RIGIDITY OF PAIRS OF RATIONAL HOMOGENEOUS SPACES OF PICARD NUMBER 1 AND ANALYTIC CONTINUATION OF GEOMETRIC SUBSTRUCTURES ON UNIRULED PROJECTIVE MANIFOLDS

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Abstract

Building on the geometric theory of uniruled projective manifolds by Hwang-Mok, which relies on the study of *varieties of minimal rational tangents* (VMRTs) from both the algebro-geometric and the differential-geometric perspectives, Mok, Hong-Mok and Hong-Park have studied standard embeddings between rational homogeneous spaces $X = G/P$ of Picard number 1. Denoting by $S \subset X$ an arbitrary germ of complex submanifold which inherits from X a geometric structure defined by taking intersections of VMRTs with tangent subspaces and modeled on some rational homogeneous space $X_0 = G_0/P_0$ of Picard number 1 embedded in $X = G/P$ as a linear section through a standard embedding, we say that (X_0, X) is *rigid* if there always exists some $\gamma \in \mathrm{Aut}(X)$ such that S is an open subset of $\gamma(X_0)$. We prove that a pair (X_0, X) of sub-diagram type is rigid whenever X_0 is nonlinear, which in the Hermitian symmetric case recovers Schubert rigidity for nonlinear smooth Schubert cycles, and which in the general rational homogeneous case goes beyond earlier works dealing with images of holomorphic maps. Our methods apply to uniruled projective manifolds $(X, \mathcal{K})$, for which we introduce a general notion of sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$, proving that they are *rationally saturated* under an auxiliary condition on the intersection $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$ and a nondegeneracy condition for substructures expressed in terms of second fundamental forms on VMRTs. Under the additional hypothesis that minimal rational curves are of degree 1 and that distributions spanned by sub-VMRTs are bracket generating, we prove that S extends to a *subvariety* $Z \subset X$. For its proof, starting with a "*Thickening Lemma*" which yields *smooth collars* around certain standard rational curves, we show that the germ of submanifold $(S; x_0)$ and hence the associated germ of sub-VMRT structure on $(S; x_0)$ can be *propagated* along chains of *"thickening"* curves issuing from x_0, and construct by *analytic continuation* a *projective* family of chains of rational curves compactifying the latter family, thereby constructing the projective completion Z of S as its image under the evaluation map.

1. Introduction, motivation and statements of main results

In a joint program starting with [HM98], Hwang and Mok have developed a geometric theory of uniruled projective manifolds. By Miyaoka-Mori [MM86], all Fano manifolds are uniruled, noting that in the Picard number 1 case any uniruled projective manifold is necessarily Fano. This theory focuses on the *variety of minimal rational tangents* (VMRT), i.e., the projective subvariety consisting of tangents to minimal rational curves passing through a general point (cf. Definition 2.1). Especially, Hwang-Mok established with very few exceptions the *Cartan-Fubini Extension Principle* for germs of local biholomorphisms between Fano manifolds of Picard number 1, according to which the germ of a *VMRT-preserving* local biholomorphism extends necessarily to a global biholomorphism provided that a certain *nondegeneracy condition* holds, which is nothing other than the generic finiteness of the Gauss map on the VMRT at a general point of the manifold (cf. [HM01]).

Cartan-Fubini extension lies at the heart of the theory of geometric structures modeled on VMRTs. It led to a proof of Ochiai's Theorem [Oc70] by means of analytic continuation along minimal rational curves, a proof which generalizes to Fano manifolds of Picard number 1 under a very mild condition. Ochiai's Theorem is essential in the proof of rigidity under Kähler deformation of irreducible Hermitian symmetric spaces of the compact type (Hwang-Mok [HM98]) and in the affirmative solution of Lazarsfeld's Problem ([HM99a]), and a solution of the latter problem based on the Cartan-Fubini Extension Principle was given in [HM04b]. Later on, Mok [Mk08a] and Hong-Mok [HoM10] generalized Cartan-Fubini extension to holomorphic immersions between Fano manifolds of Picard number 1 which are *VMRT-respecting* and which satisfy a certain non-equidimensional *nondegeneracy condition* (cf. Section 2.1 in [HoM10]). Given rational homogeneous spaces $X_0 = G_0/P_0$ and $X = G/P$ of Picard number 1, a connected open subset $U \subset X_0$, and a holomorphic immersion $f : U \to X$, the mapping f is said to be VMRT-respecting at $x \in U$ if and only if $[df]\big(\mathscr{C}_x(X_0)\big) = \mathscr{C}_{f(x)}(X) \cap \mathbb{P}(df(T_x(X_0)))$, where $\mathscr{C}_x(X_0)$ denotes the VMRT on X_0 at the point x, etc. Here and in what follows an injective linear map $\Lambda : V \to W$ between two finite-dimensional vector spaces we denote by $[\Lambda] : \mathbb{P}V \to \mathbb{P}W$ the projectivization of Λ. (This applies analogously to linear homomorphisms between vector bundles.) Hong-Mok [HoM10] characterized certain standard embeddings between rational homogeneous spaces of Picard number 1 associated to long simple roots, as follows.

Theorem 1.1. (cf. Theorem 1.2 in Hong-Mok [HoM10]). *Let $X_0 = G_0/P_0$ resp. $X = G/P$ be rational homogeneous spaces associated to long simple roots determined by marked Dynkin diagrams $(\mathscr{D}(G_0), \gamma_0)$ resp. $(\mathscr{D}(G), \gamma)$. Suppose $\mathscr{D}(G_0)$ is obtained from a sub-diagram of $\mathscr{D}(G)$ with γ_0 being identified with γ. If X_0 is nonlinear and $f : U \to X$ is a holomorphic embedding from a connected open subset $U \subset X_0$ into X which respects VMRTs at a general point $x \in U$, then f is the restriction to U of a standard embedding of X_0 into X.*

The pair (X_0, X) as in the above theorem is called a pair of *sub-diagram type*. The sub-diagram induces naturally a holomorphic embedding $i : X_0 \hookrightarrow X$, and by a standard embedding we mean an embedding $\varphi \circ i \circ \varphi_0 : X_0 \hookrightarrow X$ for any $\varphi \in \mathrm{Aut}(X)$ and any $\varphi_0 \in \mathrm{Aut}(X_0)$. (If $\mathrm{Aut}(X_0)$ is connected, $i \circ \varphi_0 = \psi \circ i$ for some $\psi \in \mathrm{Aut}(X)$, hence $\varphi \circ i \circ \varphi_0 = \varphi\psi \circ i$ and thus φ_0 may be dropped in the definition of standard embeddings.) Following [HoM10] but using projective geometry in place of root space decompositions, Hong and Park obtained the same result under the same assumptions as those in Theorem 1.1 when γ is a short root.

Theorem 1.2. (cf. Theorem 1.2 in Hong-Park [HoP11]). *The analogue of Theorem 1.1 holds when the rational homogeneous space $X = G/P$ is associated to a short simple root determined by a marked Dynkin diagram $(\mathscr{D}(G), \gamma)$, and $X_0 = G_0/P_0 \hookrightarrow G/P = X$ is nonlinear and associated to a marked sub-diagram $(\mathscr{D}(G_0), \gamma_0)$ of $(\mathscr{D}(G), \gamma)$.*

For the cases of *maximal linear subspaces* X_0, Hong-Park [HoP11] proved

Theorem 1.3. (cf. Theorem 1.3 in Hong-Park [HoP11]). *Let $X = G/P$ be a rational homogeneous space associated to a simple root and let $X_0 \subset X$ be a linear subspace. Let $f : U \to X$ be a holomorphic embedding from a connected open subset U of X_0 into X such that $\mathbb{P}(df(T_x(X_0)))$ is contained in $\mathscr{C}_{f(x)}(X)$ for any point $x \in U$. If there is a maximal linear subspace $Z_{\max}$ of X of dimension equal to* $\dim(U)$ *which is tangent to $f(U)$ at some point $f(x) \in f(U)$, then $f(U)$ is contained in $Z_{\max}$, excepting when $(Z_{\max}, X)$ is given by* (a) *X is associated to (B_ℓ, α_i), $i \leq \ell - 1$, and $Z_{\max}$ is $\mathbb{P}^{\ell-i}$;* (b) *X is associated to (C_ℓ, α_ℓ) and $Z_{\max}$ is $\mathbb{P}^1$; or* (c) *X is associated to (F_4, α_1) and $Z_{\max}$ is $\mathbb{P}^2$. In these cases the pairs $(Z_{\max}, X)$ are not rigid.*

Here we identify a rational homogeneous space $X = G/P$ of Picard number 1 with a projective submanifold by means of the first canonical embedding, i.e., $\rho : X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{O}(1))^*) =: \mathbb{P}^N$. For every point $x \in X$ a linear subspace in $\mathscr{C}_x(X)$ corresponds to a linear subspace in X passing through x. We say that a linear subspace $Z_{\max} \subset X \subset \mathbb{P}^N$ is *maximal* whenever there is no linear subspace $\Pi \subset X$ such that $Z_{\max} \subsetneq \Pi$. Obviously the analogue of Theorem 1.3 fails for (X_0, X) for non-maximal linear subspaces $X_0 \subsetneq Z_{\max}$.

We define now admissible pairs of rational homogeneous spaces of Picard number 1. Here the identity component of a complex Lie group H is denoted by H_0, and we write $H_0 =: \mathrm{Aut}_0(M)$ when $H = \mathrm{Aut}(M)$ for some complex manifold M.

Definition 1.1. *Let X_0 and X be rational homogeneous spaces of Picard number* 1*, and $i : X_0 \hookrightarrow X$ be a holomorphic embedding equivariant* *with respect to a homomorphism of complex Lie groups* $\Phi : \mathrm{Aut}_0(X_0) \to \mathrm{Aut}_0(X)$*. We say that* $(X_0, X; i)$ *is an admissible pair* (*of rational homogeneous spaces of Picard number 1*) *if and only if* (a) i *induces an isomorphism* $i_* : H_2(X_0, \mathbb{Z}) \xrightarrow{\cong} H_2(X, \mathbb{Z})$*, and* (b) *denoting by* $\mathcal{O}(1)$ *the positive generator of* $\mathrm{Pic}(X)$ *and by* $\rho : X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{O}(1))^*) =: \mathbb{P}^N$ *the first canonical projective embedding of* X*,* $\rho \circ i : X_0 \hookrightarrow \mathbb{P}^N$ *embeds* X_0 *as a* (*smooth*) *linear section of* $\rho(X)$*.*

Assumption (b) rules out examples such as the Plücker embedding of a Grassmannian of rank ≥ 2 into a projective space. As a consequence of (b) we have the following immediate corollary on varieties of minimal rational tangents which is important for the discussion in the current article.

Corollary 1.1. *Let* $(X_0, X; i)$ *be an admissible pair of rational homogeneous spaces of Picard number 1. Then, the equivariant holomorphic embedding* $i : X_0 \hookrightarrow X$ *respects VMRTs, i.e.,* $[di](\mathscr{C}_x(X_0)) = \mathscr{C}_{i(x)}(X) \cap \mathbb{P}(di(T_x(X_0)))$ *whenever* $x \in X_0$*.*

Remark Originally the VMRT-respecting property of $i : X_0 \hookrightarrow X$ was used in place of (b) to define the notion of admissible pairs $(X_0, X; i)$. Pairs $(X_0, X; i)$ of sub-diagram type are admissible in the stronger sense as in Definition 1.1, and the global condition (b) is more natural. Restricting to the cases where both X_0 and X are irreducible Hermitian symmetric spaces of the compact type, the two definitions of admissible pairs lead to the same set of pairs $(X_0, X; i)$.

In the sequel we will denote the admissible pair simply by (X_0, X), the holomorphic equivariant embedding $i : X_0 \hookrightarrow X$ being understood. If the pair (X_0, X) is of sub-diagram type as in Theorem 1.1, etc., it is naturally admissible with $i : X_0 \hookrightarrow X$ being induced by the canonical identification of the nodes of their respective Dynkin diagrams. Pairs of Grassmannians $(G(p, q), G(r, s)); 1 \leq p \leq r, 1 \leq q \leq s$; furnish prototypes of admissible pairs of sub-diagram type. On the other hand, for $n \geq 2$, denoting by (V, σ) a $2n$-dimensional complex vector space V endowed with a symplectic form σ and by $G^{III}(n, n)$ the Lagrangian Grassmannian of n-dimensional isotropic vector subspaces (Lagrangian vector subspaces), then $(G^{III}(n, n), G(n, n))$ is an admissible pair of rational homogeneous spaces which is not of sub-diagram type. In fact, if X is any Grassmannian and (X_0, X) is of sub-diagram type, then X_0 must itself be a Grassmannian.

We consider complex submanifolds $S \subset W$ of some connected open subset $W \subset X$ inheriting geometric substructures modeled on the pair (X_0, X) in some precise sense, as follows. Note that in this article a manifold is taken to be connected; likewise a submanifold of a manifold is taken to be connected. Here and in what follows, writing $X = G/P$,

$X_0 = G_0/P_0 \hookrightarrow G/P = X$ and taking $e \in P_0 \subset P$ to be the identity element of P and P_0, we fix a reference point $0 = eP_0 \in X_0$, which is identified with $eP \in X$, i.e., $0 \in X_0 \subset X$.

Definition 1.2. *Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1, $W \subset X$ be a connected open subset, and $S \subset W$ be a complex submanifold. Consider the fibered space $\pi : \mathscr{C}(X) \to X$ of varieties of minimal rational tangents on X. For every point $x \in S$ define $\mathscr{C}_x(S) := \mathscr{C}_x(X) \cap \mathbb{P}T_x(S)$ and write $\varpi : \mathscr{C}(S) \to S$ for $\varpi = \pi\big|_{\mathscr{C}(S)}, \varpi^{-1}(x) := \mathscr{C}_x(S)$ for $x \in S$. We say that $S \subset W$ inherits a sub-VMRT structure modeled on (X_0, X) if and only if for every point $x \in S$ there exists a neighborhood U of x on S and a trivialization of the holomorphic projective bundle $\mathbb{P}T(X)|_U$ given by $\Phi : \mathbb{P}T(X)|_U \xrightarrow{\cong} U \times \mathbb{P}T_0(X)$ such that* (1) $\Phi(\mathscr{C}(X)|_U) = U \times \mathscr{C}_0(X)$ *and* (2) $\Phi(\mathscr{C}(S)|_U) = U \times \mathscr{C}_0(X_0)$.

We also call $\varpi : \mathscr{C}(S) \to S$ a sub-VMRT structure modeled on $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$. The holomorphic bundle map Φ in Definition 1.2 may be referred to as a trivialization over U of the sub-VMRT structure modeled on (X_0, X). When the admissible pair (X_0, X) is of sub-diagram type, given a VMRT-respecting holomorphic embedding $f : U \to X$ from a connected open subset $U \subset X$ onto a complex submanifold $S := f(U) \subset W$ of some connected open subset $W \subset X$ and writing $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$, the canonical projection $\varpi : \mathscr{C}(S) \to S$ defines on S a sub-VMRT structure on S modeled on (X_0, X) (cf. Lemma 1.2 below and Remark (c) after the proof of Main Theorem 1 in §4).

For $i = 1, 2$ let V_i be a finite-dimensional complex vector space, and $\mathcal{A}_i \subset \mathbb{P}(V_i)$ be a subvariety. We say that $(\mathcal{A}_1 \subset \mathbb{P}(V_1))$ is projectively equivalent to $(\mathcal{A}_2 \subset \mathbb{P}(V_2))$ if and only if there exists a projective linear isomorphism $\varphi : \mathbb{P}(V_1) \xrightarrow{\cong} \mathbb{P}(V_2)$ such that $\varphi(\mathcal{A}_1) = \mathcal{A}_2$. For subvarieties $\mathcal{A}_i \subset \mathcal{B}_i \subset \mathbb{P}(V_i)$; $i = 1, 2$; we say that $(\mathcal{A}_1 \subset \mathcal{B}_1 \subset \mathbb{P}(V_1))$ is projectively equivalent to $(\mathcal{A}_2 \subset \mathcal{B}_2 \subset \mathbb{P}(V_2))$ if and only if there exists a projective linear isomorphism $\varphi : \mathbb{P}(V_1) \xrightarrow{\cong} \mathbb{P}(V_2)$ such that $\varphi(\mathcal{A}_1) = \mathcal{A}_2$ and $\varphi(\mathcal{B}_1) = \mathcal{B}_2$. If understood we may omit $\mathbb{P}(V_i)$; $i = 1, 2$.

For a finite-dimensional complex vector space E and for a subset $A \subset \mathbb{P}(E)$, we denote by $\widetilde{A} \subset E - \{0\}$ the affinization of A, i.e., $\widetilde{A} := \lambda^{-1}(A)$ for the canonical map $\lambda : E - \{0\} \to \mathbb{P}(E)$. We also write $A^\sharp := \widetilde{A} \cup \{0\}$. $A^\sharp \subset E$ is invariant under multiplication by $\lambda \in \mathbb{C}$. When $A \subset \mathbb{P}(E)$ is a subvariety, $A^\sharp \subset E$ is a subvariety which we call the cone over A.

Lemma 1.1. *Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1, $W \subset X$ be a connected open subset, and $S \subset W$ be a complex submanifold. Define $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$ and write $\varpi : \mathscr{C}(S) \to S$ for the canonical projection, $\varpi^{-1}(x) =: \mathscr{C}_x(S)$*

for any point $x \in S$. Suppose $(\mathscr{C}_x(S) \subset \mathbb{P}T_x(X))$ is projectively equivalent to $(\mathscr{C}_0(X_0) \subset \mathbb{P}T_0(X))$ for any point $x \in S$. Then, $\varpi : \mathscr{C}(S) \to S$ is a holomorphic submersion.

Proof. Taking $\mathscr{C}(S)$ to be reduced, the general fiber of $\varpi : \mathscr{C}(S) \to S$ is reduced. Since reductions of the fibers are by hypothesis homologous in $\mathbb{P}T(X)$, all fibers are reduced. Suppose $x_0 \in S$ and $q \in \mathscr{C}_{x_0}(S)$. There exist local holomorphic coordinates $(z_1, \cdots, z_m; w_1, \cdots, w_n; t_1, \cdots, t_s)$ on some neighborhood $\mathcal{O}$ of q on $\mathscr{C}(X)|_S$ such that (a) q corresponds to $(0, \cdots, 0)$ and $\mathcal{O}$ corresponds to the unit polydisk Δ^{m+n+s}; (b) $(t_1, \cdots, t_s)$ are holomorphic coordinates on a neighborhood U of x_0 on S with respect to which x_0 corresponds to $(0, \cdots, 0)$; (c) $\mathcal{O} \cap \mathscr{C}_{x_0}(S)$ corresponds to $|z_i| < 1$ and $w_1 = \cdots = w_n = t_1 = \cdots = t_s = 0$; (d) for $x \in U$ with coordinates $t = (t_1, \cdots, t_s)$, $Z_x := \mathscr{C}_x(S) \cap \mathcal{O} \subset \mathscr{C}_x(X) \cap \mathcal{O} \cong \Delta^{m+n} \times \{(t_1, \cdots, t_s)\}$ is a complex submanifold such that the coordinate projection ρ_x from Z_x into the z-polydisk Δ^m is a surjective finite proper map. By fiberwise contour integration (as in the proof of Weierstrass Preparation Theorem), the fibers of ρ_x consist of single points since $\mathscr{C}_{x_0}(S)$ is reduced, and $\varpi^{-1}(U) = \bigcup_{x \in U} Z_x$ is the graph of a vector-valued holomorphic function on $\Delta^m \times U$. Thus, ϖ is a holomorphic submersion at q. Since $q \in \mathscr{C}_{x_0}(S)$ is arbitrary, $\varpi : \mathscr{C}(S) \to S$ is a holomorphic submersion. $\square$

We have now the following pointwise characterization of sub-VMRT structures modeled on an admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1.

Lemma 1.2. *In the notation adopted in Lemma 1.1, $S \subset W \subset X$ inherits a sub-VMRT structure modeled on (X_0, X) if and only if $(\mathscr{C}_x(S) \subset \mathscr{C}_x(X))$ is projectively equivalent to $(\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X))$ for any point $x \in S$.*

Proof. The "only if" part is obvious. For the "if" part let $x_0 \in S$ and fix a holomorphic coordinate chart (V, z^i) on some neighborhood V of x_0 on X, inducing a trivialization $\alpha : \mathbb{P}T(X)|_V \xrightarrow{\cong} V \times \mathbb{P}T_{x_0}(X)$. We will prove that, shrinking V if necessary, there exist projective linear isomorphisms $\Gamma_x : \mathbb{P}T_x(X) \xrightarrow{\cong} \mathbb{P}T_{x_0}(X)$ varying holomorphically in $x \in V$ such that $\Gamma_x(\mathscr{C}_x(X)) = \mathscr{C}_{x_0}(X)$ for any $x \in V$. To see this, shrinking V if necessary, there exists $\gamma_x \in G := \mathrm{Aut}_0(X)$ varying holomorphically in $x \in V$ such that $\gamma_{x_0} = \mathrm{id}_X$ and such that $\gamma_x(x) = x_0$ for every point $x \in V$. ($\gamma : V \to G$ may be taken as a section of some P-principal bundle $F : G \to X$ for the parabolic subgroup $P \subset G$ at x_0.) Since $\pi : \mathscr{C}(X) \to X$ is invariant under $G^{\mathbb{C}}$, we have $[d\gamma_x](\mathscr{C}_x(X)) = \mathscr{C}_{x_0}(X)$. Thus, defining $\Gamma_x([v]) := [d\gamma_x(v)]$ and $\Gamma(x, [v]) := (x, \Gamma_x([v])) \in \mathscr{C}_x(X)$, and identifying $\mathbb{P}T(X)|_V$ with $V \times \mathbb{P}T_{x_0}(X)$ by means of α we have a bundle isomorphism $\Gamma : \mathbb{P}T(X)|_V \xrightarrow{\cong} V \times \mathbb{P}T_{x_0}(X)$ such that $\Gamma(\mathscr{C}(X)|_V) =$

$V \times \mathscr{C}_{x_0}(X)$. Restricting Γ to $U := V \cap S$ we have $\Psi : \mathbb{P}T(X)|_U \xrightarrow{\cong} U \times \mathbb{P}T_{x_0}(X)$ such that $\Psi(\mathscr{C}(X)|_U) = U \times \mathscr{C}_{x_0}(X)$. For $x \in U$ define $\mathscr{S}_x := \Gamma_x(\mathscr{C}_x(S)) \subset \mathscr{C}_{x_0}(X)$. Note that $\mathscr{S}_{x_0} = \mathscr{C}_{x_0}(S)$ since $\gamma_{x_0} = \mathrm{id}_X$.

From now on, by means of the trivialization $\Psi : \mathbb{P}T(X)|_U \xrightarrow{\cong} U \times \mathbb{P}T_{x_0}(X)$ we identify $\mathscr{C}(X)|_U$ with $U \times \mathscr{C}_{x_0}(X)$ and the fibers $\mathscr{C}_x(S)$ of $\varpi : \mathscr{C}(S) \to S$ with $\{x\} \times \mathscr{S}_x$. Denote by $H \subset \mathbb{P}\mathrm{GL}(T_{x_0}(X))$ the Lie subgroup of all projective linear transformations $\varphi \in \mathbb{P}\mathrm{GL}(T_{x_0}(X))$ satisfying $\varphi(\mathscr{C}_{x_0}(X)) = \mathscr{C}_{x_0}(X)$, and by $J \subset H$ the Lie subgroup consisting of $\varphi \in H$ satisfying the further condition $\varphi(\mathscr{C}_{x_0}(X_0)) = \mathscr{C}_{x_0}(X_0)$. By hypothesis, for every point $x \in U$ there exists $\theta_x \in \mathbb{P}\mathrm{GL}(T_{x_0}(X))$ such that $\theta_x(\mathscr{C}_{x_0}(X)) = \mathscr{C}_{x_0}(X)$ and such that $\theta_x(\mathscr{S}_x) = \mathscr{S}_{x_0} = \mathscr{C}_{x_0}(S)$. Given one such isomorphism θ^0_x the set of all θ_x satisfying the same requirements is given by $J_x := J \cdot \theta^0_x$. To show that $\varpi : \mathscr{C}(S) \to S$, $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$, is indeed a sub-VMRT structure in the sense of Definition 1.2, the problem is (†) to find $\Theta_x \in J_x$ such that Θ_x varies holomorphically in $x \in U$, shrinking U if necessary. To this end recall that $\mathscr{C}_{x_0}(X)^\sharp$ resp. $\mathscr{C}_{x_0}(X_0)^\sharp$ denotes the cone over $\mathscr{C}_{x_0}(X)$ resp. $\mathscr{C}_{x_0}(X_0)$. The affinization $\widetilde{H} \subset \mathrm{GL}(T_{x_0}(X))$ resp. $\widetilde{J} \subset \mathrm{GL}(T_{x_0}(X))$ of $H \subset \mathbb{P}\mathrm{GL}(T_{x_0}(X))$ resp. $J \subset \mathbb{P}\mathrm{GL}(T_{x_0}(X))$ is a linear subgroup, $\widetilde{J} \subset \widetilde{H}$. There is a positive integer m and a finite number of homogeneous polynomials $Q_1, \cdots, Q_N$ on $T_{x_0}(X)$ of degree m whose common zero set is precisely $\mathscr{C}_{x_0}(S)^\sharp$. Since $\varpi : \mathscr{C}(S) \to S$ is a submersion, its lifting $\widetilde{\varpi} : \widetilde{\mathscr{C}(S)} \to S$ to the affinization is also a submersion. Hence, for any $v_0 \in \widetilde{\mathscr{C}_{x_0}}(S)$, shrinking U if necessary, there exists a biholomorphism $\sigma : U \times \Delta^r \xrightarrow{\cong} \mathcal{W}$, $r = \dim(\widetilde{\mathscr{C}_{x_0}}(S))$, onto an open subset $\mathcal{W} \subset \widetilde{\mathscr{C}}(S)$ such that $\widetilde{\varpi}(\sigma(x,t)) = x$. For each $t \in \Delta^r$, $1 \le k \le N$, the zero set of the holomorphic function $Q_k(T(\sigma(x,t))$ in $(x,T) \in U \times \widetilde{H}$ is a subvariety $\widetilde{Z}(t,k) \subset U \times \widetilde{H}$, which descends to $Z(t,k) \subset U \times H$. Consider the intersection $Z := \bigcap \big\{ Z(t,k) : t \in \Delta^r, 1 \le k \le N \big\}$ and write its fiber over $x \in U$ as $Z_x = \{x\} \times Z'_x$. Then $Z \subset U \times H$ is a subvariety and $Z'_x \subset H$ consists precisely of all projective linear transformations $h \in H$ satisfying $h(\mathcal{O}_x) \subset \mathscr{S}_{x_0}$ for a certain given nonempty open subset $\mathcal{O}_x \subset \mathscr{S}_x$. By the Identity Theorem for holomorphic functions it follows that $Z'_x = J_x = J \cdot \theta^0_x$ for some $\theta^0_x \in H$. Define $\mathcal{N} := J \backslash H$, then $Z \subset U \times H$ descends to a subvariety $\mathcal{Z} \subset U \times \mathcal{N}$. For $x \in U$ the fibers $\mathcal{Z}_x = \mathcal{Z} \cap (\{x\} \times \mathcal{N})$ consists of a single point, hence $\mathcal{Z}$ is the graph of a holomorphic map $\tau : U \to \mathcal{N}$. Shrinking U if necessary, τ lifts to a holomorphic map $\Theta : U \to H$ and (†) is solved with $\Theta_x := \Theta(x)$ for $x \in U$. The proof of Lemma 1.2 is complete. $\square$

Remarks

(a) For the given proof of Lemma 1.2, in place of Lemma 1.1 it is sufficient to know that $\varpi : \mathscr{C}(S) \to S$ is a submersion at a general point on any fiber $\mathscr{C}_x(S)$, $x \in U$.

(b) Assume that Ψ has been constructed. For $h \in H$ write $\nu(h) := Jh \in J \setminus H =: \mathcal{N}$. Consider $\mathcal{S} := \{(\nu(h), w) : h \in H, w \in h^{-1}(\mathscr{S}_{x_0})\} \subset \mathcal{N} \times \mathscr{C}_{x_0}(X)$. The canonical projection $\beta : \mathcal{S} \to \mathcal{N}$ realizes $\mathcal{S}$ as the total space of a locally trivial holomorphic fiber bundle. Denote by $\mathscr{Q}$ the Chow component of $\mathscr{C}_{x_0}(X)$ containing the reduced cycle $[\mathscr{S}_0]$ as a member and by $\mathscr{N} \subset \mathscr{Q}$ the orbit of $[\mathscr{S}_0]$ under H. Then, there exists a universal family $\rho : \mathscr{A} \to \mathscr{Q}$, $\mathscr{A} \subset \mathscr{Q} \times \mathscr{C}_{x_0}(X)$, over $\mathscr{Q}$ and a classifying map $f : U \to \mathscr{Q}$ such that $f(U) \subset \mathscr{N}$ and such that $\varpi|_U : \mathscr{C}(S)|_U \to U$ is isomorphic to $f^*\rho : f^*\mathscr{A} \to U$. Noting that $\rho|_{\mathscr{N}} : \mathscr{A}|_{\mathscr{N}} \to \mathscr{N}$ is isomorphic as a holomorphic fiber bundle to $\beta : \mathcal{S} \to \mathcal{N}$ we obtain local trivializations of $\varpi : \mathscr{C}(S) \to S$ as a sub-VMRT structure modeled on (X_0, X). As the construction of Ψ also follows from the same argument, we have a quick proof of Lemma 1.2. However, in view of the elementary character of the results we have chosen to give a self-contained proof of Lemma 1.2 using Lemma 1.1.

(c) For the prototype of Grassmannians, $S \subset W \subset G(p,q)$ inherits a sub-VMRT structure modeled on $(G(r,s), G(p,q))$ if and only if $\varphi : \mathscr{C}(S) \to S$ is a Grassmann structure on S modeled on $G(r,s)$, and the latter holds if and only if $\mathscr{C}_x(S) \cong \zeta(\mathbb{P}^{r-1} \times \mathbb{P}^{s-1})$, ζ being the Segre embedding.

In view of the rigidity results Theorem 1.1, Theorem 1.2 and Theorem 1.3 for VMRT-respecting germs of holomorphic immersions $f : (X_0; x_0) \to (X; f(x_0))$ when the admissible pair (X_0, X) is of sub-diagram type, we formulate the following notion of rigidity for sub-VMRT structures modeled on admissible pairs (X_0, X) of rational homogeneous spaces of Picard number 1. A manifold is taken to be connected. Likewise a submanifold of a manifold is taken to be connected.

Definition 1.3. *An admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1 is said to be rigid if and only if for any connected open subset $W \subset X$, any complex submanifold $S \subset W$ inheriting a sub-VMRT structure modeled on (X_0, X) must necessarily be an open subset of $\gamma(X_0) \subset X$ for some $\gamma \in \mathrm{Aut}(X)$. We also say equivalently that sub-VMRT structures modeled on (X_0, X) are rigid.*

We note that in Theorem 1.3 (from Hong-Park [HoP11]), where $X_0 \subset X$ is a linear subspace, the condition for $f : U \to X$ to be VMRT-respecting (where $U \subset X_0$ is a connected open subset), i.e., $df(\mathbb{P}T(U)) \subset \mathscr{C}(X)|_S$, is actually a condition on $S := f(U) \subset X$, i.e., the condition $\mathbb{P}T(S) \subset \mathscr{C}(X)|_S$. Hence, *any* biholomorphism $f : U \xrightarrow{\cong} S \subset W$ is trivially VMRT-respecting. As such Theorem 1.3 is actually a result on rigidity of sub-VMRT structures modeled on (X_0, X) for the linear case. For the nonlinear case there is a big difference between germs of VMRT-respecting holomorphic immersions and sub-VMRT structures modeled on admissible pairs. For instance, when X_0 is Hermitian symmetric the VMRT-respecting property implies that the image germ of complex submanifold S is intrinsically flat, i.e., that a number of curvature-type invariants are equal to 0 (cf. Guillemin [Gu65]).

The problem of rigidity of admissible pairs originated with the works of Walters [Wa97] and Bryant [Br01] in connection with certain rigidity problems in the Hermitian symmetric case concerning Schubert cycles. This includes in particular the problem of Schur rigidity of a smooth Schubert cycle $X_0 \subset X$ (corresponding necessarily to an admissible pair $(X_0; X)$ of sub-diagram type), which asks whether an irreducible algebraic cycle Z homologous to rX_0, r being a positive integer, is necessarily a sum $\gamma_1(X_0) + \cdots + \gamma_r(X_0)$ of translates of X_0 by $\gamma_i \in \mathrm{Aut}(X)$, $1 \le i \le r$. The problem was converted to a question on uniqueness of integral varieties of a certain differential system called Schubert differential system, which is exactly the question of rigidity of the pair $(X_0; X)$ in Definition 1.3. The latter problem was solved by Hong [Ho07] using methods of Lie algebra cohomology. It appears nonetheless a daunting task to generalize the same arguments to the general case of rational homogeneous spaces $X = G/P$ of Picard number 1.

[Mk08a], [HoM10] and [HoP11] rely on local differential geometry. In Hong-Mok [HoM13] the methodology was further applied to rigidity problems on (not necessarily homogeneous) smooth Schubert cycles. In this article we further develop the differential-geometric approach of the aforementioned articles to study rigidity of admissible pairs primarily for the nonlinear case, and we prove the following Main Theorem 1 which gives sufficient conditions for the rigidity of (X_0, X).

Main Theorem 1. *Let (X_0, X) be an admissible pair of sub-diagram type of rational homogeneous spaces of Picard number 1 marked at a simple root. Suppose $X_0 \subset X$ is nonlinear. Then, (X_0, X) is rigid.*

Combining Main Theorem 1 with Theorem 1.3 for the maximal linear case, the question on rigidity of geometric structures modeled on admissible pairs (X_0, X) of Picard number 1 is completely settled.

Corollary 1.2. *An admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1 of sub-diagram type is a rigid pair excepting when $X_0 \subset X$ is a non-maximal linear subspace, or $X_0 \subset X$ is a maximal linear subspace $Z_{\max}$ given by* (a) *X is associated to (B_ℓ, α_i), $i \le \ell - 1$, and $Z_{\max}$ is $\mathbb{P}^{\ell-i}$;* (b) *X is associated to (C_ℓ, α_ℓ) and $Z_{\max}$ is $\mathbb{P}^1$; or* (c) *X is associated to (F_4, α_1) and $Z_{\max}$ is $\mathbb{P}^2$.*

Results of [Mk08a], [HoM10] and [HoP11] for the characterization of standard embeddings defined by marked Dynkin sub-diagrams subsume under the more general phenomenon of rigidity of certain pairs of admissible rational homogeneous spaces of Picard number 1 covered by Main Theorem 1. Thus, Main Theorem 1 strengthens [Mk08a], [HoM10] and [HoP11] by removing the assumption that the sub-VMRT structures under consideration are images of VMRT-respecting maps.

On the rational homogeneous space $X = G/P$ of Picard number 1 there is a rank one distribution $\mathscr{F} \subset T(\mathscr{C}(X))$ corresponding to the tautological foliation $\mathcal{F}$ of leaf dimension 1 on $\mathscr{C}(X)$. Local leaves of $\mathcal{F}$ are tautological liftings (cf. second paragraph after Theorem 5.1) to $\mathscr{C}(X)$ of connected open subsets of minimal rational curves on X. Note that the minimal rational curves on X are precisely the projective lines lying on X when the latter is embedded as a projective submanifold by the first canonical embedding. Crucial to our study is the question whether $\mathscr{F}$ restricts to $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$ over S, i.e., whether $\mathscr{F}|_{\mathscr{C}(S)} \subset T(\mathscr{C}(S))$, when S inherits a sub-VMRT structure modeled on (X_0, X). If the latter holds we say that S is *linearly saturated.* Equivalently, S is linearly saturated if and only if any projective line ℓ tangent to S at some point must necessarily lie on S. In §2 we develop first of all a method which gives sufficient conditions for S to be linearly saturated. This method introduces a notion of *nondegeneracy for substructures* for the pair $(\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X))$ which is *a priori* different from the notion of *nondegeneracy* introduced in Mok [Mk08a] and Hong-Mok [HoM10] for mappings when X_0 is nonlinear. In §3 we compare the two notions of nondegeneracy for admissible pairs of rational homogeneous spaces of Picard number 1, proving their equivalence in cases of sub-diagram type by means of root space decompositions and Grothendieck splitting over minimal rational curves. In §4 we conclude the proof of Main Theorem 1 in a formulation that incorporates both the long-root and short-root cases. In §5 we consider uniruled projective manifolds $(X, \mathcal{K})$ endowed with minimal rational components. Denoting by $\pi : \mathscr{C}(X) \to X$ the accompanying VMRT structure of $(X, \mathcal{K})$, we introduce a general notion for sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$ of $\pi : \mathscr{C}(X) \to X$ defined by taking intersections of VMRTs with projectivizations of tangent subspaces, and a general notion of *nondegeneracy for substructures* to be given in Definition 5.3 for a *proper pair* $(\mathcal{B}, \mathcal{A})$ of subvarieties (cf. Definition 5.2) of a projective space. Our methods, combined with the method of analytic continuation developed in Hwang-Mok [HM01], [HM04b], Mok [Mk08a] and Hong-Mok [HoM10], will yield in §5 the following result giving sufficient conditions for a sub-VMRT structure on a uniruled projective manifold $(X, \mathcal{K})$ to be *rationally saturated.* For its formulation we need a condition on the sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$, called Condition (T), concerning the intersection

$\mathscr{C}(S) = \mathscr{C}(X) \cap \mathbb{P}T(S)$, to be given in Definition 5.4. For the precise meaning of terms in the following result we refer the reader to §5. We have

Theorem 1.4. *Let $(X, \mathcal{K})$ be an ordered pair consisting of a uniruled projective manifold X and a minimal rational component $\mathcal{K}$ on X and denote by $\pi : \mathscr{C}(X) \to X$ the VMRT structure on X associated to $(X, \mathcal{K})$. Assume that at a general point $x \in X$, the VMRT $\mathscr{C}_x(X)$ is irreducible. Write $B' \subset X$ for the enhanced bad locus of $(X, \mathcal{K})$. Let $W \subset X - B'$ be a connected open subset, and $S \subset W$ be a complex submanifold such that, writing $\mathscr{C}(S) := \mathscr{C}(X)|_S \cap \mathbb{P}T(S)$ and $\varpi := \pi\big|_{\mathscr{C}(S)}$, $\varpi : \mathscr{C}(S) \to S$ is a sub-VMRT structure satisfying* Condition (T). *Suppose furthermore that for a general point x on S and for each of the irreducible components $\Gamma_{k,x}$ of $\mathscr{C}_x(S)$, $1 \le k \le m$, the pair $(\Gamma_{k,x}, \mathscr{C}_x(X))$ is nondegenerate for substructures. Then, S is rationally saturated with respect to $(X, \mathcal{K})$.*

When X is of Picard number 1, by a line ℓ on X we mean a rational curve ℓ such that $c_1(\mathcal{O}(1)) \cdot \ell = 1$, where $\mathcal{O}(1)$ denotes the positive generator of $\mathrm{Pic}(X) \cong \mathbb{Z}$. We say that $(X, \mathcal{K})$ is a uniruling by lines to mean that members of $\mathcal{K}$ are lines. For $(X, \mathcal{K})$ uniruled by lines we prove a sufficient condition for the *algebraicity* of germs of sub-VMRT structures on them. For the formulation a holomorphic distribution $\mathcal{D}$ on a complex manifold M is said to be bracket generating if and only if, defining inductively $\mathcal{D}_1 = \mathcal{D}$, $\mathcal{D}_{k+1} = \mathcal{D}_k + [\mathcal{D}, \mathcal{D}_k]$, $\mathcal{D}_m|_U = T(U)$ on a neighborhood of U of a general point $x \in M$ for m sufficiently large. By a distribution we will mean a coherent subsheaf of the tangent sheaf.

Main Theorem 2. *In Theorem 1.4 suppose furthermore that $(X, \mathcal{K})$ is a projective manifold of Picard number 1 uniruled by lines and that the distribution $\mathcal{D}$ on S defined by $\mathcal{D}_x := \mathrm{Span}(\widetilde{\mathscr{C}_x}(S))$ is bracket generating. Then, there exists an irreducible subvariety $Z \subset X$ such that $S \subset Z$ and such that $\dim(Z) = \dim(S)$.*

Main Theorem 2 is proved by the process of adjoining rational curves. Unlike the *a priori* algebraic reconstruction of a uniruled projective manifold by the adjunction process, in the construction of a uniruled projective subvariety starting with a germ of sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$, the basic issue is to prove *algebraicity* starting with a germ of *complex submanifold S*. Under the assumption that minimal rational curves are of degree 1 we resolve the problem by means of techniques of analytic continuation. In §6 we prove the *Thickening Lemma* by which we construct a *collar* (an immersed complex submanifold) around a general standard rational curve ℓ issuing from S, allowing the sub-VMRT structure to be *propagated* . In §7 we prove results on Hartogs and Thullen extension to be applied to sub-VMRT structures. In §8 we devise an iterative scheme for the construction of a projective completion of a germ of sub-VMRT structure, and give a proof of Main Theorem 2 using extension results of §6 and §7. Central to the arguments is the construction of a certain projective "universal family" of chains of rational curves emanating from a base point of S.

As applications of analytic continuation of sub-VMRT structures, we study in §9 the Recognition Problem on sub-VMRT structures on ambient uniruled projective manifolds which are complete intersections on rational homogeneous spaces of Picard number 1.

Main Theorem 1 in the long-root cases was established in the thesis of Yunxin Zhang written under the supervision of Ngaiming Mok. At the same time, Zhang [Zh14] has in his thesis given a complete classification of admissible pairs of irreducible Hermitian symmetric spaces of the compact type, furnishing in particular new examples for the general theory.

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2. Linearly saturated sub-VMRT structures on rational homogeneous spaces of Picard number 1

We will be using some basic notions and terminology in the geometric theory of uniruled projective manifolds modeled on VMRTs. General references are Hwang-Mok [HM99b], Hwang [Hw01] and Mok [Mk08b], but the relevant notions will be recalled in §5. Consider a uniruled projective manifold X endowed with a minimal rational component $\mathcal{K}$. The VMRT structure on X is given by $\pi : \mathscr{C}(X) \to X$, $\mathscr{C}(X) \subset \mathbb{P}T(X)$, in which the fibers $\mathscr{C}_x(X) \subset \mathbb{P}T_x(X)$ are the VMRTs of $(X, \mathcal{K})$. Here $\mathscr{C}_x(X)$ is defined only for x lying on the Zariski open subset $X - B$ of X, where $B \subset X$ is the bad locus of $(X, \mathcal{K})$. Note that $B = \emptyset$ when X is a rational homogeneous space of Picard number 1.

We introduced in [HoM10, Proposition 2.5] the notion of privileged coordinate charts. Here we give a simplified but equivalent definition.

Definition 2.1. *Let $(X, \mathcal{K})$ be a uniruled projective manifold endowed with a minimal rational component and denote by $\pi : \mathscr{C}(X) \to X$ the associated VMRT structure. Let (U, z^i) be a holomorphic coordinate* *chart on X, where U is disjoint from the bad locus B of $(X, \mathcal{K})$. (U, z^i) is said to be privileged if and only if for any minimal rational curve ℓ passing through U, the set $\ell \cap U$ described in terms of the coordinates (z^i) is an open subset of an affine line.*

Remark Let (U, z^i) be a holomorphic coordinate chart on X. Write $T(X)|_U \cong U \times V$ for the trivialization of the holomorphic tangent bundle $T(X)$ over U in terms of a standard basis of $T(X)|_U$ consisting of $\{\frac{\partial}{\partial z_i}\}$, and identify hence $T_x(U)$ for every point $x \in U$ with the fixed complex vector space V. In the definition of privileged coordinate charts in [HoM10], an additional assumption was imposed, viz., that the VMRTs of X along $\ell \cap U$ are tangentially constant, i.e., there exists some fixed complex vector subspace $P_\ell \subset V$ such that $P_{\alpha_x} := T_{\alpha_x}(\widetilde{\mathscr{C}_x}(X)) \equiv P_\ell$ for every point $x \in \ell \cap U$, $T_x(\ell) := \mathbb{C}\alpha_x$. We observe here that the latter condition is automatically satisfied. To see this, let (U, z^i) be a privileged coordinate chart in the sense of Definition 2.1, ℓ be a minimal rational curve such that $\ell \cap U \neq \emptyset$, $x_0 \in \ell \cap U$, $T_{x_0}(\ell) =: \mathbb{C}\alpha_{x_0}$. In terms of (z^i) write $F(s,t) = z(x_0) + s\alpha(t); s \in \mathbb{C}, t \in \Delta^p$; for an effective holomorphic parametrization of affine lines passing through x_0 such that $[\alpha(t)] \in \mathscr{C}_{x_0}(X)$ for $t \in \Delta^p$ and $\alpha(0) = \alpha$. Denote the image by Σ. Then, for $x \in \ell \cap U, x \neq x_0$, we have $P_{\alpha_x} = T_x(\Sigma)$ by the deformation theory of rational curves (cf. §5). Hence, $P_{\alpha_x} = \mathrm{Span}\left\{\alpha_x, \frac{\partial \alpha}{\partial t_1}\big|_{t=0} \cdots, \frac{\partial \alpha}{\partial t_p}\big|_{t=0}\right\} =: P_\ell$ for $x \in \ell - \{x_0\}$. Replacing the base point $x_0 \in \ell \cap U$ by $x_1 \in (\ell \cap U) - \{x_0\}$ we conclude that $P_{\alpha_{x_0}} = P_\ell$ too, i.e., P_{α_x} is tangentially constant as x travels along $\ell \cap U$.

