prod_{j=1}^n e^{-2(v_{\sigma(j)+1} - v_{\sigma(j)})}\Psi(\xi_{\sigma(j)+1} + \cdots + \xi_{\sigma(j)}) \mu(d\xi).
\]

we have

\[
\int_{[0,t]^n} A_t^2(u)du = n! \int_{[0,u_1<\cdots<u_n=t]} \int_{\mathbb{R}^d} (\Psi(\xi_1 + \cdots + \xi_n))^{\alpha_2} \prod_{j=1}^n e^{-2(u_{j+1} - u_j)}\Psi(\xi_1 + \cdots + \xi) \mu(d\xi)du
\]

\[
\leq n! \sup_{z \in \mathbb{R}^d} \int_0^t \int_{\mathbb{R}^d} (\Psi(\xi_n + z))^{\alpha_2} e^{-2(t-u_n)}\Psi(\xi_n + z) \mu(d\xi_n)du
\]

\[
\times \int_{[0,u_1<\cdots<u_{n-1}<t]} \int_{\mathbb{R}^{(n-1)d}} \prod_{j=1}^{n-1} e^{-2(u_{j+1} - u_j)}\Psi(\xi_1 + \cdots + \xi) \mu(d\xi)du
\]

\[
\leq n! C \int_{[0,u_1<\cdots<u_{n-1}<t]} \int_{\mathbb{R}^{(n-1)d}} \prod_{j=1}^{n-1} e^{-2(u_{j+1} - u_j)}\Psi(\xi_j) \mu(d\xi)du
\]

\[
\leq n! C^{n+1} T^n \Gamma(n\alpha_2 + 1),
\]

where the last second step follows from Lemma 3.7, Hypothesis (T2), Remark 5.8 and Lemma 4.9, and the last step follows by a similar argument for (5.30). Now we have the estimation for \( B_n \),

\[
B_n \leq (t-s)^{\alpha_2} \frac{1}{(n!)^2} \int_{[0,t]^n \times [0,s]^n} (A_t^2(u) + A_s^2(v)) \prod_{j=1}^n |u_j - v_j|^{-\beta_0}dvdu
\]

\[
\leq 2(t-s)^{\alpha_2} \frac{1}{(n!)^2} D_t^n \int_{[0,t]^n} A_t^2(u)du
\]

\[
\leq 2(t-s)^{\alpha_2} \frac{1}{n!} D_t^n C^{n+1} T^n \Gamma(n\alpha_2 + 1).
\]

**Step 2(c).** Therefore, combining (5.32) and (5.33), we have that there exists a constant \( C \) depending on the measure \( \mu \), \( T \), \( \alpha_2 \) and \( \beta_0 \) such that

\[
\sum_{n=1}^{\infty} (p-1)^{n/2} \sqrt{n!} \sqrt{D_n} = \sum_{n=1}^{\infty} (p-1)^{n/2} \sqrt{n!} \sqrt{A_n + B_n} \leq C(t-s)^{[\alpha_2(1-\beta_0)]/2}.
\]
Note that we can get estimation for $D'_n$ analogous to (5.34), by an argument similar as the above for $D_n$. Finally, by (5.22), (5.23) and (5.34), we have

$$
\|u(t, x) - u(s, x)\|_{L^p} \leq \sum_{n=1}^{\infty} (p-1)^{n/2}\sqrt{n!}\|h_n(\cdot, t, x) - h_n(\cdot, s, y)\|_{H^n}
$$

$$
\leq \sum_{n=1}^{\infty} (p-1)^{n/2}\sqrt{n!}\sqrt{D_n + D'_n} \leq C(t - s)^{[\alpha^2(1-\beta_0)]/2}.
$$

The Hölder continuity in time now is concluded by the Kolmogorov’s criterion.

References


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Jian Song
Department of Mathematics and Department of Statistics & Actuarial Science
The University of Hong Kong, Hong Kong
txjsong@hku.hk

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