Let $X \subset \mathbb{P}^N$ be a projective manifold uniruled by projective lines, and denote by $\mathcal{K}$ a minimal rational component consisting of projective lines, by $B \subset X$ the bad locus of $(X, \mathcal{K})$. Then, for the VMRT structure $\pi : \mathscr{C}(X) \to X$ associated to $(X, \mathcal{K})$, $\mathscr{C}(X) \subset \mathbb{P}T(X - B)$ is nonsingular, and π is a submersion (cf. Mok [Mk08b, Lemma 3]). For $x \in X - B$, using affine coordinate charts on $\mathbb{P}^N$ and linear projections into Euclidean spaces, a germ of open subset of a projective line passing through x is projected to a germ of open subset on an affine line. Thus, privileged coordinate charts always exist in such cases. In particular, this is the case of a rational homogeneous space $X = G/P$ embedded as a projective submanifold of $\mathbb{P}(\Gamma(X, \mathcal{O}(1))^*)$, where $\mathcal{O}(1)$ denotes the positive generator of the Picard group $\mathrm{Pic}(X) \cong \mathbb{Z}$. The Harish-Chandra coordinates on a compact Hermitian symmetric space, on which the minimal rational curves appear as affine lines and the VMRTs form a constant family, serve as a special case of privileged coordinates.

In terms of *any* holomorphic coordinate chart (U, z^i) we have a holomorphic basis $\left\{\frac{\partial}{\partial z_i} : 1 \leq i \leq n\right\}$ of $T(U)$ dual to the holomorphic basis $\left\{dz^i : 1 \leq i \leq n\right\}$ of $T^*(U)$. Correspondingly, we have coordinates

$(z^i; w^j) = (z^1, \cdots, z^n; w^1, \cdots, w^n)$ on $T(U)$, where $(z^i; w^j)$ denotes the vector $w^1\frac{\partial}{\partial z_1} + \cdots + w^n\frac{\partial}{\partial z_n}$ at a point on U with coordinates (z^i). For any point $x \in U$ we have $T(U) \cong U \times T_x(U) \cong U \times \mathbb{C}^n$. With respect to the described trivialization of $T(U)$, we sometimes use the notation $\eta \in T_x(U)$ also for the vector $(y, \eta) \in T_y(U)$ for $y \in U$. Wherever it is necessary to make a reference to the base point we will write η_y for (y, η), etc.

We proceed to consider the holomorphic tangent bundle $T(T(U))$ of $T(U)$. In the coordinates $(z^i; w^j)$ on $T(U)$ we have a holomorphic basis $\left\{\frac{\partial}{\partial z_i}, \frac{\partial}{\partial w_j} : 1 \leq i,\ j \leq n\right\}$ of $T(T(U))$ dual to the basis $\left\{dz^i, dw^j : 1 \leq i, j \leq n\right\}$ of $T^*(T(U))$. Depending on the context, $\frac{\partial}{\partial z_i}$ denotes either a certain tangent vector belonging to $T_x(U)$ for some $x \in U$, or a certain tangent vector at some $(x, \eta) \in T(U)$. Write $\check{T}_{x,\eta}(U) \subset T_{x,\eta}(T(U))$ for the vector subspace of 'horizontal' vectors spanned by $\frac{\partial}{\partial z_i}$, $1 \leq i \leq n$, and $\hat{T}_{x,\eta}(U) \subset T_{x,\eta}(T(U))$ for the vector subspace of (intrinsically defined) vertical tangent vectors spanned by $\frac{\partial}{\partial w_i}$, $1 \leq i \leq n$. There is a natural isomorphism $\check{\varphi}_{x,\eta} : T_x(U) \xrightarrow{\cong} \check{T}_{x,\eta}(U)$ given by $\check{\varphi}_{x,\eta}\big(\frac{\partial}{\partial z_i}\big) = \frac{\partial}{\partial z_i}$ for $1 \leq i \leq n$. (Here the notation $\frac{\partial}{\partial z_i}$ carries two different meanings, as explained above.) At the same time, there is a natural isomorphism $\hat{\varphi}_{x,\eta} : T_x(U) \xrightarrow{\cong} \hat{T}_{x,\eta}(U)$ given by $\hat{\varphi}\big(\frac{\partial}{\partial z_i}\big) = \frac{\partial}{\partial w_i}$ for $1 \leq i \leq n$. With these identifications, we have $T_{x,\eta}(T(U)) = \check{T}_{x,\eta}(U) \oplus \hat{T}_{x,\eta}(U)$. When $\eta \in T_x(U)$ is understood we also write $\check{T}_x(U)$ for $\check{T}_{x,\eta}(U)$ and $\hat{T}_x(U)$ for $\hat{T}_{x,\eta}(U)$. Moreover, for a vector subspace $E \subset T_x(U)$ and for $\eta \in T_x(U)$ we have a vector subspace $\check{E} \oplus \hat{E} \subset T_{x,\eta}(T(U))$, where $\check{E} = \check{\varphi}_{x,\eta}(E)$ and $\hat{E} = \hat{\varphi}_{x,\eta}(E)$. We also write $\check{\varphi}_{x,\eta} = \check{\varphi}_{\eta_x}$ and $\hat{\varphi}_{x,\eta} = \hat{\varphi}_{\eta_x}$. We note that the isomorphism $\hat{\varphi}_{x,\eta} : T_x(U) \xrightarrow{\cong} \hat{T}_{x,\eta}(U)$ is intrinsically defined while the image of $\check{\varphi}_{x,\eta} : T_x(U) \xrightarrow{\cong} \check{T}_{x,\eta}(U) \subset T_{x,\eta}(T(U))$ depends on (z^i).

Denote by $0(X) \subset T(X)$ the zero section and by $\lambda : T(X) - 0(X) \to \mathbb{P}T(X)$ the canonical projection. For a distribution $\mathscr{E}$ on a smooth open subset $\mathcal{O} \subset \mathscr{C}(X)$ we have a distribution $\widetilde{\mathscr{E}}$ on $\widetilde{\mathcal{O}} = \lambda^{-1}(\mathcal{O})$ given by $\widetilde{\mathscr{E}} := (d\lambda)^{-1}(\mathscr{E})$.

Let $\mathcal{O} \subset \mathscr{C}(X)$ be a connected open subset such that $\pi|_{\mathcal{O}} : \mathcal{O} \to X$ is a submersion. There is a distribution $\mathscr{P}$ on $\mathcal{O}$ defined by assigning to any $[\alpha] \in \mathcal{O}$, $\pi([\alpha]) =: x \in X - B$, the vector subspace $\mathscr{P}_{[\alpha]} \subset T_{[\alpha]}(\mathscr{C}(X))$ given by $\mathscr{P}_{[\alpha]} := (d\pi)^{-1}(P_\alpha)$, where $P_\alpha \subset T_x(X)$ is defined by $\hat{P}_\alpha := T_\alpha(\widetilde{\mathscr{C}_x}(X))$. (For the original definition of P_α in terms of Grothendieck splitting as in Hwang-Mok [HM04b, §3] we defer to §5, paragraph preceding Lemma 5.2.) Assuming now the existence of privileged coordinates, the distribution $\widetilde{\mathscr{P}}$ on $\widetilde{\mathscr{C}}(X)$ can be identified, as follows.

Lemma 2.1. *Let $(X, \mathcal{K})$ be a uniruled projective manifold X of dimension n endowed with a minimal rational component $\mathcal{K}$, and $B \subset X$ be the bad locus of $(X, \mathcal{K})$, and assume that the associated VMRT structure $\pi : \mathscr{C}(X) \to X$ is a regular family over $X - B$. Let (U, z^i) be a privileged coordinate chart on a connected open subset $U \subset X - B$. Suppose $x \in U$ and $\alpha \in \widetilde{\mathscr{C}_x}(X)$. Then, in terms of the coordinates $(z^1, \cdots, z^n; w^1, \cdots, w^n)$ on $T(U) = U \times T_x(U) = U \times \mathbb{C}^n \subset \mathbb{C}^n \times \mathbb{C}^n$ arising from the privileged coordinates as described in the above, we have $\widetilde{\mathscr{P}_\alpha} = \check{P}_\alpha \oplus \hat{P}_\alpha$.*

Proof. Let ℓ be the line on $\mathbb{C}^n$ passing through x with $T_x(\ell) = \mathbb{C}\alpha$. We may take U to be convex so that $U \cap \ell$ is connected. Let $y \in U \cap \ell$ be distinct from x and consider a holomorphic pencil of lines $\{\ell(t) : |t| < \epsilon\}$, $\epsilon > 0$, where $\ell(t)$ are minimal rational curves on X passing through y, $\ell(0) = \ell$. Replacing α by some proportional vector, we may assume that $x = y + \alpha$ in the privileged coordinates, and that there is a holomorphic map $F : \Delta(2) \times \Delta(\epsilon) \to U$ for some $\epsilon > 0$ given by $F(s,t) = y + s\alpha(t)$ such that $\alpha(0) = \alpha$, $F(\Delta(2) \times \{t\}) \subset \ell(t) \cap U$ for $|t| < \epsilon$ and such that F is a holomorphic embedding on some neighborhood $\mathcal{O}$ of $\{1\} \times \Delta(\epsilon)$. Thus $\Sigma := F(\mathcal{O})$ is a complex surface containing the germ of ℓ at x. Now, it follows from $\alpha(t) = \frac{\partial F}{\partial s}(1,t)$ that $\alpha(t) \in T_{x(t)}(\ell(t))$ at $x(t) = F(1,t)$. As t varies, $\chi(t) := (x(t), \alpha(t)) \in U \times T_x(U)$ describes a holomorphic curve Γ lying on $\widetilde{\mathscr{C}}$, $\chi(t) \in \widetilde{\mathscr{C}}_{x(t)}(X)$. Writing $\varpi(t) := \frac{\partial \chi}{\partial t}(t) = (\frac{\partial F}{\partial t}(1,t), \alpha'(t))$, by definition $\varpi(t) \in T_{\chi(t)}(\Gamma) \subset T_{\chi(t)}(\widetilde{\mathscr{C}})$. We have $\varpi(t) = (\alpha'(t), \alpha'(t))$. In particular, at $t = 0$ we have $\varpi(0) = (\alpha'(0), \alpha'(0)) \in T_\alpha(\widetilde{\mathscr{C}})$. Writing $\widetilde{\pi} : \widetilde{\mathscr{C}}(X) \to X - B$ for the canonical projection, we have $\widetilde{\mathscr{P}_\alpha} = (d\widetilde{\pi})^{-1}(P_\alpha)$. From $\alpha'(0) \in T_x(\Sigma) \subset P_\alpha$ and $(\alpha'(0), \alpha'(0)) \in T_\alpha(\widetilde{\mathscr{C}})$ it follows that $(\alpha'(0), \alpha'(0)) \in \widetilde{\mathscr{P}_\alpha}$. Since $\hat{P}_\alpha \subset \widetilde{\mathscr{P}_\alpha}$, we have $(0, \alpha'(0)) \in \widetilde{\mathscr{P}_\alpha}$. Since also $(\alpha'(0), \alpha'(0)) \in \widetilde{\mathscr{P}_\alpha}$, it follows that $(\alpha'(0), 0) \in \widetilde{\mathscr{P}_\alpha}$. Given that $\alpha'(0) \in P_\alpha$ is arbitrary excepting that $\alpha'(0) \notin \mathbb{C}\alpha$ (so that F is an immersion at $(1,0)$), we conclude that $\check{P}_\alpha \subset \widetilde{\mathscr{P}_\alpha}$, Hence, $\widetilde{\mathscr{P}_\alpha} = \check{P}_\alpha \oplus \hat{P}_\alpha$, as asserted. □

For a VMRT structure $\pi : \mathscr{C}(X) \to X$, $\mathscr{C}_x(X) \subset \mathbb{P}T_x(X)$ may be linearly degenerate. In this case there is a meromorphic distribution D on X where D_x is the linear span of $\widetilde{\mathscr{C}_x}(X)$ at a general point x on X. For instance, when X is a non-symmetric rational homogeneous space of Picard number 1 marked at a long simple root, $D \subsetneq T(X)$ is the minimal nonzero distribution invariant under $\mathrm{Aut}(X)$.

The main result in this section is Proposition 2.3, which gives a sufficient condition for a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ modeled on an admissible pair (X_0, X) to be linearly saturated. In the proof here we make use of the existence of privileged coordinate charts, and a generalization will be given in §5 where the latter assumption is dropped.

For the proofs of Proposition 2.3 and its generalization in §5 we will need the following result concerning the distribution D established in Hwang-Mok [HM98]. For later references we state the general result for a uniruled projective manifold X endowed with a minimal rational component $\mathcal{K}$.

Proposition 2.1. *Let $(X, \mathcal{K})$ be a uniruled projective manifold endowed with a minimal rational component, and $\pi : \mathscr{C}(X) \to X$ be the associated VMRT structure. Let B be the bad locus of $(X, \mathcal{K})$ and assume that the VMRTs $\mathscr{C}_x(X)$ are irreducible for $x \in X - B$. Let $U \subset X - B$ be a connected open subset and assume that the linear span D_x of $\widetilde{\mathscr{C}_x}(X)$ is of dimension independent of $x \in U$, thus defining a holomorphic distribution $D \subset T(U)$ over U. Let $\widetilde{\alpha} \in \Gamma(U, T(U))$ be such that $\widetilde{\alpha}(x)$ is a smooth point of $\widetilde{\mathscr{C}_x}(X)$ for any point $x \in U$, and $\widetilde{\xi} \in \Gamma(U, T(U))$ be such that $\widetilde{\xi}(x) \in T_{\widetilde{\alpha}(x)}\big(\widetilde{\mathscr{C}_x}(X)\big)$. Then, the Lie bracket $[\widetilde{\alpha}, \widetilde{\xi}] \in \Gamma(U, D)$.*

Remark Proposition 2.1 results from the deformation theory of rational curves, from which it follows that a surface Σ swiped out by a pencil of minimal rational curves emanating from a point $x \in U$ is an integral surface of D.

Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1. Recall that there is an implicitly understood equivariant holomorphic embedding $i : X_0 \hookrightarrow X$. We identify X_0 with $i(X_0)$, hence as a complex submanifold of X. Writing $\pi : \mathscr{C}(X) \to X$ for the VMRT structure on X, and $\pi_0 : \mathscr{C}(X_0) \to X_0$ for the VMRT structure on X_0, from the definition of admissible pairs note that $\mathscr{C}_y(X_0) = \mathscr{C}_y(X) \cap \mathbb{P}T_y(X_0)$ for any point $y \in X_0$, i.e., $\widetilde{\mathscr{C}_y}(X_0) = \widetilde{\mathscr{C}_y}(X) \cap T_y(X_0)$. Moreover, for $\alpha \in \widetilde{\mathscr{C}_y}(X)$, we have (†) $T_\alpha(\widetilde{\mathscr{C}_y}(X_0)) = T_\alpha(\widetilde{\mathscr{C}_y}(X)) \cap T_y(X_0)$. The latter follows readily in the case of admissible pairs of subdiagram type in the long-root cases by interpreting the VMRTs $\mathscr{C}_y(X_0)$ and $\mathscr{C}_y(X)$ as orbits. This remains the case for admissible pairs in general and in fact in a much more general setting, where we formulate in Definition 5.4 a general condition called Condition (T) for generalized sub-VMRT structures on uniruled projective manifolds. From now on we assume known the fact that (†) holds for admissible pairs (X_0, X) and refer the reader to Lemma 5.5 for a proof in the general setting.

Let now $S \subset W$ be a complex submanifold of a connected open subset W of X inheriting a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ modeled on (X_0, X). We will need to apply the condition (†) above on VMRTs to $\varpi : \mathscr{C}(S) \to S$. For a fixed reference point $0 \in X_0$ and for any $\alpha \in \widetilde{\mathscr{C}_0}(X_0)$, we have (†) $T_\alpha(\widetilde{\mathscr{C}_0}(X_0)) = T_\alpha(\widetilde{\mathscr{C}_0}(X)) \cap T_0(X_0)$. For any point $x \in S$, by definition $\big(\mathscr{C}_x(S) \subset \mathscr{C}_x(X)\big)$ is projectively equivalent to $\big(\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X)\big)$. The condition (†) translates into a condition for the pair $(\mathscr{C}_x(S), \mathscr{C}_x(X))$ which will bear the same name, viz., for any $x \in S$ and for any $[\alpha] \in \mathscr{C}_x(S)$, we have (†) $T_\alpha(\widetilde{\mathscr{C}_x}(S)) = T_\alpha(\widetilde{\mathscr{C}_x}(X)) \cap T_x(S)$.

To examine linear saturation we need to consider the variation of $\mathscr{C}_x(S)$ as x varies over S. For this purpose we need to consider certain holomorphic vector fields adapted to the sub-VMRT structure, and the following lemma furnishes the necessary starting point. For a neighborhood U of $0 \in \mathbb{C}^m$ and a vector-valued function $f(z,t)$ on $U \times \Delta(\epsilon)$, $\epsilon > 0$, and for an integer $k \geq 0$ we write $f(z,t) = O(t^k)$ to mean that, shrinking U and ϵ if necessary, there exists a positive constant C such that $\|f(z,t)\| \leq C|t|^k$ on $U \times \Delta(\epsilon)$. In what follows recall that for any point $x \in X$ and for any tangent vector $\alpha \in \widetilde{\mathscr{C}_x}(X)$, we have $P_\alpha = T_\alpha(\widetilde{\mathscr{C}_x}(X))$. ($P_\alpha$ consists of vertical vectors at α, and $\hat{P}_\alpha$ would be a more precise notation as in previous paragraphs. We will however refrain from such and similar notation whenever the meaning is clear from the context.) We have

Lemma 2.2. *Suppose $x \in S$, $\alpha \in \widetilde{\mathscr{C}_x}(S)$, and $\epsilon > 0$. Let $\{\alpha_t : |t| < \epsilon\}$ be a holomorphic arc on $\widetilde{\mathscr{C}_x}(S)$ such that $\alpha_0 = \alpha$. Then, shrinking ϵ if necessary there exists a neighborhood U of x on X and a holomorphic map $A : U \times \Delta(\epsilon) \to \widetilde{\mathscr{C}}(X)$ such that* (a) *$A(x,t) = \alpha_t$ for $t \in \Delta(\epsilon)$,* (b) *$A(z,t) \in \widetilde{\mathscr{C}_z}(X)$ whenever $z \in U$ and* (c) *$A(z',t) \in \widetilde{\mathscr{C}_{z'}}(S)$ whenever $z' \in U \cap S$. Hence, for $t \in \Delta(\epsilon)$, $\widetilde{\alpha}_t(z) := A(z,t)$ defines a holomorphic section of $\widetilde{\pi} : \widetilde{\mathscr{C}}(X) \to X$ over U such that $\widetilde{\alpha}_t|_{U\cap S}$ is a section of $\widetilde{\varpi} : \widetilde{\mathscr{C}}(S) \to S$ over $U \cap S$. We have $\widetilde{\alpha}_t = \widetilde{\alpha} + t\widetilde{\xi} + O(t^2)$, where $\widetilde{\alpha}(z) := A(z,0)$, $\widetilde{\xi}(z) \in P_{\widetilde{\alpha}(z)}$ for $z \in U$, and $\widetilde{\xi}(z') \in P_{\widetilde{\alpha}(z')} \cap T_{z'}(S)$ for $z' \in U \cap S$.*

Proof. By hypothesis $\varpi : \mathscr{C}(S) \to S$ is a sub-VMRT structure modeled on (X_0, X). Hence, by Definition 1.2, for every point $x \in S$ there exists a neighborhood U of x on S and a trivialization of the holomorphic projective bundle $\mathbb{P}T(X)|_U$ given by $\Phi : \mathbb{P}T(X)|_U \xrightarrow{\cong} \mathbb{P}T_0(X) \times U$ such that (1) $\Phi(\mathscr{C}(X)|_U) = \mathscr{C}_0(X) \times U$ and (2) $\Phi(\mathscr{C}(S)|_U) = \mathscr{C}_0(X_0) \times U$. Shrinking U if necessary Φ can be lifted to a holomorphic bundle isomorphism $\Phi' : T(X)|_U \xrightarrow{\cong} U \times T_0(X)$ such that (1) $\Phi'(\widetilde{\mathscr{C}}(X)|_U) = U \times \widetilde{\mathscr{C}_0}(X)$ and (2) $\Phi'(\widetilde{\mathscr{C}}(S)|_U) = U \times \widetilde{\mathscr{C}_0}(X_0)$. To extend $A(x,t) := \alpha_t$ to a holomorphic section $A(z,t)$ of $\widetilde{\pi}|_U \times \mathrm{id}_{\Delta(\epsilon)} : \widetilde{\mathscr{C}}(X) \times \Delta(\epsilon) \to U \times \Delta(\epsilon)$ such that $A(z',t) \in \widetilde{\mathscr{C}_{z'}}(S)$ whenever $z' \in U \cap S$ and $|t| < \epsilon$, we may as an example define $A(z,t)$ by setting $\Phi'(A(z,t)) = (z; A(x,t))$. In other words, for each $t \in \Delta(\epsilon)$, we may take $A(z,t)$ to be a "constant" section in z over U with respect to the trivialization over U of the affinized sub-VMRT structure modeled on (X_0, X) given by Φ'. For *any* choice of $\widetilde{\alpha}_t(z) := A(z,t)$ satisfying (a), (b) and (c), defining $\widetilde{\xi}(z) := \frac{\partial}{\partial t}\big|_{t=0} A(z,t)$ we have $\widetilde{\xi}(z) \in T_{\widetilde{\alpha}(z)}(\widetilde{\mathscr{C}}(X)) = P_{\widetilde{\alpha}(z)}$, and it follows that $\widetilde{\alpha}_t = \widetilde{\alpha} + t\widetilde{\xi} + O(t^2)$. Finally, $\widetilde{\xi}|_{U\cap S}$ is a holomorphic section of $T(S)$ over $U \cap S$, hence $\widetilde{\xi}(z') \in P_{\widetilde{\alpha}(z')} \cap T_{z'}(S)$, proving Lemma 2.2. □

Our study of the geometry of VMRT structures and sub-VMRT structures will involve local differential geometry in terms of the flat connection on Euclidean spaces. Let V be a finite-dimensional complex vector space and denote by ∇ the flat Euclidean connection on V. For a complex submanifold A on some connected open subset of V we have the second fundamental form $\sigma := \sigma_{A|V}$. For $\alpha \in A$ and $\xi, \eta \in T_\alpha(A)$ we have $\sigma_\alpha(\xi,\eta) := \nabla_\xi \widetilde{\eta} \bmod T_\alpha(A)$, where $\widetilde{\eta}$ is a holomorphic vector field defined on some open neighborhood U of α on A such that $\widetilde{\eta}(\alpha) = \eta$. $\sigma_\alpha(\xi,\eta)$ thus defined is independent of the choice of holomorphic vector field $\widetilde{\eta}$ extending η. Moreover $\sigma_\alpha(\xi,\eta) = \sigma_\alpha(\eta,\xi)$ by the torsion-freeness of the flat connection. Thus, we have the linear map $\sigma_\alpha : S^2T_\alpha(A) \to V/T_\alpha(A) =: N_{A|V,\alpha}$. For our study typically we have a projective variety $E \subset \mathbb{P}V$, and $A \subset V - \{0\}$ is the smooth locus of the affinization $\widetilde{E}$ of E. Then, A is invariant under multiplication by nonzero complex numbers. In this case, any vector $\xi \in T_\alpha(A)$ extends to a constant vector field $\widetilde{\xi}$ (in terms of Euclidean coordinates) along the punctured line $\Lambda_\alpha := \mathbb{C}\alpha - \{0\}$, and it follows that $\sigma_\alpha(\alpha,\xi) = 0$. Thus, considered as a vector-valued symmetric bilinear form, the kernel of σ_α always contains $\mathbb{C}\alpha$. For $[\alpha] \in \mathrm{Reg}(E)$, noting that $T_{[\alpha]}(E) = V/\mathbb{C}\alpha$, σ_α descends to $\sigma'_{[\alpha]} : S^2T_{[\alpha]}(E) \to V/T_\alpha(A) \cong (V/\mathbb{C}\alpha)/(T_\alpha(A)/\mathbb{C}\alpha) = T_{[\alpha]}(\mathbb{P}V)/T_{[\alpha]}(E) =: N_{\mathrm{Reg}(E)|\mathbb{P}V,[\alpha]}$, thus defining the projective second fundamental form $\sigma' = \sigma'_{\mathrm{Reg}(E)|\mathbb{P}V}$ on $\mathrm{Reg}(E)$.

Starting with Lemma 2.2, using the Euclidean flat connection on privileged coordinate charts and applying Lemma 2.1 and Proposition 2.1, we obtain the following result on sub-VMRT structures expressed in terms of second fundamental forms on affinizations of VMRTs.

Proposition 2.2. *In the notation adopted in Lemma 2.2, define* $\beta_t := (\nabla_{\widetilde{\alpha}_t}\widetilde{\alpha}_t)(x)$, $\beta := \beta_0$, *where* ∇ *is the Euclidean flat connection on a privileged coordinate chart* U *on* X. *Then, letting* D *be the distribution on* X *spanned by* $\widetilde{\mathscr{C}}(X)$, *we have* $\frac{d\beta_t}{dt}\big|_{t=0} \in P_\alpha + (D_x \cap T_x(S))$. *As a consequence, denoting by* ν_α *the canonical projection from* $T_x(X)/P_\alpha$ *into* $T_x(X)/(P_\alpha + (D_x \cap T_x(S)))$, *and writing* $\tau_\alpha := \nu_\alpha \circ \sigma_\alpha$, *the tangent vector* $\beta \in T_\alpha(\widetilde{\mathscr{C}}_x(X))$ *satisfies* $\tau_\alpha(\beta,\xi) := \nu_\alpha(\sigma_\alpha(\beta,\xi)) = 0$ *for every* $\xi \in P_\alpha \cap T_x(S)$, *where* σ_α *stands for the second fundamental form at* α *of* $\widetilde{\mathscr{C}}_x(X)$ *in* $T_x(X)$ *with respect to the Euclidean flat connection* ∇ *on the vector space* $T_x(X)$.

Proof. By Lemma 2.2 we have a holomorphic family of sections $\{\widetilde{\alpha}_t : |t| < \epsilon\}$ of $\widetilde{\mathscr{C}}(X)$ over U, $\widetilde{\alpha}_t = \widetilde{\alpha} + t\widetilde{\xi} + O(t^2)$. Write $\Gamma_t = \{\widetilde{\alpha}_t(z) : z \in U\} \subset \widetilde{\mathscr{C}}(X)$, so that $\Gamma_t \subset \widetilde{\mathscr{C}}(X)$ is a complex submanifold. Hence, for $|t| < \epsilon$, $(\alpha_t, \beta_t) = (\alpha_t, \nabla_{\widetilde{\alpha}_t}\widetilde{\alpha}_t(x)) \in T_{\alpha_t}(\Gamma_t) \subset T_{\alpha_t}(\widetilde{\mathscr{C}}(X))$.

(Here α_t denotes in $(\alpha_t, \nabla_{\widetilde{\alpha}_t}\widetilde{\alpha}_t(x))$ *a tangent vector in* $T_x(X)$ while the same notation denotes in $T_{\alpha_t}(\Gamma_t)$ and $T_{\alpha_t}(\widetilde{\mathscr{C}}(X))$ *a point on* $\widetilde{\mathscr{C}}(X)$.) Since $\alpha_t \in P_{\alpha_t}$, we have $T_{\alpha_t}(\Gamma_t) \subset (d\widetilde{\pi})^{-1}(P_{\alpha_t}) = \widetilde{\mathscr{P}_{\alpha_t}}$. By Proposition 2.1, $\widetilde{\mathscr{P}_{\alpha_t}} = \check{P}_{\alpha_t} \oplus \hat{P}_{\alpha_t}$, hence $\beta_t \in P_{\alpha_t}$. Expanding in t we have

$$\beta_t = \nabla_{\widetilde{\alpha}_t}\widetilde{\alpha}_t(x) = \nabla_{\widetilde{\alpha}}\widetilde{\alpha}(x) + t(\nabla_{\widetilde{\xi}}\widetilde{\alpha}(x) + \nabla_{\widetilde{\alpha}}\widetilde{\xi}(x)) + O(t^2).$$

Hence, $\frac{d\beta_t}{dt}\big|_{t=0} = \nabla_{\widetilde{\xi}}\widetilde{\alpha}(x) + \nabla_{\widetilde{\alpha}}\widetilde{\xi}(x)$. From $\Gamma_t \subset \widetilde{\mathscr{C}}(X)$ it follows that $(\xi, \nabla_{\widetilde{\xi}}\widetilde{\alpha}(x)) \in T_\alpha(\widetilde{\mathscr{C}}(X))$. Again, since $\xi \in P_\alpha$ and $\widetilde{\mathscr{P}_\alpha} = \check{P}_\alpha \oplus \hat{P}_\alpha$ we deduce that $\nabla_{\widetilde{\xi}}\widetilde{\alpha}(x) \in P_\alpha$. Now, by the torsion-freeness of the flat connection ∇ on U we have

$$\nabla_{\widetilde{\xi}}\widetilde{\alpha}(x) - \nabla_{\widetilde{\alpha}}\widetilde{\xi}(x) - [\widetilde{\alpha}, \widetilde{\xi}](x) = 0.$$

Since by construction $\widetilde{\alpha}$ is a holomorphic field of minimal rational tangents on U and $\widetilde{\xi}$ is a holomorphic vector field on U such that $\widetilde{\xi}(z) \in P_{\widetilde{\alpha}(z)}$ for $z \in U$, $\widetilde{\alpha}$ and $\widetilde{\xi}$ satisfy the hypothesis in Proposition 2.1, hence the Lie bracket $[\widetilde{\alpha}, \widetilde{\xi}] \in \Gamma(U, D)$. On the other hand, by definition both $\widetilde{\alpha}|_{U\cap S}$ and $\widetilde{\xi}|_{U\cap S}$ are holomorphic vector fields on $U \cap S$, so that $[\widetilde{\alpha}, \widetilde{\xi}]\big|_{U\cap S} \in \Gamma(U \cap S, T(S))$. Hence, $[\widetilde{\alpha}, \widetilde{\xi}](x) \in D_x \cap T_x(S)$. As a consequence, we have

$$\begin{aligned}\frac{d\beta_t}{dt}\Big|_{t=0} = \nabla_{\widetilde{\xi}}\widetilde{\alpha}(x) + \nabla_{\widetilde{\alpha}}\widetilde{\xi}(x) = 2\nabla_{\widetilde{\xi}}\widetilde{\alpha}(x) - \big(\nabla_{\widetilde{\xi}}\widetilde{\alpha}(x) - \nabla_{\widetilde{\alpha}}\widetilde{\xi}(x)\big) = \\ 2\nabla_{\widetilde{\xi}}\widetilde{\alpha}(x) - [\widetilde{\alpha}, \widetilde{\xi}](x) \in P_\alpha + (D_x \cap T_x(S)).\end{aligned}$$

Finally, denoting by ∇' the Euclidean flat connection on $T_x(X)$, we may interpret $\beta(t) := \beta_t$ as a holomorphic vector field along the holomorphic arc $\{\alpha_t\} \subset \widetilde{\mathscr{C}_x}(S)$ and conclude that $\nabla'_{\xi}\beta(0) = \frac{d\beta_t}{dt}\big|_{t=0} \in P_\alpha + (D_x \cap T_x(S))$. We have hence $\tau_\alpha(\beta, \xi) := \nu_\alpha(\sigma_\alpha(\beta, \xi)) = \nabla'_{\xi}\beta(0) \bmod P_\alpha + (D_x \cap T_x(S)) = 0$. Since the holomorphic arc $\{\alpha_t\}$ on $\widetilde{\mathscr{C}_x}(S)$ is arbitrary except that $\alpha_0 = \alpha$, $\xi = \frac{d}{dt}\big|_{t=0}\alpha_t \in T_\alpha(\widetilde{\mathscr{C}_x}(S))$ is nonzero but otherwise arbitrary. We have $T_\alpha(\widetilde{\mathscr{C}_0}(X_0)) = P_\alpha \cap T_0(X)$ by the condition (†), and Proposition 2.2 follows. □

By means of Proposition 2.2 we obtain a sufficient condition for a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ modeled on (X_0, X) to be linearly saturated.

Proposition 2.3. *Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1, and $\pi : \mathscr{C}(X) \to X$ be the associated VMRT structure. Denote by $D \subset T(X)$ the invariant distribution spanned by $\widetilde{\mathscr{C}}(X)$, and by $\mathscr{F} \subset T(\mathscr{C}(X))$ the rank one distribution on $\mathscr{C}(X)$ corresponding to the tautological foliation on $\mathscr{C}(X)$. At a reference point $0 \in X_0$, for $\alpha \in \widetilde{\mathscr{C}_0}(X_0)$ denote by $\sigma_\alpha : S^2P_\alpha \to T_0(X)/P_\alpha$, $P_\alpha = T_\alpha(\widetilde{\mathscr{C}_0}(X))$, the second fundamental form of $\widetilde{\mathscr{C}_0}(X) \subset T_0(X)$ at α, and by ν_α the canonical projection $\nu_\alpha : T_0(X)/P_\alpha \to T_0(X)/(P_\alpha +$*

$(D_0 \cap T_0(X_0)))$. *Write* $\tau_\alpha := \nu_\alpha \circ \sigma_\alpha$. *Suppose for a general point* $\alpha \in \widetilde{\mathscr{C}_0}(X_0)$, *the following statement* ($\sharp$) *holds true*: ($\sharp$) $\tau_\alpha(\eta, \xi) = 0$ *for all* $\xi \in P_\alpha \cap T_0(X_0)$ *implies* $\eta \in P_\alpha \cap T_0(X_0)$. *Let now* $S \subset W$ *be a complex submanifold of some connected open subset* W *of* X *inheriting a sub-VMRT structure* $\varpi : \mathscr{C}(S) \to S$ *modeled on* (X_0, X). *Then,* $\varpi : \mathscr{C}(S) \to S$ *is linearly saturated, i.e.,* $\mathscr{F}\big|_{\mathscr{C}(S)} \subset T(\mathscr{C}(S))$. *As a consequence, for any point* $x \in S$, *and any projective line* ℓ *passing through* x *and tangent to* S *at* x, *the germ of* ℓ *at* x *must lie on* S.

Proof. Let $x \in S$ be an arbitrary point. For $\gamma \in \widetilde{\mathscr{C}_x}(X)$ denote also by σ_γ the second fundamental form of $\widetilde{\mathscr{C}_x}(X)$ in $T_x(X)$ at γ, and by $\nu_\gamma : T_x(X)/P_\gamma \to T_x(X)/(P_\gamma + (D_x \cap T_x(S)))$ the canonical linear projection, where $P_\gamma = T_\gamma(\widetilde{\mathscr{C}_x}(X))$ and $D_x = \mathrm{Span}(\widetilde{\mathscr{C}_x}(X))$. From Definition 1.2 it follows readily that there exists a linear isomorphism $\lambda : T_0(X) \xrightarrow{\cong} T_x(X)$ such that $\lambda(\widetilde{\mathscr{C}_0}(X)) = \widetilde{\mathscr{C}_x}(X)$ and $\lambda(\widetilde{\mathscr{C}_0}(X_0)) = \widetilde{\mathscr{C}_x}(S)$. Since second fundamental forms on Euclidean spaces with respect to the flat connection are obviously invariant under linear transformations, and since $\lambda(D_0) = \mathrm{Span}(\lambda(\widetilde{\mathscr{C}_0}(X)) = \mathrm{Span}(\widetilde{\mathscr{C}_x}(X)) = D_x$, $\lambda(P_\alpha) = \lambda(T_\alpha(\widetilde{\mathscr{C}_0}(X))) = T_{\lambda(\alpha)}(\widetilde{\mathscr{C}_x}(X)) = P_{\lambda(\alpha)}$, writing $\lambda(\theta) = \theta' \in T_0(X)$ for $\theta \in T_x(X)$, for any $\alpha \in \widetilde{\mathscr{C}_0}(X_0)$ and any $\eta, \xi \in T_\alpha(\widetilde{\mathscr{C}_0}(X))$, we have $\tau_\alpha(\eta, \xi) = 0$ if and only if $\tau_{\alpha'}(\eta', \xi') = 0$. It follows from the hypothesis in the proposition on $\tau_\alpha : S^2 P_\alpha \to T_0(X)/(P_\alpha + (D_0 \cap T_0(X_0)))$ that at $x \in S$, and for $\gamma \in \widetilde{\mathscr{C}_x}(S), \delta \in P_\gamma$, we also have $(\sharp)'$ $\tau_\gamma(\delta, \epsilon) = 0$ *for all* $\epsilon \in P_\gamma \cap T_x(S)$ *implies* $\delta \in P_\gamma \cap T_x(S)$.

For local computations we make use of a privileged coordinate chart (U, z^i) at x and the Euclidean coordinates $(z^i; w^j) = (z^1, \cdots, z^n; w^1, \cdots, w^n)$ as used in Lemma 2.1 and explained in the paragraphs preceding it. By Proposition 2.2, for $\alpha \in \widetilde{\mathscr{C}_x}(S)$, $\beta_t := \nabla_{\widetilde{\alpha_t}} \widetilde{\alpha}_t(x)$ defined as in Lemma 2.2 and Proposition 2.2, and for $\beta := \beta_0 \in P_\alpha$, we have $\tau_\alpha(\beta, \xi) = 0$ for $\xi = \frac{d}{dt}\big|_{t=0} \alpha_t \in T_\alpha(\widetilde{\mathscr{C}_x}(S))$. Recall that $\{\alpha_t\}$ is an arbitrary holomorphic arc on $\widetilde{\mathscr{C}_x}(S)$ at $\alpha = \alpha_0$. By the condition ($\dagger$) $T_\alpha(\widetilde{\mathscr{C}_x}(S)) = T_\alpha(\widetilde{\mathscr{C}_x}(X)) \cap T_x(S)$, ξ is an arbitrary nonzero vector on $P_\alpha \cap T_x(S)$, and it follows from $(\sharp)'$ that $\beta \in P_\alpha \cap T_x(S)$ for a general $\alpha \in \widetilde{\mathscr{C}_x}(X)$.

We may assume that $A(z, t) := \widetilde{\alpha}_t(z)$ is defined on $U \times \Delta(\epsilon_0)$ for some $\epsilon_0 > 0$, giving therefore a holomorphic map $A : U \times \Delta(\epsilon_0) \to \mathscr{C}(X)|_U := \pi^{-1}(U)$ which satisfies $\pi(A(z, t)) = z$. Let $\Sigma \subset \mathscr{C}(X)|_U$ be the image of $U \times \{0\}$ under A so that $\alpha = A(x, 0) \in \Sigma$. The differential $d\pi$ induces at α a linear isomorphism $d\pi(\alpha) : T_\alpha(\Sigma) \xrightarrow{\cong} T_x(U)$. Let $\epsilon > 0$ and $\kappa : \Delta(\epsilon) \to X$ be a parametrized holomorphic curve on X such that $\kappa(0) = x$ and such that $\kappa'(0) = \alpha$. (Here and henceforth $\kappa'(0)$ means $d\kappa(0)(\frac{\partial}{\partial s})$, etc., which is independent of local holomorphic coordinates at $x \in X$.) Let $\gamma : \Delta(\epsilon) \to \widetilde{\mathscr{C}}(X)$ be given by $\gamma(s) = (\kappa(s), \widetilde{\alpha}(\kappa(s)))$ in terms of the Euclidean coordinates $(z^i; w^j)$ in the above. We have $\Gamma := \gamma(\Delta(\epsilon)) \subset \Sigma$. Furthermore, $\gamma'(0) \in T_\alpha(\Sigma)$ is the unique tangent vector such that $d\pi(\gamma'(0)) = \kappa'(0)$. Hence, $\gamma'(0)$ is independent of the choice of (U, z^i) and it is moreover independent of the choice of the holomorphic curve $\kappa : \Delta(\epsilon) \to X$ satisfying $\kappa'(0) = \alpha$. In the coordinates chosen we have $\gamma'(0) = (\alpha, \nabla_{\widetilde{\alpha}}\widetilde{\alpha}(0))$. Since $\alpha \in T_x(S)$ we may choose κ with image lying on S, so that Γ is a germ of smooth holomorphic curve on $\widetilde{\mathscr{C}}(S)$ since $\widetilde{\alpha}|_{U\cap S}$ takes values in $\widetilde{\mathscr{C}}(S)|_{U\cap S} := \widetilde{\varpi}^{-1}(U \cap S)$ by our choice of $\widetilde{\alpha}$ (as in the conclusion of Lemma 2.2) so that $(\alpha, \beta) \in T_\alpha(\Gamma) \subset T_\alpha(\widetilde{\mathscr{C}}(S))$. Since $\beta \in P_\alpha \cap T_x(S)$, by the condition (†) we have $(0, \beta) \in T_\alpha(\widetilde{\mathscr{C}_x}(S)) \subset T_\alpha(\widetilde{\mathscr{C}}(S))$. As a consequence $(\alpha, 0) = (\alpha, \beta) - (0, \beta) \in T_\alpha(\widetilde{\mathscr{C}}(S))$.

Let ℓ be the projective line on X passing through x such that $T_x(\ell) = \mathbb{C}\alpha$ and denote by $\ell^\sharp$ the tautological lifting of ℓ to $\mathscr{C}(X)$. Assume without loss of generality that $w^n(\alpha) \neq 0$ and write $(u^1, \cdots u^{n-1})$ for the inhomogeneous coordinates $u^i = \frac{w^i}{w^n}$ on a coordinate chart on $\mathbb{P}(V)$. Then, in terms of the Euclidean coordinates $(z^1, \cdots, z^n; u^1, \cdots u^{n-1})$ at $[\alpha]$, we have $T_{[\alpha]}(\ell^\sharp) = \mathbb{C}(\alpha, 0)$. From the last paragraph we have $(\alpha, 0) \in T_\alpha(\widetilde{\mathscr{C}}(S))$ (where $0 \in \mathbb{C}^n$), hence $(\alpha, 0) \in T_{[\alpha]}(\mathscr{C}(S))$ (where $0 \in \mathbb{C}^{n-1}$). As a result $\ell^\sharp$ is tangent to $\mathscr{C}(S)$ at $[\alpha]$, and for the rank one distribution $\mathscr{F}$ on $\mathscr{C}(X)$ corresponding to the tautological foliation $\mathcal{F}$ we have $\mathscr{F}_{[\alpha]} = T_{[\alpha]}(\ell^\sharp) \subset T_{[\alpha]}(\mathscr{C}(S))$, i.e., $\mathscr{F}|_{\mathscr{C}(S)} \subset T(\mathscr{C}(S))$. Thus, the germ of $\ell^\sharp$ at $[\alpha]$, being the germ of an integral curve of $\mathcal{F}$ at $[\alpha]$, must necessarily lie on $\mathscr{C}(S)$, and the germ of ℓ at x, which is the image of $\ell^\sharp$ under the canonical projection $\varpi : \mathscr{C}(S) \to S$, must lie on S. The proof of Proposition 2.3 is complete. □

3. Nondegeneracy conditions for pairs (X_0, X) of rational homogeneous spaces of Picard number 1

The study of admissible pairs of sub-diagram type started with Mok [Mk08a] and [HoM10] in the context of germs of VMRT-respecting holomorphic immersions $f : (X_0; x_0) \to (X; f(x_0))$ (cf. §1, esp. Theorem 1.1), where a nondegeneracy condition was introduced for the study of non-equidimensional Cartan-Fubini extension. We recall the notion here and formulate it for admissible pairs (X_0, X).

Definition 3.1. (cf. [HoM10]). *Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1. Let $0 \in X_0 \subset X$ be a reference point and $\alpha \in \widetilde{\mathscr{C}_0}(X_0)$ be arbitrary. Write $P_\alpha = T_\alpha(\widetilde{\mathscr{C}_0}(X))$, and denote by $\sigma_\alpha : S^2P_\alpha \to T_0(X)/P_\alpha$ the second fundamental form of $\widetilde{\mathscr{C}_0}(X) \subset T_0(X) - \{0\}$ at α with respect to the flat connection on $T_0(X)$*

as a Euclidean space. Define

$$\operatorname{Ker}\sigma_\alpha(\,\cdot\,, T_\alpha(\widetilde{\mathscr{C}_0}(X_0))) := \Big\{\eta \in T_\alpha(\widetilde{\mathscr{C}_0}(X)) : \sigma_\alpha(\eta,\xi) = 0$$

$$\text{for every } \xi \in T_\alpha(\widetilde{\mathscr{C}_0}(X_0)) = P_\alpha \cap T_0(X_0).\Big\}.$$

We say that the pair of subvarieties $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ of $\mathbb{P}T_0(X)$ is nondegenerate if and only if $\operatorname{Ker}\sigma_\alpha(\,\cdot\,, T_\alpha(\widetilde{\mathscr{C}_0}(X_0))) = \mathbb{C}\alpha$ at a general point $\alpha \in \widetilde{\mathscr{C}_0}(X_0)$. We also say that (X_0, X) is nondegenerate to mean that $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ is nondegenerate.

Remark In place of the pair $(\widetilde{\mathscr{C}_0}(X_0), \widetilde{\mathscr{C}_0}(X))$ we may also refer equivalently to the pair $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ in the definition of nondegeneracy. Denoting by $\sigma_{[\alpha]}$ the projective second fundamental form at $[\alpha]$ of $\mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$ the analogous nondegeneracy condition is $\operatorname{Ker}\sigma_{[\alpha]}(\,\cdot\,, T_{[\alpha]}(\mathscr{C}_0(X_0))) = 0$.

In view of Proposition 2.3 we introduce a new notion of nondegeneracy for (X_0, X), called *nondegeneracy for substructures*, formulated also in terms of the second fundamental form for the pair $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$.

Definition 3.2. *In the notation adopted in Definition 3.1 denote by* $\nu_\alpha : T_0(X)/P_\alpha \to T_0(X)/(P_\alpha + (D_0 \cap T_0(X_0)))$ *the canonical projection. Writing* $\tau_\alpha := \nu_\alpha \circ \sigma_\alpha$*, so that* $\tau_\alpha : S^2P_\alpha \to T_0(X)/(P_\alpha + (D_0 \cap T_0(X_0)))$*, define* $\operatorname{Ker}\tau_\alpha(\,\cdot\,, T_\alpha(\widetilde{\mathscr{C}_0}(X_0))$ *as in the definition of* $\operatorname{Ker}\sigma_\alpha(\,\cdot\,, T_\alpha(\widetilde{\mathscr{C}_0}(X_0)))$ *except that* σ_α *is replaced by* τ_α*. We say that* $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ *is nondegenerate for substructures if and only if for a general point* $\alpha \in \widetilde{\mathscr{C}_0}(X_0)$ *we have* $\operatorname{Ker}\tau_\alpha(\,\cdot\,, T_\alpha(\widetilde{\mathscr{C}_0}(X_0))) = T_\alpha(\widetilde{\mathscr{C}_0}(X_0))$*, which is the same as* $P_\alpha \cap T_0(X_0)$*. We also say that* (X_0, X) *is nondegenerate for substructures to mean that* $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ *is nondegenerate for substructures.*

We note first of all the following relation between the two notions of nondegeneracy in Definition 3.1 and Definition 3.2. To emphasize the distinction between the two notions we will from now on refer to the first notion of nondegeneracy in Definition 3.1 as *nondegeneracy for mappings*. We have

Lemma 3.1. *Let* (X_0, X) *be an admissible pair of rational homogeneous spaces of Picard number* 1*. Suppose* $X_0 \subset X$ *is nonlinear and* (X_0, X) *is nondegenerate for substructures. Then,* (X_0, X) *is also nondegenerate for mappings.*

Proof. We have $\operatorname{Ker}\sigma_\alpha(\,\cdot\,, T_\alpha(\widetilde{\mathscr{C}_0}(X_0))) \subset \operatorname{Ker}\tau_\alpha(\,\cdot\,, T_\alpha(\widetilde{\mathscr{C}_0}(X_0)))$ since $\tau_\alpha = \nu_\alpha \circ \sigma_\alpha$. Since (X_0, X) is nondegenerate for substructures, we have $\operatorname{Ker}\tau_\alpha(\,\cdot\,, T_\alpha(\widetilde{\mathscr{C}_0}(X_0))) \subset T_\alpha(\widetilde{\mathscr{C}_0}(X_0))$, hence $\operatorname{Ker}\sigma_\alpha(\,\cdot\,, T_\alpha(\widetilde{\mathscr{C}_0}(X_0)))$

$\subset T_\alpha(\widetilde{\mathscr{C}_0}(X_0)) = P_\alpha \cap T_0(X_0)$. The VMRT of a projective submanifold uniruled by projective lines is always nonsingular (cf. Mok [Mk08b, Lemma 3]). We observe that $\mathscr{C}_0(X_0) \subset \mathbb{P}T_0(X_0)$ is nonlinear since $X_0 \subset X$ is by assumption nonlinear. (Supposing otherwise the VMRTs would span an integrable holomorphic distribution $\mathscr{D}$ on X_0 of rank $s := (\dim \mathscr{C}_0(X_0)) + 1 < n := \dim(X_0)$, $s \geq 2$, such that the maximal integral submanifolds are $\cong \mathbb{P}^s$, so that X_0 would be the total space of a regular family of $\mathbb{P}^s$ over a projective manifold of dimension > 0, contradicting the assumption that X_0 is of Picard number 1.) It follows that the second fundamental form of $\widetilde{\mathscr{C}_0}(X_0) \subset T_0(X_0) - \{0\}$ is nondegenerate at a general $\alpha \in \widetilde{\mathscr{C}_0}(X_0)$, i.e., $\mathrm{Ker}\, \sigma_\alpha(\,\cdot\,, T_\alpha(\widetilde{\mathscr{C}_0}(X_0))) = \mathbb{C}\alpha$ (cf. Griffiths-Harris [GH79]). Hence (X_0, X) is also nondegenerate for mappings. $\square$

For admissible pairs (X_0, X) of rational homogeneous spaces of Picard number 1, our focus is on those of sub-diagram type. These were studied for the rigidity problem for VMRT-respecting holomorphic maps in [Mk08a], [HoM10] and [HoP11], where nondegeneracy was established using root space decomposition in the long-root cases in [HoM10] and using projective geometry in the short-root cases in [HoP11]. For sub-VMRT structures we will show that for admissible pairs of sub-diagram type the algebraic statement in terms of second fundamental forms on nondegeneracy for mappings in the nonlinear case actually implies nondegeneracy for substructures. To prove the latter we will give a uniform formulation using root space decomposition incorporating both the long-root and short-root cases.

Let G be a complex simple Lie group and denote by $\mathfrak{g}$ its Lie algebra. Let Φ be the set of all roots of G with respect to a Cartan subalgebra $\mathfrak{h}$ and let Δ be a simple root system of G. For any root $\rho \in \Phi$ we write $\rho = \sum_{\delta \in \Delta} n_\delta(\rho)\delta$, $n_\delta(\rho) \in \mathbb{Z}$ being either all non-negative or all non-positive. Each root space g_ρ is 1-dimensional and we write $\mathfrak{g}_\rho = \mathbb{C}E_\rho$, where, E_ρ stands for any nonzero vector belonging to $\mathfrak{g}_\rho$. Denote by $\Phi^+ \subset \Phi$ the subset of positive roots, and $\Phi^- = -\Phi^+$ the set of negative roots. For the Lie algebra $\mathfrak{g}$ of G we have the root space decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \sum_{\rho \in \Phi^+} \mathfrak{g}_\rho \oplus \sum_{\rho \in \Phi^-} \mathfrak{g}_\rho\,.$$

To a simple root $\gamma \in \Delta$ we associate a maximal parabolic subgroup $P \subset G$ such that the Lie algebra $\mathfrak{p} \subset \mathfrak{g}$ of P is given by

$$\mathfrak{p} = \mathfrak{h} \oplus \sum \left\{ \mathfrak{g}_\rho : n_\gamma(\rho) \leq 0 \right\}.$$

$X := G/P$ is a rational homogeneous space of Picard number 1. Let $G_c \subset G$ be a compact real form such that $\mathfrak{h} = \mathfrak{h}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ for some real form $\mathfrak{h}_\mathbb{R}$ of $\mathfrak{h}$ lying on $\mathfrak{g}_c$, the Lie algebra of G_c. We endow X with a Kähler metric invariant under G_c. The pair $(\mathfrak{g}, \gamma)$ is a graded

Lie algebra when we define $\mathfrak{g}^k = \sum \{\mathfrak{g}_\rho : n_\gamma(\rho) = k\}$ for $k \neq 0$ and $\mathfrak{g}^0 = \mathfrak{h} \oplus \sum \{\mathfrak{g}_\rho : n_\gamma(\rho) = 0\}$, noting that $[\mathfrak{g}^k, \mathfrak{g}^\ell] \subset \mathfrak{g}^{k+\ell}$ for integers k, ℓ. Define $L \subset G$ by $L := \{g \in G : \mathrm{Ad}_g(\mathfrak{g}^k) \subset \mathfrak{g}^k$ for all $k\}$. Then, $L \subset G$ is a Levi subgroup preserving each $\mathfrak{g}^k$, and the L-action on $\mathfrak{g}^k$ is irreducible for $k \neq 0$. There is a unique element $H \in \mathfrak{h}$ such that $[H, \eta^k] = k\eta^k$ for any $\eta^k \in \mathfrak{g}^k$, and we call $H \in \mathfrak{h}$ the element defining the canonical structure of $(\mathfrak{g}, \gamma)$ as a graded Lie algebra. (See Yamaguchi [Ya93] concerning basic facts on $(\mathfrak{g}, \gamma)$ as a graded Lie algebra.) For $k \in \mathbb{Z}$ we denote by $\Phi^k \subset \Phi$ the set of roots ρ such that $\mathfrak{g}_\rho \subset \mathfrak{g}^k$. At $0 = eP \in G/P$ we have $T_0(X) = \sum \{\mathfrak{g}_\rho : n_\gamma(\rho) > 0\} = \sum_{k \geq 1} \mathfrak{g}^k =: \mathfrak{g}^+$. We say that $X = G/P$ is of diagram type $(\mathscr{D}(\mathfrak{g}), \gamma)$ marked at a long simple root γ, i.e., marked at the node corresponding to $\gamma \in \Delta$ of the Dynkin diagram $\mathscr{D}(\mathfrak{g})$ of $\mathfrak{g}$, and also that $X = G/P$ is of type $(\mathscr{D}(\mathfrak{g}), \gamma)$. Let G_0 be a simple complex Lie group, and denote by $\mathfrak{g}_0$ the Lie algebra of G_0. Suppose G_0 can be embedded and hence identified as a Lie subgroup of G in such a way that the Dynkin diagram $\mathscr{D}(\mathfrak{g}_0)$ is identified with a Dynkin sub-diagram of $\mathscr{D}(\mathfrak{g})$ containing the node γ_0 which is identified as γ in the inclusion $\mathscr{D}(\mathfrak{g}_0) \subset \mathscr{D}(\mathfrak{g})$. Defining $\mathfrak{p}_0 := \mathfrak{p} \cap \mathfrak{g}_0$ we have the parabolic subalgebra $\mathfrak{p}_0 \subset \mathfrak{g}_0$. Writing $P_0 \subset G_0$ for the corresponding parabolic subgroup we have the rational homogeneous space $X_0 := G_0/P_0 \hookrightarrow G/P = X$, and (X_0, X) is an admissible pair of rational homogeneous spaces of Picard number 1 of sub-diagram type marked at (γ_0, γ).

When X is of diagram type $(\mathscr{D}(\mathfrak{g}), \gamma)$ marked at a long simple root γ, at $0 \in X_0 \subset X$ the VMRT $\mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$ is the highest weight orbit $\mathcal{W}_0 \subset \mathbb{P}D^1_0$, where $D^1 \subset T(X)$ is the minimal nonzero $\mathrm{Aut}(X)$-invariant distribution on X. We have the identification $D^1_0 = \mathfrak{g}^1$, considered as an irreducible L-representation space of the reductive Lie group L. For the short-root cases, identifying rational homogeneous spaces which are biholomorphic to each other, we only need to consider X of type (C_n, α_k), $2 \leq k < n$, (F_4, α_3) and (F_4, α_4). To see this, other than those listed there remain the cases of (B_n, α_n), $n \geq 3$, (C_n, α_1), $n \geq 2$ and (G_2, α_1). For (B_n, α_n), $n \geq 3$, the underlying rational homogeneous space X is biholomorphic to the orthogonal Grassmannian $G^{II}(n, n)$ of isotropic subspaces of a $2n$-dimensional complex vector space (V, q) equipped with a non-degenerate complex symmetric bilinear form. Up to biholomorphism X is also the underlying space of the rational homogeneous space of type (D_n, α_n). Consider an admissible pair (X_0, X) with X of type (B_n, α_n) and X_0 of type (B_ℓ, α_ℓ), $1 \leq \ell < n$, corresponding to the marked sub-diagram of (B_n, α_n) by deleting the first $n - \ell$ nodes. Likewise consider the admissible pair (X'_0, X') defined by replacing B_n by D_n, and B_ℓ by D_ℓ in the definition of (X_0, X). Then, there is a biholomorphism $\Phi : X \xrightarrow{\cong} X'$ such that $\Phi' : X_0 \xrightarrow{\cong} X'_0$. Thus, the problem of rigidity of (X_0, X) reduces to that of (X'_0, X'), which is of sub-diagram type marked at a long root. The case of (C_n, α_1), which is that of an odd-dimensional projective space as a contact homogeneous manifold, is irrelevant for our rigidity problem. The underlying space of (G_2, α_1) is the 5-dimensional hyperquadric Q^5, and the only sub-diagram marked at α_1 is (A_1, α_1) corresponding to a minimal rational curve identified as $\mathbb{P}^1$, and the pair $(\mathbb{P}^1, Q^5)$ is obviously not a rigid pair since $\mathbb{P}^1 \subset Q^5$ is a non-maximal linear subspace.

In each of the cases (C_n, α_k), $2 \le k < n$, (F_4, α_3) and (F_4, α_4) the VMRT $\mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$ is the Zariski closure of the orbit under $P = L\cdot U$ of a highest weight vector η of the L-representation space $\mathfrak{g}^2$ (cf. the description of VMRTs in the short-root cases in Hwang-Mok [HM04a] [HM05]).

Consider now an admissible pair (X_0, X) of sub-diagram type, where $X_0 = G_0/P_0 \hookrightarrow G/P = X$. Since $(\mathfrak{g}_0, \mathfrak{g})$ is of sub-diagram type we have a Cartan subalgebra $\mathfrak{h}_0 \subset \mathfrak{g}_0$ and a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{h}_0 = \mathfrak{h} \cap \mathfrak{g}_0$, and a root decomposition of $\mathfrak{g}_0$ in the same way as that of $\mathfrak{g}$, such that the roots (resp. the root vectors) of $\mathfrak{g}_0$ are identified naturally as roots (resp. root vectors) of $\mathfrak{g}$. $\Delta_0 \subset \Delta$ will stand for the set of simple roots of the Dynkin diagram of $\mathfrak{g}_0$, and $\Phi_0 \subset \Phi$ for the set of roots of $\mathfrak{g}_0$, etc. Note that $(\mathfrak{g}_0, \gamma_0)$ carries the structure of a graded Lie algebra, where $\mathfrak{g}_0^k = \mathfrak{g}^k \cap \mathfrak{g}_0$ for $k \in \mathbb{Z}$. From root space decompositions, we have $P_0 = L_0 \cdot U_0$ for the Levi decomposition of P_0 with $L_0 \subset P_0$ being a Levi factor, $U_0 \subset P_0$ being the unipotent radical, such that $L_0 = L \cap G_0$ and $U_0 = U \cap G_0$.

In the long-root cases at $0 \in X_0 \subset X$ we have $\widetilde{\mathscr{C}_0}(X_0) \subset \mathbb{P}\mathfrak{g}_0^1$, $\widetilde{\mathscr{C}_0}(X) \subset \mathbb{P}\mathfrak{g}^1$. To describe $\big(\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X)\big)$ we choose $\alpha = E_\mu$ for the lowest weight $\mu = \gamma_0$ in $\mathfrak{g}_0^1$ as an irreducible L_0-representation space. Since $(\mathfrak{g}_0, \gamma_0)$ is a marked sub-diagram of $(\mathfrak{g}, \gamma)$, with γ_0 being identified with γ, μ is also the lowest weight in $\mathfrak{g}$ as an irreducible L-representation space. $\mathscr{C}_0(X_0)$ is the orbit of $[\alpha] = [E_\mu]$ under L_0, and $\widetilde{\mathscr{C}_0}(X)$ is the orbit of $[\alpha]$ under L, and they are the same as the respective highest weight orbits. Since the unipotent radical $U_0 \subset P_0$ resp. $U \subset P$ acts trivially on $T_0(X_0)$ resp. $T_0(X)$, for the description of VMRTs, $\mathscr{C}_0(X_0) \subset \mathbb{P}\mathfrak{g}_0^1$ is equivalently the orbit of $[\alpha]$ under the parabolic subgroup $P_0 \subset G_0$, and $\widetilde{\mathscr{C}_0}(X) \subset \mathbb{P}\mathfrak{g}^1$ is the orbit of $[\alpha]$ under the parabolic subgroup $P \subset G$.

In the short-root cases (where X is of type $\mathfrak{g} = C_n, n \ge 3$, or F_4), there are minimal rational curves $\ell \subset X$, $0 \in \ell$, such that $T_0(\ell) \subset T_0(X) \cong \mathfrak{g}^+$ are spanned by root vectors $\alpha = E_\mu \in \mathfrak{g}^2$. In general it may happen that $\mathfrak{g}_0^2 = 0$. This is the case when $X_0 \subset X$ is a maximal linear subspace, or when $X = S_{k,n}$ is a symplectic Grassmannian, where $2 \le k < n$, and $X_0 = G(\ell, m)$ is some Grassmannian of rank ≥ 2 (cf. Proposition 4.2(a)). When however $\mathfrak{g}_0^2 \ne 0$ we can deal with $\widetilde{\mathscr{C}_0}(X)$ in a way analogous to the long-root cases.

In the long-root cases a formula based on "the cubic expansion" in Hwang-Mok [HM99b, 4.2] was used in Hong-Mok [HoM10] for the second fundamental form σ on $\widetilde{\mathscr{C}_0}(X)$. Since details were lacking in [HM99b] and we need a more general formula, for completeness we derive here a formula for σ with a uniform formulation and proof for both the long-root and short-root cases.

Lemma 3.2. *Let $X = G/P$ be a rational homogeneous space of Picard number 1, $0 = eP$ and $\pi : \mathscr{C}(X) \to X$ be the VMRT structure on X. In the long-root cases let μ be the lowest weight in $\mathfrak{g}^1$ (i.e., the simple root γ). In the short-root cases assume $\mathfrak{g}^2 \neq 0$ and let μ be the lowest weight in $\mathfrak{g}^2$. Let $\Psi_\mu \subset \Phi^+$ be the subset of all roots $\nu \in \Phi^+$ such that $\nu - \mu =: \kappa \in \Phi$ and $E_\kappa \in \mathfrak{p}$, and $\Theta_\mu \subset \Phi^0 \cup \Phi^-$ be the subset of all such roots κ. Then, for $\alpha = E_\mu$, $P_\alpha := T_\alpha(\widetilde{\mathscr{C}_0}(X))$ is the linear span of $\mathbb{C}\alpha$ and $\mathcal{H}_\alpha := \mathrm{Span}\{E_\nu : \nu \in \Psi_\mu\}$, and the second fundamental form $\sigma_\alpha : S^2 P_\alpha \to T_0(X)/P_\alpha$ of $\widetilde{\mathscr{C}_0}(X) \subset T_0(X) - \{0\}$ at α is given by*

$$\sigma_\alpha(E_\nu, E_{\nu'}) = [E_\kappa, [E_{\kappa'}, E_\mu]] \mod (P_\alpha + \mathfrak{p})$$

for $\nu = \mu + \kappa, \nu' = \mu + \kappa' \in \Psi_\mu$.

Proof. The affinized VMRT $\widetilde{\mathscr{C}_0}(X)$ is the P-orbit of α under the isotropy representation of P on $T_0(X)$, which is induced by the restriction to P of the adjoint action of G on $\mathfrak{g}$. Write $p = \mathrm{Card}(\Theta_\mu) = \mathrm{Card}(\Psi_\mu)$. For $\chi \in \mathfrak{p}$, $\exp(\mathrm{ad}\chi)$ is a linear automorphism of $\mathfrak{g}$ which preserves $\mathfrak{p}$, and it descends to a linear automorphism of $\mathfrak{g}/\mathfrak{p} \cong T_0(X)$ which is the same as the Jacobian $\mathrm{d}(\exp(\chi))(0)$ at 0. From the Taylor expansion of the adjoint action of G on $\mathfrak{g}$, the orbit $P.\alpha$ is of dimension $1 + p$, and it contains as an open subset the image mod $\mathfrak{p}$ of E_μ under $\exp(\mathrm{ad}U)$ for a sufficiently small open neighborhood U of 0 in $\mathfrak{p}$. We have $P_\alpha = T_\alpha(\widetilde{\mathscr{C}_0}(X)) = [\mathfrak{p}, E_\mu]$. Write $\Theta_\mu = \{\kappa_1, \cdots, \kappa_p\}$. We have $\mathbb{C}[E_{\kappa_i}, E_\mu] = \mathbb{C}E_{\nu_i}$ for $1 \le i \le p$, $[\mathfrak{h}, E_\mu] = \mathbb{C}E_\mu$, while $[E_\rho, E_\mu] = 0$ whenever $E_\rho \subset \mathfrak{p}$ and $\rho \notin \Theta_\mu$. Hence, $P_\alpha = \mathbb{C}\alpha \oplus \mathcal{H}_\alpha$. Write $\xi = z_0 E_\mu + z_1 E_{\nu_1} + \cdots + z_p E_{\nu_p}$, $\nu_i := \mu + \kappa_i$. Let $H \in \mathfrak{h}$ be the element defining the canonical structure on $(\mathfrak{g}, \gamma)$ as a graded Lie algebra, and write $\chi = z_0 H + z_1 E_{\kappa_1} + \cdots + z_p E_{\kappa_p}$ in the long-root cases, $\chi = \frac{z_0}{2} H + z_1 E_{\kappa_1} + \cdots + z_p E_{\kappa_p}$ in the short-root cases, so that $[\chi, E_\mu] = \xi$. Again from the Taylor expansion of the adjoint action, $P.\alpha$ contains as an open subset the image mod $\mathfrak{p}$ of E_μ under $\exp(\mathrm{ad}(U'))$, where U' is some open neighborhood of 0 in $\mathrm{Span}\{H, E_{\kappa_1}, \cdots, E_{\kappa_p}\}$. Writing $c = 1$ resp. $\frac{1}{2}$ in the long-root resp. short-root cases, expanding

$\exp(\mathrm{ad}\chi)(E_\mu)$ we have

$$\exp(\mathrm{ad}\chi)(E_\mu) = E_\mu + \big(z_0E_\mu + z_1E_{\nu_1} + \cdots + z_pE_{\nu_p}\big) + \frac{c}{2}\sum_{1\le i\le p} z_0z_i\big([H,[E_{\kappa_i},E_\mu]] + [E_{\kappa_i},[H,E_\mu]]\big) + \frac{1}{2}\sum_{1\le i,j\le p} z_iz_j\big([E_{\kappa_i},[E_{\kappa_j},E_\mu]] + [E_{\kappa_j},[E_{\kappa_i},E_\mu]]\big) + O(\|z\|^3)\,.$$

Since $[E_{\kappa_i},E_\mu]\in\mathbb{C}E_{\nu_i}$, $[H,E_{\nu_i}]\in\mathbb{C}E_{\nu_i}$ and $[H,E_\mu]\in\mathbb{C}E_\mu$, we deduce

$$\exp(\mathrm{ad}\chi)(E_\mu) \equiv \frac{1}{2}\sum_{1\le i,j\le p} z_iz_j\big([E_{\kappa_i},[E_{\kappa_j},E_\mu]] + [E_{\kappa_j},[E_{\kappa_i},E_\mu]]\big) + O(\|z\|^3) \mod P_\alpha\,,$$

where $O(\|z\|^3)$ is the tail of the Taylor expansion which is the sum of terms of degree ≥ 3 in $(z_0,\cdots,z_p)$. To prove Lemma 3.2 it suffices to show that

$$(\sharp)\qquad [E_{\kappa_i},[E_{\kappa_j},E_\mu]] = [E_{\kappa_j},[E_{\kappa_i},E_\mu]] \mod \mathfrak{p}\,.$$

Obviously $(\sharp)$ holds when $\mu+\kappa_i+\kappa_j\notin\Phi$ since both sides are 0, or when $\mu+\kappa_i+\kappa_j\in\Phi^0\cup\Phi^-$ (which is *a priori* possible in the short-root cases), and it remains to consider the case where $\tau := \mu+\kappa_i+\kappa_j\in\Phi^+$. By the Jacobi Identity, $(\sharp)$ holds whenever $[E_\mu,[E_{\kappa_i},E_{\kappa_j}]]=0$. It remains therefore to show that $[E_{\kappa_i},E_{\kappa_j}]=0$, equivalently that $\kappa_i+\kappa_j\notin\Phi$. We claim that the latter is indeed the case.

For the proof of the claim consider the projective line $\ell\subset X$ such that $T_0(\ell)=\mathbb{C}\alpha=\mathbb{C}E_\mu$. By Grothendieck splitting we have $T(X)|_\ell\cong\mathcal{O}(2)\oplus(\mathcal{O}(1))^p\oplus\mathcal{O}^q$, where $1+p+q=\dim(X)$. Let $\mathfrak{g}(\mu)$ be the Lie subalgebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ spanned by unit root vectors $E_\mu, E_{-\mu}$ and $H_\mu := [E_\mu,E_{-\mu}]$ and denote by $G(\mu)\subset G$ the corresponding Lie subgroup isomorphic to $\mathbb{P}SL(2,\mathbb{C})$. For $\nu\in\Psi_\mu$, $[E_{-\mu},E_\nu]\in\mathfrak{p}$ as can be seen from $[\mathfrak{g}^{-1},\mathfrak{g}^1]\subset\mathfrak{g}^0$ in the long-root cases and from $[\mathfrak{g}^{-2},\mathfrak{g}^1\oplus\mathfrak{g}^2]\subset\mathfrak{g}^{-1}\oplus\mathfrak{g}^0$ in the short-root cases. It follows that for any point $x\in\ell$, $d\varphi(E_\nu)$ are proportional vectors at x for any choice of $\varphi\in G(\mu)$ such that $\varphi(0)=x$. Hence, the union of images of $\mathbb{C}E_\nu$ under the action of $G(\mu)$ gives a holomorphic line subbundle $\Lambda_\nu\subset T(X)|_\ell$. Observing that the fiber of Λ_ν at 0 is spanned by $E_\nu\in P_\alpha-\mathbb{C}\alpha$, by the invariance of Λ_ν under $G(\mu)$, $\Lambda_\nu\subset T(X)|_\ell$ is transversal to $T_\ell=\mathcal{O}(2)$ and must lie on the positive part $\mathcal{O}(2)\oplus(\mathcal{O}(1))^p\subset T(X)|_\ell$. Hence $\Lambda_\nu\cong\mathcal{O}(1)$, and $T(\ell)\oplus(\Lambda_{\nu_1}\oplus\cdots\oplus\Lambda_{\nu_p})\cong\mathcal{O}(2)\oplus(\mathcal{O}(1))^p$ is precisely the positive part of $T(X)|_\ell$ in the Grothendieck splitting. It follows that $[H_\mu,E_\nu]=E_\nu$, i.e., $\nu(H_\mu)=1$. Note that $\mu(H_\mu)=2$ since E_μ generates $T_\ell\cong\mathcal{O}(2)$. To prove the claim by contradiction assume that $\kappa_i+\kappa_j\in\Phi$. Now for $\kappa\in\Theta_\mu$ we have $\kappa=\nu-\mu$ for some $\nu\in\Psi_\mu$ so that $\kappa(H_\mu)=\nu(H_\mu)-\mu(H_\mu)=-1$. Thus, for $i,j\in\{1,\cdots,p\}$ we have $(\kappa_i+\kappa_j)(H_\mu)=-2$, hence $\tau(H_\mu) = 0$ for $\tau = \mu + (\kappa_i + \kappa_j)$. But by assumption $\tau \in \Phi^+$ while $\kappa_i + \kappa_j \in \Phi^0 \cup \Phi^-$, so that $\tau \in \Psi_\mu$ and hence $\tau(H_\mu) = 1$, a plain contradiction, proving Lemma 3.2, as desired. □

In order to apply Lemma 3.2 we observe

Lemma 3.3. *Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1 of sub-diagram type marked at a short root, where $X = G/P$ is of type $\mathfrak{g} = C_n, n \geq 3$, or F_4 associated to a marked Dynkin diagram $(\mathfrak{g}, \gamma)$, and $X_0 = G_0/P_0$ is defined by a marked Dynkin sub-diagram $(\mathfrak{g}_0, \gamma_0)$. Assume that $\mathfrak{g}_0^2 \neq 0$. Then, taking $\alpha = E_\mu$ for the lowest weight μ in $\mathfrak{g}_0^2$, at $0 = eP_0$, $\mathscr{C}_0(X_0)$ is the closure of the orbit of $[\alpha] \in \mathbb{P}(\mathfrak{g}_0^1 + \mathfrak{g}_0^2)$ under P_0, while $\mathscr{C}_0(X)$ is the closure of the orbit of $[\alpha] \in \mathbb{P}(\mathfrak{g}^1 + \mathfrak{g}^2)$ under P.*

Proof. Since $(\mathfrak{g}_0, \gamma_0)$ is a marked Dynkin sub-diagram of $(\mathfrak{g}, \gamma)$, the lowest weight μ in $\mathfrak{g}_0^2$ is also the lowest weight of $\mathfrak{g}^2$. By the description of VMRTs in Hwang-Mok [HM04a] [HM05b], $\mathscr{C}_0(X)$ is the closure of the orbit of $[\alpha]$ in $\mathbb{P}(\mathfrak{g}^1 + \mathfrak{g}^2)$ under P. As for $\mathscr{C}_0(X_0)$, in the notation adopted in Lemma 3.3 and its proof we have $T_\alpha(\widetilde{\mathscr{C}_0}(X)) = \mathrm{Span}\{E_\mu, E_{\nu_1}, \cdots, E_{\nu_p}\}$. Enumerate ν_i in such a way that $\Psi_{\mu,0} := \Psi_\mu \cap \Phi_0 = \{\nu_1, \cdots, \nu_{p_0}\}$, $1 \leq p_0 < p$ (assuming $X_0 \neq X$). By (†), $T_\alpha(\widetilde{\mathscr{C}_0}(X_0)) = T_\alpha(\widetilde{\mathscr{C}_0}(X)) \cap T_0(X_0)$. Since $E_{\nu_i} \perp T_0(X_0)$ whenever $i > p_0$, we have $T_\alpha(\widetilde{\mathscr{C}_0}(X_0)) = \mathrm{Span}\{E_\mu, E_{\nu_1}, \cdots, E_{\nu_{p_0}}\}$. Hence $\mathscr{C}_0(X_0)$ agrees with the closure of the orbit $P_0.[\alpha]$ of $[\alpha]$ under P_0, completing the proof of the lemma. □

Remark Lemma 3.3 also follows from Landsberg-Manivel [LM03, §4].

In the long-root nonlinear cases $(\mathfrak{g}_0, \mathfrak{g})$ of sub-diagram type, by Lemma 3.2 for these cases Hong-Mok [HoM10] gave a proof that these pairs are nondegenerate for mappings. Using Lemma 3.2 and Lemma 3.3, by identical arguments as in [HoM10] one can show the same in the nonlinear short-root cases satisfying the hypothesis of Lemma 3.3, giving a uniform proof of nondegeneracy for mappings and hence for substructures in the sequel. Since a verification of the short-root cases has been given in [HoP11] by projective geometry, we will not repeat the arguments of [HoM10], and proceed to argue assuming [HoP11].

Lemma 3.4. *Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1 of sub-diagram type. Assume that $\mathscr{C}_0(X_0)$ contains a smooth point of the VMRT $\mathscr{C}_0(X)$ at $0 \in X_0$. Then, the admissible pair (X_0, X) is degenerate for substructures if and only if there exists some positive root $\nu \in \Psi_\mu - \Psi_{\mu,0}$ such that $\tau_\alpha(E_\nu, \xi) = 0$ for every $\xi \in T_\alpha(\widetilde{\mathscr{C}_0}(X_0))$, i.e., $E_\nu \in \mathrm{Ker}\, \tau_\alpha(\,\cdot\,, T_0(\widetilde{\mathscr{C}_0}(X_0))$.*

Proof. By Lemma 3.3 we may take the general point of $\mathscr{C}_0(X_0)$ to be $[\alpha] = [E_\mu]$. For the proof of Lemma 3.4 only the "only if" part requires an argument. Suppose (X_0, X) is degenerate for substructures. Then, there exists a nonzero vector $\eta \in P_\alpha$, $\eta \perp P_{\alpha,0}$, such that $\tau_\alpha(\eta, \xi) = 0$ for every $\xi \in P_{\alpha,0}$. (Here $P_\alpha = T_\alpha(\widetilde{\mathscr{C}_0}(X))$, also $P_{\alpha,0} := T_\alpha(\widetilde{\mathscr{C}_0}(X_0)) = P_\alpha \cap T_0(X_0)$.) Relabeling the roots $\nu_i \in \Psi_\mu$ if necessary, write $\eta = a_1 E_{\nu_1} + \cdots + a_s E_{\nu_s}$, where for $1 \le i \le s$, $a_i \neq 0$, and ν_i are distinct roots in $\Psi_\mu - \Psi_{\mu,0}$. For any $\lambda \in \Psi_{\mu,0}$, we have

$$\begin{aligned} 0 = \tau_\alpha(\eta, E_\lambda) &= \tau_\alpha(a_1 E_{\nu_1} + \cdots + a_s E_{\nu_s}, E_\lambda) \\ &= a_1 \tau_\alpha(E_{\nu_1}, E_\lambda) + \cdots + a_s \tau_\alpha(E_{\nu_s}, E_\lambda). \end{aligned}$$

Let $\Gamma = \big\{k : \tau_\alpha(E_{\nu_k}, E_\lambda) \neq 0\big\} \subset \{1, ..., s\}$. Write $\zeta_k := \nu_k + \lambda - \mu \in \Phi^+$ for $k \in \Gamma$. Since $\tau_\alpha(E_{\nu_k}, E_\lambda) \neq 0$, by Lemma 3.2 the mutually orthogonal root vectors E_{ζ_k}, $k \in \Gamma$, are orthogonal to $P_\alpha + (D_0 \cap T_0(X_0))$ and hence $\tau_\alpha(E_{\nu_k}, E_\lambda)$, $1 \le k \le s$, are linearly independent, i.e., $\Gamma = \emptyset$. Thus, $\tau_\alpha(E_{\nu_k}, E_\lambda) = 0$ for every k, $1 \le k \le s$. As $\lambda \in \Psi_{\mu,0}$ is arbitrary, $E_{\nu_k} \in \operatorname{Ker} \tau_\alpha(\,\cdot\,, P_{\alpha,0})$ for $1 \le k \le s$, proving the lemma. □

Lemma 3.5. *Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1 of sub-diagram type, $X_0 = G_0/P_0 \hookrightarrow G/P = X$. Assume that $\mathscr{C}_0(X_0)$ intersects nontrivially with the unique open P-orbit of $\mathscr{C}_0(X)$ at $0 = eP$ (identified with eP_0). Suppose (X_0, X) is nondegenerate for mappings. Then, it must necessarily be nondegenerate for substructures.*

Proof. By assumption, choosing $\alpha = E_\mu \in \widetilde{\mathscr{C}_0}(X_0)$, for every $\nu \in \Psi_\mu$ there exists $\nu_0 \in \Psi_{\mu,0}$ such that $\sigma_\alpha(E_\nu, E_{\nu_0}) \neq 0$, i.e., $\zeta := \nu + \nu_0 - \mu \in \Phi^+$, hence $\sigma_\alpha(E_\nu, E_{\nu_0}) = cE_\zeta$, $c \neq 0$. To prove that (X_0, X) is nondegenerate for substructures we have to show that $\operatorname{Ker} \tau_\alpha(\,\cdot\,, P_{\alpha,0}) = P_{\alpha,0}$. By Lemma 3.4, it suffices to show that for any $\nu \in \Psi_\mu - \Psi_{\mu,0}$, $E_\nu \notin \operatorname{Ker} \tau_\alpha(\,\cdot\,, P_{\alpha,0})$. Suppose otherwise. Then, $\tau_\alpha(E_\nu, E_{\nu_0}) = 0$, hence $E_\zeta \in P_\alpha + (D_0 \cap T_0(X_0))$. From $\zeta(H_\mu) = \nu(H_\mu) + \nu_0(H_\mu) - \mu(H_\mu) = 1 + 1 - 2 = 0$ (cf. proof of Lemma 3.2) it follows that $E_\zeta \perp P_\alpha$, hence $E_\zeta \in D_0 \cap T_0(X_0) \subset T_0(X_0)$, i.e., $\zeta \in \Phi_0$. But then $\nu = \zeta + \mu - \nu_0 \in \Phi_0$, a plain contradiction, proving that (X_0, X) is nondegenerate for substructures, as desired. □

4. Proof of Main Theorem 1

Let $(X, \mathcal{K})$ be a rational homogeneous space of Picard number 1 endowed with the uniruling by projective lines, and (X_0, X) be an admissible pair of sub-diagram type where $X_0 \subset X$ is nonlinear. Main Theorem 1 asserts that (X_0, X) is a rigid pair. For the proof we collect here relevant results from Hong-Mok [HoM10] in the long-root cases, from Hong-Park [HoP11] in the short-root cases, and from Hong-Mok

[HoM13] on the parallel transport of VMRTs along minimal rational curves applicable to both the long-root and the short-root cases. The starting point is a proof of nondegeneracy for substructures of (X_0, X), which by §3 reduces to nondegeneracy for mappings. In the long-root cases we have the following result by Hong-Mok [HoM10, Proposition 3.4].

Proposition 4.1. (Hong-Mok [HoM10]). *Let (X_0, X) be an admissible pair of nonlinear rational homogeneous spaces of Picard number 1 and of sub-diagram type marked at a long root. Then, (X_0, X) is nondegenerate for mappings. Moreover, given any linear section $\mathcal{B}$ of $\mathscr{C}_0(X)$ such that $\big(\mathcal{B} \subset \mathscr{C}_0(X)\big)$ is projectively equivalent to $\big(\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X)\big)$, there exists $\gamma \in P$ such that $\mathcal{B} = \gamma(\mathscr{C}_0(X_0))$.*

Identifying rational homogeneous spaces X biholomorphic to each other, for the proof of Main Theorem 1 it suffices to consider the long-root cases and the short-root cases pertaining to $C_n, n \geq 3$, and F_4 (cf. third paragraph after the proof of Lemma 3.1). For the case of C_n the relevant short-root cases are (C_n, α_k) where $2 \leq k \leq n-1$, which corresponds to the symplectic Grassmannian $S_{k,n}$ of isotropic k-dimensional vector subspaces of a symplectic vector space (W, ω) of dimension $2n$. There are precisely two short-root cases of type F_4, viz., the cases of (F_4, α_3) and (F_4, α_4). Here the convention in the labeling of nodes of $\mathscr{D}(F_4)$ is such that α_3 and α_4 are the short roots and the longest root is $2\alpha_1+3\alpha_2+4\alpha_3+2\alpha_4$. We have the following enumeration of relevant short-root cases taken from [HoP11, Lemma 3.1 and Lemma 4.1].

Proposition 4.2. (Hong-Park [HoP11]). *The following list enumerates all admissible pairs (X_0, X) of sub-diagram type of rational homogeneous spaces of Picard number 1 of type C_n or F_4 marked at a short root such that $X_0 \subset X$ is nonlinear.*

(a) *$X = S_{k,n}$ (i.e., X is of type (C_n, α_k)), $2 \leq k \leq n-1$, $X_0 \subset X$ is given by $\{[E] \in S_{k,n} : F_1 \subset E \subset F_2\}$, where $F_1 \subset F_2 \subset W$ are isotropic subspaces of the symplectic vector space (W, ω) of dimension $2n$, and $0 < \dim(F_1) \leq k-2, \dim(F_2) \geq k+2$. ($X_0$ is of type (A_{m-1}, α_ℓ), where $m = \dim F_2 - \dim F_1$, $\ell = k - \dim F_1$.)*

(b) *$X = S_{k,n}$, and $X_0 \subset X$ is given by $\{[E] \in S_{k,n} : F \subset E\}$ where F is an isotropic subspace of (W, ω) as in* (a), *and $0 < \dim F \leq k-2$. (X_0 is of type (C_m, α_ℓ), where $m = n - \dim F$, $\ell = k - \dim F$.)*

(c) *X is of type (F_4, α_3) and $X_0 \subset X$ is of type (B_3, α_3), obtained by deleting $\{\alpha_4\}$ from $\mathscr{D}(F_4)$.*

(d) *X is of type (F_4, α_3) and $X_0 \subset X$ is of type (C_3, α_2), obtained by deleting $\{\alpha_1\}$ from $\mathscr{D}(F_4)$.*

The following proposition is taken from [HoP11, Lemma 3.1, Lemma 4.1, Proposition 3.4 and Proposition 4.4].

Proposition 4.3. (Hong-Park [HoP11]). *For each short-root case (X_0, X) of sub-diagram type as listed in Proposition 4.2, the admissible pair (X_0, X) is nondegenerate for mappings. Moreover, for (X_0, X) belonging to classes* (b), (c) *or* (d), *given any linear section $\mathcal{B}$ of $\mathscr{C}_0(X)$ such that $\big(\mathcal{B} \subset \mathscr{C}_0(X)\big)$ is projectively equivalent to $\big(\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X)\big)$, there exists $\gamma \in P$ such that $\mathcal{B} = \gamma(\mathscr{C}_0(X_0))$.*

Regarding the method of parallel transport of Mok [Mk08a] and Hong-Mok [HoM11] [HoM13] the following general result applicable to both the long-root and short-root cases was established in [HoM13, Proposition 3.3] by complex-analytic methods using the theory of exceptional sets of Grauert [Gr65].

Proposition 4.4. (Hong-Mok [HoM13]). *Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1 of sub-diagram type. Suppose $\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X)$ contains a general point $[\alpha] \in \mathscr{C}_0(X)$. Assume that there exists $h \in P$ such that $h([\alpha]) = [\alpha]$ and such that $h(\mathscr{C}_0(X_0))$ and $\mathscr{C}_0(X_0)$ are tangent to each other at $[\alpha]$. Then, $h(\mathscr{C}_0(X_0)) = \mathscr{C}_0(X_0)$.*

For the proof of Main Theorem 1 we need finally a lemma regarding varieties swept out by minimal rational curves in relation to a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ modeled on an admissible pair of rational homogeneous spaces (X_0, X) of Picard number 1.

Lemma 4.1. *Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1, and $S \subset W$, $\dim(S) =: s$, be a complex submanifold of some connected open subset $W \subset X$ such that $\varpi : \mathscr{C}(S) \to S$, $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$, defines a sub-VMRT structure modeled on (X_0, X). Suppose $S \subset X$ is linearly saturated. Let $x \in S$ and denote by $\Pi(x, S) \subset X$ the union of all minimal rational curves ℓ of X emanating from x such that $[T_x(\ell)] \in \mathscr{C}_x(S)$. Suppose now ℓ_0 is a minimal rational curve on X such that $[T_x(\ell_0)] \in \mathscr{C}_x(S)$, and let $y \in \ell_0 \cap S$ be distinct from x, so that $\Pi(x, S)$ is smooth at y. Write $T_y(\ell_0) = \mathbb{C}\alpha_y$. Then, $T_y(\Pi(x, S)) = T_{\alpha_y}(\widetilde{\mathscr{C}_y}(S))$.*

Remark Strictly speaking, $T_y(\Pi(x, S)) \subset T_y(X)$, while $T_{\alpha_y}(\widetilde{\mathscr{C}_y}(S)) \subset T_{\alpha_y}(T_y(X)) \cong T_y(X)$. The last isomorphism is canonical, and we make use of it in the statement of Lemma 4.1 to identify vertical tangent vectors at α_y with tangent vectors of X at y (cf. paragraphs following Lemma 2.1).

In the rest of §4 we will need to cite basic results from the geometric theory of uniruled projective manifolds based on VMRTs stated in §5. For the statements and proofs of such results we refer the reader to §5 and the references given there.

Proof of Lemma 4.1. For any $z \in X$ we write $\mathcal{K}_z \subset \mathcal{K}$ for the subset consisting of minimal rational curves passing through z, and define $\mathcal{V}(z) := \bigcup \{\ell : [\ell] \in \mathcal{K}_z\}$. For $x \in S$ as given in the lemma we have $\Pi(x,S) = \bigcup \{\ell : [\ell] \in \mathcal{K}_x, [T_x(\ell)] \in \mathscr{C}_x(S)\} \subset \mathcal{V}(x)$. Shrinking W (and hence S) if necessary and taking W for instance to be a convex open set in a privileged coordinate chart, we may assume for convenience that any nonempty intersection of a minimal rational curve ℓ on X with W is connected. By assumption S is linearly saturated. For any $[\ell] \in \mathcal{K}_x$ such that $[T_x(\ell)] \in \mathscr{C}_x(S)$, the connected set $\ell \cap W$ lies on S, hence $\Pi(x,S) \cap W \subset S$.

Fix now $[\ell_0] \in \mathcal{K}_x$ such that $[T_x(\ell_0)] \in \mathscr{C}_x(S)$, and let $y \in \ell_0 \cap W \subset S$, $y \neq x$, $T_y(\ell_0) =: \mathbb{C}\alpha_y$. From the smoothness of $\mathscr{C}_x(S)$ it follows that $\Pi(x,S)$ is smooth at y and we have $T_y(\Pi(x,S)) \subset T_y(S)$. On the other hand (cf. Hwang-Mok [HM08]), we have ($\sharp$) $T_y(\mathcal{V}(x)) = P_{\ell_0,y}$, recalling that P_{ℓ_0} is the positive part of the Grothendieck decomposition $T(X)|_{\ell_0} \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^p \oplus \mathcal{O}^q$. For completeness we include a proof of ($\sharp$), as follows. There exists a holomorphic map $F : \mathbb{P}^1 \times \Delta^p \to X$ such that $F(0,t) = x$ for any $t \in \Delta^p$, $F(1,0) = y$, and such that, writing $f_t(w) := F(w,t)$, $f_t : \mathbb{P}^1 \to X$ is a biholomorphism onto a minimal rational curve $\ell(t)$, $\ell(0) = \ell_0$, in such a way that $\varphi(t) := [\ell(t)]$ defines a biholomorphism of Δ^p onto a neighborhood of $[\ell_0]$ in $\mathcal{K}_x$. We have $T_{[\ell_0]}(\mathcal{K}_x) \cong \Gamma(\ell_0, N_{\ell_0|X} \otimes \mathfrak{m}_x)$, where $\mathfrak{m}_x$ stands for the maximal ideal sheaf at $x \in \ell_0$. For $1 \leq k \leq p$ define $\sigma_k = \frac{\partial F}{\partial t_k} \in \Gamma(\mathbb{P}^1, f_0^*T(X) \otimes \mathfrak{m}_0) \cong \Gamma(\ell_0, T(X) \otimes \mathfrak{m}_x)$, where $\mathfrak{m}_0$ stands for the maximal ideal sheaf at $0 \in \mathbb{P}^1$, we have $\mathrm{Span}\{\sigma_1(y), \cdots, \sigma_p(y), \alpha_y\} = T_y(\mathcal{V}(x)) \cong \mathbb{C}^{p+1}$. Hence, $T_y(\mathcal{V}(x)) = P_{\ell_0,y}$, proving ($\sharp$).

By ($\sharp$) we have $T_y(\Pi(x,S)) \subset T_y(\mathcal{V}(x)) = P_{\ell_0,y}$. Since also $T_y(\Pi(x,S)) \subset T_y(S)$, we have $T_y(\Pi(x,S)) \subset T_y(\mathcal{V}(x)) \cap T_y(S) = P_{\ell_0,y} \cap T_y(S)$. By Lemma 5.1, $P_{\ell_0,y} =: P_{\alpha_y} = T_{\alpha_y}(\widetilde{\mathscr{C}_y}(X))$, and it follows that $T_y(\Pi(x,S)) \subset T_{\alpha_y}(\widetilde{\mathscr{C}_y}(X)) \cap T_y(S) = T_{\alpha_y}(\widetilde{\mathscr{C}_y}(S))$, by the condition ($\dagger$) stated in the second paragraph after the proof of Proposition 2.1. Finally, from $\dim(T_y(\Pi(x,S))) = \dim(\mathscr{C}_x(S)) + 1 = \dim(\mathscr{C}_y(S)) + 1$ we conclude that actually $T_y(\Pi(x,S)) = T_{\alpha_y}(\widetilde{\mathscr{C}_y}(S))$, proving Lemma 4.1. $\square$

Proof of Main Theorem 1. Let (X_0, X) be an admissible pair of nonlinear rational homogeneous spaces of Picard number 1 of sub-diagram type, $W \subset X$ be a connected open subset and $S \subset W$ be a complex submanifold admitting a sub-VMRT structure modeled on (X_0, X). By Proposition 4.1 and Proposition 4.3, (X_0, X) is always nondegenerate for mappings. By Lemma 3.5, (X_0, X) is also nondegenerate for substructures provided that, writing $X = G/P$ and denoting by $0 = eP_0 \in X_0$ a reference point (identified with $0 = eP \in X$), $\mathscr{C}_0(X_0)$ intersects the unique open P-orbit of $\mathscr{C}_0(X)$ nontrivially. In the long-root cases the latter condition is always satisfied since the VMRT $\mathscr{C}_0(X)$ is homogeneous. In the 4 classes of short-root cases enumerated as (a) - (d) in Proposition 4.1, the latter condition is valid with the exception of the class (a) consisting of Grassmannians of rank ≥ 2 in symplectic Grassmannian. Thus, Lemma 3.5 applies excepting the short-root cases of class (a). For the time being we exclude the short-root cases of class (a) in the statement of Proposition 4.2. Start with a base point $x \in S$. By definition, $\big(\mathscr{C}_x(S) \subset \mathscr{C}_x(X)\big)$ is projectively equivalent to $\big(\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X)\big)$. By Proposition 4.1 and Proposition 4.3, there exists some $\gamma \in \mathrm{Aut}(X)$ such that $\mathscr{C}_x(S) = \mathscr{C}_x(\gamma(X_0))$.

Write $Z := \gamma(X_0)$. Any minimal rational curve ℓ on Z is nonsingular and standard (cf. paragraph preceding Lemma 5.1) in the sense that $T(Z)|_\ell \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^a \oplus \mathcal{O}^b$ for some integers $a, b \geq 0$ (independent of ℓ). Fix such a minimal rational curve ℓ on Z passing through x, $T_x(\ell) =: \mathbb{C}\alpha_x$. Denoting by $Q_\ell = \mathcal{O}(2) \oplus (\mathcal{O}(1))^a \subset T(Z)|_\ell$ the positive part of $T(Z)|_\ell$, for any $y \in \ell$, we have $T_{[\alpha_y]}(\mathscr{C}_y(Z)) = Q_{\ell,y}/\mathbb{C}\alpha_y, T_y(\ell) =: \mathbb{C}\alpha_y$ (cf. Lemma 5.1), where $T(\ell) \subset T(Z)|_\ell$ is the $\mathcal{O}(2)$ component, so that $Q_\ell \supset T(\ell)$. Let $\mathcal{V}(x, Z) \subset Z$ be the union of minimal rational curves ℓ on Z emanating from x. $\mathcal{V}(x, Z)$ is nonsingular at any point $y \in \ell$ distinct from x. By the deformation theory of rational curves $T_y(\mathcal{V}(x, Z)) = \big\{\sigma(y) : \sigma \in \Gamma(\ell, T(Z)|_\ell) \text{ and } \sigma(x) = 0\big\}$. Hence, $T_y(\mathcal{V}(x, Z)) = Q_{\ell,y}$.

As in the proof of Lemma 4.1 we may assume that for any minimal rational curve ℓ on X, either $\ell \cap W = \emptyset$, or $\ell \cap W$ is connected. By Proposition 2.3, S is linearly saturated, so that $\mathcal{V}(x, Z) \cap W \subset S$. From now on ℓ stands for a minimal rational curve on X such that $x \in \ell$ and $\ell \cap W \subset S$. By our choice of the rational homogeneous subspace $Z \subset X$, we have $\mathscr{C}_x(S) = \mathscr{C}_x(Z)$. Let now $y \in \ell \cap W, y \neq x$. We claim that (††) *$\mathscr{C}_{[\alpha_y]}(S)$ and $\mathscr{C}_{[\alpha_y]}(Z)$ are tangent to each other at $[\alpha_y]$.* To see this, writing $\Pi(x, S) := \bigcup\big\{\ell : [\ell] \in \mathcal{K}_x, [T_x(\ell)] \in \mathscr{C}_x(S)\big\}$, we have $\Pi(x, S) \cap W \subset S$ by linear saturation of S. For $y \in \ell \cap W$ by Lemma 4.1 we have $T_y(\Pi(x, S)) = T_{\alpha_y}(\widetilde{\mathscr{C}_y}(S))$. On the other hand, for the union of minimal rational curves $\mathcal{V}(x, Z)$ on Z emanating from x, from $\mathscr{C}_x(S) = \mathscr{C}_x(Z)$ we have $\mathcal{V}(x, Z) = \Pi(x, S)$, hence $\mathcal{V}(x, Z) \cap W = \Pi(x, S) \cap W \subset S$. We also have $T_y(\mathcal{V}(x, Z)) = T_{\alpha_y}(\widetilde{\mathscr{C}_y}(Z))$ (which already follows from the statement (♯) in the proof of Lemma 4.1, applied to Z). It follows that $T_{\alpha_y}(\widetilde{\mathscr{C}_y}(S)) = T_{\alpha_y}(\widetilde{\mathscr{C}_y}(Z))$, implying that $\mathscr{C}_y(Z)$ and $\mathscr{C}_y(S)$ are tangent to each other at $[\alpha_y]$, proving the claim (††).

Recall that $x \in \ell \cap S$ and $\ell \cap W \subset S$, and that $\mathscr{C}_x(S) = \mathscr{C}_x(Z)$. By the property (††), $\mathscr{C}_x(Z)$ and $\mathscr{C}_x(S)$ are tangent to each other at any point $y \in \ell \cap S$. By Proposition 4.1, Proposition 4.3 and Proposition 4.4, we know that in fact († † †) *$\mathscr{C}_y(S)$ and $\mathscr{C}_y(Z)$ are identical for $y \in \ell \cap S$.* (Note that Proposition 4.4 fails in general for admissible pairs not of sub-diagram type.) Thus, the methods of parallel transport along minimal rational curves and adjunction of minimal rational curves of [Mk08a], [HoM10] and [HoM13] apply to conclude that S is an open subset of $Z = \gamma(X_0)$. In part to make the proof of Main Theorem 1 self-contained and in part in anticipation of the more elaborate method of constructing moduli spaces of chains of rational curves for general sub-VMRT structures to be given in §8 we give below a schematization of the procedure of adjoining minimal rational curves to recover Z and hence to identify S as an open subset of Z. On the rational homogeneous space Z of Picard number 1 denote by $\mathcal{Q}$ the Chow component of 1-cycles consisting of minimal rational curves on Z. Let $\rho : \mathcal{U} \to \mathcal{Q}$ be the universal $\mathbb{P}^1$-bundle over $\mathcal{Q}$. Since $\mathcal{Q}$ is isomorphic to the minimal rational component of projective lines on the rational homogeneous space Z, we treat $\rho : \mathcal{U} \to \mathcal{Q}$ as the universal family of a minimal rational component as in §5. Since Z is homogeneous and uniruled by projective lines, the tangent map $\tau : \mathcal{U} \xrightarrow{\cong} \mathscr{C}(Z) \subset \mathbb{P}T(Z)$ is a biholomorphism. From now on we write $\rho : \mathscr{C}(Z) \to \mathcal{Q}$ for the universal $\mathbb{P}^1$-bundle over $\mathcal{Q}$, identifying $\mathscr{C}(Z)$ with $\mathcal{U}$ via the inverse of the tangent map. In what follows all bundles, sections and maps are understood to be holomorphic.

Scheme 4.1. *There exist projective manifolds $\mathscr{S}_j$ and $\mathcal{W}_j$, $0 \le j \le m$, constructed iteratively in the order $\mathscr{S}_0, \mathcal{W}_0, \mathscr{S}_1, \mathcal{W}_1, \cdots, \mathscr{S}_m, \mathcal{W}_m$, starting with $\mathscr{S}_0 := \mathscr{S} := \mathscr{C}_x(Z)$, $\kappa_0 : \mathscr{S}_0 \to \mathcal{Q}$ given by $\kappa_0 := \rho|_{\mathscr{C}_x(Z)}$, such that*

(a) *for $0 \le j \le m, \mathscr{S}_j$ is equipped with a classifying map $\kappa_j : \mathscr{S}_j \to \mathcal{Q}$;*
(b) *for $0 \le j \le m$, $\mathcal{W}_j$ is the total space of a $\mathbb{P}^1$-bundle $\gamma_j : \mathcal{W}_j \to \mathscr{S}_j$, obtained by pulling back the universal $\mathbb{P}^1$-bundle $\rho : \mathscr{C}(Z) \to \mathcal{Q}$ by $\kappa_j : \mathscr{S}_j \to \mathcal{Q}$, which is equipped with a tautological section $\mathfrak{s}_{\gamma_j} : \mathscr{S}_j \to \mathcal{W}_j$ and an accompanying evaluation map $\lambda_j : \mathcal{W}_j \to X$;*
(c) *for $0 \le j \le m-1$, $\mathscr{S}_{j+1}$ is the total space of a fiber bundle $\delta_{j+1} : \mathscr{S}_{j+1} \to \mathcal{W}_j$ with fibers isomorphic to $\mathscr{S}$ obtained by pulling back the VMRT-structure $\pi : \mathscr{C}(Z) \to Z$ by $\lambda_j : \mathcal{W}_j \to X$, which is equipped with a tautological section $\mathfrak{s}_{\delta_{j+1}} : \mathcal{W}_j \to \mathscr{S}_{j+1}$ (and with a classifying map $\kappa_{j+1} : \mathscr{S}_{j+1} \to \mathcal{Q}$ as given in* (a)*); and*
(d) *$\lambda_m(\mathcal{W}_m) = Z$.*

Thus, sequentially $\mathscr{S}_0 := \mathscr{S} := \mathscr{C}_x(Z)$, $\mathcal{W}_0$ is the underlying space of a $\mathbb{P}^1$-bundle $\gamma_0 : \mathcal{W}_0 \to \mathscr{S}_0$, $\mathscr{S}_1$ is the underlying space of an $\mathscr{S}$-bundle $\delta_1 : \mathscr{S}_1 \to \mathcal{W}_0$, $\cdots$, iteratively until we obtain a $\mathbb{P}^1$-bundle $\gamma_m : \mathcal{W}_m \to \mathscr{S}_m$ such that $\mathcal{W}_m$ covers Z under the evaluation map $\lambda_m : \mathcal{W}_m \to Z$. The $\mathbb{P}^1$-bundle $\gamma_0 : \mathcal{W}_0 \to \mathscr{S}_0$ is the pull-back of the universal $\mathbb{P}^1$-bundle $\rho : \mathscr{C}(Z) \to \mathcal{Q}$ by $\kappa_0 := \rho|_{\mathscr{C}_x(Z)}$. The $\mathbb{P}^1$-bundle $\gamma_0 : \mathcal{W}_0 \to \mathscr{S}_0$ is equipped with a canonical section $\mathfrak{s}_{\gamma_0} : \mathscr{S}_0 \to \mathcal{W}_0$ where, for a minimal rational curve ℓ_0 passing through $x_0 := x$ and for $\zeta_0 := [T_x(\ell_0)] \in \mathscr{C}_x(Z) =: \mathscr{S}_0$, $\mathfrak{s}_{\gamma_0}(\zeta_0)$ is the point on $\rho^{-1}([\ell_0])$

corresponding to the point $x_0 \in \ell_0$. (Here and henceforth we denote by $[\ell] \in \mathcal{Q}$ the minimal rational curve ℓ regarded as a member of the Chow component $\mathcal{Q}$.) The classifying map $\kappa_j : \mathscr{S}_j \to \mathcal{Q}$ and the evaluation map $\lambda_j : \mathcal{W}_j \to X$ are also defined inductively. For $0 \le j \le m$, a point in $\mathcal{W}_j$ corresponds to a chain of $j+1$ minimal rational curves $\ell_0, \cdots, \ell_j$ and a sequence of $j+2$ points $x_0, \cdots, x_{j+1}$ on X, $x_0 = x$, such that $x_i, x_{i+1} \in \ell_i$ for $0 \le i \le j$. Suppose $w_j \in \mathcal{W}_j$ is associated to $(x_0, \cdots, x_{j+1})$ and $(\ell_0, \cdots, \ell_j)$ this way, we define $\lambda_j(w_j) = x_{j+1}$. Then, the tautological section $\mathfrak{s}_{\delta_{j+1}} : \mathcal{W}_j \to \mathscr{S}_{j+1}$ is defined by $\mathfrak{s}_{\delta_{j+1}}(w_j) = [\lambda_j^* T_{x_{j+1}}(\ell_j)] \in \lambda_j^* \mathscr{C}_{x_{j+1}}(Z) =: \mathscr{S}_{j+1,w_j}$, which sits over w_j. Similarly for $1 \le j \le m$ and for ζ_j a point in $\mathscr{S}_j$, $\delta_j(\zeta_j) := w_{j-1} \in \mathcal{W}_{j-1}$, corresponding to sequences $(x_0, \cdots, x_j)$ and $(\ell_0, \cdots, \ell_{j-1})$ accompanied by $[\alpha_j] \in \mathscr{C}_{x_j}(Z)$, the latter being identified with $\mathscr{S}_{j,w_{j-1}}$ as above. Since $[\alpha_j]$ determines uniquely a line ℓ_j passing through x_j such that $T_{x_j}(\ell_j) = \mathbb{C}\alpha_j$, we may equivalently think of ζ_j as being associated to $(x_0, \cdots, x_j)$ and $(\ell_0, \cdots, \ell_j)$, and we define $\kappa_j : \mathscr{S}_j \to \mathcal{Q}$ by $\kappa_j(\zeta_j) := [\ell_j] \in \mathcal{Q}$. There is a unique point $v_j \in \mathcal{W}_{j,\zeta_j} = \kappa_j^*(\rho^{-1}([\ell_j])$ corresponding to the point $x_j \in \ell_j$, and we define the tautological section $\mathfrak{s}_{\gamma_j} : \mathscr{S}_j \to \mathcal{W}_j$ by $\mathfrak{s}_{\gamma_j}(\zeta_j) := v_j$. Identifying $\mathscr{S}_j$ (resp. $\mathcal{W}_j$) with the image of the tautological section $\mathfrak{s}_{\gamma_j}$ (resp. $\mathfrak{s}_{\delta_{j+1}}$) we may write $\mathscr{S}_0 \subset \mathcal{W}_0 \subset \mathscr{S}_1 \subset \mathcal{W}_1 \subset \cdots \subset \mathscr{S}_j \subset \mathcal{W}_j \subset \cdots$. Writing $Z_j := \lambda_j(\mathcal{W}_j)$, $Z_0 \subset Z_1 \subset \cdots \subset Z_j \subset \cdots$. By the Proper Mapping Theorem $Z_j \subset Z$ is an irreducible subvariety. By dimension count there exists some m such that $Z_m = Z_{m+1} = \cdots$. By assumption, Z is of Picard number 1, and it follows from Proposition 5.1 (cf. Hwang-Mok [HM98]) that $Z_m = Z$. This completes the description of the iterative scheme for constructing $\mathscr{S}_j$ and $\mathcal{W}_j$ as total spaces of iterated fiber bundles. For more details and explanation on the iterative scheme (in the context of more general sub-VMRT structures) we refer the reader to §8.

It remains to show $S \subset Z$. Recall that from $(\dagger\dagger\dagger)$, starting with any point x lying on S, for any minimal rational curve ℓ passing through x such that $\ell \cap W \subset S$, we have $\mathscr{C}_y(S) = \mathscr{C}_y(Z)$ for $y \in \ell \cap S$. For $0 \le j \le m$ define the subset $\mathcal{O}_j^\sharp \subset \mathcal{W}_j$ encompassing every point $w_j \in \mathcal{W}_j$ associated to a sequence of points $(x_0, \cdots, x_{j+1})$ on Z and a sequence of minimal rational curves $(\ell_0, \cdots, \ell_j)$ on Z satisfying the additional requirements that $\ell_i \cap W \subset S$ for $0 \le i \le j-1$ and that $x_i \in S$ for $0 \le i \le j$. From $(\dagger\dagger\dagger)$ it follows that for $w_j \in \mathcal{O}_j^\sharp$ we also have $\ell_j \cap W \subset S$. Define now $\mathcal{O}_j := \{w_j \in \mathcal{O}_j^\sharp : x_{j+1} \in S\}$. Then, by induction $\mathcal{O}_j \subset \mathcal{W}_j$ is a nonempty open subset and we have $\lambda_j(\mathcal{O}_j) \subset S \cap Z$. In particular, $\lambda_m(\mathcal{O}_m) \subset S \cap Z$. Since $\lambda_m : \mathcal{W}_m \to Z$ is surjective, it is a submersion on $\mathcal{W}_m - E$ for some subvariety $E \subsetneq \mathcal{W}_m$, hence there exists $w_m \in \mathcal{O}_m - E$ such that λ_m is a submersion at w_m. It follows that Z contains a non-empty open subset of S and hence $S \subset Z$ by the

Identity Theorem for holomorphic functions, proving Main Theorem 1 with one class of exceptions given by (a) in Proposition 4.2.

For the remaining class (a), where $(X_0, X) = (G(\ell, m), S_{k,n})$, $\gamma \in \mathrm{Aut}(S_{k,n})$ need not exist. By considering $S_{k,n}$ as a submanifold of the Grassmannian $G(k, 2n)$ of all k-dimensional vector subspaces in $W \cong \mathbb{C}^{2n}$ we have $G(\ell, m) \subset S_{k,n} \subset G(k, 2n)$ as in Hong-Park [HoP11], and Main Theorem 1 for the admissible pair $(G(\ell, m), S_{k,n})$ follows readily from the rigidity statement for the admissible pair $(G(\ell, m), G(k, 2n))$, one of the long root cases, for which the rigidity result has been established. The proof of Main Theorem 1 is complete. $\square$

Remarks

(a) For the proof of Main Theorem 1 for $(X_0, X) = (G(\ell, m), S_{k,n})$, as in [HoP11] it is necessary to make use of the embedding $G(\ell, m) \subset S_{k,n} \subset G(k, 2n)$ because the analogue of Proposition 4.3 fails, even though it remains the case that $(G(\ell, m), S_{k,n})$ is nondegenerate for substructures.

(b) For the short-root cases (X_0, X) other than those of class (a) one can also check using Lemma 3.2 that (X_0, X) is nondegenerate for substructures.

(c) Consider all admissible pairs (X_0, X) in Main Theorem 1 other than the short-root cases $(X_0, X) = (G(\ell, m), S_{k,n})$, and let $S \subset W$ be a complex submanifold of some connected open subset W of X. If we define $\mathscr{C}(S)$ as $\mathscr{C}(X) \cap \mathbb{P}T(S)$, $\varpi := \pi|_{\mathscr{C}(S)}$, then, denoting by E_x the linear span of $\widetilde{\mathscr{C}_x}(S)$ for $x \in S$ and by D_0 the linear span of $\widetilde{\mathscr{C}_0}(X_0)$, the assumption that $\big(\mathscr{C}_x(S) \subset \mathbb{P}E_x\big)$ is projectively equivalent to $\big(\mathscr{C}_0(X_0) \subset \mathbb{P}D_0\big)$ for $x \in S$ already implies that $\varpi : \mathscr{C}(S) \to S$ is a sub-VMRT structure modeled on (X_0, X) (cf. [HoM10, Proposition 3.4] and [HoP11, Proposition 2.2]). Even when $(X_0, X) = (G(\ell, m), S_{k,n})$, the assumption that $\varpi : \mathscr{C}(S) \to S$ defines a $G(\ell, m)$-structure also implies that $\varpi : \mathscr{C}(S) \to S$ is a sub-VMRT structure modeled on $(G(\ell, m), S_{k,n})$, cf. Hwang-Mok [HM05, Proposition 4.2.1]. Thus, Main Theorem 1 incorporates and strengthens both Theorem 1.1 and Theorem 1.2.

(d) Zhang [Zh14] classified all admissible pairs (X_0, X) in which both X_0 and X are Hermitian symmetric spaces. New examples of nonrigid pairs are found, but it is also found that there are admissible pairs (X_0, X) not of sub-diagram type such that (X_0, X) are nondegenerate for substructures while the analogue of Proposition 4.4 fails so that the method of parallel transport of VMRTs does not apply.

5. Sub-VMRT structures satisfying Condition (T) and a nondegeneracy condition on the second fundamental form

In this section we consider germs of complex submanifolds on uniruled projective manifolds. First of all, we are interested in finding sufficient conditions to guarantee that a complex submanifold S on some connected open subset $W \subset X$ is rationally saturated with respect to $(X, \mathcal{K})$ (cf. paragraph after Proposition 5.2 here).

We recall here some basic notions and facts concerning minimal rational curves on uniruled projective manifolds and refer the reader to Hwang-Mok [HM99b], Hwang [Hw01] and Mok [Mk08b] for overviews, and to Kollár [Ko96] for a standard reference on rational curves. Let X be a uniruled projective manifold, and L be an ample line bundle on X. A rational curve $f : \mathbb{P}^1 \to X$ is said to be *free* if and only if the holomorphic vector bundle $f^*T(X)$ on $\mathbb{P}^1$ is semipositive, i.e., $f^*T(X)$ is a direct sum of line bundles $\mathcal{O}(a_k)$ of degree $a_k \geq 0$. (Given a rational curve $f : \mathbb{P}^1 \to X$, we denote by $[f]$ the same map regarded as a member of $\mathrm{Hom}(\mathbb{P}^1, X)$.) When $[f] \in \mathrm{Hom}(\mathbb{P}^1, X)$ is a free rational curve, $\mathrm{Hom}(\mathbb{P}^1, X)$ is nonsingular at $[f]$. A tangent vector $\sigma \in T_{[f]}(\mathrm{Hom}(\mathbb{P}^1, X))$ corresponds to the infinitesimal deformation at $[f]$ of a one-parameter family of rational curves $\{f_t\}$ defined by a holomorphic map $F : \mathbb{P}^1 \times \Delta(\epsilon) \to X$ for some $\epsilon > 0$ such that $f_t(w) := F(w, t)$ and $f_0 \equiv f$, given by $\sigma(w) = \frac{\partial}{\partial t}\big|_{t=0} F(w, t) \in \Gamma(\mathbb{P}^1, f^*T(X))$. From the freeness of $[f]$ there is no obstruction to lifting any $\sigma \in \Gamma(\mathbb{P}^1, f^*T(X))$ to a holomorphic one-parameter family given by some F as above, and in fact $T_{[f]}(\mathrm{Hom}(\mathbb{P}^1, X))$ is naturally identified with the vector space $\Gamma(\mathbb{P}^1, f^*T(X))$ of global holomorphic sections of $f^*T(X)$ over $\mathbb{P}^1$.

There is a maximal Zariski open subset $\mathcal{H}$ of an irreducible component of $\mathrm{Hom}(\mathbb{P}^1, X)$ such that each $[f] \in \mathcal{H}$ is a free rational curve on X, and such that $\deg(f^*L)$ is minimal among all free rational curves on X. By minimality each $[f] \in \mathcal{H}$ is generically injective. $\mathcal{H}$ is smooth since each $[f] \in \mathcal{H}$ is free. Moreover, $\mathrm{Aut}(\mathbb{P}^1)$ acts effectively on $\mathcal{H}$ since each $[f] \in \mathcal{H}$ is generically injective, and $\mathcal{K} := \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1)$ is quasi-projective and nonsingular. We call $\mathcal{K}$ a minimal rational component on X. Given $[f] \in \mathcal{H}$, $\epsilon > 0$, and a holomorphic curve $\{\gamma_t : |t| < \epsilon\}$ on $\mathrm{Aut}(\mathbb{P}^1)$ with $\gamma(0) = \mathrm{id}_{\mathbb{P}^1}$, we have a holomorphic curve $\{[f_t] : |t| < \epsilon\}$ on $\mathcal{H}$ given by $f_t = f \circ \gamma_t$, thus infinitesimal deformations of $f : \mathbb{P}^1 \to X$ induced by reparametrization correspond to the vector subspace $df(\Gamma(\mathbb{P}^1, T(\mathbb{P}^1))) \subset \Gamma(\mathbb{P}^1, f^*T(X))$, where the differential df of $f : \mathbb{P}^1 \to X$ is interpreted as a homomorphism $df : T(\mathbb{P}^1) \to f^*T(X)$. Denoting by $\kappa \in \mathcal{K}$ the equivalence class of $[f]$ modulo the action of $\mathrm{Aut}(\mathbb{P}^1)$, we have canonically $T_\kappa(\mathcal{K}) = \Gamma(\mathbb{P}^1, f^*T(X))/df(\Gamma(\mathbb{P}^1, T(\mathbb{P}^1)))$. (Writing $\ell := f(\mathbb{P}^1) \subset X$, we will also denote κ by $[\ell] \in \mathcal{K}$ in the sequel.)

We have a universal $\mathbb{P}^1-$bundle $\rho : \mathcal{U} \to \mathcal{K}$ called the universal family of $\mathcal{K}$. Denoting by $\mathrm{Aut}(\mathbb{P}^1; 0)$ the isotropy subgroup of $\mathrm{Aut}(\mathbb{P}^1)$ at $0 \in \mathbb{P}^1$, we have $\mathcal{U} = \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1; 0)$, and $\rho : \mathcal{U} \to \mathcal{K} = \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1)$ is the canonical projection realizing $\mathcal{U}$ as the total space of a holomorphic fiber bundle with fibers isomorphic to $\mathrm{Aut}(\mathbb{P}^1)/\mathrm{Aut}(\mathbb{P}^1; 0) \cong \mathbb{P}^1$. We have canonically the evaluation map $\mu : \mathcal{U} \to X$, and we write $\mathcal{U}_x := \mu^{-1}(x)$. The minimal rational component $\mathcal{K}$ can be naturally identified with a dense Zariski open subset of the normalization $\widehat{\mathcal{Q}}$ of an irreducible component $\mathcal{Q}$ of $\mathrm{Chow}(X)$, the Chow space of the projective manifold X, where a general member of $\mathcal{Q}$ is the irreducible and reduced 1-cycle corresponding to a free rational curve. There is a smallest variety $B \subset X$, called the *bad locus* of $(X, \mathcal{K})$, such that, for any $x \in X - B$, any member of $\mathcal{Q}$ passing through x is the 1-cycle corresponding to a free rational curve. For $x \in X - B$, $\mathcal{U}_x$ is a projective manifold.

At a general point $x \in X$ we have a rational map $\tau_x : \mathcal{U}_x \dashrightarrow \mathbb{P}T_x(X)$ called the tangent map, defined as follows. For $u \in \mathcal{U}_x$ corresponding to $f : \mathbb{P}^1 \to X$ with a marking at x (i.e., $f(0) = x$) and immersed at 0, we define $\tau_x(u) = [df(T_0(\mathbb{P}^1))] \in \mathbb{P}T_x(X)$. Suppose $x \in X - B$ and a general point on each irreducible component of $\mathcal{U}_x$ is immersed at x. Then, $\tau_x : \mathcal{U}_x \dashrightarrow \mathbb{P}T_x(X)$ is defined as a rational map, hence a morphism on $\mathcal{U}_x - A$ for some nowhere dense subvariety $A \subset \mathcal{U}_x$. The variety $\mathscr{C}_x(X) := \overline{\tau_x(\mathcal{U}_x - A)} \subset \mathbb{P}T_x(X)$ is called the variety of minimal rational tangents (VMRT) of $(X, \mathcal{K})$ at x. Collecting the VMRTs $\mathscr{C}_x(X) \subset \mathbb{P}T_x(X)$ where defined we have $\pi : \mathscr{C}(X) \to X$ yielding what we call the VMRT structure in this article.

In what follows for any holomorphic vector bundle $\gamma : V \to \mathbb{P}^1$, the *positive part* V' of V is defined to be the direct sum of summands of degree > 0 in the Grothendieck decomposition of V into a direct sum of holomorphic line subbundles, noting that $V' \subset V$ is independent of the particular choice of Grothendieck decomposition. A rational curve $f : \mathbb{P}^1 \to X$ is said to be *standard* if and only if $f^*T(X) \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some $p, q \geq 0$, $1 + p + q = n := \dim(X)$. When $\ell = f(\mathbb{P}^1)$ is smooth, we have $T(X)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$, and the vector subbundle $P_\ell = \mathcal{O}(2) \oplus \mathcal{O}(1)^p \subset T(X)|_\ell$ is the positive part of $T(X)|_\ell$. For $x \in \ell$, writing $T_x(\ell) = \mathbb{C}\alpha$, we define $P_\alpha := P_{\ell,x}$ to be the fiber of P_ℓ at x. In general $f : (\mathbb{P}^1, 0) \to (X, x)$ is only immersed. When ℓ is smooth at x, we still write $P_\alpha \subset T_x(X)$ for the vector subspace corresponding to $(\mathcal{O}(2) \oplus \mathcal{O}(1)^p)_0 \subset f^*(T_x(X))$. We have (cf. Mok [08b, esp. Lemma 1 and Lemma 2])

Lemma 5.1. *Let $(X, \mathcal{K})$ be a uniruled projective manifold endowed with a minimal rational component. For a general point $x \in X$, a general minimal rational curve passing through x is standard. Suppose $x \in X - B$, and $u \in \mathcal{U}_x$ corresponds to a marked rational curve $f : (\mathbb{P}^1; 0) \to (X; x)$ immersed at 0. Then, the tangent map τ_x :*

$\mathcal{U}_x \dashrightarrow \mathbb{P}T_x(X)$ is a holomorphic immersion at u if and only if the underlying minimal rational curve $\rho(u) = [\ell]$ is standard. Moreover, when ℓ is smooth at x and $\mathscr{C}_x(X)$ is smooth at $[\alpha] = [T_x(\ell)]$, we have $T_{[\alpha]}(\mathscr{C}_x(X)) \cong P_\alpha/\mathbb{C}\alpha$.

When $X \subset \mathbb{P}^n$ is uniruled by projective lines, τ_x is an isomorphism at a general point $x \in X$ (cf. Mok [Mk08b, Lemma 3]). For an arbitrary $(X, \mathcal{K})$, at a general point $x \in X$ the tangent map is known to be birational under a nondegeneracy condition on the Gauss map by Hwang-Mok [HM01] and holomorphic by Kebekus [Ke02]. Finally we have

Theorem 5.1. (Hwang-Mok [HM04b, Corollary 1]). *Let $(X, \mathcal{K})$ be a uniruled projective manifold endowed with a minimal rational component, B be the bad locus of $(X, \mathcal{K})$, and $\pi : \mathscr{C}(X) \to X$ be the associated VMRT structure. Then, at a general point $x \in X - B$, the tangent map $\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$ is a birational finite morphism onto $\mathscr{C}_x(X)$, i.e., $\tau_x : \mathcal{U}_x \to \mathscr{C}_x(X)$ is the normalization.*

By Theorem 5.1 there exists a smallest subvariety $B' \subsetneq X$ of X such that $B' \supset B$ and such that for any $x \in X - B'$, $\tau_x : \mathcal{U}_x \to \mathscr{C}_x(X) \subset \mathbb{P}T_x(X)$ is a birational finite morphism. We call $B' \subset X$ the *enhanced bad locus* of $(X, \mathcal{K})$.

On the universal family $\rho : \mathcal{U} \to \mathcal{K}$ the relative tangent bundle T_ρ defines a rank one (holomorphic) distribution on $\mathcal{U}$. By Theorem 5.1 there is a maximal dense Zariski open subset $\mathcal{O} \subset \mathcal{U}|_{X-B'}$ such that the tangent map $\tau|_{\mathcal{O}} : \mathcal{O} \to \mathscr{C}(X)|_{X-B'}$ is a biregular morphism onto some dense Zariski open subset $\mathcal{W} \subset \mathscr{C}(X)|_{X-B'}$. The rank one distribution $T_\rho \subset T(\mathcal{U})$ translates via the tangent map to a rank one distribution $\mathscr{F}$ on $\mathcal{W}$, yielding the *tautological foliation* $\mathcal{F}$ on $\mathcal{W}$ whose leaves are *tautological liftings* of Zariski open subsets of standard minimal rational curves. By the tautological lifting of a nonsingular curve $\Gamma \subset X$ to $\mathbb{P}T(X)$ we mean the nonsingular curve $\Gamma' \subset \mathbb{P}T(X)$ given by $\Gamma' = \big\{[T_x(\Gamma)] \in \mathbb{P}T_x(X) : x \in \Gamma\big\}$. For an irreducible curve Γ in general, and for a dense Zariski open subset $\Gamma_0 \subset \Gamma$ consisting of smooth points, we can still define the tautological lifting Γ'_0 of Γ_0 in the same way. In the situation under study, since we are only concerned with tautological liftings of standard rational curves to $\mathscr{C}(X)|_{X-B'}$, for such a curve $\ell \not\subset B'$, $\ell \cap (X - B')$ is smooth, and the tautological lifting of ℓ to $\mathscr{C}(X)|_{X-B'}$ is taken to be $\widehat{\ell} = \big\{[T_x(\ell)] \in \mathbb{P}T_x(X) : x \in \ell - B'\big\}$.

Let $\mathscr{P}$ be the distribution on $\mathcal{W}$ where $\mathscr{P}_{\tau(u)} \subset T_{\tau(u)}(\mathcal{W})$ consists of all $\eta \in T_{\tau(u)}(\mathcal{W})$, $\eta = d\tau(\xi)$, such that $d\mu(\xi) \in P_\alpha$ (cf. Hwang-Mok [HM04b, §3]). $\mathscr{P}$ translates via $\tau|_{\mathcal{O}} : \mathcal{O} \xrightarrow{\cong} \mathcal{W}$ to a distribution $(d\tau)^{-1}\mathscr{P}$ on $\mathcal{O}$. Denote by $\mathcal{K}_{\mathrm{st}} \subset \mathcal{K}$ the dense Zariski open subset consisting of standard rational curves, and write $P_f \subset f^*T(X)$ for the positive part $\mathcal{O}(2) \oplus \mathcal{O}(1)^p \subset \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q \cong f^*T(X)$. We have

$df(\Gamma(\mathbb{P}^1, T(\mathbb{P}^1))) \subset \Gamma(\mathbb{P}^1, P_f) \subset \Gamma(\mathbb{P}^1, f^*T(X)) \cong T_{[f]}(\mathcal{H})$. Defining now $\mathscr{D}_{[\ell]} := \Gamma(\mathbb{P}^1, P_f)/df(\Gamma(\mathbb{P}^1, T(\mathbb{P}^1))) \subset \Gamma(\mathbb{P}^1, f^*T(X))/df(\Gamma(\mathbb{P}^1, T(\mathbb{P}^1))) \cong T_{[\ell]}(\mathcal{K})$, and observing that $\dim(\mathscr{D}_{[\ell]}) = \dim(\mathbb{P}^1, (\mathcal{O}(1))^p \cong \mathbb{C}^{2p}$, we have defined a holomorphic distribution $\mathscr{D}$ on $\mathcal{K}_{st}$. Moreover, we have the following result relating $\mathscr{D}$ to $\mathscr{P}$.

Lemma 5.2. (Hwang-Mok [HM04b, Proposition 8]). *Let $\mathscr{D}$ be the distribution on $\mathcal{K}_{st}$ given by $\mathscr{D}_{[\ell]} = \Gamma(\mathbb{P}^1, P_f)/df(\Gamma(\mathbb{P}^1, T(\mathbb{P}^1)))$. Then, over $\mathcal{O}$ we have $(d\tau)^{-1}\mathscr{P} = (d\rho)^{-1}\mathscr{D}$. As a consequence, $\tau^*(\mathscr{P}/\mathscr{F}) \cong \rho^*\mathscr{D}$ on $\mathcal{O}$ for the tautological foliation $\mathscr{F}$.*

In the ensuing discussion for simplicity we assume that for each $u \in \mathcal{O}$, $\rho(u) = [\ell]$ is a smooth standard rational curve, the general case where ℓ is an immersed standard rational curve being similar with obvious modifications. For a complex submanifold M in a complex manifold Y we write $N_{M|Y}$ for the normal bundle of M in Y. For $u \in \mathcal{O}$, $T_{[\ell]}(\mathcal{K}) = \Gamma(\ell, N_{\ell|X})$ where $N_{\ell|X} \cong \mathcal{O}(1)^p \oplus \mathcal{O}^q$, and $\mathscr{D}_{[\ell]} := \Gamma(\ell, P_\ell/T(\ell))$, where $P_\ell/T(\ell) \cong \mathcal{O}(1)^p \subset \mathcal{O}(1)^p \oplus \mathcal{O}^q$ is the *positive part* of the normal bundle $N_{\ell|X}$. Thus, $(\tau^*\mathscr{P}/\mathscr{F})_u$ can be identified naturally with $\Gamma(\ell, \mathcal{O}(1)^p)$ which is the direct sum of p copies of $\Gamma(\ell, \mathcal{O}(1)) \cong \mathbb{C}^2$. In the case where privileged coordinates exist, the decomposition $\mathscr{P}_{[\alpha]}/\mathscr{F}_{[\alpha]} = \check{P}_\alpha/\mathbb{C}\check{\alpha} \oplus \hat{P}_\alpha/\mathbb{C}\hat{\alpha}$ as implied by Lemma 2.1 gives a splitting at u of the short exact sequence

$$(\dagger) \qquad 0 \longrightarrow A \longrightarrow \Gamma(\ell, \mathcal{O}(1)^p) \cong \mathbb{C}^{2p} \xrightarrow{\mathrm{ev}_x} (\mathcal{O}(1)^p)_x \cong \mathbb{C}^p \longrightarrow 0\,.$$

where $\mathrm{ev}_x : \Gamma(\ell, \mathcal{O}(1)^p) \to (\mathcal{O}(1)^p)_x$ is the evaluation map $\mathrm{ev}_x(s) = s(x)$ for $s \in \Gamma(\ell, \mathcal{O}(1)^p)$. Thus $\mathrm{Ker}(\mathrm{ev}_x) = A$ is intrinsic and corresponds to the vector space of vertical vectors $\hat{P}_\alpha/\mathbb{C}\hat{\alpha} \cong \mathbb{C}^p$, while the vector space of horizontal vectors $\check{P}_\alpha/\mathbb{C}\check{\alpha} \subset \mathscr{P}_{[\alpha]}/\mathscr{F}_{[\alpha]}$ is complementary to $\hat{P}_\alpha/\mathbb{C}\hat{\alpha}$, and its definition depends on the choice of privileged coordinates, and cannot be made intrinsic, since the short exact sequence $(\dagger)$ restricted to ℓ is a direct sum of p copies of the basic sequence $0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathcal{O} \oplus \mathcal{O} \longrightarrow \mathcal{O}(1) \longrightarrow 0$ on $\mathbb{P}^1$, which does not split holomorphically.

The VMRTs on $(X, \mathcal{K})$ can be used to reconstruct X in the following way. For $x \in X - B$ we denote by $\mathcal{V}(x) := \mathcal{V}_1(x)$ the union of lines emanating from x. For $k \geq 0$ we define inductively $\mathcal{V}_{k+1}(x) = \bigcup\big\{\mathcal{V}(y) : y \in \mathcal{V}_k(x) - B\big\}$. The process stops after a finite number of steps. In fact, writing $\dim(X) =: n$ we must have $\mathcal{V}_m(x) = \mathcal{V}_n(x) := \mathcal{L}(x)$ whenever $m \geq n$. For $y \in \mathcal{V}_k(x)$, we have $\mathcal{V}_{m-k}(x) \subset \mathcal{V}_m(y) \subset \mathcal{V}_{m+k}(x)$ whenever $m > k$. Applying to $m \geq n + k$ we conclude that $\mathcal{L}(y) = \mathcal{L}(x)$, and we have defined a meromorphic foliation $\mathcal{E}$ on X such that the leaf through a general point $x \in X - B$ compactifies to a projective subvariety $\mathcal{L}(x)$ of X. By Hwang-Mok [HM98, Proposition 12] a general minimal rational curve ℓ is disjoint from the locus of indeterminacies of $\mathcal{E}$. In case the uniruled projective manifold X is of Picard number 1 we have from the proof of Hwang-Mok [HM98, Proposition 13] the following result.

Proposition 5.1. (Hwang-Mok [HM98]). *Let $x \in X - B$ be a general point. Define $\mathcal{V}_1(x) := \mathcal{V}(x)$, and for $k \geq 1$ define inductively $\mathcal{V}_{k+1}(x) := \overline{\bigcup \big\{\mathcal{V}(y) : y \in \mathcal{V}_k(x) - B\big\}}$. Assume that X is of Picard number 1. Then, $X = \mathcal{V}_m(x)$ for m sufficiently large.*

Returning to a uniruled projective manifold $(X, \mathcal{K})$, we consider a complex submanifold $S \subset W$ of a connected open subset $W \subset X - B'$, where $B' \subset X$, $B' \supset B$, is the enhanced bad locus of $(X, \mathcal{K})$ (cf. paragraph after Theorem 5.1). In the proof of Main Theorem 1 on the rigidity of certain admissible pairs (X_0, X) of rational homogeneous spaces of Picard number 1, X_0 was identified with a model $Z \subset X$ of Picard number 1, and $S \subset W$ was reconstructed as an open subset of Z by means of Proposition 5.1. In general, subject to conditions to be imposed on a possibly variable *"geometric substructure"* inherited on $S \subset W$ from X, we are interested in proving that S is rationally saturated and in reconstructing S as a subset of some *irreducible projective variety* of the same dimension. Especially, we will prove an *algebraicity theorem* given by Main Theorem 2 of the article.

Given a uniruled projective manifold $(X, \mathcal{K})$ with enhanced bad locus $B' \subset X$, a connected open subset $W \subset X - B'$ and a complex submanifold $S \subset W$, for $x \in S$ we define $\mathscr{C}(S) := \mathscr{C}_x(X) \cap \mathbb{P}T(S)$, $\mathscr{C}_x(S) := \mathscr{C}_x(X) \cap \mathbb{P}T_x(S)$. Write $\varpi := \pi|_{\mathscr{C}(S)} : \mathscr{C}(S) \to S$. We have

Definition 5.1. *We say that $\varpi := \pi|_{\mathscr{C}(S)} : \mathscr{C}(S) \to S$ is a sub-VMRT structure on $(X, \mathcal{K})$ if and only if*

(a) *the restriction of ϖ to each irreducible component of $\mathscr{C}(S)$ is surjective, and*
(b) *at a general point $x \in S$ and for any irreducible component Γ_x of $\mathscr{C}_x(S)$, we have $\Gamma_x \not\subset \mathrm{Sing}(\mathscr{C}_x(X))$.*

By a general point on S we mean any point outside some complex-analytic subvariety $A \subsetneq S$. Then, for a general point $x \in S$ and for some integer $m \geq 1$

(c) *the fiber $\mathscr{C}_x(S)$ of $\varpi : \mathscr{C}(S) \to S$ has exactly m irreducible components;*
(d) *for each irreducible component $\Gamma_{k,x}$ of $\mathscr{C}_x(S)$, $1 \leq k \leq m$, $\varpi : \mathscr{C}(S) \to S$ is a holomorphic submersion at a general point χ_k of $\Gamma_{k,x}$.*

After passing to normalizations, (c) follows from Stein factorization of proper holomorphic maps (cf. Grauert-Remmert [GR84, pp.212-214]). (Given a surjective proper holomorphic map $\alpha : \mathscr{X} \to B$ between irreducible normal complex spaces, there exists an irreducible normal complex space $\mathscr{Z}$, a holomorphic map $\beta : \mathscr{X} \to \mathscr{Z}$ with *connected fibers*, and a *finite holomorphic map* $\gamma : \mathscr{X} \to B$ such that $\alpha = \gamma \circ \beta$. For a general point $b \in B$ the number of irreducible components of $\alpha^{-1}(b)$ agrees with the sheeting number of γ.) For a surjective proper holomorphic map $\varphi : \mathscr{X} \to B$ between complex manifolds, the locus where $d\varphi$ is of rank $< \dim(B)$ is a subvariety $\mathscr{S} \subsetneq \mathscr{X}$. When X is singular, taking $\sigma : \mathscr{X}^\sharp \to \mathscr{X}$ to be a desingularization, and $\mathscr{S}^\sharp \subset \mathscr{X}^\sharp$ to be the locus where $d(\varphi \circ \sigma)$ fails to be of maximal rank, then φ is a submersion outside of $\nu(\mathscr{S}) \subsetneq \mathscr{X}$, implying (d).

In the study in §2 on sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$ modeled on an admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1, starting with a holomorphic arc $\{\widetilde{\alpha_t}\}$ on $\widetilde{\mathscr{C}_x}(S)$ at a base point $x \in S$, Lemma 2.2 gives the existence of a holomorphic family of sections $\widetilde{\alpha}(z,t)$ of $\widetilde{\varpi} : \widetilde{\mathscr{C}}(X) \to X$ at x which restricts over S to sections of $\widetilde{\mathscr{C}}(S)$. Towards a generalization of Proposition 2.3 on linear saturation we will need a generalization of the latter lemma to sub-VMRT structures in the sense of Definition 5.1. We have

Lemma 5.3. *Let $\varpi : \mathscr{C}(S) \to S$, $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$, be a sub-VMRT structure on $S \subset W \subset X - B'$ as in Definition 5.1. Suppose $x \in S$, $\alpha \in \widetilde{\mathscr{C}_x}(S)$, and $\epsilon > 0$. Let $\{\alpha_t : |t| < \epsilon\} =: \Gamma$ be a holomorphic arc on $\widetilde{\mathscr{C}_x}(S)$ such that $\alpha_0 = \alpha$, and such that $\varpi : \mathscr{C}(S) \to S$ is a submersion at α. Then, the conclusion analogous to that of Lemma 2.2 holds.*

Proof. Since $\widetilde{\pi} : \widetilde{\mathscr{C}}(X) \to X$ is a submersion at $\alpha \in \widetilde{\mathscr{C}_x}(X)$, there exists a coordinate neighborhood U of x on X, a neighborhood W of α on $\widetilde{\mathscr{C}}(X)$, a domain Ω in some Euclidean space, and a biholomorphism $\varphi : \mathcal{W} \xrightarrow{\cong} U \times \Omega$ such that $\widetilde{\pi} \circ \varphi^{-1} : U \times \Omega \to U$ is the canonical projection. Write $n := \dim(X), s := \dim(S), a := \dim(\widetilde{\mathscr{C}_x}(S)), b := \dim(\widetilde{\mathscr{C}_x}(X)), c := b - a$. We may take U to be a coordinate neighborhood identified with the unit polydisk Δ^n, such that $U \cap S$ is identified with $\Delta^s \times \{0\}$ and x with 0, and we write $z = (z'; z'')$ for such holomorphic coordinates, $z' := (z_1, \cdots, z_s)$, $z'' := (z_{s+1}, \cdots, z_n)$. We may take Ω to be the unit polydisk Δ^b in the Euclidean coordinates $w = (w'; w'')$, $w' := (w_1, \cdots, w_a), w'' := (w_{a+1}, \cdots, w_b)$ such that, writing $\varphi(\zeta) = (z; w)$, we have $\varphi(\alpha) = (0; 0)$.

Since $\widetilde{\varpi} : \widetilde{\mathscr{C}}(S) \to S$ is a submersion at α, without loss of generality we may assume $T_\alpha(\widetilde{\mathscr{C}}(S)) = \mathrm{Span}\{\frac{\partial}{\partial z_1}, \cdots, \frac{\partial}{\partial z_s}; \frac{\partial}{\partial w_1}, \cdots, \frac{\partial}{\partial w_a}\}$ and $T_\alpha(\widetilde{\mathscr{C}_x}(S)) = \mathrm{Span}\{\frac{\partial}{\partial w_1}, \cdots, \frac{\partial}{\partial w_a}\}$. For $0 < \delta \leq 1$ we write $\mathcal{W}(\delta) := \varphi^{-1}(\Delta^{n+b}(\delta))$. Thus, for $\delta > 0$ sufficiently small $\mathcal{W}(\delta) \cap \widetilde{\mathscr{C}}(S)$ is given in the coordinates $(z; w)$ as the graph of a vector-valued function $h : \Delta^{s+a} \to \Delta^c$, i.e., $\varphi^{-1}((z', 0; w', w'')) \in \mathcal{W}(\delta) \cap \widetilde{\mathscr{C}}(S)$ if and only if $(w_{a+1}, \cdots, w_b) = (h_1(z', w'), \cdots, h_c(z', w'))$. For $1 \leq k \leq c = b - a$

write $f_k(z,w) = w_{a+k} - h_k(z',w')$. Then, $\mathcal{W}(\delta) \cap \widetilde{\mathscr{C}}(S)$ is the common zero set of $\{z_{s+1}, \cdots, z_n, f_1, \cdots, f_c\}$. Modify now the fiber coordinates for $\widetilde{\pi} : \widetilde{\mathscr{C}}(X) \to X$ on $\mathcal{W}(\delta)$ to $\omega = (\omega_1, \cdots, \omega_b)$ by setting $\omega_i = w_i$ for $1 \le i \le a$ and $\omega_i = f_{i-a}$ for $a+1 \le i \le b$. Then $\mathcal{W}(\delta) \cap \mathscr{C}(S)$ is described by $\omega_{a+1} = 0, \cdots, \omega_b = 0$. Define $\varphi^\sharp : \mathcal{W}(\delta) \to \Delta^{n+a}(\delta) \times \mathbb{C}^c$ by $\varphi^\sharp(\zeta) = (z_1, \cdots, z_n; \omega_1, \cdots, \omega_b) = (z', z''; w', w'') - (0, 0; 0, h_1(z', w'), \cdots, h_c(z', w'))$. For $\delta^\flat > 0$ sufficiently small there exists a neighborhood $\mathcal{W}^\flat \Subset \mathcal{W}(\delta)$ of α such that, writing $\psi := \varphi^\sharp|_{\mathcal{W}^\flat}$, $\psi : \mathcal{W}^\flat \xrightarrow{\cong} U^\flat \times \Delta^b(\delta^\flat)$, $U^\flat$ being a neighborhood of x on S identified with $\Delta^n(\delta^\flat)$.

For the holomorphic arc $\{\alpha_t : |t| < \epsilon\} =: \Gamma$ given in the hypothesis of the lemma, shrinking ϵ if necessary we may assume that $\Gamma \subset \mathcal{W}^\flat \cap \widetilde{\mathscr{C}_x}(S)$. We have a holomorphic section $A(z,t)$ of $\widetilde{\pi} \times \mathrm{id}_{\Delta(\epsilon)} : \widetilde{\mathscr{C}}(X) \times \Delta(\epsilon) \to \Delta^n(\delta^\flat) \times \Delta(\epsilon)$ such that $A(x,t) = \alpha_t$ for $|t| < \epsilon$, and such that $A(z', t) \in \widetilde{\mathscr{C}_x}(S)$ for $z' \in U^\flat \cap S$ (identified with $\Delta^s(\delta^\flat) \times \{0\}$), e.g., for $z \in \Delta^n(\delta^\flat)$ we may define $A(z,t)$ by setting $\psi(A(z,t)) = (z, \psi(A(x,t)) \in \Delta^{n+a}(\delta^\flat) \times \{0\}$. The rest of the proof is identical to that given in the proof of Lemma 2.2. $\square$

With an aim to generalizing Proposition 2.3 on linear saturation we proceed to introduce a general notion of nondegeneracy for substructures. To start with we define the notion of proper pairs of projective subvarieties.

Definition 5.2. *Let V be a Euclidean space and $\mathcal{A} \subset \mathbb{P}(V)$ be an irreducible subvariety. We say that $(\mathcal{B}, \mathcal{A})$ is a proper pair if and only if $\mathcal{B}$ is a linear section of $\mathcal{A}$ and for each irreducible component Γ of $\mathcal{B}$, $\Gamma \not\subset \mathrm{Sing}(\mathcal{A})$.*

For a uniruled projective manifold X and a complex submanifold $S \subset W \subset X - B'$ inheriting a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ as in Definition 5.1, at a general point $x \in S$, $(\mathscr{C}_x(S), \mathscr{C}_x(X))$ is a proper pair of subvarieties. Recall that for a complex vector space V and a subset $Z \subset \mathbb{P}V$, we denote by $\widetilde{Z}$ the affinization of Z, i.e., $Z = \lambda^{-1}(Z)$ for the canonical projection $\lambda : V - \{0\} \to \mathbb{P}V$.

Definition 5.3. *Let V be a finite-dimensional vector space, $E \subsetneq V$ be a vector subspace and $(\mathcal{B}, \mathcal{A})$ be a proper pair of projective subvarieties in $\mathbb{P}(V)$, $\mathcal{B} := \mathcal{A} \cap \mathbb{P}(E) \subset \mathcal{A} \subset \mathbb{P}(V)$. Assume that $\mathcal{A}$ is irreducible. Let $\xi \in \widetilde{\mathcal{B}}$ be a smooth point of both $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$, and let $\sigma : S^2T_\xi(\widetilde{\mathcal{A}}) \to V/T_\xi(\widetilde{\mathcal{A}})$ be the second fundamental form of $\widetilde{\mathcal{A}}$ in V with respect to the Euclidean flat connection on V. Write $V' \subset V$ for the linear span of $\widetilde{\mathcal{A}}$ and define $E' := E \cap V'$. Let $\nu : V/T_\xi(\widetilde{\mathcal{A}}) \to V/(T_\xi(\widetilde{\mathcal{A}}) + E')$ be the canonical projection and define $\tau : S^2T_\xi(\widetilde{\mathcal{A}}) \to V/(T_\xi(\widetilde{\mathcal{A}}) + E')$ by $\tau := \nu \circ \sigma$. We say that $(\mathcal{B}, \mathcal{A}; E)$ is nondegenerate for substructures* *if and only if for each irreducible component Γ of $\mathcal{B}$ and for a general point $\chi \in \Gamma$, we have*

$$\Big\{\eta \in T_\chi(\widetilde{\mathcal{A}}) : \tau(\eta,\xi) = 0 \text{ for any } \xi \in T_\chi(\widetilde{\mathcal{B}})\Big\} = T_\chi(\widetilde{\mathcal{B}}) .$$

Remarks

(a) In case $E' = E \cap V'$ is the same as the linear span of $\widetilde{\mathcal{B}}$ we will drop the reference to E, with the understanding that the projection map ν is defined by using the linear span of $\widetilde{\mathcal{B}}$ as E'.

(b) Definition 5.3 extends readily when $\mathcal{A}$ is allowed to be reducible. Denoting by $\mathcal{A}_i$ the irreducible components of $\mathcal{A}$, we still assume that for each irreducible component $\mathcal{B}_k$ of $\mathcal{B}$, $\mathcal{B}_k \not\subset \mathrm{Sing}(\mathcal{A})$. In particular, $\mathcal{B}_k \subset \mathcal{A}_{i(k)}$ for a unique $i(k)$. To define nondegeneracy for substructures one needs to introduce ν_i and hence τ_i for each $\mathcal{A}_i$, i.e., in terms of the linear span V_i' of each $\widetilde{\mathcal{A}}_i$. We say that $(\mathcal{B}, \mathcal{A}; E)$ is nondegenerate for substructures if and only if for each irreducible component $\mathcal{B}_k$ of $\mathcal{B}$ and for a general point $\chi_k \in \mathcal{B}_k$, we have
$$\Big\{\eta \in T_{\chi_k}(\widetilde{\mathcal{A}}_{i(k)}) : \tau(\eta,\xi) = 0 \text{ for any } \xi \in T_{\chi_k}(\widetilde{\mathcal{B}_k})\Big\} = T_{\chi_k}(\widetilde{\mathcal{B}_k}) .$$

For the proof of Theorem 1.4 one follows the same line of arguments as that in the special case of Proposition 2.3 for rational homogeneous spaces $X = G/P$ and sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$ modeled on admissible pairs (X_0, X) of sub-diagram type of rational homogeneous spaces of Picard number 1, which relies on Lemma 2.1, Lemma 2.2 and Proposition 2.2. In Lemma 2.1, in terms of privileged coordinates we have the decomposition $\widetilde{\mathscr{P}}_\alpha = \check{P}_\alpha \oplus \hat{P}_\alpha$ for the distribution $\widetilde{\mathscr{P}}$ on $\widetilde{\mathscr{C}}(S)$ (defined in the paragraph preceding Lemma 2.1). We proceed now to generalize Lemma 2.1. Let $(X, \mathcal{K})$ be a uniruled projective manifold X of dimension n endowed with a minimal rational component $\mathcal{K}$, $\pi : \mathscr{C}(X) \to X$ be the associated VMRT structure. Let $[\alpha] \in \mathscr{C}(X)$ be a smooth point, $\pi(\alpha) := x \in X - B'$, where B' is the enhanced bad locus of $(X, \mathcal{K})$. Recall that $\tau|_{\mathcal{O}} : \mathcal{O} \xrightarrow{\cong} \mathcal{W}$ and recall the distribution $\mathscr{P}$ on $\mathcal{W}$ as in the paragraph preceding Lemma 5.2. Since $[\alpha] \in \mathscr{C}(X)$ be a smooth point, by Theorem 5.1 there is a unique standard minimal rational curve ℓ passing through x such that ℓ is smooth at x and $T_x(\ell) = \mathbb{C}\alpha$. Let V be a neighborhood of x in $X{-}B'$ such that $\ell \cap V$ is connected and such that, denoting by $\ell^\sharp$ the tautological lifting of ℓ to $\mathbb{P}T(X)$, we have $\ell^\sharp \cap \pi^{-1}(V) \subset \mathcal{W}$. We have

Lemma 5.4. *Let $y \in \ell \cap V$ be distinct from x. Suppose (U, z^i) is a (holomorphic) coordinate chart on a neighborhood U of x, $U \Subset V$, such that for any minimal rational curve ℓ' passing through y sufficiently close to ℓ, $\ell' \cap U$ is a connected open set on an affine line with respect to the coordinates (z^i). Then, in terms of Euclidean coordinates*

$(z^1, \cdots, z^n; w^1, \cdots, w^n)$ *on* $T(U) = U \times T_x(U) = U \times \mathbb{C}^n \subset \mathbb{C}^n \times \mathbb{C}^n$ *arising from* (U, z^i) *in the standard way, we have* $\widetilde{\mathscr{P}_\alpha} = \check{P}_\alpha \oplus \hat{P}_\alpha$.

Proof. The proof of Lemma 5.4 is the same as that of Lemma 2.1. To see this, observe that in place of requiring (U, z^i) to be a privileged coordinate chart a weaker requirement is enough, viz., letting ℓ be the minimal rational curve joining x to a nearby point $y \neq x$, it suffices that in the coordinate chart (U, z^i) the intersection of U with any minimal rational curve ℓ' *emanating from* y and sufficiently close to ℓ is a connected open subset of an affine line. The full hypothesis in the definition of a privileged coordinate chart that *all* minimal rational curves intersecting U nontrivially appear as portions of affine lines was never used in the proof. $\square$

We call (U, z^i) in the proof above a coordinate chart at x adapted to (y, ℓ). Note that it is not required that $y \in U$. The existence of such a coordinate chart (U, z^i) can be seen using "polar coordinates" at y as in [HoM10], which we make more precise here, as follows. Let V be some coordinate neighborhood of x so that all maps into V can be viewed as a map into a complex Euclidean space. We also denote by $\Delta(a; r)$ the open disk on $\mathbb{C}$ centered at a and of radius $r > 0$, write $\Delta(r) := \Delta(0; r)$, and write $\Delta^m(r)$ for the Cartesian product of m copies of $\Delta(r)$.

Choose now y to be a point distinct from x and lying on the connected component of $\ell \cap V$ containing x, and let $\{\alpha_t\}$, $t \in \Delta^p(\epsilon)$, be a holomorphic family of minimal rational tangents $\alpha_t \in \widetilde{\mathscr{C}_y}(X)$, $\alpha_0 \in T_y(\ell)$ such that $\varphi(t) := [\alpha_t] \in \mathscr{C}_y(X)$ is a biholomorphism of $\Delta^p(\epsilon)$ onto a smooth open neighborhood $\mathcal{O}$ of $[T_y(\ell)]$ in $\mathscr{C}_y(X)$. If y is further chosen to be sufficiently close to x, then, shrinking $\epsilon > 0$ if necessary, there exists a holomorphic map $F : \Delta(2) \times \Delta^p(\epsilon) \to V$ such that $F(0,0) = y$, $F(1,0) = x$, $\frac{\partial F}{\partial s}(0,t) = \alpha_t \in \widetilde{\mathscr{C}_y}(X)$, $F|_{\Delta(2)\times\{t\}}$ is a biholomorphism onto an open subset of the minimal rational curve ℓ'_t passing through y, $T_y(\ell'_t) = \mathbb{C}\alpha_t$, and $F|_{\Delta^*(2)\times\Delta^p(\epsilon)}$ is a holomorphic embedding onto an open subset $\Sigma_0 \subset \mathcal{V}(y) - \{y\}$, $x \in \Sigma_0$. Let Σ be $F(\Delta(1;\delta) \times \Delta^p(\epsilon))$ for some $\delta > 0$ sufficiently small, and define $\Phi : \Sigma \to \mathbb{C}^n$ by $\Phi(F(s,t)) := s\alpha_t$. Then, Φ extends to a biholomorphism of some neighborhood U of x in X onto a neighborhood D of α in $\mathbb{C}^n$, with Euclidean coordinates $(z_1, \cdots, z_n)$, giving a coordinate chart (U, z^i) adapted to (x, ℓ). In order to generalize Proposition 2.3 on rational saturation to the general setting of sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$, we introduce a condition called Condition (T) on such structures, as follows.

Definition 5.4. *Let* $\varpi : \mathscr{C}(S) \to S$, $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$, *be a sub-VMRT structure on* $S \subset W \subset X - B'$ *as in Definition 5.1. For a point* $x \in S$, *and* $[\alpha] \in \mathrm{Reg}(\mathscr{C}_x(S)) \cap \mathrm{Reg}(\mathscr{C}_x(X))$, *we say that* $(\mathscr{C}_x(S), [\alpha])$, *or equivalently* $(\widetilde{\mathscr{C}_x}(S), \alpha)$, *satisfies* Condition (T) *if and* *only if* $T_\alpha(\widetilde{\mathscr{C}_x}(S)) = T_\alpha(\widetilde{\mathscr{C}_x}(X)) \cap T_x(S)$. *We say that* $\varpi : \mathscr{C}(S) \to S$ *satisfies* Condition (T) *at* x *if and only if* $(\mathscr{C}_x(S), [\alpha])$ *satisfies* Condition (T) *for a general point* $[\alpha]$ *of each irreducible component of* $\mathrm{Reg}(\mathscr{C}_x(S)) \cap \mathrm{Reg}(\mathscr{C}_x(X))$. *We say that* $\varpi : \mathscr{C}(S) \to S$ *satisfies* Condition (T) *if and only if it satisfies the condition at a general point* $x \in S$.

Condition (T) is hence a hypothesis imposed on tangent spaces of VMRTs and their linear sections. In the study of rigidity of admissible pairs of (X_0, X) of rational homogeneous spaces of Picard number 1, we may regard the VMRT structure $\pi_0 : \mathscr{C}(X_0) \to X_0$ on X_0 also as a sub-VMRT structure of the VMRT structure $\pi : \mathscr{C}(X) \to X$ of the ambient manifold X. Regarding Condition (T) we have the following general result applicable to (X_0, X).

Lemma 5.5. *Let* $(X, \mathcal{K})$, $X \subset \mathbb{P}^N$, *be a uniruled projective manifold endowed with a minimal rational component consisting of projective lines, and denote by* $\pi : \mathscr{C}(X) \to X$ *the VMRT structure on* X. *Let* $Z \subset X$ *be a linear section of* X *such that* Z *is irreducible as a variety and uniruled by projective lines belonging to* $\mathcal{K}$, *and denote by* $\varpi : \mathscr{C}(Z) \to \mathrm{Reg}(Z)$, $\mathscr{C}(Z) = \mathscr{C}(X) \cap \mathbb{P}T(\mathrm{Reg}(Z))$, *the sub-VMRT structure on* $\mathrm{Reg}(Z)$. *Assume that there exists a member of* $\mathcal{K}$ *lying on* $\mathrm{Reg}(Z)$ *which is a free rational curve on* $\mathrm{Reg}(Z)$. *Then, for a general point* $z \in \mathrm{Reg}(Z)$ *and a general smooth point* $[\alpha] \in \mathscr{C}_z(Z)$, $(\mathscr{C}_z(Z), [\alpha])$ *satisfies* Condition (T).

Proof. Write $Z^0 := \mathrm{Reg}(Z)$. Denote by $\mathcal{J} \subset \mathcal{K}$ the variety of projective lines belonging to $\mathcal{K}$ and lying on Z. By assumption there exists $[\ell_0] \in \mathcal{J}$ such that $\ell_0 \subset Z^0$ is a free rational curve, i.e., $T(Z^0)|_{\ell_0}$ is semipositive. Denote by $\mathcal{H}_0 \subset \mathcal{J}$ the irreducible component containing $[\ell_0]$ as a member, and let $\mathcal{H} \subset \mathcal{H}_0$ be the Zariski open subset consisting free rational curves on Z^0. From the semipositivity of $N_{\ell|Z}$ and from $N_{\ell|Z} \subset N_{\ell|\mathbb{P}^N} \cong \mathcal{O}(1)^{N-1}$ it follows that $\ell \subset Z^0$ is a standard rational curve. We have $T(Z^0)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^a \oplus \mathcal{O}^b$ for some $a, b \geq 0$, while $T(X)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some $p, q \geq 0$. Write $T_z(\ell) =: \mathbb{C}\alpha$, $Q_\ell := \mathcal{O}(1)^a \oplus \mathcal{O}^b$ for the positive part of $T(Z^0)|_\ell$ and recall that $P_\ell = \mathcal{O}(1)^p \oplus \mathcal{O}^q$ is the positive part of $T(X)|_\ell$. Denote by $Q_\alpha \subset T_z(Z^0)$ the fiber at z of Q_ℓ and recall that P_α is the fiber at z of P_ℓ. We have $T_\alpha(\widetilde{\mathscr{C}_z}(Z)) = Q_\alpha$ while $T_\alpha(\widetilde{\mathscr{C}_z}(X)) = P_\alpha$.

Consider the injective bundle homomorphism $\varphi : T(Z^0)|_\ell \hookrightarrow T(X)|_\ell$. Noting that $\Gamma(\ell, \mathrm{Hom}(\mathcal{O}(a), \mathcal{O}(b))) = 0$ whenever $a > b$, φ must induce an isomorphism $\varphi|_{\mathcal{O}(2)} : \mathcal{O}(2) \xrightarrow{\cong} \mathcal{O}(2)$, which induces a bundle homomorphism $\psi : \mathcal{O}(1)^a \oplus \mathcal{O}^b \to \mathcal{O}(1)^p \oplus \mathcal{O}^q$. Since $\Gamma(\ell, \mathrm{Hom}(\mathcal{O}(1)^p, \mathcal{O}^q)) = 0$, ψ descends to $\overline{\psi} : \mathcal{O}^b \to \mathcal{O}^q$. To prove that $Q_\alpha = P_\alpha \cap T_z(Z^0)$ it is equivalent to show that the homomorphism $\overline{\psi} : \mathcal{O}^b \to \mathcal{O}^q$ is injective. By assumption $Z = \Pi \cap X$ for some projective linear subspace

$\Pi \subset \mathbb{P}^N$, $\dim(\Pi) := s$. Hence, $N_{Z^0|X}|_\ell \subset N_{\Pi|\mathbb{P}^N}|_\ell \cong \mathcal{O}(1)^{N-s}$. On the other hand, the normal bundle $N_{\ell|X} = \mathcal{O}(1)^p \oplus \mathcal{O}^q$ is semipositive, and $N_{Z^0|X}|_\ell = N_{\ell|X}/N_{\ell|Z}$ must be semipositive as a quotient bundle of $N_{\ell|X}$. It follows that $N_{Z^0|X}|_\ell \cong \mathcal{O}(1)^c \oplus \mathcal{O}^d$ for some $c, d \geq 0$. If $\overline{\psi} : \mathcal{O}^b \to \mathcal{O}^q$ has a nontrivial kernel, there must be some trivial line subbundle $\mathcal{O} \subset \mathcal{O}^b$ such that $\psi(\mathcal{O}) \subset \mathcal{O}(1)^p$, in which case the normal bundle $N_{Z^0|X}|_\ell \cong \mathcal{O}(1)^c \oplus \mathcal{O}^d$ must contain a subbundle of degree ≥ 2, a plain contradiction, proving the lemma. □

Remarks

(a) Suppose (X_0, X) is an admissible pair of rational homogeneous spaces of Picard number 1 and $\varpi : \mathscr{C}(S) \to S$, $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$, is a sub-VMRT structure modeled on (X_0, X). Then, by Definition 1.1 and Lemma 5.5, Condition (T) is satisfied at every point $x \in S$ by $(\mathscr{C}_x(S), [\alpha])$ for any point $[\alpha] \in \mathscr{C}_x(S)$.

(b) Let X be an irreducible Hermitian symmetric space of the compact type of rank ≥ 2 , $x \in X$, and denote by $\mathcal{V}(x) \subset X$ the union of projective lines emanating from x. Then, $Z := \mathcal{V}(x)$ is a linear section of X with respect to the minimal embedding. Moreover, Z is smooth except for an isolated singularity at the point x. On $Z^0 = Z - \{x\}$ we have $\mathscr{C}_z(Z) := \mathscr{C}_z(X) \cap \mathbb{P}T_z(Z)$ for each $z \in Z^0$. If $X = G^{III}(n, n), n \geq 2$, the Lagrangian Grassmannian of rank n (cf. second last paragraph preceding Definition 1.2), then $\mathscr{C}_z(Z) = [\alpha_z]$, where $T_z(\ell_0) = \mathbb{C}\alpha_z$ for the unique projective line ℓ_0 on Z joining z to x, hence $T_{\alpha_z}(\widetilde{\mathscr{C}_z}(X)) \cap T_z(Z) = P_{\alpha_z} \cap P_{\alpha_z} = P_{\alpha_z} \supsetneq \mathbb{C}\alpha_z = T_{\alpha_z}(\widetilde{\mathscr{C}_z}(Z))$, violating Condition (T). For $X \neq G^{III}(n, n), n \geq 2$, $\mathscr{C}_z(Z)$ is uniruled by projective lines. ($\mathscr{C}_z(Z)$ is irreducible except in the case where X is a Grassmannian, in which case there are two irreducible components.) Let $\Lambda \subset \mathscr{C}_z(X)$ be a projective line containing $[\alpha_z]$. Then, there exists a projective plane $\Pi \subset Z$ containing x such that $\mathbb{P}T_z(\Pi) = \Lambda$. Any projective line $\ell \subset \Pi$ avoiding x lies on Z^0 and all such projective lines ℓ are standard rational curves on Z^0. By the proof of Lemma 5.5, $(\mathscr{C}_z(Z), [T_z(\ell)])$ satisfies Condition (T).

Proposition 5.2. *Let $(X, \mathcal{K})$ be a uniruled projective manifold endowed with a minimal rational component, $B' \subset X$ be its enhanced bad locus. Let $W \subset X - B'$ be a connected open set, and $S \subset W$ be a complex submanifold inheriting a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ which satisfies* Condition (T), *$\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$. Suppose at a general point $x \in X$, $(\mathscr{C}_x(S), \mathscr{C}_x(X))$ is nondegenerate for substructures. Then, S is rationally saturated with respect to $(X, \mathcal{K})$.*

Here $(S, \mathscr{C}(S))$ is rationally saturated with respect to $(X, \mathcal{K})$ (cf. Mok [Mk08a, Definition 5]) if and only if the statement (♯) below holds. (♯) *For any point $x \in S$ and any minimal rational curve ℓ on X such that*

$x \in \ell$ and $T_x(\ell) \subset T_x(S)$, the germ of holomorphic curve $(\ell; x)$ must lie on S.

Proof of Proposition 5.2. Proposition 5.2 is a generalization of Proposition 2.3. By Lemma 5.3 (which replaces Lemma 2.2) we obtain a holomorphic family of special vector fields $\{\widetilde{\alpha}_t(z)\}$ on a neighborhood of x. Given Lemma 5.4 (which replaces Lemma 2.1) and noting that Condition (T) replaces the condition (†) in the proof of Proposition 2.2, the analogue of the latter proposition applies *verbatim* to the given sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$. By the proof of Proposition 2.3, to generalize the latter proposition to sub-VMRT structures it suffices to prove tangential constancy of $\mathcal{V}(y)$ along $\ell \cap U$ for a coordinate chart (U, z^i) adapted to (x, ℓ), viz., denoting by P_{α_u} the positive part of $T(X)|_\ell$ at $u \in \ell \cap U$, P_{α_u} are constant on $\ell \cap U$ in these coordinates. But on (U, z^i) the desired property follows readily when we parametrize a neighborhood of x on $\mathcal{V}(y)$ by $\Phi(s,t) = s\alpha_t$, $t = (t_1, \cdots, t_p)$, $t \in \Delta^p$, from which one sees that P_{α_u} is the linear span of $\alpha = \alpha_0$ and $\frac{\partial \alpha_t}{\partial t_i}\Big|_{t=0}$, $1 \le i \le p$, for any $u \in \ell \cap U$. Proposition 5.2 follows. □

Proof of Theorem 1.4. Theorem 1.4 follows from Proposition 5.2. □

6. The Thickening Lemma for the propagation of sub-VMRT structures

To prove Main Theorem 2, which gives a sufficient condition for the algebraicity of germs of sub-VMRT structures, we need to have a method of analytic continuation of $\varpi : \mathscr{C}(S) \to S$, where $\mathscr{C}(S) := \mathscr{C}(X)|_S \cap \mathbb{P}T(S)$. Unlike the proofs of Cartan-Fubini Extension Principle in the equidimensional (Hwang-Mok [HM01]) and non-equidimensional cases (Hong-Mok [HoM10]), which are results on the analytic continuation of mappings, we need to tackle the more delicate issue of *analytic continuation of subvarieties.*

We give now an outline of our strategy for constructing a projective completion $S \subset Z$. Adjoining rational curves as in Proposition 5.1, we consider chains of rational curves issuing from S. The process consists of an iterative construction of fibered spaces of sub-VMRTs, followed by $\mathbb{P}^1$-bundles over their normalizations, until we reach a point where the total space $\mathcal{W}_k$ of a $\mathbb{P}^1$-bundle over an iterated fibered space $\mathscr{S}_k$ projects onto a subvariety $Z \subset X$ which is already saturated, giving a projective subvariety which extends S. The prototype of such a construction in the case of sub-VMRT structures modeled on an admissible pair (X_0, X) of sub-diagram type of rational homogeneous spaces of Picard number 1 was explained in the proof of Main Theorem 1 in §4.

We will in fact construct a certain "*universal family*" of chains of rational curves belonging to a compactification $\mathcal{Q}$ of the minimal rational component $\mathcal{K}$. To initiate the process we prove using deformation theory of rational curves that sub-VMRT structures can be *propagated* along certain standard minimal rational curves. This allows us to do analytic continuation on a dense open subset of some projective parameter space $\mathscr{S}_k$ of members of $\mathcal{Q}$. *A priori* a locally closed hypersurface over which the sub-VMRT structure is undefined can be the source of *essential singularities*. We overcome the latter difficulty by using methods on hulls of holomorphy in Several Complex Variables to prove Thullen-type extension of sub-VMRT structures to further cut down the set of *inadmissible points* on some of these divisors to subvarieties of codimension ≥ 2 in X, leaving behind divisors which are shown to be immaterial for the problem of analytic continuation by means of Hartogs extension for fibered spaces.

For the initial step of analytic continuation we have the following "*Thickening Lemma*" on certain standard rational curves arising from $\mathscr{C}(S)$.

Proposition 6.1. *Let $(X, \mathcal{K})$ be a uniruled projective manifold endowed with a minimal rational component,* $\dim(X) =: n$*, and $\varpi : \mathscr{C}(S) \to S$ be a sub-VMRT structure as in Theorem 1.4,* $\dim(S) =: s$*. Let $[\alpha] \in \mathscr{C}(S)$ be a smooth point of both $\mathscr{C}(S)$ and $\mathscr{C}(X)$ such that $\varpi : \mathscr{C}(S) \to S$ is a submersion at $[\alpha]$, $\varpi([\alpha]) =: x$, and $[\ell] \in \mathcal{K}$ be the minimal rational curve* (*which is smooth at x*) *such that $T_x(\ell) = \mathbb{C}\alpha$, and $f : \mathbf{P}_\ell \to \ell$ be the normalization of ℓ, $\mathbf{P}_\ell \cong \mathbb{P}^1$. Suppose $(\mathscr{C}_x(S), [\alpha])$ satisfies* Condition (T) *in Definition 5.4. Then, there exists an s-dimensional complex manifold $\mathbf{E}_\ell$, $\mathbf{P}_\ell \subset \mathbf{E}_\ell$, and a holomorphic immersion $F : \mathbf{E}_\ell \to X$ such that $F|_{\mathbf{P}_\ell} \equiv f$ and $F(\mathbf{E}_\ell)$ contains a neighborhood of x on S.*

For the proof of Proposition 6.1 and further discussion on VMRT structures and sub-VMRT structures we slightly modify the set-up. Recall that $B' \subsetneq X, B' \supset B$ is the enhanced bad locus of $(X, \mathcal{K})$ (cf. paragraph after Theorem 5.1). In Definition 5.1, we assume that $S \subset X - B'$ for a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$. By a *tame* sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ we will mean a sub-VMRT structure satisfying the strengthened forms (b'), (c') and (d') of (b), (c) and (d) in which the conditions are assumed to hold at *every* point $x \in S$ (in place of a *general* point on S), together with a further condition which contains (a) in Definition 5.1, viz., (e) $\mathscr{C}(S) = \mathscr{C}_1(S) \cup \cdots \cup \mathscr{C}_m(S)$, where for $1 \leq k \leq m$, $\varpi|_{\mathscr{C}_k(S)} : \mathscr{C}_k(S) \to S$ is a surjective holomorphic map with irreducible fibers. Starting with a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ and restricting to some nonempty connected open subset $S' \subset S$ we obtain a tame sub-VMRT structure $\varpi|_{\mathscr{C}(S')} : \mathscr{C}(S') \to S'$ on $(X, \mathcal{K})$.

Proof of Proposition 6.1. Without loss of generality we may assume that $\varpi : \mathscr{C}(S) \to S$ is a tame sub-VMRT structure of $\pi : \mathscr{C}(X) \to X$. Recall that $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ is the universal family, where $\rho : \mathcal{U} \to \mathcal{K}$ is a $\mathbb{P}^1$-bundle and $\mu : \mathcal{U} \to X$ is the evaluation map. Let $x \in S$ be an arbitrary point. Since $S \subset X - B'$, the tangent map $\tau_x : \mathcal{U}_x \to \mathscr{C}_x(X)$ is a birational finite morphism. For each irreducible component $\mathscr{C}_{k,x}(S)$ of $\mathscr{C}_x(S)$, $1 \leq k \leq m$, we have $\mathscr{C}_{k,x}(S) \not\subset \mathrm{Sing}(\mathscr{C}_x(X))$. Let $\chi_k \in \mathscr{C}_{k,x}(S)$ be a smooth point of both $\mathscr{C}_x(S)$ and $\mathscr{C}_x(X)$. By tameness of the sub-VMRT structure, $\varpi : \mathscr{C}(S) \to S$ is a submersion at χ_k. Now fix k and write $[\alpha] := \chi_k$. Let $u \in \mathcal{U}_x$ be the unique point such that $\tau_x(u) = [\alpha]$. By the choice of $[\alpha] = \chi_k$, there is a unique minimal rational curve $\ell \subset X$ belonging to $\mathcal{K}$ such that ℓ is smooth at x and such that $T_x(\ell) = \mathbb{C}\alpha$. Since τ_x is a morphism and $[\alpha]$ is a smooth point of $\mathscr{C}_x(X)$, ℓ is in fact an immersed standard rational curve and τ_x is a local biholomorphism at u.

In what follows we will be treating X as a complex manifold, and "open subsets" will be defined in terms of the complex topology unless otherwise specified. There exists a nonempty connected smooth open subset $\mathcal{O} \subset \tau^{-1}(\mathscr{C}(S)) \subset \mathcal{U}$ such that for every $u \in \mathcal{O}$, $\rho(u) \in \mathcal{K}$ is a standard rational curve and $\varpi \circ \tau : \mathcal{O} \to S$ is a submersion at u, and such that $\tau|_{\mathcal{O}} : \mathcal{O} \to \mathscr{C}(S)$ is a biholomorphism onto an open subset $\mathcal{W} \subset \mathrm{Reg}(\mathscr{C}(S))$. Writing $\rho_\flat$ for the restriction of $\rho : \mathcal{U} \to \mathcal{K}$ to $\mathcal{O}$, consider the $\mathbb{P}^1$-bundle $\xi : \rho_\flat^*\mathcal{U} \to \mathcal{O}$ obtained by pulling back the universal $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$ by $\rho_\flat$. Hence, the fiber of ξ over $u \in \mathcal{O}$ is a copy of $\mathbb{P}^1$ and it is naturally mapped onto $\ell(u) := \mu(\rho^{-1}(\rho(u)))$ by $\nu := \xi^*\mu$ for the evaluation map $\mu : \mathcal{U} \to X$. (We also call $\nu : \rho_\flat^*\mathcal{U} \to X$ the evaluation map.) Thus, any smooth point z on $\ell(u)$ lifts to a unique point $\zeta \in (\rho_\flat^*\mathcal{U})_u$, $z = \nu(\zeta)$. In particular, $u \in \mathcal{U}_x$, $x := \mu(u) \in \ell(u) \subset X$ and it lifts to a unique point $u^\flat \in (\rho_\flat^*\mathcal{U})_u$. Writing $\mathfrak{s}_\xi(u) := u^\flat$, $\mathfrak{s}_\xi : \mathcal{O} \to \rho_\flat^*\mathcal{U}$ defines a tautological holomorphic section of $\xi : \rho_\flat^*\mathcal{U} \to \mathcal{O}$. Since $\rho(u)$ is a standard rational curve for every $u \in \mathcal{O}$, $\nu|_{\xi^{-1}(u)} : \xi^{-1}(u) \to X$, $\xi^{-1}(u) \cong \mathbb{P}^1$, is a holomorphic immersion. It is the normalization $f : \mathbf{P}_\ell \to \ell$ of the immersed but possibly singular standard rational curve ℓ.

Recall that $s := \dim(S)$. Let a_k be the dimension of $\mathscr{C}_x(S)$ at $[\alpha] := \chi_k \in \mathrm{Reg}(\mathscr{C}_{k,x}(S)) \cap \mathrm{Reg}(\mathscr{C}_x(X))$, $[\alpha] = \tau(u)$. By Theorem 1.4, $\varpi : \mathscr{C}(S) \to S$ is rationally saturated, hence $\nu : \rho_\flat^*\mathcal{U} \to X$ maps the germ $(\rho_\flat^*\mathcal{U}; u^\flat)$ into the germ of $(S; x)$. We claim that ($\sharp$) *ν is of maximal rank everywhere on $\rho_\flat^*\mathcal{U}$.* Assuming the latter, by a standard covering argument there exists a complex manifold $\mathbf{E}_\ell$ containing $\mathbf{P}_\ell$, a holomorphic submersion $\omega : \rho_\flat^*\mathcal{U} \to \mathbf{E}_\ell$ and a holomorphic immersion $F : \mathbf{E}_\ell \to X$ such that $F|_{\mathbf{P}_\ell} \equiv f$ and such that $\nu = F \circ \omega$. In particular, $F(\mathbf{E}_\ell)$ must contain a neighborhood of x on S since $\mathrm{rank}(d\nu(u^\flat)) = s$ by the claim ($\sharp$).

To prove the Thickening Lemma (Proposition 6.1) it remains therefore to establish the claim ($\sharp$). It is straightforward that $d\nu(\zeta)$ is of rank s at $\zeta = u^\flat$ for $u \in \mathcal{O}$. In fact, $\varpi|_{\mathcal{W}} : \mathcal{W} \to S$ is a submersion. For $[\alpha] \in \mathcal{W}$ letting $(\mathscr{S}; [\alpha]) \subset (\mathcal{W}; [\alpha])$ be a germ of complex submanifold at $[\alpha] \in \mathcal{W}$ *transversal* to the submersion $\varpi|_{\mathcal{W}} : \mathcal{W} \to S$, the mapping $\varpi|_{\mathscr{S}} : \mathscr{S} \to S$ is a local biholomorphism at $[\alpha]$. Hence, $\mathrm{rank}(d\nu(u^\flat)) = s$ for $u \in \mathcal{O}$ such that $\tau(u) = [\alpha]$. The main point of Proposition 6.1 is to prove that the property $\mathrm{rank}(d\nu(\zeta)) = s$ propagates as $\zeta = u^\flat$ travels along the tautological lifting $\ell^\dagger(u) := \xi^{-1}(u) = (\rho_\flat^*\mathcal{U})_u$ of the standard rational curve $\ell(u)$. *In what follows we will assume for simplicity that all standard rational curves are embedded.* The general situation is very similar, and the necessary modification amounts to considering local smooth branches of immersed standard rational curves. Let now ℓ be a standard rational curve passing through a base point $x \in S$ such that $T_x(\ell) = \mathbb{C}\alpha$. Recall that for $z \in X$ we define $\mathcal{V}(z) := \bigcup\big\{\ell : [\ell] \in \mathcal{K}_z\big\}$. Let $y \in \ell \cap S$ be distinct from x and consider the cone of curves $\Pi(y, S) \subset \mathcal{V}(y)$ which is the union of minimal rational curves ℓ' passing through y whose germ at y lies on S. We write $u(\ell', y)$ for the point on $\mathcal{U}_y$ corresponding to the minimal rational curve ℓ' with a marking at y. Recall that $[\alpha] = \chi_k \in \mathrm{Reg}(\mathscr{C}_{k,x}(S))$, $\dim(\mathscr{C}_{k,x}(S)) =: a_k$. Denote by $\Sigma(y, S) \subset \Pi(y, S)$ the set obtained by requiring $u(\ell', y)$ to lie on a sufficiently small neighborhood of $u(\ell, y)$, so that $\Sigma(y, S)$ is a germ of $(a_k + 1)$-dimensional complex manifold at $z \in \ell - \{y\}$, in particular at the base point x. By an adaptation of the proof of Lemma 4.1, we have $T_x(\Sigma(y, S)) = T_\alpha(\widetilde{\mathscr{C}_x}(S))$. Here in the adaptation it suffices to replace the condition ($\dagger$) for (X_0, X) by the hypothesis in Proposition 6.1 that $(\mathscr{C}_x(S), [\alpha])$ satisfies Condition (T), i.e., $T_\alpha(\widetilde{\mathscr{C}_x}(S)) = T_\alpha(\widetilde{\mathscr{C}_x}(X)) \cap T_x(S)$.

Let now $Z \subset U \cap S$ be a complex submanifold of $U \cap S$ for some sufficiently small open neighborhood U of x in X such that $T_x(Z) \cap T_x(\Sigma(y, S)) = 0$ and $T_x(Z) + T_x(\Sigma(y, S)) = T_x(S)$. Shrinking U and hence Z if necessary there exists a holomorphic section $\widetilde{\alpha}(z)$ of $\widetilde{\varpi}|_Z : \widetilde{\mathscr{C}}(S)|_Z \to Z$, $\widetilde{\mathscr{C}}(S)|_Z := \widetilde{\varpi}^{-1}(Z)$, such that $\widetilde{\alpha}(x) = \alpha$. Write $\mathcal{Z} := \big\{[\widetilde{\alpha}(z)] : z \in Z\big\}$, and $\mathcal{D}$ for a smooth neighborhood $\mathcal{Z}$ in $\mathscr{C}(S)|_Z \cap \mathcal{O}$. We have $\dim(\mathcal{Z}) = \dim(Z) = s-(a_k+1)$, and $\dim(\mathcal{D}) = \dim(\mathcal{Z})+a_k = s-1$. Using the biholomorphism $\varphi := \tau|_{\mathcal{O}} : \mathcal{O} \xrightarrow{\cong} \mathcal{W}$ we define $\widehat{\mathcal{Z}} = \varphi^{-1}(\mathcal{Z})$, $\widehat{\mathcal{D}} = \varphi^{-1}(\mathcal{D})$, $\widehat{\mathcal{Z}} \subset \widehat{\mathcal{D}}$. Write $\beta := \rho|_{\widehat{\mathcal{D}}}$. Pulling back the universal $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$ by the classifying map $\beta : \widehat{\mathcal{D}} \to \mathcal{K}$ we obtain a $\mathbb{P}^1$-bundle $\gamma : \beta^*\mathcal{U} \to \widehat{\mathcal{D}}$ which is obtained from $\xi : \rho_\flat^*\mathcal{U} \to \mathcal{O}$ by restricting the base to $\widehat{\mathcal{D}} \subset \mathcal{O}$, $\dim(\beta^*\mathcal{U}) = s$. We write $\mathfrak{s}_\gamma : \widehat{\mathcal{D}} \to \beta^*\mathcal{U}$ for the restriction $\mathfrak{s}_\xi|_{\widehat{\mathcal{D}}}$ of the tautological section $\mathfrak{s}_\xi : \mathcal{O} \to \rho_\flat^*\mathcal{U}$. Consider the restriction $\nu^\sharp := \nu|_{\beta^*\mathcal{U}}$ of the evaluation map $\nu : \rho_\flat^*\mathcal{U} \to X$ to $\beta^*\mathcal{U}$. Recall that ℓ is the standard rational curve passing through x such that

$T_x(\ell) = \mathbb{C}\alpha$. Denote by $\widehat{\ell}$ the lifting of ℓ to $\mathcal{U}$. Let $z \in \ell - \{x\}$ and write ζ_0 for the unique point in $\widehat{\ell} \cap \mathcal{U}_z$. ζ_0 corresponds to a unique point $\zeta \in \beta^*\mathcal{U}$ such that $\nu^\sharp(\zeta) = z$ and ζ belongs to the fiber of the $\mathbb{P}^1$-bundle $\gamma : \beta^*\mathcal{U} \to \widehat{\mathcal{D}}$ lying over the single point $\widehat{\ell} \cap \mathcal{U}_x$ on $\widehat{\mathcal{D}}$. We have

Lemma 6.1. *The evaluation map $\nu^\sharp : \beta^*\mathcal{U} \to X$ is an immersion at ζ.*

Proof. Suppose for argument by contradiction that there exists a nonzero vector $\eta \in \mathrm{Ker}(d\nu^\sharp(\zeta))$. For $w \in Z$ write $\mathcal{D}_w := \mathcal{D} \cap \mathscr{C}_w(S)$, etc. Identifying $\widehat{\mathcal{D}}_x$ with its image $\mathfrak{s}_\gamma(\widehat{\mathcal{D}}_x)$ in $\mathcal{E}_x$ by the tautological section $\mathfrak{s}_\gamma : \widehat{\mathcal{D}} \to \beta^*\mathcal{U}$, the evaluation map $\nu^\sharp\big|_{\mathcal{E}_x} : \mathcal{E}_x \to X$ is a holomorphic immersion outside $\mathfrak{s}_\gamma(\widehat{\mathcal{D}}_x)$, the latter being collapsed to the point x, giving $\Sigma(x, S) \subset \Pi(x, S)$ which is immersed outside of x, $\ell \subset \Sigma(x, S)$, such that $d\nu^\sharp(T_\zeta(\mathcal{E}_x)) = T_z(\Sigma(x, S))$. (For $z = y \in \ell \cap S$ sufficiently close to x, $T_y(\Sigma(x, S)) = P_{\alpha_y} \cap T_y(S)$.) Write $u \in \widehat{\mathcal{D}}_x$ for the point such that $\tau(u) = [\alpha]$. Writing $H_\zeta \subset T_\zeta(\beta^*\mathcal{U})$ for the $(a_k + 1)$-dimensional vector subspaces defined by $H_\zeta = (d\gamma)^{-1}(T_u(\widehat{\mathcal{D}}_x))$, from the above $d\nu^\sharp(\zeta)$ is injective on H_ζ. The evaluation map $\mu : \mathcal{U} \to X$ is a holomorphic submersion at u, and $T_u(\widehat{\mathcal{D}}_x) \subset T_u(\widehat{\mathcal{D}})$ consists of vertical tangent vectors, i.e., $d\mu(T_u(\widehat{\mathcal{D}}_x)) = 0$.

Suppose $\eta \in \mathrm{Ker}(d\nu^\sharp(\zeta)), \eta \neq 0$. (Note that $\nu^\sharp(\zeta) = \nu(\zeta) = z$.) Let $h : \Delta \to \beta^*\mathcal{U}$ be an immersed holomorphic curve such that $h(0) = \zeta$ and $h'(0) = \eta$. Then, for $\gamma \circ h : \Delta \to \widehat{\mathcal{D}}$, we have $(\gamma \circ h)(0) = u$ (where $\tau_x(u) = [\alpha]$) and $(\gamma \circ h)'(0) \notin T_u(\widehat{\mathcal{D}}_x)$ since $d\nu^\sharp(\zeta)$ is injective on $H_\zeta := (d\gamma)^{-1}(T_u(\widehat{\mathcal{D}}_x))$. Thus, $d\mu(d\gamma(\eta)) \neq 0$ (noting that $\mu(\gamma(\zeta)) = x$). Now $d\mu(T_u(\widehat{\mathcal{D}})) = d\mu(T_u(\mathcal{Z})) = T_x(Z)$. Thus, $d\mu(d\gamma(\eta)) \in T_x(S)$ is tangent at x to Z, hence transversal to $P_\alpha \cap T_x(S)$. In other words, $g(t) := \mu(\gamma(h(t)))$ is a holomorphic curve on S such that $g(0) = x$ and $g'(0) \in T_x(S) - P_\alpha$. Thus $\gamma \circ h$ is a holomorphic section of a holomorphic 1-parameter family of standard rational curves $\{\ell_t : t \in \Delta\}$ such that $\ell_0 = \ell$, $g(t) \in \ell_t$ and the lifting $\ell_t^\dagger$ to $\rho_\flat^*\mathcal{U}$ lies on $\beta^*\mathcal{U}$, i.e., the tautological lifting $\widehat{\ell} \subset \mathcal{U}$ passes through $\widehat{\mathcal{D}}$. Since any holomorphic $\mathbb{P}^1$-bundle over the unit disk is holomorphically trivial, the family $\{\ell_t : t \in \Delta\}$ of curves is parametrized by a holomorphic map $F : \mathbb{P}^1 \times \Delta$ such that, writing $f_t(w) := F(w, t)$ for $w \in \mathbb{P}^1$, we have $f_t(\mathbb{P}^1) = \ell_t$ and $f_t(0) = g(t)$ and $f_t(\infty) = \mu(h(t))$. Defining $\sigma(w) = \frac{\partial}{\partial t}\big|_{t=0} F(w, t)$, we have $\sigma \in \Gamma(\mathbb{P}^1, f_0^*T(X))$ such that $\sigma(0) = g'(0) \in T_x(S) - P_\alpha$ while $\sigma(\infty) = d\mu(h'(0)) = d\nu^\sharp(\eta) = 0$ by the assumption on η. Recall that on the standard rational curve ℓ (assumed embedded for simplicity as in the above) we have $T(X)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$, where $p = \dim(\mathscr{C}_x(X))$ and $1 + p + q = n$. Clearly any nonzero holomorphic section of $T(X)|_\ell$ over $\ell \cong \mathbb{P}^1$ which is 0 at some point must necessarily take values in the positive part $P_\ell = \mathcal{O}(2) \oplus \mathcal{O}(1)^p$. Hence $\sigma(0) \in P_{\ell,x} = P_\alpha$, contradicting with $\sigma(0) = g'(0) \in T_x(S) - P_\alpha$, yielding $\mathrm{Ker}(d\nu^\sharp(\zeta)) = 0$, as desired. □

Proof of Proposition 6.1 cont. Since $\dim(\beta^*\mathcal{U}) = s$, it follows from Lemma 6.1 that $\mathrm{rank}(d\nu(\zeta)) = s$. We can move the base point x in Lemma 6.1 slightly to $y \in (\ell - \{x\}) \cap S$ and conclude that $\mathrm{rank}(d\nu(\zeta')) = s$ also at a corresponding point $\zeta' \in \rho_\flat^*\mathcal{U}$ lying over x. As $\rho_\flat = \rho|_{\mathcal{O}}$ and all fibers of $\xi : \rho_\flat^*\mathcal{U} \to \mathcal{O}$ correspond to standard rational curves on X, Lemma 6.1 implies that $\mathrm{rank}(d\nu) = s$ everywhere, from which Proposition 6.1 follows. □

From the proof of Proposition 6.1 we obtain

Corollary 6.1. *Under the assumptions of Proposition 6.1 and in the notation adopted there,* $\mathbf{P}_\ell \subset \mathbf{E}_\ell$ *is a standard rational curve.*

Proof. Since $\mathrm{rank}(d\nu) = s$ everywhere on $\rho_\flat^*\mathcal{U}$, $T(\mathbf{E}_\ell)|_{\mathbf{P}_\ell}$ is semipositive. On the other hand, $T(\mathbf{E}_\ell)|_{\mathbf{P}_\ell} \subset F^*T(X)|_{\mathbf{P}_\ell} \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some $p, q \geq 0$. Since $\mathcal{O}(2) \cong T(\mathbf{P}_\ell) \hookrightarrow T(\mathbf{E}_\ell)|_{\mathbf{P}_\ell} \subset F^*T(X)|_{\mathbf{P}_\ell}$ and all Grothendieck direct summands of $T(\mathbf{E}_\ell)|_{\mathbf{P}_\ell}$ are of degree ≥ 0, it follows that $T(\mathbf{E}_\ell)|_{\mathbf{P}_\ell} \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^a \oplus \mathcal{O}^b$ for some $a, b \geq 0$, i.e., $\mathbf{P}_\ell \subset \mathbf{E}_\ell$ is a standard rational curve, as desired. □

7. Hartogs and Thullen extension for sub-VMRT structures

In this section we prove results for extending holomorphic objects applicable to our problem of analytic continuation of sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$ with an aim to provide the analytic tools for the proof of Main Theorem 2. We start with the following "*Thullen Extension Lemma*" for fibered spaces of projective subvarieties.

Lemma 7.1. *Let* $\mathscr{B}$ *be an irreducible complex space,* $\nu : \mathscr{P} \to \mathscr{B}$ *be a holomorphic projective bundle. Let* $E \subsetneq \mathscr{B}$ *be a complex-analytic subvariety, and* $E_i, 1 \leq i \leq m$, *be the irreducible components of* E *of codimension 1 in* $\mathscr{B}$. *Let* $\mathcal{W} \subset \mathscr{B}$ *be an open subset such that* $\mathcal{W} \supset \mathscr{B} - E_1 - \cdots - E_m$ *and such that* $\mathcal{W} \cap E_i \neq \emptyset$ *for* $1 \leq i \leq m$. *Let* $\mathcal{A} \subset \nu^{-1}(\mathcal{W})$ *be an irreducible subvariety such that* $\nu(\mathcal{A}) = \mathcal{W}$. *Then, the topological closure* $\overline{\mathcal{A}} \subset \mathscr{P}$ *of* $\mathcal{A}$ *in* $\mathscr{P}$ *is a complex-analytic subvariety. If* $\mathscr{B}$ *is a projective variety, then* $\overline{\mathcal{A}} \subset \mathscr{P}$ *is projective.*

Proof. The last statement follows readily from the rest of the lemma. Let $\{\mathscr{B}_\alpha : \alpha \in A\}$ be an open covering of $\mathscr{B}$. If the lemma excepting the last statement holds for $\nu|_{\mathscr{B}_\alpha} : \mathscr{P}|_{\mathscr{B}_\alpha} \to \mathscr{B}_\alpha$ for each $\alpha \in A$, it also holds for $\nu : \mathscr{P} \to \mathscr{B}$. Thus, we may assume that the projective bundle $\nu : \mathscr{P} \to \mathscr{B}$ is holomorphically trivial, i.e., $\mathscr{P} = \mathscr{B} \times \mathbb{P}^N$, and $\nu : \mathscr{B} \times \mathbb{P}^N \to \mathscr{B}$ is the canonical projection, so that the fibers of $\lambda := \nu|_{\mathcal{A}} : \mathscr{A} \to \mathscr{B}$ as cycles are identified as elements in the Chow space of $\mathbb{P}^N$. Let $\mathscr{Q} \subset \mathrm{Chow}(\mathbb{P}^N)$ be the Chow component such that $[\lambda^{-1}(w)] \in \mathscr{Q}$ for a general point $w \in \mathcal{W}$, and $\mathcal{A} \subset \mathscr{P}|_{\mathcal{W}}$ corresponds to a meromorphic map $h : \mathcal{W} \dashrightarrow \mathscr{Q}$ defined by $h(w) := [\lambda^{-1}(w)] \in \mathscr{Q}$. By Thullen extension (cf. Siu [Si74]), h extends meromorphically to $h^\flat : \mathscr{B} \dashrightarrow \mathscr{Q}$. Equivalently, this means that there exists some modification $\mu : \mathscr{B}^\sharp \to \mathscr{B}$ and a holomorphic map $h^\sharp : \mathscr{B}^\sharp \to \mathscr{Q}$ such that $h^\sharp = h^\flat \circ \mu$. Let $\lambda : \mathscr{U} \to \mathscr{Q}$ be the restriction of the universal family over $\mathrm{Chow}(\mathbb{P}^N)$ to the Chow component $\mathscr{Q}$. Write $\mu' = (\mu, \mathrm{id}_{\mathbb{P}^N})$. We have $(h^\sharp)^*\mathscr{U} \subset \mathscr{B}^\sharp \times \mathbb{P}^N$, and $\mathcal{Z} := \mu'((h^\sharp)^*\mathscr{U}) \subset \mathscr{B} \times \mathbb{P}^N = \mathscr{P}$ gives an extension of $\mathcal{A} \subset \mathcal{W} \times \mathbb{P}^N$. Obviously $\mathcal{Z} \subset \mathscr{P}$ is the topological closure $\overline{\mathcal{A}}$ of $\mathcal{A}$ in $\mathscr{P}$. $\square$

Next we prove Thullen-type extension for germs of complex submanifolds. Lemma 7.2 is a preparatory statement on domains of holomorphy.

Lemma 7.2. *Let $m, s \geq 1$ be integers. Write $H := \Delta^m - (\{0\} \times \Delta^{m-1})$. Let Ω_0 be a neighborhood of $H \times \{0\}$ in $H \times \Delta^s$, and consider a domain $\Omega \subset \Delta^{m+s}$ given by $\Omega = \Omega_0 \cup \Delta^{m+s}(\epsilon)$ for some ϵ such that $0 < \epsilon < 1$. Then, there exists a neighborhood $\mathscr{D}$ of $\Delta^m \times \{0\}$ in Δ^{m+s} such that the germ at 0 of any holomorphic function f on Ω extends holomorphically to $\mathscr{D}$.*

Proof. Let $\pi : \widetilde{\Omega} \to \mathbb{C}^{m+s}$ be the hull of holomorphy of $\Omega \subset \Delta^{m+s}$. Write $\Omega^\flat$ for the connected component of $\pi^{-1}(\mathbb{C}^m \times \{0\})$ containing $(H \cup \Delta^m(\epsilon)) \times \{0\}$. Consider the Riemann domain $\pi^\flat : \Omega^\flat \to \mathbb{C}^m$, $\pi^\flat := \pi|_{\Omega^\flat}$. Then, the complex submanifold $\Omega^\flat \subset \widetilde{\Omega}$ of the Riemann domain of holomorphy $\widetilde{\Omega}$ must be Stein. Hence, $\pi^\flat : \Omega^\flat \to \mathbb{C}^m$ is a Riemann domain of holomorphy. Since the hull of holomorphy of $H \cup \Delta^m(\epsilon)$ is precisely Δ^m, the latter must embed into $\Omega^\flat$, and we identify its image as a subset $\Delta^m \times \{0\} \subset \widetilde{\Omega}$. We claim that there exists a neighborhood $\mathscr{D}$ of $\Delta^m \times \{0\}$ in $\widetilde{\Omega}$ such that $\pi|_{\mathscr{D}} : \mathscr{D} \to \mathbb{C}^{m+s}$ is an open embedding. Assuming the claim and identifying $\mathscr{D}$ with $\pi(\mathscr{D})$, the lemma follows.

To prove the claim observe that for any $x \in \Delta^m$ there exists a neighborhood $U(x; 3r(x))$ of x in $\Omega^\flat$ such that $\pi^\flat|_{U(x;3r(x))} : U(x; 3r(x)) \xrightarrow{\cong} B^{m+s}(x; 3r(x))$ is a biholomorphism and such that $U(x; 3r(x)) \cap (\Delta^m \times \{0\}) \Subset \Delta^m \times \{0\}$. Define now an open subset $W \subset \Omega^\flat$ by $W := \bigcup \big\{U(x; r(x)) : x \in \Delta^m \times \{0\}\big\}$. We proceed to show that the local biholomorphism $\pi^\flat|_W : W \to \mathbb{C}^{m+s}$ is injective, hence an open embedding. Suppose $a, b \in W$ are such that $\pi^\flat(a) = \pi^\flat(b)$. By the definition of W there exist $x, y \in \Delta^m \times \{0\}$ such that $a \in U(x; r(x))$ and $b \in U(y; r(y))$. We may assume that $r(x) \geq r(y)$. Then, $\pi^\flat(a) = \pi^\flat(b) \in B^{m+s}(x; r(x)) \cap B^{m+s}(y; r(y))$, hence $\|x - y\| < r(x) + r(y) \leq 2r(x)$, so that $U(x;r(x)) \cup U(y;r(y)) \subset U(x;3r(x))$. Since $\pi^\flat|_{U(x;3r(x))}$ is injective, it follows from $\pi^\flat(a) = \pi^\flat(b)$ that $a = b$, proving the claim and hence Lemma 7.2. $\square$

Lemma 7.3. *Let $m, s, n \geq 1$ be integers, $1 \leq s \leq n$. Write $H := \Delta^m - (\{0\} \times \Delta^{m-1})$. For $0 < r < 1$ write $\mathcal{H}(r) := H \cup \Delta^m(r)$. Suppose $0 < \epsilon < 1$ and let $\mathcal{N} \subset \mathcal{H}(\epsilon) \times \Delta^n$ be an $(m+s)$-dimensional locally closed complex submanifold such that $\mathcal{N} \supset \mathcal{H}(\epsilon) \times \{0\}$ and such that the canonical projection $\varpi : \mathcal{N} \to \mathcal{H}(\epsilon)$ is a submersion. Then, there exists a subvariety $E \subset \{0\} \times \Delta^{m-1}$ and an $(m+s)$-dimensional locally closed complex submanifold $\mathscr{X} \subset (\Delta^m - E) \times \Delta^n$, $\mathscr{X} \supset (\Delta^m - E) \times \{0\}$ such that $\mathscr{X} \supset \mathcal{N}$ and the canonical projection $\mathrm{pr}_1 : \mathscr{X} \to \Delta^m - E$ is a submersion.*

Proof of Lemma 7.3. For $x \in \mathcal{H}(\epsilon)$, write $\mathcal{N}_x = \mathcal{N} \cap (\{x\} \times \Delta^n)$. The assignment $x \mapsto T_{(x,0)}(\mathcal{N}_x)$ for $x \in \mathcal{H}(\epsilon)$ defines a holomorphic map $\Phi : \mathcal{H}(\epsilon) \to \mathrm{Gr}(s, \mathbb{C}^n)$ into the Grassmannian of s-planes in $\mathbb{C}^n$. By Thullen extension Φ extends meromorphically to $\Phi^\sharp : \Delta^m \dashrightarrow \mathrm{Gr}(s, \mathbb{C}^n)$. Denote by $A \subset \Delta^m$ the set of indeterminacies of $\Phi^\sharp$. Note that $A \subset \Delta^m$ is a subvariety of codimension ≥ 2 and $A \cap \mathcal{H}(\epsilon) = \emptyset$ by assumption. Hence, $A \subsetneq \{0\} \times \Delta^{m-1}$ is a subvariety and $A \cap (\{0\} \times \Delta^{m-1}(\epsilon)) = \emptyset$.

Assume that the following holds: (†) *Lemma 7.3 can be established in the special case where $A = \emptyset$ with the stronger conclusion that $E = \emptyset$.* We claim that the full lemma will have followed. To see this let $y \in (\{0\} \times \Delta^{m-1}) - A$ be an arbitrary given point, $y = (y_1, \cdots, y_m), y_1 = 0, y =: (0, y')$. Consider the problem of analytic continuation of $\varpi : \mathcal{N} \to \mathcal{H}(\epsilon)$ along a continuous path γ joining 0 to y on $(\{0\} \times \Delta^{m-1}) - A$. Let $\delta > 0$ be sufficiently small (depending on y), and write $\Delta^m(y;\delta) := \Delta(y_1;\delta) \times \cdots \times \Delta(y_m;\delta)$, etc. Suppose there exists a parametrized family $\sigma : \mathscr{E} \to \mathcal{H}(\epsilon) \cup \Delta^m(y;\delta)$ of s-dimensional complex submanifolds, $\mathscr{E} \subset (\mathcal{H}(\epsilon) \cup \Delta^m(y;\delta)) \times \Delta^n$ such that $\mathscr{E} \supset (\mathcal{H}(\epsilon) \cup \Delta^m(y;\delta)) \times \{0\}$ and such that $\mathscr{E} \supset \mathcal{N}$. Then, by the Identity Theorem for holomorphic functions and by the connectedness of $\Delta^m(y;\delta) - H$, the germ $\mathscr{E}_y$ is uniquely determined. Thus, to prove Lemma 7.3 it suffices to perform analytic continuation of $\mathcal{N}$ to an arbitrary given point y on $(\{0\} \times \Delta^{m-1}) - A$ along any continuous path γ joining 0 to y on $(\{0\} \times \Delta^{m-1}) - A$. Now given $y \in (\{0\} \times \Delta^{m-1}) - A$, for some ϵ' satisfying $0 < \epsilon' \leq \epsilon$ there exists a connected and simply connected open subset $\mathcal{O}$ on the complex plane and a connected open subset $U \subset \Delta^{m-1}$ biholomorphic to Δ^{m-1} such that $y \in \mathcal{O} \times U$, $\Delta^m(\epsilon') \subset \mathcal{O} \times U \subset \Delta^m - A$. Then, after mapping $\mathcal{O} \times U$ biholomorphically to $\Delta \times \Delta^{m-1}$ as a Cartesian product, in which $\mathcal{O}$ is mapped to Δ, U is mapped to Δ^{m-1} and $\Delta^m(\epsilon') \subset \mathcal{O} \times U$ is mapped to a neighborhood of 0 in Δ^m, $\varpi : \mathcal{N} \to \mathcal{H}(\epsilon)$ can be analytically continued to $\sigma : \mathscr{E} \to (\mathcal{H}(\epsilon) \cup \Delta^m(y;\delta))$ for some $\delta > 0$ by the assumption (†), showing that Lemma 7.3 holds with $E = A$.

To prove Lemma 7.3 it suffices therefore to establish (†). By assumption we have the locally closed complex submanifold $\mathcal{N} \subset \mathcal{H}(\epsilon) \times \Delta^n$, $\mathcal{N} \supset \mathcal{H}(\epsilon) \times \{0\}$, as a parametrized family of s-dimensional complex submanifolds $\mathcal{N}_x \subset \{x\} \times \Delta^n$. For $\epsilon \leq r \leq 1$ we write $\varpi_r : \mathcal{N}(r) \to \mathcal{H}(r)$ for a parametrized family of s-dimensional complex submanifolds $\mathcal{N}_x(r) := \varpi_r^{-1}(x), x \in \mathcal{H}(r)$, $\mathcal{N}(r) \subset \mathcal{H}(r) \times \Delta^n$, $\mathcal{N}(r) \supset \mathcal{H}(r) \times \{0\}$, such that $\mathcal{N}(r) \supset \mathcal{N}$. Note that when $\mathcal{N}(r)$ exists, the germ of $\mathcal{N}(r)$ along $\mathcal{H}(r) \times \{0\}$ is uniquely determined.

Let $I \subset [\epsilon, 1]$ be the set of all $r, \epsilon \leq r \leq 1$, such that $\mathcal{N}(r)$ exists. We are going to prove by the continuity method that $I = [\epsilon, 1]$, so that $\varpi_1 : \mathcal{N}(1) \to \mathcal{H}(1) = \Delta^m$ exists, completing the proof of Lemma 7.3.

Note that $\epsilon \in I$. Moreover, $r \in I$ implies $[\epsilon, r] \subset I$. We proceed to prove now the openness of I. Suppose $\epsilon \leq r < 1$ and $\varpi_r : \mathcal{N}(r) \to \mathcal{H}(r)$ exists. Let $y \in (\{0\} \times \partial\Delta^{m-1}(r)) \times \{0\}$, $y = (y_1, \cdots, y_m) = (0, y'; 0)$, $y_1 = 0$. Introduce Euclidean coordinates $(w_1, \cdots, w_n)$ on the Cartesian factor Δ^n such that $\Phi^\sharp(y) = \mathrm{Span}\{\frac{\partial}{\partial w_1}, \cdots, \frac{\partial}{\partial w_s}\}$. Write $\alpha : \mathbb{C}^n \to \mathbb{C}^s$ for the canonical projection given by $\alpha(w_1, \cdots, w_n) = (w_1, \cdots, w_s)$. Let $\delta > 0$ be sufficiently small so that $\alpha|_{\Phi^\sharp(z)} : \Phi^\sharp(z) \to \mathbb{C}^s$ is a linear isomorphism whenever $|z_i - y_i| < \delta$ for $1 \leq i \leq m$. Define $\mathcal{H}(r; y, \delta) := \mathcal{H}(r) \cap \Delta^m(y; \delta)$. There exists a locally closed complex submanifold $\mathcal{M} \subset \mathcal{N}(r)|_{\mathcal{H}(r;y,\delta)}, \mathcal{M} \supset \mathcal{H}(r; y, \delta) \times \{0\}$, such that the canonical projection $\beta : \mathcal{M} \to \mathbb{C}^m \times \mathbb{C}^s$ given by $\beta(z; w) = (z; \alpha(w))$ is a biholomorphism onto a neighborhood $\mathcal{G}$ of $\mathcal{H}(r; y, \delta) \times \{0\}$ in $\mathcal{H}(r; y, \delta) \times \Delta^s$. Thus, $\mathcal{M}$ can be considered as the graph of a vector-valued holomorphic function $F : \mathcal{G} \to \mathbb{C}^{n-s}$ defined by $F(z; w_1, \cdots, w_s) = (w_{s+1}, \cdots, w_n)$, where $(w_{s+1}, \cdots, w_n) \in \mathbb{C}^{n-s}$ is uniquely determined by requiring that $(z; w_1, \cdots, w_n)$ lies on $\mathcal{M}$. Choose a positive real number $a < 1$ such that for $u = ay$, $u =: (0, u')$, we have $y \in \Delta^m\big(u; \frac{\delta}{2}\big) \subset \Delta^m(y; \delta\big)$, so that $\Delta^m\big(u; \frac{\delta}{2}\big) \cap \mathcal{H}(r) \supset \big(\Delta^m(u; \frac{\delta}{2}) - \big(\{0\} \times \Delta^{m-1}\big(u'; \frac{\delta}{2}\big)\big) \cup \Delta^m(u; \varepsilon)$ for some $\varepsilon > 0$. By Lemma 7.2, the germ of F at u extends holomorphically to some neighborhood $\mathscr{G}$ of $\Delta^m\big(u; \frac{\delta}{2}\big) \times \{0\}$, hence F is defined on some neighborhood of y. Since the argument applies to any $y \in \{0\} \times \partial\Delta^{m-1}(r)$, by compactness we have obtained $\mathcal{N}(r')$ for some r', $r < r' < 1$, proving the openness of $I \subset [\epsilon, 1]$.

It remains to prove the closedness of I. Recall that $H = \Delta^m - (\{0\} \times \Delta^{m-1})$, and $\mathcal{H}(\rho) = H \cup \Delta^m(\rho)$ for $0 < \rho < 1$. Let $r_1, \cdots, r_k, \cdots$ be an increasing sequence of of positive numbers in $(\epsilon, 1)$ converging to $r \in (\epsilon, 1]$ such that $\mathcal{N}(r_k)$ exists for each k, $1 \leq k < \infty$. $\mathcal{N}(r_k) \subset \mathcal{H}(r_k) \times \Delta^n$ is a locally closed complex submanifold, $\mathcal{N}(r_k) \supset \mathcal{H}(r_k) \times \{0\}$, and $\varpi_{r_k} : \mathcal{N}(r_k) \to \mathcal{H}(r_k)$ is a holomorphic submersion. The assumptions imply that there exists some neighborhood $\mathscr{D}(r_k)$ of $\mathcal{H}(r_k) \times \{0\}$ in $\mathcal{H}(r_k) \times \Delta^n$ such that $\mathcal{N}(r_k) \cap \mathscr{D}(r_k) \subset \mathscr{D}(r_k)$ is a *complex submanifold*. Let (r'_k) be an increasing sequence, $\epsilon < r'_k < r_k$, such that r'_k also converges to r. Restricting $\mathcal{N}(r_k)$ to $\Delta^m(r'_k)$, there exists some $\varepsilon_k > 0$

such that $\mathcal{S}_k := \mathcal{N}(r_k) \cap (\Delta^m(r'_k) \times \Delta^n(\varepsilon_k))$ is a *complex submanifold* of $\Delta^m(r'_k) \times \Delta^n(\varepsilon_k)$ such that the canonical projection $\pi_k : \mathcal{S}_k \to \Delta^m(r'_k)$ is a holomorphic submersion with connected fibers. Write now $\mathcal{S} := \bigcup \big\{\mathcal{S}_k : 1 \leq k < \infty\big\}$, and $\pi : \mathcal{S} \to \Delta^m(r)$ for the canonical projection. Since $\varpi : \mathcal{N} \to \mathcal{H}(r)$ is already defined over H, taking the union of $\mathcal{N}|_H$ with $\mathcal{S}$ we obtain $\varpi_r : \mathcal{N}(r) \to \mathcal{H}(r)$, proving that $I \subset [\epsilon, 1]$ is closed. The proof of Lemma 7.3 is complete. $\square$

Remark When we analytically continue a germ of complex submanifold along a continuous path γ on $(\{0\} \times \Delta^{m-1}) - E$ joining a point x, where $\mathcal{N}_x$ is defined, to a point y, there is *a priori* the possibility of dependence of $\mathcal{N}_y$ on the path γ. This does not occur when $\mathcal{N}$ is already defined on $(\Delta - \{0\}) \times \Delta^{m-1}$ because $\Delta^m - E$ is simply connected since E is of codimension ≥ 2 in Δ^m.

We will also need a Hartogs-type extension theorem for meromorphic functions over a fibered projective variety. We regard a meromorphic function h on a complex space Y as a meromorphic map $h : Y \dashrightarrow \mathbb{P}^1$ and, denoting by $Y_0 \subset Y$ the Zariski open subset over which h is holomorphic, we write $\overline{\mathrm{Graph}(h)}$ to mean the subvariety $\overline{\mathrm{Graph}(h|_{Y^0})} \subset Y \times \mathbb{P}^1$. $\Delta(h)$ denotes the locus of indeterminacies of h. We have

Lemma 7.4. *Let B be an irreducible projective variety, and $E \subset B$ be a subvariety of codimension ≥ 2. Let $\mathcal{X}$ be an irreducible projective variety and $\alpha : \mathcal{X} \to B$ be a surjective morphism. Let $\Omega \subset B$ be an open subset in the complex topology and f be a meromorphic function on $\mathcal{X}|_{\Omega - E} := \alpha^{-1}(\Omega - E)$. Then, f extends to a meromorphic function on $\mathcal{X}|_\Omega = \alpha^{-1}(\Omega)$.*

Proof. For $t \in B$, we write $X_t := \alpha^{-1}(t)$. Write $f_t = f|_{X_t}$ for $t \in \Omega - E$ such that $X_t \not\subset \Delta(f)$. Then, the assignment $\Phi(t) := \big[\overline{\mathrm{Graph}(f_t)}\big] \in \mathrm{Chow}(\mathbb{P}^m \times \mathbb{P}^1)$ for a general point $t \in \Omega - E$ extends to a meromorphic map $\Phi : \Omega - E \dashrightarrow \mathcal{Z}$, where $\mathcal{Z} \subset \mathrm{Chow}(\mathbb{P}^m \times \mathbb{P}^1)$ is an irreducible component. Identifying $\mathcal{Z}$ via Chow coordinates as a subvariety of some $\mathbb{P}^N$, the extension of $\Phi : \Omega - E \dashrightarrow \mathcal{Z}$ to a meromorphic map $\Phi^\sharp : \Omega \dashrightarrow \mathcal{Z}$ results from Hartogs extension of meromorphic functions. Finally, the meromorphic extension $f^\sharp : \mathcal{X}|_\Omega \dashrightarrow \mathbb{P}^1$ of the function f on $\mathcal{X}|_{\Omega - E}$ is obtained by pulling back via $\Phi^\sharp$ the universal family $\mathscr{U} \subset \mathcal{Z} \times (\mathbb{P}^m \times \mathbb{P}^1)$ over $\mathcal{Z}$ and by projecting to the $\mathbb{P}^1$-factor. $\square$

8. Proof of Main Theorem 2

We refer the reader to the starting paragraphs in §6 for an overview on our strategy for the proof of Main Theorem 2. Starting with the rational saturation of $\varpi : \mathscr{C}(S) \to S$ with respect to $(X, \mathcal{K})$, by adjoining minimal rational curves it is standard from the bracket generating property that some nonempty open subset of S is "*rationally* *connected*", i.e., covered by chains of open subsets of minimal rational curves. The key issue is to show that the adjunction process is *algebraic*, more precisely, that one can construct an irreducible subvariety $Z \subset X$ such that $\dim(Z) = \dim(S)$ and $S \subset Z$. We will do this inductively using the "Thickening Lemma" along certain standard rational curves, Thullen extension and Hartogs-type extension as given in §6 and §7.

Let now $\varpi : \mathscr{C}(S) \to S$ be the sub-VMRT structure on $(X, \mathcal{K})$ in Main Theorem 2. Without loss of generality we assume that the sub-VMRT structure on S is *tame* (cf. the paragraph preceding Proof of Proposition 6.1). Recall that $S \subset X - B'$, where $B' \subset X$ is the enhanced bad locus of $(X, \mathcal{K})$, so that the tangent map $\tau_x : \mathcal{U}_x \to \mathscr{C}_x(X)$ is a normalization for *every* point $x \in S$. Moreover, writing $\mathscr{C}(S) = \mathscr{C}_1(S) \cup \cdots \cup \mathscr{C}_m(S)$ for the decomposition into irreducible components, for $1 \le k \le m$, the fiber $\mathscr{C}_{k,x}(S)$ over *every* point $x \in S$ is irreducible, $\mathscr{C}_{k,x}(S) \not\subset \mathrm{Sing}(\mathscr{C}_x(X))$, and $\varpi|_{\mathscr{C}_k(S)} : \mathscr{C}_k(S) \to S$ is a submersion at a general point of $\mathscr{C}_{k,x}(S)$. For a point $x \in X - B$, where $B \subset X$ is the bad locus of $(X, \mathcal{K})$, recall that $\mathcal{V}(x)$ is the union of minimal rational curves emanating from x (cf. §5, paragraph preceding Proposition 5.1), and for $x \in S$, $\Pi(x, S) \subset \mathcal{V}(x)$ is the union of minimal rational curves emanating from x whose germs at x lie on S (cf. 4th paragraph in Proof of Proposition 6.1). We call $\mathcal{V}(x)$ the $\mathscr{C}(X)$-cone of minimal rational curves at x and $\Pi(x, S)$ the $\mathscr{C}(S)$-cone of minimal rational curves at x.

By assumption $\varpi : \mathscr{C}(S) \to S$ satisfies Condition (T) in Definition 5.4. Choose now $x_0 \in S$ such that $\mathscr{C}_{x_0}(S)$ satisfies Condition (T), and define $\mathcal{V}_0(S) = \{x_0\}$, $\mathcal{V}_1(S) = \Pi(x_0, S)$. For $x \in S$ define $\mathscr{S}_0 := \tau_x^{-1}(\mathrm{Reg}(\mathscr{C}_x(S))$ and and denote by $\nu_0 : \check{\mathscr{S}}_0 \to \mathscr{S}_0$ the normalization. Consider the $\mathbb{P}^1$-bundle $\gamma_0 : \mathcal{W}_0 \to \check{\mathscr{S}}_0$, where, writing $\kappa_0 : \check{\mathscr{S}}_0 \xrightarrow{\nu_0} \mathscr{S}_0 \xrightarrow{\rho|_{\mathscr{S}_0}} \mathcal{K}$, we have $\mathcal{W}_0 := \kappa_0^*(\mathcal{U})$, where $\mathcal{U}$ is regarded as the total space of the universal $\mathbb{P}^1$-bundle $\rho : \mathcal{U} \to \mathcal{K}$, $\gamma_0 := \kappa_0^*(\rho)$ (cf. Remarks preceding Proposition 8.1 below on the notation). Identifying $\mathcal{W}_0$ with $\big\{(a, b) \in \check{\mathscr{S}}_0 \times \mathcal{U} : \kappa_0(a) = \rho(b)\big\}$, the evaluation map $\mu : \mathcal{U} \to X$ induces $\lambda_0 : \mathcal{W}_0 \to X$, which we also call the evaluation map. We have $\mathcal{V}_1(S) = \lambda_0(\mathcal{W}_0)$. By Chow's Theorem, $\mathcal{V}_1(S) \subset X$ is a subvariety. We note that $\gamma_0 : \mathcal{W}_0 \to \check{\mathscr{S}}_0$ admits a tautological section $\mathfrak{s}_{\gamma_0} : \check{\mathscr{S}}_0 \to \mathcal{W}_0$ such that $\lambda_0(\mathfrak{s}_{\gamma_0}(\check{\mathscr{S}}_0)) = \{x_0\}$.

We proceed to enlarge $\mathcal{V}_1(S)$ by the process of adjunction of minimal rational curves. We will do this inductively to obtain $\{x_0\} = \mathcal{V}_0(S) \subsetneq \mathcal{V}_1(S) \subsetneq \mathcal{V}_2(S) \subsetneq \cdots \subsetneq \mathcal{V}_r(S) = \mathcal{V}_{r+1}(S)$ by the adjunction of minimal rational curves such that $\mathcal{V}_r(S) \subset X$ is a subvariety of dimension $s = \dim(S)$ which contains a nonempty open subset of S. Taking Z to be an irreducible component of $\mathcal{V}_r(S)$ of dimension s containing S we will have proven Main Theorem 2. The critical issue is to prove inductively the *projectivity* of $\mathcal{V}_i(S) \subset X$, $1 \le i \le r$, constructed in a specific way by adjoining certain minimal rational curves. To inductively enlarge $\mathcal{V}_i(S)$ we will need to adjoin subsets of $\mathscr{C}(X)$-cones of minimal rational curves at points y lying outside S, and here lies the first difficulty of the problem. To prove projectivity of the constructed set in fact one has even to adjoin cones of rational curves passing through points lying on the bad locus B, and for this reason we need to compactify the universal family $\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$, as follows.

On the uniruled projective manifold $(X, \mathcal{K})$ denote by $\mathcal{Q}_0$ the Chow component whose general member is a reduced irreducible 1-cycle which is the image of a parametrized minimal rational curve $f : \mathbb{P}^1 \to X$ belonging to $\mathcal{H} \subset \mathrm{Hom}(\mathbb{P}^1, X)$, $\mathcal{K} = \mathcal{H}/\mathrm{Aut}(\mathbb{P}^1)$, and denote by $\rho_0 : \mathcal{U}_0 \to \mathcal{Q}_0$ the universal family over $\mathcal{Q}_0$, $\mathcal{U}_0 \subset \mathcal{Q}_0 \times X$. We write $\mathcal{Q}$ for the normalization of $\mathcal{Q}_0$, and $\rho' : \mathcal{U}' \to \mathcal{Q}$ for the normalized universal family over $\mathcal{Q}$. Since $\mathcal{Q}$ is normal and every fiber of $\rho' : \mathcal{U}' \to \mathcal{Q}$ is an irreducible reduced 1-cycle and the general fiber is a (smooth) rational curve, $\rho' : \mathcal{U}' \to \mathcal{Q}$ is in fact a holomorphic $\mathbb{P}^1$-bundle (cf. Kollár [Ko96, Theorem 2.8]). The complex manifold $\mathcal{K}$ will be naturally identified as the Zariski open subset of $\mathcal{Q}$ consisting of minimal rational curves (which are by definition free rational curves), and $\rho' : \mathcal{U}' \to \mathcal{Q}$ is a compactification of $\rho : \mathcal{U} \to \mathcal{K}$. Composing the normalization $\eta : \mathcal{U}' \to \mathcal{U}_0$ with the canonical projection of $\mathcal{U}_0 \subset \mathcal{Q} \times X$ into X we obtain the evaluation map $\mu' : \mathcal{U}' \to X$, which extends the evaluation map $\mu : \mathcal{U} \to X$. Recall that $\mathscr{C}(X) \subset \mathbb{P}T(X - B')$, where $B' \subset X$ is the enhanced bad locus of $(X, \mathcal{K})$. Compactifying $\mathcal{K}$ by the normalized Chow component $\mathcal{Q}$ and extending the tangent map to a rational map on the extended universal family $\rho' : \mathcal{U}' \to \mathcal{Q}$ it follows readily that the topological closure $\mathscr{C}'(X) \subset \mathbb{P}T(X)$ of $\mathscr{C}(X)$ is a subvariety, and we have the dominant proper map $\pi' : \mathscr{C}'(X) \to X$.

The starting point of our method for the analytic continuation of sub-VMRT structures is the Thickening Lemma given by Proposition 6.1. Some of the problems and their solutions will appear in the next step in the construction, viz., that of $\mathcal{V}_2(S)$, as follows. We have the $\mathbb{P}^1$-bundle $\gamma_0 : \mathcal{W}_0 \to \check{\mathscr{S}}_0$ accompanied by the evaluation map $\lambda_0 : \mathcal{W}_0 \to X$, $\mathcal{V}_1(S) = \lambda_0(\mathcal{W}_0)$. Recall that $\mathcal{W}_0 = \kappa_0^*(\mathcal{U})$, $\gamma_0 = \kappa_0^*(\rho)$, where $\kappa_0 : \check{\mathscr{S}}_0 \xrightarrow{\nu_0} \mathscr{S}_0 \xrightarrow{\rho|_{\mathscr{S}_0}} \mathcal{K}$. Consider $\mu' : \mathcal{U}' \to X$ as a holomorphic fibration. Define $\mathcal{E}_0 = \lambda_0^*(\mathcal{U}')$. There is naturally a holomorphic map $\chi_0 : \mathcal{E}_0 \to \mathcal{W}_0$, $\chi_0 = \lambda_0^*(\mu')$, and a tautological holomorphic section $\mathfrak{s}_{\chi_0} : \mathcal{W}_0 \to \mathcal{E}_0$. $\mathcal{E}_0$ also comes equipped with a tautological map $\sigma_0 : \mathcal{E}_0 \to \mathcal{Q}$ where $\sigma_0 := \lambda_0^*(\rho')$. Thus, for each point $w \in \mathcal{W}_0$ we attach $\mathcal{U}'_{\lambda_0(w)}$ to $u = \mathfrak{s}_{\chi_0}(w)$. For $x \in S$, $\tau_x : \mathcal{U}_x \to \mathscr{C}_x(X)$ is a normalization. To relate to the sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$, when $x := \lambda_0(w) \in S$ we will attach $\mathcal{U}_x(S) \subset \mathcal{U}_x = \mathcal{U}'_x$ to w, where $\mathcal{U}_x(S) := \mathcal{U}_{x,1}(S) \cup \cdots \cup \mathcal{U}_{x,m}(S)$, and $\mathcal{U}_{k,x}(S) := \overline{\tau_x^{-1}(\mathrm{Reg}(\mathscr{C}_{k,x}(S)))}$ for $1 \le k \le m$. Denote by $\check{\mathcal{U}}_{k,x}(S)$

the normalization of $\mathcal{U}_{k,x}(S)$. Then $\check{\mathscr{S}}_0$ is the disjoint union of $\check{\mathcal{U}}_{k,x_0}(S)$, $1 \le k \le m$. Identifying $\check{\mathscr{S}}_0$ as a subvariety of $\mathcal{W}_0$ by means of the tautological section $\mathfrak{s}_{\gamma_0} : \check{\mathscr{S}}_0 \to \mathcal{W}_0$, for $1 \le k \le m$ let $\mathscr{D}_{0,k}$ be a neighborhood of $\check{\mathcal{U}}_{k,x_0}(S)$ in $\mathcal{W}_0$ such that $\lambda_0(\mathscr{D}_{0,k}) \subset S$, and write $\mathscr{D}_0 := \mathscr{D}_{0,1} \cup \cdots \cup \mathscr{D}_{0,m}$. This defines a subvariety $\mathscr{S}_1^\flat \subset \chi_0^{-1}(\mathscr{D}_0) = \mathcal{E}_0|_{\mathscr{D}_0}$ such that $\mathscr{S}_1^\flat \cap \mathcal{E}_{0,w} = \lambda_0^*(\mathcal{U}_{\lambda_0(w)}(S))$ for $w \in \mathscr{D}_0$. The problem is to extend $\mathscr{S}_1^\flat$ to a subvariety $\mathscr{S}_1 \subset \mathcal{E}_0$.

Remarks

(a) Note that $\kappa_0^*(\rho)$, $\lambda_0^*(\mu')$ etc. signify the pull-backs of canonical projections of fibered spaces, and they serve as canonical projections of pulled back fibered spaces. Such constructions will be carried out repeatedly. Typically, there is a holomorphic map $\alpha : Z \to \mathcal{U}'$, where Z is some complex space. $\mathcal{U}'$ is the total space of a double fibration $\rho' : \mathcal{U}' \to \mathcal{K}$, $\mu' : \mathcal{U}' \to X$, and new fibered spaces are $\mathbb{P}^1$-bundles obtained by pulling back by $\rho' \circ \alpha$ or fibered spaces obtained by pulling back by $\mu' \circ \alpha$.

(b) Strictly speaking, one should write for instance $\kappa_0^*(\mathcal{U}_\rho)$ for $\mathcal{W}_0 = \kappa_0^*(\mathcal{U})$, where $\mathcal{U}_\rho$ means $\mathcal{U}$ equipped with the map $\rho : \mathcal{U} \to \mathcal{K}$. We refrain from such notation, noting that this is implicit since $\kappa_0 : \check{\mathscr{S}}_0 \to \mathcal{K}$ maps to the base of $\mathcal{U}_\rho$.

(c) The constructed fibered space comes equipped with a tautological holomorphic section. In fact, for $z \in Z$, $\alpha(z)$ serves both as a point in $\mathcal{U}'$ and as the parameter for a fiber of either $\rho' : \mathcal{U}' \to X$ or $\mu' : \mathcal{U}' \to \mathcal{K}$. For instance, when we pull back $\rho' : \mathcal{U}' \to \mathcal{Q}$ as a $\mathbb{P}^1$-bundle by $\rho' \circ \alpha$, the mapping which associates $z \in Z$ to $\alpha(z) \in \rho'^{-1}(\rho'(\alpha(z)))$ defines such a section.

For the construction of $\mathscr{S}_1$ we have

Proposition 8.1. *There exists a subvariety $\mathscr{S}_1 \subset \mathcal{E}_0$ such that $\mathscr{S}_1^\flat = \mathscr{S}_1 \cap \chi_0^{-1}(\mathscr{D}_0)$ and such that each irreducible component of $\mathscr{S}_1$ projects under χ_0 onto a connected component of the normal projective variety $\mathcal{W}_0$. As a consequence, writing $\nu_1 : \check{\mathscr{S}}_1 \to \mathscr{S}_1$ for the normalization, $\check{\mathscr{S}}_1$ is the total space of the fibered space $\delta_1 : \check{\mathscr{S}}_1 \xrightarrow{\nu_1} \mathscr{S}_1 \xrightarrow{\chi_0|_{\mathscr{S}_1}} \mathcal{W}_0$. Moreover, defining $\kappa_1 : \check{\mathscr{S}}_1 \to \mathcal{Q}$ by $\kappa_1 = \nu_1^*(\sigma_0)$, writing $\gamma_1 : \mathcal{W}_1 \to \check{\mathscr{S}}_1$ for the pull-back of the $\mathbb{P}^1$-bundle $\rho' : \mathcal{U}' \to \mathcal{Q}$ by κ_1, and defining $\lambda_1 : \mathcal{W}_1 \to X$ by $\lambda_1 = (\sigma_0 \circ \nu_1)^*\mu$, $\mathcal{V}_2(S) := \lambda_1(\mathcal{W}_1) \subset X$ is a subvariety containing $\mathcal{V}_1(S)$ and all $\mathscr{C}(S)$-cones of minimal rational curves $\Pi(x,S)$ emanating from points x lying on $\lambda_0(\mathscr{D}_0) \subset S$ for the open subset $\mathscr{D}_0 \subset \mathcal{W}$.*

Proof. For $x \in S \subset X - B'$ recall that $\mathcal{U}_x(S) := \overline{\tau_x^{-1}(\mathrm{Reg}(\mathscr{C}_x(S))}$. Define $\mathscr{S}_0^\dagger \subset \mathcal{U}_{x_0}(S) = \mathscr{S}_0$ to be the Zariski open subset consisting of points $u \in \mathrm{Reg}(\mathcal{U}_{x_0}(S))$ such that τ_{x_0} is immersive at u, i.e., $d\tau_{x_0} : T_u(\mathcal{U}_{x_0}) \to T_{\tau_{x_0}(u)}(\mathbb{P}T_{x_0}(X))$ is injective, equivalently that $\rho(u) \in \mathcal{K}$ is a standard rational curve, such that the canonical projection $\psi : \mathcal{U}(S) \to S$ is a submersion at u, and such that $(\mathscr{C}_{x_0}(S), \tau_{x_0}(u))$ satisfies Condition (T). Define $\mathcal{W}_0^\dagger := (\nu_0 \circ \gamma_0)^{-1}(\mathscr{S}_0^\dagger) \subset \mathcal{W}_0$. Let ℓ be a standard rational curve passing through x_0 such that $[\ell] \in \mathscr{S}_0^\dagger$. By the proof of Proposition 6.1 (cf. Remarks below) and in the notation used there, there exists an s-dimensional complex manifold $\mathbf{E}_\ell \supset \mathbf{P}_\ell$ and a holomorphic immersion $F : \mathbf{E}_\ell \to X$ such that $F|_{\mathbf{P}_\ell} : \mathbf{P}_\ell \to \ell$ is the normalization of ℓ, and $N_\ell = F(\mathbf{E}_\ell)$ contains the germ of S at x_0. Thus, N_ℓ may be regarded as an analytic continuation of S along ℓ. We call N_ℓ a $\mathscr{C}(S)$-thickening of ℓ. For brevity we will also call N_ℓ a "*collar*" around ℓ.

Consider $\varphi : \mathcal{W}_0 \to \mathcal{W}_0 \times X$ defined by $\varphi(w) := (w, \lambda_0(w))$ and the map $\mathcal{W}_0 \xrightarrow{\gamma_0} \check{\mathscr{S}}_0 \xrightarrow{\nu_0} \mathscr{S}_0$. By Proposition 6.1 and its proof, there exists a complex submanifold $\mathcal{N}_0$ on some open subset of $\mathcal{W}_0^\dagger \times X$ such that $\mathcal{N}_0$ contains $\varphi(\mathcal{W}_0^\dagger) = \mathrm{Graph}\big(\lambda_0|_{\mathcal{W}_0^\dagger}\big)$ and such that for any marked standard rational curve ℓ at x_0 belonging to $\mathscr{S}_0^\dagger$, any $w \in \widehat{\ell}$, the tautological lifting of ℓ to $\mathcal{W}_0$ (cf. Remarks below), $\mathcal{N}_0 \cap (\{w\} \times X)$ contains a neighborhood of $\varphi(w) = (w, \lambda_0(w))$ in $\{w\} \times N_\ell$ for some $\mathscr{C}(S)$-thickening N_ℓ of ℓ. For the fibered space $\chi_0 : \mathcal{E}_0 \to \mathcal{W}_0$, $\mathcal{E}_0 = \lambda_0^*(\mathcal{U}')$, $\chi_0 = \lambda_0^*(\mu')$, arising from $\lambda_0 : \mathcal{W}_0 \to X$, we have a tautological holomorphic section $\mathfrak{s}_{\chi_0} : \mathcal{W}_0 \to \mathcal{E}_0$. Identifying $\mathcal{W}_0$ with $\mathfrak{s}_{\chi_0}(\mathcal{W}_0) \subset \mathcal{E}_0$, we have $\mathcal{W}_0 \subset \mathcal{E}_0 = \lambda_0^*(\mathcal{U}')$. Using $\mathcal{N}_0 \supset \mathcal{W}_0^\dagger$, which may be regarded as a parametrized $\mathscr{C}(S)$-thickening over certain standard rational curves whose germs at x_0 lie on S, we obtain an extension of $\mathscr{S}_1^\flat \subset \chi_0^{-1}(\mathscr{D}_0) \subset \mathcal{E}_0$ to $\mathscr{S}_1^\sharp \subset \chi_0^{-1}(\mathcal{W}_0^\dagger \cup \mathscr{D}_0) \subset \mathcal{E}_0$, as follows. Write $\mathrm{pr}_1 : \mathcal{W}_0 \times X \to \mathcal{W}_0$ for the canonical projection onto the first factor. From the description of $\mathcal{N}_0 \supset \varphi(\mathcal{W}_0^\dagger)$ there is a holomorphic vector bundle $V \subset (\lambda_0 \circ \mathrm{pr}_1)^* T(X)$ on $\varphi(\mathcal{W}_0^\dagger) \subset \mathcal{W}_0 \times X$ such that $V|_{\varphi(\widehat{\ell})}$ is naturally identified with $T(N_\ell)|_\ell$ for $[\ell] \in \rho(\mathscr{S}_0^\dagger)$. Let $\mathcal{Z} \subset \mathcal{W}_0$ be the dense Zariski open subset of points w such that $\tau_x : \mathcal{U}_x \to \mathscr{C}_x(X)$ is a normalization at $x = \lambda_0(w)$. For $w \in (\mathcal{W}_0^\dagger \cup \mathscr{D}) \cap \mathcal{Z}$ corresponding to a standard rational curve ℓ emanating from x_0 and passing through x, write $\mathscr{C}_x(S;\ell) := \mathbb{P}T_x(N_\ell) \cap \mathscr{C}_x(X)$, and define $\mathcal{U}_x(S;\ell) := \overline{\tau_x^{-1}(\mathscr{C}_x(S;\ell))}$, $\Theta_w := \chi_0^*(\mathcal{U}_x(S;\ell)) \subset \mathcal{E}_{0,w}$. Define Θ to be the topological closure of $\bigcup \big\{\Theta_x : x \in (\mathcal{W}_0^\dagger \cup \mathscr{D}) \cap \mathcal{Z}\big\}$ in $\mathcal{E}_0|_{\mathcal{W}_0^\dagger \cup \mathscr{D}}$. By the rationality of the tangent map, $\Theta \subset \mathcal{E}_0|_{\mathcal{W}_0^\dagger \cup \mathscr{D}} = \chi_0^{-1}(\mathcal{W}_0^\dagger \cup \mathscr{D})$ is a subvariety. Over $\mathcal{W}_0^\dagger \cap \mathscr{D}_0$, Θ agrees with $\mathscr{S}_1^\flat$.

For any irreducible component Ψ of $\Theta \subset \mathcal{E}_0|_{\mathcal{W}_0^\dagger \cup \mathscr{D}}$, by the Proper Mapping Theorem $\chi_0(\Psi) \subset \mathcal{W}_0^\dagger \cup \mathscr{D}$ is a subvariety. Let $\mathscr{S}_1^\sharp \subset \Theta$ be the union of the finitely many irreducible components Ψ of Θ such that $\chi_0(\Psi) = \mathcal{W}_0^\dagger \cup \mathscr{D}$. For the subvariety $\mathscr{I} := \mathcal{W}_0 - \mathcal{W}_0^\dagger \subset \mathcal{W}_0$ let $\mathscr{I} = \mathscr{I}_1 \cup \cdots \cup \mathscr{I}_N$ be the decomposition of $\mathscr{I}$ into irreducible components. Each $\mathscr{I}_k$, $1 \le k \le N$, is the union of a variety of minimal rational curves emanating from x_0, hence $\mathscr{I} \cap \chi_0^{-1}(\mathscr{D}_0)$ contains a nonempty open subset of each irreducible component $\mathscr{I}_k$. As a consequence, Thullen extension as in Lemma 7.1 applies to give a subvariety $\mathscr{S}_1 = \overline{\mathscr{S}_1^\sharp} \subset \mathcal{E}_0 = \lambda_0^*(\mathcal{U}')$. We have thus obtained a subvariety $\mathscr{S}_1 \subset \mathcal{E}_0$ such that each irreducible component of $\mathscr{S}_1$ dominates $\mathcal{W}_0$, from which the rest of the proposition follows, noting that $\mathcal{V}_2(S) := \lambda_1(\mathcal{W}_1) \subset X$ is a subvariety, by Chow's Theorem. $\square$

Remarks

(a) In Proposition 6.1 for convenience we assume $[\alpha] \in \mathrm{Reg}(\mathscr{C}_{x_0}(S)) \cap \mathrm{Reg}(\mathscr{C}_{x_0}(X))$, and that $(\mathscr{C}_{x_0}(S), [\alpha])$ satisfies Condition (T), and construct a collar around the unique standard minimal rational curve ℓ emanating from x_0 such that $T_{x_0}(\ell) = \mathbb{C}\alpha$. For the first assumption, from the proof it is clear we need only that $[\alpha] \in \mathscr{C}_{x_0}(S)$ belongs to a smooth local irreducible branch at $[\alpha]$ of both $\mathscr{C}_{x_0}(S)$ and $\mathscr{C}_{x_0}(X)$, equivalently that ℓ is a standard rational curve. When this occurs, we can replace the second assumption by a corresponding generalized Condition (T) on $(\mathscr{C}_{x_0}(S), [\alpha])$ adapted to the local branches in question.

(b) For a rational curve ℓ on X belonging to $\mathcal{Q}$ we write $\widehat{\ell}$ for its tautological lifting to $\mathcal{U}'$, or to $\alpha^*(\mathcal{U}')$ for a holomorphic map $\alpha : Z \to \mathcal{Q}$, where Z is some complex space, when such a map is understood in the given context. Here for the notation $\widehat{\ell}$ in the proof, the classifying map $\alpha : \mathscr{S}_0^\dagger \to \mathcal{Q}$ is given by the inclusion $\mathscr{S}_0^\dagger \subset \mathcal{U}_x \xrightarrow{\rho|_{\mathcal{U}_x}} \mathcal{K} \subset \mathcal{Q}$, and $\alpha^*\mathcal{U}$ agrees with a Zariski open subset of $\mathcal{W}_0$.

(c) Note that the $\mathbb{P}^1$-bundle $\gamma_1 : \mathcal{W}_1 \to \check{\mathscr{S}}_1$ also comes equipped with a tautological section $\mathfrak{s}_{\gamma_1} : \check{\mathscr{S}}_1 \to \mathcal{W}_1$.

The procedure in constructing $\mathcal{V}_1(S)$ and $\mathcal{V}_2(S)$ can be iterated, but the arguments of Thullen-type extension of sub-VMRT structures have to be reinforced. For the proof of Main Theorem 2 we will implement the following iterative scheme, cf. a simpler version in Scheme 4.1 in the case of sub-VMRT structures modeled on an admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1. In what follows all bundles, sections and maps are understood to be holomorphic.

Scheme 8.1. *Starting with a tame sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ satisfying the hypothesis of Main Theorem 2, we will construct iteratively for $j \geq 0$*

(a) *a projective variety $\mathscr{S}_j$, its normalization $\nu_j : \check{\mathscr{S}}_j \to \mathscr{S}_j$, together with a classifying map $\kappa_j : \check{\mathscr{S}}_j \to \mathcal{Q}$;*

(b) *a $\mathbb{P}^1$-bundle $\gamma_j : \mathcal{W}_j \to \check{\mathscr{S}}_j$ such that the general fiber corresponds to a standard rational curve on X belonging to $\mathcal{K}$, together with a tautological section $\mathfrak{s}_{\gamma_j} : \check{\mathscr{S}}_j \to \mathcal{W}_j$, and an accompanying evaluation map $\lambda_j : \mathcal{W}_j \to X$;*

(c) *a projective fibered space* $\chi_j : \mathcal{E}_j \to \mathcal{W}_j$, $\mathcal{E}_j := \lambda_j^*(\mathcal{U}')$, *and a tautological section* $\mathfrak{s}_{\chi_j} : \mathcal{W}_j \to \mathcal{E}_j$;

(d) *a projective fibered subspace* $\chi_j|_{\mathscr{S}_{j+1}} : \mathscr{S}_{j+1} \to \mathcal{W}_j$ *of* $\chi_j : \mathcal{E}_j \to \mathcal{W}_j$, *where for each connected component* $\mathcal{W}_{j,q}$ *of* $\mathcal{W}_j$ *there exists a nonempty open subset* $\mathscr{D}_{j,q} \subset \mathcal{W}_{j,q}$ *satisfying* $\lambda_j(\mathscr{D}_{j,q}) \subset S$ *such that* $\mathscr{S}_{j+1,w} = \lambda_j^*(\mathcal{U}_{\lambda_j(w)}(S))$ *for* $w \in \mathscr{D}_{j,q}$.

Reduction in the implementation of the scheme. Write $(\sharp)_j$ for Scheme 8.1 for a specific value of the integer $j \geq 0$, and $(\sharp(\mathrm{a}))_j$ for Part (a) of $(\sharp)_j$, etc. We have defined sequentially (1) $\mathscr{S}_0 = \mathcal{U}_{x_0}(S) = \overline{\tau_{x_0}^{-1}(\mathrm{Reg}(\mathscr{C}_{x_0}(S)))}$ together with the classifying map $\kappa_0 : \mathscr{S}_0 \to \mathcal{K} \subset \mathcal{Q}$; (2) the $\mathbb{P}^1$-bundle $\gamma_0 : \mathcal{W}_0 \to \check{\mathscr{S}}_0$ obtained by pulling back the universal family $\rho' : \mathcal{U}' \to \mathcal{Q}$ by κ_0, accompanied by the evaluation map $\lambda_0 : \mathcal{W}_0 \to X$ and the tautological section $\mathfrak{s}_{\gamma_0} : \check{\mathscr{S}}_0 \to \mathcal{W}_0$; (3) $\mathcal{E}_0 = \lambda_0^*(\mathcal{U}')$ with canonical projection $\chi_0 : \mathcal{E}_0 \to \mathcal{W}_0$, $\chi_0 = \lambda_0^*(\mu')$, accompanied by the tautological section $\mathfrak{s}_{\chi_0} : \mathcal{W}_0 \to \mathcal{E}_0$; (4) $\mathscr{S}_1 \subset \mathcal{E}_0$ with fibers $\lambda_0^*(\mathcal{U}_x(S))$, where $x = \lambda_0(w) \in S$, for w belonging to some open subset $\mathscr{D}_0 \subset \mathcal{W}_0$ having a nonempty intersection with each connected component of $\mathcal{W}_0$, and (5) the normalization $\delta_1 : \check{\mathscr{S}}_1 \to \mathcal{W}_0$ of $\chi_0|_{\mathscr{S}_0} : \mathscr{S}_1 \to \mathcal{W}_0$. This gives $(\sharp)_0$.

To proceed inductively for the construction of the objects in (a) – (d), assume $(\sharp)_{j-1}$ has been implemented. Then $\chi_{j-1}|_{\mathscr{S}_j} : \mathscr{S}_j \to \mathcal{W}_{j-1}$ has been constructed. We have the normalization $\nu_j : \check{\mathscr{S}}_j \to \mathscr{S}_j$. From $\chi_{j-1} : \mathcal{E}_{j-1} \to \mathcal{W}_{j-1}$, where $\mathcal{E}_{j-1} = \lambda_{j-1}^*(\mathcal{U}')$ and $\chi_{j-1} = \lambda_{j-1}^*(\mu')$, we have a canonical map $\alpha_{j-1} : \mathcal{E}_{j-1} = \lambda_{j-1}^*(\mathcal{U}') \subset \mathcal{W}_{j-1} \times \mathcal{U}' \to \mathcal{U}'$, where $\lambda_{j-1}^*(\mathcal{U}')$ is identified with $\big\{(w,u) \in \mathcal{W}_{j-1} \times \mathcal{U}' : \lambda_{j-1}(w) = \rho'(u)\big\}$. The universal family $\rho' : \mathcal{U}' \to \mathcal{Q}$ defines canonically a classifying map $\sigma_{j-1} : \mathcal{E}_{j-1} \to \mathcal{Q}$ where $\sigma_{j-1} = \rho' \circ \alpha_{j-1}$, and we denote the normalization of $\sigma_{j-1}|_{\mathscr{S}_j}$ by $\kappa_j : \check{\mathscr{S}}_j \to \mathcal{Q}$, $\kappa_j = \sigma_{j-1} \circ \nu_j$. This gives $(\sharp(\mathrm{a}))_j$. Given this, we write $\gamma_j : \mathcal{W}_j \to \check{\mathscr{S}}_j$ for the $\mathbb{P}^1$-bundle $\kappa_j^*(\rho') : \kappa_j^*(\mathcal{U}') \to \check{\mathscr{S}}_j$. For $\zeta \in \mathscr{S}_j$, $\alpha_{j-1}(\zeta) \in \rho'^{-1}(\sigma_{j-1}(\zeta)) \subset \mathcal{U}'$, and for $\eta \in \check{\mathscr{S}}_j$, $\mathfrak{s}_{\gamma_j}(\eta) := \kappa_j^*(\alpha_{j-1}(\nu_j(\eta))) \in \kappa_j^*(\sigma_{j-1}(\nu_j(\eta))$ defines $\mathfrak{s}_{\gamma_j} : \check{\mathscr{S}}_j \to \mathcal{W}_j = \kappa_j^*(\mathcal{U}')$. Moreover, the evaluation map $\mu' : \mathcal{U}' \to X$ leads trivially to the evaluation map $\kappa_j^*(\mu') : \kappa_j^*(\mathcal{U}') \to X$, giving $\lambda_j : \mathcal{W}_j \to X$, completing $(\sharp(\mathrm{b}))_j$. For $(\sharp(\mathrm{c}))_j$, it remains only to define $\mathfrak{s}_{\chi_j} : \mathcal{W}_j \to \mathcal{E}_j$, which we omit as it is very similar to the existence of $\mathfrak{s}_{\gamma_j} : \check{\mathscr{S}}_j \to \mathcal{W}_j$.

Given $(\sharp)_{j-1}$, from preceding arguments we obtain $(\sharp)_j$ by an algebraic procedure excepting $(\sharp(\mathrm{d}))_j$, where difficulties lie. We note here only that any $\chi_j|_{\mathscr{S}_{j+1}} : \mathscr{S}_{j+1} \to \mathcal{W}_j$ of $\chi_j : \mathcal{E}_j \to \mathcal{W}_j$ is uniquely determined by the requirement that $\mathscr{S}_{j+1,w} = \lambda_j^*(\mathcal{U}_{\lambda_j(w)}(S))$ for $w \in \mathscr{D}_j$, $\mathscr{D}_j \subset \mathcal{W}_j$ being some open subset intersecting each connected component of $\mathcal{W}_j$ and $\lambda_j(\mathscr{D}_j) \subset S$. $\square$

For any $j \geq 0$, once $(\sharp(\mathrm{d}))_j$ is established, we define by normalization $\delta_{j+1} : \check{\mathscr{S}}_{j+1} \to \mathcal{W}_j$, which is equipped again with a tautological section $\mathfrak{s}_{\delta_{j+1}} : \mathcal{W}_j \to \check{\mathscr{S}}_{j+1}$ arising from $\mathfrak{s}_{\chi_j} : \mathcal{W}_j \to \mathcal{E}_j$. We give here a brief description and preliminary remarks on the implementation of $(\sharp(\mathrm{d}))_j$ in the inductive scheme.

(1) Starting with $x_0 \in S$, $(S; x_0)$ equipped with a sub-VMRT structure can be analytically continued along a chain of standard rational curves for which the "Thickening Lemma" applies. We call such curves *thickening* curves.
(2) When collars are available, the sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ is analytically continued along a chain of thickening curves by taking intersections of $\mathscr{C}(X)$ with tangent spaces of collars surrounding such curves.
(3) For other rational curves belonging to $\mathcal{Q}$, analytic continuation of the subvarieties $\mathscr{C}_x(S) \subset \mathbb{P}T_x(S)$ is still possible (as in the case of the construction of $\mathscr{S}_1 \subset \mathcal{E}_1$) by Thullen extension along a chain of rational curves $\ell_0, \cdots, \ell_j$ where all but the last leg ℓ_j are thickening curves.
(4) A new problem arises in going further with (3). We need stronger results on Thullen extension (Lemma 7.3) to analytically continue the germ of complex manifold $(S; x)$ along more general chains of rational curves.
(5) The existence of $\chi|_{\mathscr{S}_{j+1}} : \mathscr{S}_{j+1} \to \mathcal{W}_j$ relies on extension across subvarieties of $\mathcal{W}_j$ over which the fibration is not *a priori* defined. For a hypersurface in $\mathcal{W}_j$ arising from points on $\mathcal{W}_{j-1}$ not covered by (3), we resort to Hartogs extension for fibered spaces (Lemma 7.4).
(6) Since $\mathscr{C}_x(S)$ need not be irreducible, we have to work with all irreducible components of each $\mathcal{W}_i$, $0 \leq i \leq j$, at the same time.

As will be seen, the fibered space $\gamma_j : \mathcal{W}_j \to \check{\mathscr{S}}_j$ may be understood as a compactification of a "universal family" of rational curves belonging to $\mathcal{Q}$ which are the last legs of chains of length $j+1$ of members of $\mathcal{Q}$ issuing from a fixed base point $x_0 \in S$ and lying on germs of s-dimensional complex submanifolds of X obtained from S by analytic continuation. For $j \geq 0$ we say that $\mathcal{W}_j$ has been constructed to mean that $(\sharp)_i$ has been implemented for $0 \leq i \leq j-1$, and $(\sharp(\mathrm{a}))_j$, $(\sharp(\mathrm{b}))_j$ and $(\sharp(\mathrm{c}))_j$ have been implemented according to Scheme 8.1 and the "Reduction of the implementation of the scheme" immediately following it.

Definition 8.1. *Let $j \geq 1$ and suppose the projective fibered space $\gamma_j : \mathcal{W}_j \to \check{\mathscr{S}}_j$ has been constructed. For a point $w = w_j \in \mathcal{W}_j$ and for $0 \leq i \leq j$ define inductively* (*with decreasing indices*) *$\zeta_i := \gamma_i(w_i) \in \check{\mathscr{S}}_i$, $w_{i-1} := \delta_i(\zeta_i)$, $[\ell_i] := \kappa_i(\zeta_i)$, and $x_{i+1} := \lambda_i(w_i)$. We call $(\ell_0, \cdots, \ell_j)$ the chain of rational curves subordinate to $w \in \mathcal{W}_j$, and $(x_0, \cdots, x_{j+1})$ the sequence of marked points subordinate to $w \in \mathcal{W}_j$, where $x_0 \in S$*

stands for the base point we started with in defining $\mathcal{W}_0$. We say that $(\ell_0, \cdots, \ell_j)$ is a chain of rational curves linking $(x_0, \cdots, x_{j+1})$.

Remark Given ℓ_i there are finitely many possible choices of $\zeta_i \in \check{\mathscr{S}}_{i,x_i}$ such that $\kappa_i(\zeta_i) = [\ell_i]$; it is implicit above that such choices of ζ_i have been made when we talk of a chain of rational curves $(\ell_0, \cdots, \ell_j)$ linking $(x_0, \cdots, x_{j+1})$.

Here the unparametrized rational curves ℓ we encounter are images of $f : \mathbb{P}^1 \to X$ which are only generically injective. By a marked point x on ℓ we mean a point x on ℓ together with a specification of one of the finitely many local irreducible branches of ℓ passing through x, equivalently the specification of one of the points in the finite set $f^{-1}(x)$. To avoid clumsy expressions the markings are implicit and will not be indicated. To say that $(x_0, \cdots, x_{j+1})$ is a sequence of marked points subordinate to $w \in \mathcal{W}_j$ we mean that for $1 \le i \le j$, x_i is regarded both as a marked point on ℓ_{i-1} and as a marked point on ℓ_i, while $x_0 \in \mathrm{Reg}(\ell_0)$ is equipped with the unique marking, and x_{j+1} is a marked point of ℓ_j.

We introduce the notion of analytic continuation of the sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ along a minimal rational curve emanating from $x_0 \in S$.

Definition 8.2. *Let ℓ_0 be a minimal rational curve passing through $x_0 \in S$, $x_0 = \lambda_0(v_0), x_1 = \lambda_0(w_0)$, $v_0, w_0 \in \widehat{\ell}_0$. Defining $\varphi_0 : \mathcal{W}_0 \to \mathcal{W}_0 \times X$ by $\varphi_0(w) = (w, \lambda_0(w))$, we say that the sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ can be analytically continued from v_0 to w_0, (resp. from the marked point x_0 to the marked point x_1), along $\widehat{\ell}_0$ (resp. along ℓ_0) if and only if there exists a domain $\mathcal{O}_0$ on $\widehat{\ell}_0$, and an $(s+1)$-dimensional locally closed complex submanifold $\mathcal{N}_0 \subset \mathcal{O}_0 \times X$, $\mathcal{N}_0 \supset \varphi_0(\mathcal{O}_0)$ such that the canonical projection from $\mathcal{N}_0$ to $\mathcal{O}_0$ is a submersion and such that for u belonging to the connected component of $\widehat{\ell}_0 \cap \lambda_0^{-1}(S)$ containing v_0, $y := \lambda_0(u)$, writing $\mathcal{N}_{0,\varphi_0(u)} := \{\varphi_0(u)\} \times D_y$, $D_y \cap S$ contains a neighborhood of y on S.*

Let $j \ge 1$ and suppose the projective fibered space $\gamma_j : \mathcal{W}_j \to \check{\mathscr{S}}_j$ has been constructed. Let $w = w_j \in \mathcal{W}_j$ and denote by $(\ell_0, \cdots, \ell_j)$ (resp. $(x_0, \cdots, x_{j+1})$) the chain of rational curves (resp. the sequence of marked points) subordinate to $w \in \mathcal{W}_j$. For $0 \le i \le j$ write $v_i = \mathfrak{s}_{\gamma_i}(\zeta_i) \in \mathcal{W}_i$, $x_{i+1} := \lambda_i(w_i)$, noting that $\lambda_i(v_i) = x_i$ and define $\varphi_i : \mathcal{W}_i \to X$ by $\varphi_i(w) = (w, \lambda_i(x))$. For the notion of analytic continuation of sub-VMRT structures along a chain of rational curves belonging to $\mathcal{Q}$ we break it up into two parts, as follows.

Definition 8.3(a). *We say that $\varpi : \mathscr{C}(S) \to S$ can be analytically continued as a sub-VMRT structure along $(\ell_0, \cdots, \ell_j)$ through $(x_0, \cdots, x_{j+1})$ if and only if for any i, $0 \le i \le j$, there is a domain*

$\mathcal{O}_i \subset \widehat{\ell}_i$ containing both v_i and w_i, an $(s+1)$-dimensional locally closed complex submanifold $\mathcal{N}_i \subset \mathcal{O}_i \times X$, $\mathcal{N}_i \supset \varphi_i(\mathcal{O}_i)$ such that the canonical projection from $\mathcal{N}_i$ to $\mathcal{O}_i$ is a submersion and such that (a) *for u belonging to the connected component of $\widehat{\ell}_0 \cap \lambda^{-1}(S)$ containing v_0, writing $\mathcal{N}_{0,\varphi_0(u)} := \{\varphi_0(u)\} \times D_y$, where $y = \lambda_0(u)$, $D_y \cap S$ contains a neighborhood of y on S, and* (b) *for $1 \le i \le j$ each pair $(\mathcal{N}_{i-1}, \mathcal{N}_i)$ satisfies a further condition $(\dagger)_i$ as given below in Definition 8.3(b).*

For $1 \le i \le j$, define $S_i := \mathrm{pr}_2(\mathcal{N}_{i-1,\varphi_{i-1}(w_{i-1})}) \subset X$, $\mathrm{pr}_2(\varphi_{i-1}(w_{i-1})) = \lambda_{i-1}(w_{i-1}) = x_i$, where $\mathrm{pr}_2 : \mathcal{W}_i \times X \to X$ is the canonical projection.

Definition 8.3(b). *For $1 \le i \le j$ we say that the pair $(\mathcal{N}_{i-1}, \mathcal{N}_i)$ satisfies the condition $(\dagger)_i$ if and only if for u belonging to the connected component of $\widehat{\ell}_i \cap \lambda_i^{-1}(S)$ containing v_i, writing $\mathcal{N}_{i,\varphi_i(u)} := \{\varphi_i(u)\} \times D_y$, where $y = \lambda_i(u)$, $D_y \cap S_i$ contains a neighborhood of y on S_i.*

The condition $(\dagger)_i$ means consistency on overlaps of neighborhoods of the pair of consecutive rational curves (ℓ_{i-1}, ℓ_i). It means equivalently that the germs of complex submanifolds $(D_y; y)$ and $(S_i; y)$ at $y \in X$ agree with each other. Associated to Definition 8.3 we have the notion of accessible points on $\mathcal{W}_j$, as follows.

Definition 8.4. *Let $j \ge 0$ and suppose $\mathcal{W}_j$ has been constructed. Let $w = w_j \in \mathcal{W}_j$ be any point. Write $(\ell_0, \cdots, \ell_j)$ for the chain of rational curves and $(x_0, \cdots, x_{j+1})$ for the sequence of marked points subordinate to $w \in \mathcal{W}_j$. We say that $w \in \mathcal{W}_j$ is an accessible point if and only if the sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ can be analytically continued along $(\ell_0, \cdots, \ell_j)$ through $(x_0, \cdots, x_{j+1})$.*

Adopting the terminology and notation of Definition 8.1 and Definition 8.3, we introduce the following notions of thickening curves and perfect points of $\mathcal{W}_j$.

Definition 8.5. *Let $j \ge 1$ and suppose $\mathcal{W}_j$ has been constructed. Let $w = w_j$ be any accessible point on $\mathcal{W}_j$, $(\ell_0, \cdots, \ell_j)$ resp. $(x_0, \cdots, x_{j+1})$ be the chain of rational curves resp. the sequence of marked points subordinate to w, and write $\ell := \ell_j$. We say that $\widehat{\ell}$ is a thickening curve if and only if, writing $f : \mathbf{P}_\ell \cong \mathbb{P}^1 \to \ell$ for the normalization, $f(0) = x_j$ (as a marked point), there exists a complex manifold $\mathbf{E}_\ell \supset \mathbf{P}_\ell$ and a holomorphic immersion $F : \mathbf{E}_\ell \to X$ such that* (a) *$F|_{\mathbf{P}_\ell} \equiv f$;* (b) *$(\mathcal{N}_{j-1,\varphi_{j-1}(w_{j-1})}; \varphi_{j-1}(w_{j-1}))$ is naturally isomorphic to the image of the germ $(\mathbf{E}_\ell; 0)$ under F; and* (c) *$\mathbf{P}_\ell \subset \mathbf{E}_\ell$ is a standard rational curve.*

By abuse of language we will also say that $\ell = \ell_j$ is a thickening curve at x_j, when ℓ_j is understood in the context as the last leg of a chain of rational curves $(\ell_0, \cdots, \ell_j)$ subordinate to some point $w = w_j \in \mathcal{W}_j$.

For $j \ge 1$ recall that $(\mathcal{N}_{j-1,\varphi_{j-1}(w_{j-1})}; \varphi_{j-1}(w_{j-1})))$ is canonically isomorphic, via the projection $\mathrm{pr}_2 : \mathcal{W}_{j-1} \times X \to X$, to a germ of

s-dimensional complex submanifold $(S_j; x_j)$, where $x_j = \lambda_{j-1}(w_{j-1})$. $N_\ell := F(\mathbf{E}_\ell) \supset \ell$ is a collar around ℓ. When $F : \mathbf{E}_\ell \to X$ is an embedding, then $N_\ell \subset X$ is a locally closed submanifold containing the germ of ℓ_{j-1} at the marked point x_j, and the requirement (b) in Definition 8.5 amounts to saying that the germs $(S_j; x_j)$ and $(N_{\ell_j}; x_j)$ agree.

When we perform analytic continuation of the tame sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ along a chain of rational curves to arrive at a germ of s-dimensional complex manifold $(S'; x')$, it is no longer possible to ascertain that $S' \subset X - B'$. In fact a free rational curve ℓ may already pass over the enhanced bad locus B'. To take care of this recall that $\mathscr{C}'(X)$ is the topological closure of $\mathscr{C}(X)$ in $\mathbb{P}T(X)$ and that $\pi' : \mathscr{C}'(X) \to X$ is the canonical projection realizing $\mathscr{C}'(X)$ as a projective fibered space. Consider now the more general situation of a locally closed complex submanifold $S \subset X$ such that $S \not\subset B'$. Define $\mathscr{C}'(S) := \mathscr{C}'(X) \cap \mathbb{P}T(S)$. Denote by $\varpi' : \mathscr{C}'(S) \to S$ the canonical projection, which is a proper map. We introduce

Definition 8.6. *Assume that $\varpi'(\mathscr{C}'(S)) = S$ and let $\mathscr{C}^\flat(S)$ be the union of irreducible components of $\mathscr{C}'(S)$ which dominate S. By a generalized sub-VMRT structure over $S \not\subset B'$ we mean the holomorphically fibered space $\varpi^\flat : \mathscr{C}^\flat(S) \to S$ given by the restriction of ϖ' to $\mathscr{C}^\flat(S)$, with fibers $\mathscr{C}^\flat_x(S) := \varpi^{\flat\,-1}(x)$, $x \in S$.*

In the inductive construction of $\delta_{j+1} : \check{\mathscr{S}}_{j+1} \to \mathcal{W}_j$ a key point is to do analytic continuation of S along thickening curves and along rational curves emanating from accessible points outside a subset of codimension ≥ 2 in the ambient space. First of all we introduce a notion of perfect points (for analytic continuation) on $\mathcal{W}_j$. In what follows, for $1 \leq i \leq j$, the germ of complex submanifold $(S; x_0)$ is analytically continued along $(\ell_0, \cdots, \ell_i)$ to $(S_i; x_i)$.

Definition 8.7. *Suppose $j \geq 1$ and $\gamma_j : \mathcal{W}_j \to \check{\mathscr{S}}_j$ has been constructed. A point $w_j \in \mathcal{W}_j$ is called a perfect point on $\mathcal{W}_j$ if and only if, writing $(\ell_0, \cdots, \ell_j)$ for the sequence of standard rational curves subordinate to w_j and $v_i \in \widehat{\ell}_i$, $0 \leq i \leq j$, for the unique point such that $\lambda_i(v_i)$ is the marked point x_i on ℓ_i, for $0 \leq i \leq j$ we have* (a) $x_i \in X - B'$; (b) *writing $T_{x_i}(\ell_i) := \mathbb{C}\alpha_i$, $[\alpha_i]$ is a smooth point of both $\mathscr{C}^\flat(S_i)$ and $\mathscr{C}'(X)$;* (c) *$\varpi^\flat_i : \mathscr{C}^\flat(S_i) \to S_i$ is a submersion at $[\alpha_i]$; and* (d); *$(\mathscr{C}^\flat_{x_i}(S_i), [\alpha_i])$ satisfies* Condition (T).

By Proposition 6.1, each $\widehat{\ell}_i$, $0 \leq i \leq j$, is a thickening curve on $\mathcal{W}_i$ (cf. Definition 8.5). A point $w_j \in \mathcal{W}_j$ on a thickening curve issuing from $v_j \in \mathscr{S}_j$ and lying over a perfect point $w_{j-1} \in \mathscr{S}_{j-1}$ may fail to be perfect. To remedy for this we introduce the set of excellent points, which contains all perfect points, as follows.

Definition 8.8. *In the notation above, $w_j \in \mathcal{W}_j$ is called an excellent point if and only if* (a) *for $0 \le i \le j$, writing ℓ_i^0 for the local irreducible branch of ℓ_i corresponding to the marking at x_i given by $v_i \in \widehat{\ell}_i$, $\lambda_i(v_i) = x_i$, where $\widehat{\ell}_i$ is the lifting of ℓ_i to $\mathcal{W}_i$ given by $\widehat{\ell}_i := \gamma_i^{-1}(\zeta_i), \zeta_i \in \check{\mathscr{S}}_{i,x_i}$, defining $T_{x_i}(\ell_i^0) := \mathbb{C}\alpha_i$, and denoting by $\mathscr{C}^0(S_i)$ resp. $\mathscr{C}^0(X)$ the local irreducible branch of $\mathscr{C}^\flat(S_i)$ resp. $\mathscr{C}'(X)$ at $[\alpha_i]$ corresponding to the choice of ζ_i such that $\kappa_i(\zeta_i) = [\ell_i]$ and to the given marking on ℓ_i at x_i, $[\alpha_i]$ is a smooth point of both $\mathscr{C}^0(S_i)$ and $\mathscr{C}^0(X)$; and* (b) *$\varpi_i^\flat|_{\mathscr{C}^0(S_i)} : \mathscr{C}^0(S_i) \to S_i$ is a submersion at $[\alpha_i]$; and* (c) *$(\mathscr{C}^\flat_{x_i}(S_i), [\alpha_i])$ satisfies* Condition (T).

It will be established in Lemma 8.1 below that *the property of being an excellent point propagates along thickening curves.*

For the propagation of sub-VMRT structures, for $0 \le i \le j$ we may think of each connected component of $\mathcal{W}_i, \dim(\mathcal{W}_{i,q}) =: d_{i,q}$, as the analytic continuation of a germ of complex manifold $(\Omega_{i,q}; b_{i,q})$, as follows. Let $w \in \mathcal{W}_i$ be an arbitrary point and let $(\ell_0, \cdots, \ell_i)$ resp. $(x_0, \cdots, x_{i+1})$ be the sequence of rational curves resp. the sequence of marked points subordinate to w. For $1 \le k \le m$, choose $[\alpha_k] \in \mathrm{Reg}(\mathscr{C}_{k,x_0}(S))$ such that $\varpi : \mathscr{C}(S) \to S$ is a submersion at $[\alpha_k]$. For $1 \le q \le m^{i+1}$, where q corresponds to a choice of $(k(0), \cdots, k(i)) \in \{1, \cdots, m\}^{i+1}$, pick $b_{i,q} \in \mathcal{W}_i$ as a base point corresponding to the chain of rational curves $(\ell_0, \cdots, \ell_i)$ linking the $(i+2)$-tuple $(x_0, \cdots, x_0)$ and satisfying $[T_{x_0}(\ell_c)] = [\alpha_{k(c)}]$ for $0 \le c \le i$. Let $\Omega_{i,q} \subset \mathcal{W}_i$ be the largest connected open subset containing $b_{i,q}$ and consisting of points corresponding to chains of rational curves $(\ell_0, \cdots, \ell_i)$ linking $(x_0, \cdots, x_{i+1})$ subject to the requirement $x_c \in S$ for $0 \le c \le i+1$. For $1 \le q \le m^{i+1}$, let $\mathcal{W}_{i,q}$ be the connected component of $\mathcal{W}_i$ containing $\Omega_{i,q}$, noting that either $\mathcal{W}_{i,q'} = \mathcal{W}_{i,q''}$ or $\mathcal{W}_{i,q'} \cap \mathcal{W}_{i,q''} = \emptyset$ whenever $1 \le q', q'' \le m^{i+1}$. Thus $\mathcal{W}_i = \mathcal{W}_{i,1} \cup \cdots \cup \mathcal{W}_{i,m^{i+1}}$ is a decomposition with possible repetitions of $\mathcal{W}_i$ into connected components.

For $0 \le i \le j$ denote by $\mathcal{W}_i^\natural \subset \mathcal{W}_i$ the set of perfect points of $\mathcal{W}_i$, $\mathcal{W}_i^\dagger \subset \mathcal{W}_i$ the set of excellent points, and $\mathcal{W}_i^\sharp \subset \mathcal{W}$ the set of accessible points. We have $\mathcal{W}_i^\natural \subset \mathcal{W}_i^\dagger \subset \mathcal{W}_i^\sharp$. For $1 \le q \le m^{i+1}$ write $\mathcal{W}_{i,q}^\natural := \mathcal{W}_i^\natural \cap \mathcal{W}_{i,q}$, $\mathcal{W}_{i,q}^\dagger := \mathcal{W}_i^\dagger \cap \mathcal{W}_{i,q}$, and $\mathcal{W}_{i,q}^\sharp := \mathcal{W}_i^\sharp \cap \mathcal{W}_{i,q}$.

Fix a nonnegative integer i such that $1 \le i \le j$. Recall the holomorphic $\mathbb{P}^1$-bundle $\gamma_i : \mathcal{W}_i \to \check{\mathscr{S}}_i$ and the holomorphic fibered space $\delta_i : \check{\mathscr{S}}_i \to \mathcal{W}_{i-1}$. Define $\epsilon_i : \mathcal{W}_i \to \mathcal{W}_{i-1}$ by $\epsilon_i = \delta_i \circ \gamma_i$. For an irreducible component $\mathcal{W}_{i,q}$ of $\mathcal{W}_i$, $1 \le q \le m^{i+1}$, write $\epsilon_{i,q} := \epsilon_i|_{\mathcal{W}_{i,q}}$. For $0 \le c \le i$ define $q(c)$ inductively (with decreasing indices c) by $\epsilon_{c,q(c)}(\mathcal{W}_{c,q(c)}) = \mathcal{W}_{c-1,q(c-1)}, q(i) := q$. For $0 \le c \le i$ define $\omega_{i,c} = \omega_{i,c;q} := \epsilon_{c,q(c)} \circ \cdots \circ \epsilon_{i,q(i)}$, giving $\omega_{i,c} : \mathcal{W}_{i,q} \to \mathcal{W}_{c,q(c)}$. We have

Proposition 8.2. *Let $j \geq 0$ and suppose $\mathcal{W}_j$ has been constructed. Then, the following holds true:*

(a) *for $0 \leq i \leq j$ and for $1 \leq q \leq m^{i+1}$, $\Omega_{i,q} \subset \mathcal{W}^{\natural}_{i,q} \subset \mathcal{W}^{\dagger}_{i,q} \subset \mathcal{W}^{\sharp}_{i,q}$ are all open subsets of $\mathcal{W}_{i,q}$ in the complex topology;*

(b) *there exists an open subset $\mathscr{G}_{i,q} \subset \mathcal{W}^{\dagger}_{i,q}$ in the complex topology such that $\Omega_{i,q} \subset \mathscr{G}_{i,q} \subset \mathcal{W}^{\dagger}_{i,q}$ and such that, writing $\mathscr{A}_{i,q} := \mathcal{W}_{i,q} - \mathscr{G}_{i,q}$, we have $\mathscr{A}_{i,q} = \mathscr{A}^{(0)}_{i,q} \cup \cdots \cup \mathscr{A}^{(i)}_{i,q}$, where for $0 \leq c \leq i$, $\mathscr{A}^{(c)}_{i,q} \subset \mathcal{W}_{i,q} - \big(\mathscr{A}^{(0)}_{i,q} \cup \cdots \cup \mathscr{A}^{(c-1)}_{i,q}\big)$ is a complex-analytic subvariety, and, for $0 \leq c \leq i$, we have $\mathscr{A}^{(c)}_{i,q} := \omega^{-1}_{i,c}\big(\mathscr{A}^{(c)}_{c,q(c)}\big)$;*

(c) *there exists a subvariety $\mathscr{B}_{i,q} \subsetneq \mathcal{W}_{i,q}$ such that $\mathscr{B}_{i,q} \subset \mathscr{A}_{i,q} \cup \mathrm{Sing}(\mathcal{W}_{i,q})$ and $\mathscr{B}_{i,q} = \mathscr{B}'_{i,q} \cup \mathrm{Sing}(\mathcal{W}_{i,q}) \cup \Big(\epsilon^{-1}_{i,q}(\mathscr{B}_{i-1,q(i-1)})\Big)$ for some subvariety $\mathscr{B}'_{i,q} \subset \mathcal{W}_{i,q}$ of codimension ≥ 2, and such that $\mathcal{W}^{\sharp}_{i,q} \supset \mathcal{W}_{i,q} - \mathscr{B}_{i,q}$.*

Remark The superscript (c) in the notation $\mathscr{A}^{(c)}_{i,q}$ signifies that the set lies in $\mathcal{W}_{i,q}$ but arises from $\mathcal{W}_{c,q(c)}$.

We will make use of the following lemma on excellent points showing that the property of being an excellent point is propagated along thickening curves.

Lemma 8.1. *Suppose $j \geq 1$, $\mathcal{W}_j$ has been constructed, and $0 \leq i < j$. Let $w_i \in \mathcal{W}_i$ be an excellent point, $\zeta_{i+1} \in \check{\mathscr{S}}_{i+1,w_i}$. Write $\kappa_{i+1}(\zeta_{i+1}) =: [\ell_{i+1}]$. Assume that $\widehat{\ell}_{i+1} \subset \mathcal{W}_{i+1}$ is a thickening curve. Then, any point w_{i+1} on $\widehat{\ell}_{i+1}$ is an excellent point of $\mathcal{W}_{i+1}$, i.e., $\widehat{\ell}_{i+1} \subset \mathcal{W}^{\dagger}_{i+1,q(i+1)}$.*

Proof. Suppose $0 \leq i < j$ and let $\widehat{\ell} \subset \mathcal{W}_i$ be a thickening curve in the sense of Definition 8.5. Let now $f : \mathbf{P}_\ell \to \ell$ be the normalization of ℓ. By Proposition 6.1, there exists a holomorphic immersion $F : \mathbf{E}_\ell \to X$ of an s-dimensional complex manifold $\mathbf{E}_\ell \supset \mathbf{P}_\ell$ onto a collar N_ℓ of ℓ on X. By Corollary 6.1, $\mathbf{P}_\ell \subset \mathbf{E}_\ell$ is a standard rational curve.

Even though $\mathbf{E}_\ell$ is an open manifold, we can still define a germ of universal family of rational curves on $\mathbf{E}_\ell$ obtained from deforming $\mathbf{P}_\ell$ inside $\mathbf{E}_\ell$. There exists thus a complex manifold $\mathcal{K}^\ell$ containing $[\mathbf{P}_\ell]$, and a universal $\mathbb{P}^1$-bundle $\rho^\ell : \mathcal{U}^\ell \to \mathcal{K}^\ell$ parametrizing rational curves on $\mathbf{E}_\ell$ obtained from small deformations of $\mathbf{P}_\ell$, accompanied by an evaluation map $\mu^\ell : \mathcal{U}^\ell \to \mathbf{E}_\ell$, where $\mathcal{U}^\ell$ is only determined as a germ of complex submanifold along $\mathbf{P}_\ell$ and likewise $\rho^\ell : \mathcal{U}^\ell \to \mathcal{K}^\ell$ is determined only as a germ of holomorphic $\mathbb{P}^1$-bundle around the base point $[\mathbf{P}_\ell]$. Thus, we may assume that all points on $\mathcal{K}^\ell$ correspond to standard rational curves on $\mathbf{E}_\ell$.

Let now $\mathscr{C}(\mathbf{E}_\ell) \subset \mathbb{P}(\mu^* T(X))$ be the image of $\mathcal{U}^\ell$ under the tangent map $\tau^\ell : \mathcal{U}^\ell \to \mathbb{P}T(\mathbf{E}_\ell)$, which is an immersion as all curves belonging to $\mathcal{K}^\ell$ are standard rational curves. Define $\mathscr{C}^\flat(N_\ell) \subset \mathscr{C}'(X)|_{N_\ell}$ by an obvious modification of that of $\mathscr{C}^\flat(S_i)$ as in Definition 8.6 on generalized sub-VMRT structures. As germs along $\mathbf{P}_\ell$, $\mathcal{U}^\ell$ and $\mathscr{C}(\mathbf{E}_\ell)$, and their fibers over $z \in \mathbf{P}_\ell$ are irreducible. Then, $\mathscr{C}(\mathbf{E}_\ell) \subset F^*(\mathscr{C}^\flat(N_\ell))$ is an irreducible component. Since $F : \mathbf{E}_\ell \to X$, $F(\mathbf{E}_\ell) = N_\ell$, is an immersion, to prove Lemma 8.1, it suffices to check at each point $z \in \mathbf{P}_\ell$, that (i) $[\alpha(z)] := [T_z(\mathbf{P}_\ell)]$ is a smooth point of both $\mathscr{C}^0_z(\mathbf{E}_\ell)$ and $F^*\mathscr{C}^0_{f(z)}(X) \subset \mathbb{P}(f^*T_{f(z)}(X))$, where $\mathscr{C}^0_z(\mathbf{E}_\ell)$ refers to the local irreducible branch of $\mathscr{C}_z(\mathbf{E}_\ell)$ at $[\alpha(z)]$ being considered, etc., and, (ii) taking $\mathbf{U}$ to be a sufficiently small neighborhood of z in $\mathbf{E}_\ell$ and writing $\mathscr{C}^0(\mathbf{U})$ for the local irreducible branch of $\mathscr{C}(\mathbf{E}_\ell)$ at $[\alpha(z)]$ being considered, $\varpi^0 : \mathscr{C}^0(\mathbf{U}) \to \mathbf{U}$ for the restriction of $\varpi^\ell : \mathscr{C}^0(\mathbf{E}_\ell) \to \mathbf{E}_\ell$ to $\mathscr{C}^0(\mathbf{U})$, $\varpi^0 : \mathscr{C}^0(\mathbf{U}) \to \mathbf{U}$ is a submersion.

Since $\mathcal{K}^\ell$ consists of standard rational curves, the tangent map $\tau^\ell : \mathcal{U}^\ell \to \mathbb{P}T(\mathbf{E}_\ell)$ is an immersion, hence $[\alpha(z)]$ is a smooth point of $\mathscr{C}^0_z(\mathbf{E}_\ell)$. Since ℓ is a standard rational curve on X, writing $x := f(z)$, and ℓ^0 for the local irreducible branch of ℓ at x being considered, the tangent map $\tau_x : \mathcal{U}_x \to \mathscr{C}'_x(X)$ is an immersion on a neighborhood of $u \in \mathcal{U}_x$ corresponding to ℓ with the given marking at x, so that $[T_x(\ell^0)]$ is a smooth point of $\mathscr{C}^0_x(X)$, hence $[\alpha(z)]$ is a smooth point of $F^*\mathscr{C}^0_x(X)$, proving (i). Since every $\mathcal{K}^\ell$-curve on $\mathbf{E}_\ell$ is *a fortiori* a free rational curve, $\mu^\ell : \mathcal{U}^\ell \to \mathbf{E}_\ell$ is a submersion (cf. Hwang-Mok [HM98, Proposition 4]), hence $\varpi^0 : \mathscr{C}^0(\mathbf{U}) \to \mathbf{U}$ must be a submersion given that $\tau^\ell : \mathcal{U}^\ell \to \mathbb{P}T(\mathbf{E}_\ell)$ is an immersion, proving (ii) and hence Lemma 8.1, as desired. $\square$

Proof of Proposition 8.2. Statement (a) is obvious. For (b) and (c) we construct the sets $\mathscr{A}^{(c)}_{i,q}$ and $\mathscr{B}_{i,q}$ inductively. Although part of the proof for $i = 0$ is in the proof of Proposition 8.1, we will recall relevant parts in order to set up the inductive scheme. Recall that $\varpi : \mathscr{C}(S) \to S$ is the tame sub-VMRT we started with and $x_0 \in S$ is the base point. Let $\mathscr{O} \subset \mathscr{C}_{x_0}(S)$ be the dense Zariski open subset consisting of all $[\alpha] \in \mathscr{C}_{x_0}(S)$ such that $[\alpha]$ is a smooth point of both $\mathscr{C}_{x_0}(S)$ and $\mathscr{C}_{x_0}(X)$, the projection map $\varpi : \mathscr{C}(S) \to S$ is a submersion at $[\alpha]$ and $(\mathscr{C}_{x_0}(S), [\alpha])$ satisfies Condition (T). Let $\mathscr{S}_0 = \mathscr{S}_{0,1} \cup \cdots \cup \mathscr{S}_{0,m}$ be the decomposition of $\mathscr{S}_0$ into irreducible components. Consider the tangent map $\tau_{x_0} : \mathcal{U}_{x_0} \longrightarrow \mathscr{C}_{x_0}(S)$ and the normalization $\nu_0 : \check{\mathscr{S}}_0 \to \mathscr{S}_0$. For $1 \le p \le m$ define $E_{0,p} := \check{\mathscr{S}}_{0,p} - \nu_0^{-1}(\tau_{x_0}^{-1}(\mathscr{O}))$. $E_{0,p} \subsetneq \check{\mathscr{S}}_{0,p}$ is a subvariety containing $\mathrm{Sing}(\check{\mathscr{S}}_{0,p})$. By Proposition 6.1, for any $[\alpha] \in \check{\mathscr{S}}_{0,p} - E_{0,p}$ the unique minimal rational curve ℓ passing through x_0 and satisfying $T_{x_0}(\ell) = \mathbb{C}\alpha$ is a thickening curve. Identifying $\check{\mathscr{S}}_{0,p}$ as a subvariety of $\mathcal{W}_{0,p}$ by means of $\mathfrak{s}_{\gamma_0} : \mathcal{W}_0 \to \check{\mathscr{S}}_0$, $\check{\mathscr{S}}_{0,p} - E_{0,p}$ consists of perfect points.

For $0 \le i \le j$ and for $1 \le q \le m^{i+1}$ we write $\gamma_{i,q} : \mathcal{W}_{i,q} \to \check{\mathscr{S}}_{i,q}$ for $\gamma_{i,q} := \gamma_i|_{\mathcal{W}_{i,q}}$, and the same convention will be applied to other maps in Scheme 8.1. For $1 \le p \le m$, consider $\gamma_{0,p} : \mathcal{W}_{0,p} \to \check{\mathscr{S}}_{0,p}$. Define the subvariety $\mathscr{A}_{0,p} \subset \mathcal{W}_{0,p}$ by $\mathscr{A}_{0,p} := \gamma_{0,p}^{-1}(E_{0,p})$, and define $\mathscr{G}_{0,p} := \mathcal{W}_{0,p} - \mathscr{A}_{0,p}$. By Lemma 8.1, any thickening curve $\widehat{\ell}$ lying over $\zeta_0 \in \check{\mathscr{S}}_{0,p} - E_{0,p}$ is an excellent point, i.e., $\mathscr{G}_{0,p} \subset \mathcal{W}^{\dagger}_{0,p}$. Writing $\mathscr{A}^{(0)}_{0,p} := \mathscr{A}_{0,p}$, we have established (b) for $i = 0$. Since $\gamma_{0,p} : \mathcal{W}_{0,p} \to \check{\mathscr{S}}_{0,p}$ is a holomorphic $\mathbb{P}^1$-bundle, $\mathrm{Sing}(\mathcal{W}_{0,p}) = \gamma_{0,p}^{-1}(\mathrm{Sing}(\check{\mathscr{S}}_{0,p}))$, hence $\mathscr{A}_{0,p} \supset \mathrm{Sing}(\mathcal{W}_{0,p})$.

For $i = 0$ it remains to prove (c). For $1 \le p \le m$, $\mathscr{G}_{0,p} = \mathcal{W}_{0,p} - \mathscr{A}_{0,p} = \gamma_{0,p}^{-1}(\check{\mathscr{S}}_{0,p} - E_{0,p})$ is a union of thickening curves. By a parametrized version of Proposition 6.1, writing $\varphi_{0,p} : \mathcal{W}_{0,p} \to \mathcal{W}_{0,p} \times X$ for $\varphi_{0,p}(w) := (w, \lambda_{0,p}(w))$, there exists a complex submanifold $\mathcal{N}_{0,p}$ on $\mathcal{W}^{\dagger}_{0,p} \times X$ such that $\mathcal{N}_{0,p}$ contains $\varphi_{0,p}(\mathcal{W}^{\dagger}_{0,p}) = \mathrm{Graph}\big(\lambda_0|_{\mathcal{W}^{\dagger}_{0,p}}\big)$ and such that for $w \in \mathcal{W}^{\dagger}_{0,p}$, $[\ell] := \kappa_0(\gamma_{0,p}(w)), \mathcal{N}_{0,p} \cap (\{w\} \times X)$ contains a neighborhood of $\varphi_{0,p}(w)$ on an appropriate local irreducible component at $\lambda_{0,p}(w)$ of a collar N_ℓ of ℓ. Writing $\mathrm{pr}_1 : \mathcal{W}_{0,p} \times X \to \mathcal{W}_{0,p}$ for the canonical projection, from the description of $\mathcal{N}_{0,p} \supset \varphi_{0,p}(\mathcal{W}^{\dagger}_{0,p})$ there is a holomorphic vector subbundle $V \subset (\lambda_{0,p} \circ \mathrm{pr}_1)^* T(X)$ on $\varphi_{0,p}(\mathcal{W}^{\dagger}_{0,p}) \subset \mathcal{W}_0 \times X$ such that $V_{\varphi_{0,p}(w)}$ projects canonically to $T_{\lambda_{0,p}(w)}(N_\ell)$ at a smooth point of ℓ. We have $\mathcal{N}_{0,p} \supset \varphi_{0,p}(\mathscr{G}_{0,p})$, $\mathscr{G}_{0,p} = \mathcal{W}_{0,p} - \mathscr{A}_{0,p} \subset \mathcal{W}^{\dagger}_{0,p}$ being Zariski open in $\mathcal{W}_{0,p}$.

Note that $\lambda_{0,p}(\mathfrak{s}_{\gamma_0}(\check{\mathscr{S}}_{0,p})) = x_0$. Let $\mathscr{D}_{0,p}$ be the connected component of $\lambda_{0,p}^{-1}(S)$ containing $\mathfrak{s}_{\gamma_0}(\check{\mathscr{S}}_{0,p}) \subset \mathcal{W}_{0,p}$. Then, the holomorphic vector subbundle $V \subset (\lambda_{0,p} \circ \mathrm{pr}_1)^* T(X)$ is defined on $\varphi_{0,p}((\mathcal{W}_{0,p} - \mathscr{A}_{0,p}) \cup \mathscr{D}_{0,p})$, where each irreducible component of $\mathscr{A}_{0,p} = \gamma_{0,p}^{-1}(E_{0,p})$ intersects $\mathscr{D}_{0,p}$ nontrivially. By Thullen extension (cf. Proof of Proposition 8.1), $\mathcal{O}(V)$ extends to a unique saturated coherent subsheaf $\mathscr{F}_{0,p} \subset \mathcal{O}((\lambda_{0,p} \circ \mathrm{pr}_1)^* T(X))$. (A coherent subsheaf of a locally free sheaf is saturated whenever the quotient sheaf is torsion-free.)

We now strengthen the result on Thullen extension of V (which already implied the existence of $\mathscr{S}_1$) by using Thullen extension for sub-VMRT structures to be deduced from Lemma 7.3. Define $\mathscr{B}_{0,p} := \mathrm{Sing}(\mathscr{F}_{0,p}) \cup \mathrm{Sing}(\mathcal{W}_{0,p}) \subset \mathcal{W}_{0,p}$. Then, $\mathscr{B}_{0,p} \subset \mathcal{W}_{0,p}$ is a subvariety of codimension ≥ 2.

Consider the complex manifold $(\mathcal{W}_{0,p} - \mathscr{A}_{0,p}) \cup \mathscr{D}_{0,p}$. Let v be a smooth point of $E_{0,p}$ such that $E_{0,p}$ is of codimension 1 in $\check{\mathscr{S}}_{0,p}$ at v. Let U be a neighborhood of v on $\check{\mathscr{S}}_{0,p}$ such that the restriction of the $\mathbb{P}^1$-bundle $\gamma_{0,p} : \mathcal{W}_{0,p} \to \check{\mathscr{S}}_{0,p}$ to U is holomorphically trivial, and such that there exists a biholomorphism $f : U \xrightarrow{\cong} \Delta^{d_{0,p}-1}, f(U \cap E_{0,p}) = \{0\} \times \Delta^{d_{0,p}-2}$. Choose a trivialization $\Phi : \gamma_{0,p}^{-1}(U) \xrightarrow{\cong} U \times \mathbb{P}^1$ such that $\Phi(\mathfrak{s}_{\gamma_0}(U)) = U \times \{0\} \xrightarrow{(f,\mathrm{id})} \Delta^{d_{0,p}-1} \times \{0\}$. Shrinking U if necessarily there exists ϵ,

$0 < \epsilon < 1$, such that $\Phi(\gamma_{0,p}^{-1}(U) \cap \mathscr{D}_{0,p}) \supset \Delta^{d_{0,p}-1} \times \Delta(\epsilon) \supset \Delta^{d_{0,p}}(\epsilon)$. In the sequel we identify U with $\Delta^{d_{0,p}-1}$, and $\gamma_{0,p}^{-1}(U)$ with $U \times \mathbb{P}^1$.

We are now in a position to apply Lemma 7.3. Set $r := d_{0,p}$, $H := \Delta^r - (\{0\} \times \Delta^{r-1})$ and $\mathcal{H}(\rho) := H \cup \Delta^r(\rho)$ for $0 < \rho < 1$. Recall that $\mathcal{N}_{0,p} \subset \mathcal{W}_{0,p} \times X$, $\mathcal{N}_{0,p} \supset \varphi_{p,0}(\mathcal{W}_{0,p}^{\dagger})$ is a parametrized family of s-dimensional complex submanifolds so that for $v \in \mathcal{W}_{0,p}^{\dagger}$, $\kappa_0(\gamma_{0,p}(v)) = [\ell] \in \mathcal{K}$, the canonical projection of $\mathcal{N}_{0,p;v} := \mathcal{N}_{0,p} \cap (\{v\} \times X)$ into X agrees as a germ at $x := \lambda_{0,p}(v)$ of s-dimensional complex submanifold of X with the germ at x of the local irreducible component of the collar N_ℓ being considered. Let now $\mathcal{N}$ be the restriction of the fibered space $\mathrm{pr}_1 : \mathcal{N}_{0,p} \to \mathcal{W}_{0,p}$ to $\mathcal{H}(\epsilon) \subset U \times \Delta \subset \mathcal{W}_{0,p}$. Then, by Lemma 7.3 there exists a complex-analytic subvariety $A \subset \{0\} \times \Delta^{r-1}$ and an $(r+s)$-dimensional locally closed complex submanifold $\mathscr{X} \subset (\Delta^r - A) \times \Delta^n$, $\mathscr{X} \supset (\Delta^r - A) \times \{0\}$ such that the canonical projection $\mathrm{pr}_1 : \mathscr{X} \to \Delta^r - A$ is a submersion and such that $\mathscr{X} \supset \mathcal{N}$. From the proof of Lemma 7.3, $A \subset \Delta^r$ is precisely the locus of indeterminacies of the meromorphic extension from $\mathcal{H}(\epsilon)$ to Δ^r of the holomorphic mapping $\Phi : \mathcal{H}(\epsilon) \to \mathrm{Gr}(s, \mathbb{C}^n)$ into a Grassmannian defined by $\Phi(x) := [T_{(x,0)}(\mathcal{N}_x)]$. Noting the identification $\Delta^r = U \times \Delta \subset \gamma_{0,p}^{-1}(U) = U \times \mathbb{P}^1$, we have $A = (U \times \Delta) \cap \mathrm{Sing}(\mathscr{F}_{0,p}) \subset \gamma_{0,p}^{-1}(U) \cap \mathrm{Sing}(\mathscr{F}_{0,p})$. Covering $\mathbb{P}^1$ by the union $D' \cup D''$ of two open disks such that $0 \in D' \cap D''$ and using holomorphic coordinate charts $\psi' : D' \xrightarrow{\cong} \Delta$ and $\psi'' : D'' \xrightarrow{\cong} \Delta$ such that $\psi'(0) = \psi''(0) = 0$, Lemma 7.3 still applies to $U \times \mathbb{P}^1 \cong \gamma_{0,p}^{-1}(U)$ to show that all points on $\gamma_{0,p}^{-1}(U)$ are accessible excepting those lying on $\mathrm{Sing}(\mathscr{F}_{0,p})$, i.e., $\gamma_{0,p}^{-1}(U) - \mathcal{W}_{0,p}^{\sharp} = \gamma_{0,p}^{-1}(U) \cap \mathrm{Sing}(\mathscr{F}_{0,p})$. Since $\mathscr{F}_{0,p}$ is already defined on $\mathrm{Reg}(\mathcal{W}_{0,p})$ the same proof of analytic extension for parametrized germs of submanifolds in Lemma 7.3 holds outside of $\mathrm{Sing}(\mathscr{F}_{0,p})$ even over points $v \in \mathrm{Sing}(E_{0,p}) \subset \check{\mathscr{S}}_{0,p}$ to show that $\mathrm{Reg}(\mathcal{W}_{0,p}) - \mathcal{W}_{0,p}^{\sharp} = \mathrm{Sing}(\mathscr{F}_{0,p}) \cap \mathrm{Reg}(\mathcal{W}_{0,p})$. Hence, $\mathcal{W}_{0,p} - \mathcal{W}_{0,p}^{\sharp} \subset \mathrm{Sing}(\mathscr{F}_{0,p}) \cup \mathrm{Sing}(\mathcal{W}_{0,p}) = \mathscr{B}_{0,p}$, i.e., $\mathcal{W}_{0,p}^{\sharp} \supset \mathcal{W}_{0,p} - \mathscr{B}_{0,p}$, proving Proposition 8.2 for $i = 0$.

Suppose $0 \le i \le j$, and $1 \le q \le m^{i+1}$. We define a subset $\mathscr{G}_{i,q} \subset \mathcal{W}_{i,q}$, as follows. Let $(\ell_0, \cdots, \ell_i)$ be a chain of rational curves belonging to $\mathcal{Q}$ linking $(x_0, \cdots, x_{i+1})$ through the sequence of pairs $((v_0, w_0), (v_1, w_1), \cdots, (v_i, w_i))$ where $v_c, w_c \in \widehat{\ell_c}$ for $0 \le c \le i$. We defined $\mathscr{G}_{0,p}$ to be the subset where $v_0 \in \mathcal{W}_{0,p}^{\natural}$ and $\widehat{\ell_0} \subset \mathcal{W}_{0,p}$ is a thickening curve, and noted that $\widehat{\ell_0} \subset \mathcal{W}_{0,p}^{\dagger}$ by Lemma 8.1. For $1 \le c \le i$ by means of the tautological sections $\mathfrak{s}_{\epsilon_c} : \mathcal{W}_{c-1,q(c-1)} \to \mathcal{W}_{c,q}$ we identify $\mathcal{W}_{c-1,q(c-1)}$ as a subvariety of $\mathcal{W}_{c,q(c)}$ to obtain an ascending chain of varieties $\mathcal{W}_{0,q(0)} \subset \mathcal{W}_{1,q(1)} \subset \cdots \subset \mathcal{W}_{i,q(i)} = \mathcal{W}_{i,q}$. Let now $\mathscr{G}_{c,q(c)} \subset \mathcal{W}_{c,q(c)}$ be the open subset with respect to the complex topology consisting of points w_c in the notation above such that for

$0 \leq e \leq c$, $v_e \in \mathcal{W}^{\natural}_{e,q(e)}$ and $\widehat{\ell}_e \subset \mathcal{W}_{e,q(e)}$ is a thickening curve. By Lemma 8.1, $\mathscr{G}_{e,q(e)} \subset \mathcal{W}^{\dagger}_{e,q(e)}$, and (b) follows when we define inductively $\mathscr{A}^{(c)}_{c,q(c)} := \mathcal{W}_{c,q(c)} - \mathscr{G}_{c,q(c)} - \Big(\omega^{-1}_{c,0}\big(\mathscr{A}^{(0)}_{0,q(0)}\big) \cup \cdots \cup \omega^{-1}_{c,c-1}\big(\mathscr{A}^{(c-1)}_{c-1,q(c-1)}\big)\Big)$. In other words $\mathscr{A}^{(c)}_{c,q(c)} \subset \mathcal{W}_{c,q(c)}$ is the locally closed complex-analytic subvariety corresponding to a chain of $\mathcal{Q}$-curves $(\ell_0, \cdots, \ell_c)$ linking $(x_0, \cdots, x_{c+1})$ such that $(\widehat{\ell}_0, \cdots, \widehat{\ell}_{c-1})$ is a chain of thickening curves $\widehat{\ell}_e$ on $\mathcal{W}_{e,q(e)}$, $0 \leq e \leq c-1$, emanating from perfect points $v_e \in \mathcal{W}_{e,q(e)}$ and such that $v_c \in \mathcal{W}_{c,q(c)}$ is not a perfect point. For the proof of (c) we formulate the following lemma on the inductive construction of parametrized families of sub-VMRT structures.

Lemma 8.2. *Suppose $0 \leq i \leq j$ and $1 \leq q \leq m^{i+1}$. Writing $\varphi_{i,q} : \mathcal{W}_{i,q} \to \mathcal{W}_{i,q} \times X$ for the holomorphic map $\varphi_{i,q}(w) := (w, \lambda_{i,q}(w))$ and denoting by $\mathrm{pr}_1 : \mathcal{W}_{i,q} \times X \to \mathcal{W}_{i,q}$ the canonical projection onto the first factor, the holomorphic vector bundle $V := (\lambda_{i,q} \circ \mathrm{pr}_1)^*T(S) \subset (\lambda_{i,q} \circ \mathrm{pr}_1)^*T(X)$ admits an extension as a saturated coherent subsheaf to $\mathscr{F}_{i,q}$ defined on $\mathcal{W}_{i,q}$. Moreover, writing $\mathscr{B}_{i,q} \subsetneq \mathcal{W}_{i,q}$ for the subvariety given recursively by $\mathscr{B}_{i,q} = \mathrm{Sing}(\mathscr{F}_{i,q}) \cup \mathrm{Sing}(\mathcal{W}_{i,q}) \cup \epsilon^{-1}_{i,q}(\mathscr{B}_{i-1,q(i-1)})$, $\mathscr{B}_{0,q(0)} := \mathrm{Sing}(\mathscr{F}_{0,q(0)}) \cup \mathrm{Sing}(\mathcal{W}_{0,q(0)})$ as already been defined, there is a $(d_{i,q}+s)$-dimensional locally closed complex submanifold $\mathcal{N}_{i,q} \subset (\mathcal{W}_{i,q} - \mathscr{B}_{i,q}) \times X$ such that:*

(a) *the canonical projection $\varpi_{i,q} : \mathcal{N}_{i,q} \to \mathcal{W}_{i,q} - \mathscr{B}_{i,q}$ is a submersion;*
(b) *$\mathcal{N}_{i,q} \supset \varphi_{i,q}(\mathcal{W}_{i,q} - \mathscr{B}_{i,q})$;*
(c) *$\mathcal{N}_{i,q}$ contains the germ of $(\Omega_{i,q} \times S; (b_{i,q}, x_0))$ for the base points $b_{i,q} \in \mathcal{W}_{i,q}$ and $x_0 \in S$ corresponding to a lifting of the germ of the s-dimensional complex submanifold $(S; x_0)$ of $(X; x_0)$.*

Proof. A point $w_i \in \mathcal{W}_{i,q}$ can equivalently be described as follows. Let $(\ell_0, \cdots, \ell_i)$ be a chain of rational curves belonging to $\mathcal{Q}$ linking $(x_0, \cdots, x_{i+1})$ through a sequence $((v_0, w_0), (v_1, w_1), \cdots, (v_i, w_i))$ of pairs of points where $v_c, w_c \in \widehat{\ell}_c$ for $0 \leq c \leq i$, $v_c \in \mathfrak{s}_{\gamma_c}(\check{\mathscr{S}}_{c,q(c)}) \subset \mathcal{W}_{c,q(c)}$ and $w_c \in \mathcal{W}_{c,q(c)}$. The point $w_i \in \mathcal{W}_{i,q}$ is an accessible point, i.e., $w_i \in \mathcal{W}^{\sharp}_{i,q}$, if and only if $v_c, w_c \notin \mathrm{Sing}(\mathscr{F}_{c,q(c)})$ for $0 \leq c \leq i$.

We will prove Lemma 8.2 by induction. The case of $i = 0$ has been established. For $1 \leq i \leq j$ and for $1 \leq q \leq m^{i+1}$ assume that Lemma 8.2 has been established when (i, q) is replaced by $(c, q(c))$ for $0 \leq c \leq i-1$. The main issue is the existence of $\mathscr{F}_{i,q}$. Granted this, the existence of a $(d_{i,q}+s)$-dimensional locally closed complex submanifold $\mathcal{N}_{i,q} \subset (\mathcal{W}_{i,q} - \mathscr{B}_{i,q}) \times X$ satisfying the requirements (a) $-$ (c) follows from an application of Lemma 7.3 exactly as in the proof of Lemma 8.2 for $i = 0$.

It remains therefore to prove that, under the inductive hypothesis above, for the holomorphic vector bundle $V = (\lambda_{i,q} \circ \mathrm{pr}_1)^*T(S) \subset$

$(\lambda_{i,q} \circ \mathrm{pr}_1)^* T(X) := T_{i,q}$ of rank s defined on $\Omega_{i,q}$, the locally free sheaf $\mathcal{O}(V)$ admits an extension as a saturated coherent subsheaf $\mathscr{F}_{i,q} \subset \mathscr{T}_{i,q} := \mathcal{O}(T_{i,q})$, noting that $T_{i,q}$ is a holomorphic vector bundle defined on $\mathcal{W}_{i,q}$. First of all, by the parametrized version of Proposition 6.1 (the Thickening Lemma), for $0 \le c \le i$, $V \subset T_{i,q}$ admits an extension as a holomorphic vector bundle to the open subset $\mathcal{W}^\sharp_{c,q(c)} \subset \mathcal{W}_{c,q(c)}$ consisting of accessible points. By the induction hypothesis we have $\mathcal{W}^\sharp_{i-1,q(i-1)} \supset \mathcal{W}_{i-1,q(i-1)} - \mathscr{B}_{i-1,q(i-1)}$, where $\mathscr{B}_{i-1,q(i-1)} = \mathrm{Sing}(\mathscr{F}_{i-1,q(i-1)}) \cup \mathrm{Sing}(\mathcal{W}_{i-1,q(i-1)}) \cup \epsilon^{-1}_{i-1,q(i-1)}(\mathscr{B}_{i-2,q(i-2)})$. Recall that a point $w_i \in \mathcal{W}_{i,q}$ corresponds to a chain of length $i+1$ of $\mathcal{Q}$-curves $(\ell_0, \cdots, \ell_i)$ linking the chain of $i+2$ points $(x_0, \cdots, x_{i+1})$ on X through an ordered $(i+1)$-tuple of pairs of points $((v_0, w_0), \cdots, (v_i, w_i))$ such that for $0 \le c \le i$, $v_c, w_c \in \widehat{\ell}_c \subset \mathcal{W}_{c,q(c)}$, $v_c \in \widehat{\ell}_c \cap \mathfrak{s}_{\gamma_c}(\check{\mathscr{S}}_{c,q(c)})$. For $0 \le c \le i$ and $1 \le t \le m^{c+1}$ write $\mathscr{H}_{c,t} := \mathcal{W}_{c,t} - \mathscr{B}_{c,t}$. Consider the dense Zariski open subset $\mathscr{O}_{i,q} := \epsilon^{-1}_{i,q}(\mathscr{H}_{i-1,q(i-1)}) \subset \mathcal{W}_{i,q}$. Let $\mathscr{J}_{i,q} \subsetneq \mathscr{O}_{i,q}$ be the complex-analytic subvariety defined by requiring that $v_c \notin \mathcal{W}^\natural_{i,q}$, i.e., $v_c \in \mathcal{W}_{i,q}$ is not a perfect point, so that $\mathcal{W}^\sharp_{i,q} \supset \mathscr{O}_{i,q} - \mathscr{J}_{i,q}$ by the Thickening Lemma. From the definition $\mathscr{J}_{i,q} = \gamma^{-1}_{i,q}(\mathscr{L}_{i,q})$ for some complex-analytic subvariety $\mathscr{L}_{i,q} \subset \check{\mathscr{S}}_{i,q}$. Let $\mathscr{L}_{i,q;s}$, $1 \le s \le N(i,q)$, where $0 \le N(i,q) \le +\infty$, be an enumeration of all irreducible components of $\mathscr{L}_{i,q}$, and write $\mathscr{J}_{i,q;s} := \gamma^{-1}_{i,q}(\mathscr{L}_{i,q;s})$. We have $\mathscr{O}_{i,q} = \gamma^{-1}_{i,q}(\delta^{-1}_{i,q}(\mathscr{H}_{i-1,q(i-1)}))$. Since $Z_{i,q} := \mathfrak{s}_{\gamma_i}(\delta^{-1}_{i,q}(\mathscr{H}_{i-1,q(i-1)}))$ consists of accessible points, and the condition for accessibility is an open condition in the complex topology, there exists a neighborhood $\mathscr{D}_{i,q}$ of $Z_{i,q}$ in $\mathscr{O}_{i,q}$ such that $\mathscr{D}_{i,q} \subset \mathcal{W}^\sharp_{i,q}$. In particular, for $1 \le s \le N(i,q)$, $\mathscr{J}_{i,q;s} \cap \mathscr{D}_{i,q} \ne \emptyset$. By Thullen extension for meromorphic functions as explained in the case of $i = 0$, the locally free subsheaf $\mathcal{O}(V) \subset \mathscr{T}_{i,q}|_{\mathcal{W}^\dagger_{i,q}}$ defined on $\mathcal{W}^\dagger_{i,q}$ extends to a saturated coherent subsheaf $\mathscr{F}^0_{i,q} \subset \mathscr{T}_{i,q}|_{\mathscr{O}_{i,q}}$ defined on $\mathscr{O}_{i,q}$.

To complete the proof of Lemma 8.2 it remains to prove that $\mathscr{F}^0_{i,q}$ extends to a saturated coherent subsheaf $\mathscr{F}_{i,q} \subset \mathscr{T}_{i,q}$ on $\mathcal{W}_{i,q}$. Embedding the total space of the Grassmann bundle of $\pi : \mathrm{Gr}(s, T_{i,q}) \to \mathcal{W}_{i,q}$ into a projective space the problem is reduced to showing that any meromorphic function on $\mathscr{O}_{i,q}$ extends meromorphically to $\mathcal{W}_{i,q}$. Now $\mathcal{W}_{i,q} - \mathscr{O}_{i,q} = \mathcal{W}_{i,q} - \epsilon^{-1}_{i,q}(\mathscr{H}_{i-1,q(i-1)}) := \epsilon^{-1}_{i,q}(\mathscr{B}_{i-1,q(i-1)})$. By the recursive definition of $\mathscr{B}_{c,q(c)}$ for $0 \le c \le i-1$, we have $\epsilon^{-1}_{i,q}(\mathscr{B}_{i-1,q(i-1)}) = \omega^{-1}_{i,0}(\mathscr{B}'_{0,q(0)}) \cup \cdots \cup \omega^{-1}_{i,i-1}(\mathscr{B}'_{i-1,q(i-1)})$, where $\mathscr{B}'_{0,q(0)} := \mathscr{B}_{0,q(0)}$ and $\mathscr{B}'_{c,q(c)} = \overline{\mathscr{B}_{c,q(c)} - \epsilon^{-1}_{c,q(c)}(\mathscr{B}_{c-1,q(c-1)})}$ for $1 \le c \le i-1$. For $0 \le c \le i$, write $\mathscr{O}^{(c)}_{i,q} := \mathcal{W}_{i,q} - \big(\omega^{-1}_{i,0}(\mathscr{B}'_{0,q(0)}) \cup \cdots \cup \omega^{-1}_{i,c-1}(\mathscr{B}'_{c-1,q(c-1)})\big)$. We have

$\mathscr{O}_{i,q} = \mathscr{O}^{(i)}_{i,q} \subset \mathscr{O}^{(i-1)}_{i,q} \subset \cdots \subset \mathscr{O}^{(0)}_{i,q} = \mathcal{W}_{i,q}$, giving an increasing sequence of Zariski open subsets. Writing $\mathscr{B}^0_{c,q(c)} := \mathscr{B}_{c,q(c)} - \epsilon^{-1}_{c,q(c)}(\mathscr{B}_{c-1,q(c-1)})$, we also have $\mathscr{O}^{(c)}_{i,q} = \mathscr{O}_{i,q} \cup \omega^{-1}_{i,i-1}(\mathscr{B}^0_{i-1,q(i-1)}) \cup \cdots \cup \omega^{-1}_{i,c}(\mathscr{B}^0_{c,q(c)})$, where the union is disjoint.

We claim that for $0 \le c \le i-1$, any meromorphic function f on $\mathscr{O}^{(c+1)}_{i,q}$ extends meromorphically to $\mathscr{O}^{(c)}_{i,q}$. We have $\mathscr{O}^{(c)}_{i,q} = \mathscr{O}^{(c+1)}_{i,q} \cup \omega^{-1}_{i,c}(\mathscr{B}^0_{c,q(c)})$, $\mathscr{O}^{(c+1)}_{i,q} = \mathscr{O}^{(c)}_{i,q} - \omega^{-1}_{i,c}(\mathscr{B}^0_{c,q(c)})$. While $\mathscr{B}_{c,q(c)}$ is of codimension ≥ 2 in $\mathcal{W}_{c,q(c)}$, the dimension of fibers of $\omega_{i,c} : \mathcal{W}_{i,q} \to \mathcal{W}_{c,q(c)}$ may jump, and some irreducible components of $\omega^{-1}_{i,c}(\mathscr{B}^0_{c-1,q(c-1)})$ may be of codimension 1, for which reason Hartogs extension does not apply immediately. To prove meromorphic extension of f we resort to Lemma 7.4, which is a Hartogs extension theorem across "*relatively exceptional*" divisors for surjective morphisms between projective varieties. In the notation adopted in the latter lemma, let $\pi : \mathcal{X} \to B$ be $\omega_{i,c} : \mathcal{W}_{i,q} \to \mathcal{W}_{c,q(c)}$, Ω be the Zariski open subset $\mathcal{W}_{c,q(c)} - \epsilon^{-1}_{c,q(c)}(\mathscr{B}_{c-1,q(c-1)}) \subset \mathcal{W}_{c,q(c)}$ and the subvariety $E \subset \Omega$ be $\mathscr{B}^0_{c,q(c)} = \mathscr{B}_{c,q(c)} - \epsilon^{-1}_{c,q(c)}(\mathscr{B}_{c-1,q(c-1)}) \subset \mathcal{W}_{c,q(c)} - \epsilon^{-1}_{c,q(c)}(\mathscr{B}_{c-1,q(c-1)}) = \Omega$. Lemma 7.4 applies to prove that f extends meromorphically from $\mathscr{O}^{(c+1)}_{i,q}$ to $\mathscr{O}^{(c)}_{i,q}$, hence by induction f extends meromorphically from $\mathscr{O}^{(i)}_{i,q} = \mathscr{O}_{i,q}$ to $\mathscr{O}^{(0)}_{i,q} = \mathcal{W}_{i,q}$. As a consequence the locally free subsheaf $\mathscr{F}^0_{0,q} \subset \mathscr{T}_{0,q}|_{\mathcal{W}^\dagger_{i,q}}$ extends to a saturated coherent subsheaf $\mathscr{F}_{i,q} \subset \mathscr{T}_{i,q}$ on $\varphi_{i,q}(\mathcal{W}_{i,q})$, proving Lemma 8.2. □

Proof of Proposition 8.2 cont. Proposition 8.2 follows readily from Lemma 8.2. For the only missing statement $\mathscr{B}_{i,q} \subset \mathscr{A}_{i,q} \cup \mathrm{Sing}(\mathcal{W}_{i,q})$, it suffices to observe that in the definition of $\mathscr{B}_{i,q}$, at a smooth point $w \in \mathcal{W}_{i,q}$, the analytic continuation of the sub-VMRT structure on S through a chain of $\mathcal{Q}$-curves fails at w precisely when the subsheaf $\mathscr{F}_{i,q} \subset \mathscr{T}_{i,q}$ fails to be locally free at w. In particular, w cannot lie on a thickening curve, hence $\mathscr{B}_{i,q} \subset \mathscr{A}_{i,q} \cup \mathrm{Sing}(\mathcal{W}_{i,q})$, as desired. □

Proof of Main Theorem 2. By Lemma 8.2 and in the notation there, for $1 \le q \le m^{j+1}$ we have constructed a $(d_{j,q} + s)$-dimensional locally closed complex submanifold $\mathcal{N}_{j,q} \subset (\mathcal{W}_{j,q} - \mathscr{B}_{j,q}) \times X$ such that conditions (a) $-$ (c) in Lemma 8.2 hold when (i, q) is replaced by (j, q). Recall the holomorphic map $\varphi_{j,q} : \mathcal{W}_{j,q} \times X \to \mathcal{W}_{j,q}$ defined by $\varphi_{j,q}(w) = (w, \lambda_{j,q}(w))$, and denote by $\mathrm{pr}_1 : \mathcal{W}_{j,q} \times X \to \mathcal{W}_{j,q}$ the canonical projection. For $w \in \mathcal{W}_{j,q} - \mathscr{B}_{j,q}$ define $\mathscr{S}_{j,q;w} := \mathbb{P}T_w(\mathcal{N}_{j,q;w}) \cap \lambda^*_{j,q}(\mathcal{U}'_w)$.

On $\mathcal{W}_{j,q} - \mathrm{Sing}(\mathscr{F}_{j,q})$ we have $\mathscr{F}|_{\mathcal{W}_{j,q}-\mathrm{Sing}(\mathscr{F}_{j,q})} = \mathcal{O}(V)$ for a certain holomorphic vector subbundle $V \subset T_{j,q}|_{\mathcal{W}_{j,q}-\mathrm{Sing}(\mathscr{F}_{j,q})}$. The existence of $\mathscr{F}_{j,q} \subset \mathscr{T}_{j,q}$ is equivalent to the statement that there exists an irreducible fibered subspace $\mathscr{Z} \subset \mathbb{P}(\lambda^*_{j,q}(T(X)))$ with canonical projection $\nu : \mathscr{Z} \to \mathcal{W}_{j,p}$ such that $\mathrm{pr}_1^*(\mathscr{Z}_w) = \mathbb{P}V_w$ at a point $w \in \mathcal{W}_{j,q} - \mathrm{Sing}(\mathscr{F}_{j,q})$. It follows that, denoting by $\mathscr{R}^0 := \mathscr{Z} \cap \lambda^*_{j,q}(\mathcal{U}')$, and by $\mathscr{R} \subset \mathscr{R}^0$ the union of irreducible components of $\mathscr{R}^0$ which dominate $\mathcal{W}_{j,q}$, we have defined the fibered subspace $\chi_{j,q}|_{\mathscr{R}} : \mathscr{R} \to \mathcal{W}_{j,q}$. There are m indices t, $1 \leq t \leq m^{j+2}$, satisfying the requirement that $\Omega_{j+1,t}$ sits over $\Omega_{j,q}$. The fibered space $\chi_{j,q}|_{\mathscr{R}} : \mathscr{R} \to \mathcal{W}_{j,q}$ decomposes into irreducible components giving m fibered spaces, with possible repetitions, to be denoted by $\delta_{j+1,t} : \check{\mathscr{S}}_{j+1,t} \to \mathcal{W}_{j,q}$. (The same convention applies in the notation $\gamma_{j+1,t} : \mathcal{W}_{j+1,t} \to \check{\mathscr{S}}_{j+1,t}$, etc.) Here $\mathcal{W}_{j+1,t}$ are the m possibly repeated irreducible components of $\mathcal{W}_{j+1}$ where $\epsilon_{j+1}(\mathcal{W}_{j+1,t}) = \mathcal{W}_{j,q}$, noting that $\mathcal{W}_{j+1,t}$ contains $\Omega_{j+1,t}$ as an open subset.

Define now $\mathcal{V}_{j+1}(S) := \lambda_j(\mathcal{W}_j)$. Then, we have $\mathcal{V}_0(S) \subset \mathcal{V}_1(S) \subset \cdots$. Under the assumption in Main Theorem 2 that the distribution $\mathcal{D}$ on S spanned at a general point $x \in S$ by $\widetilde{\mathscr{C}_x}(S)$ is bracket generating, we claim that $\dim(\mathcal{V}_{j+1}(S)) = s$ for some $j \geq 0$. For $i \geq 0$, $1 \leq q \leq m^{i+1}$, write $\mathcal{V}_{i+1,q} = \lambda_i(\mathcal{W}_{i,q})$. Suppose the maximal dimension of all $\mathcal{V}_{i+1,p}$, $0 \leq i < \infty$, is equal to $d < s$. Let (j,q) be such that $\dim(\lambda_{j,q}(\mathcal{W}_{j,q})) = d$. Then, $\mathcal{V}_{j+1,t} = \mathcal{V}_{j,q}$ whenever $\epsilon_{j+1}(\mathcal{W}_{j+1,t}) = \mathcal{W}_{j,q}$. In other words, for $1 \leq k \leq m$ the adjunction of rational curves belonging to $\mathscr{C}_{k,x}(S)$, $x \in S$, does not enlarge $\mathcal{V}_{j,q}$. Note that $x_0 \in \mathcal{V}_{j,q} \cap S$. Denote by $\mathscr{E}_k$ the distribution on $\mathrm{Reg}(\mathcal{V}_{j,q} \cap S)$ spanned at a general point x by $\widetilde{\mathscr{C}_{k,x}}(S)$. Then, $\mathscr{E}_{k,x} \subset T_x(\mathcal{V}_{j,q})$ and so $\mathcal{D}_x = \mathscr{E}_{1,x} + \cdots + \mathscr{E}_{m,x} \subset T_x(\mathcal{V}_{j,q})$. Since the tangent bundle $\mathcal{T} := \mathcal{O}(T(\mathrm{Reg}(\mathbb{V}_{j,q})))$ is integrable, we must have $[\mathcal{D}, \mathcal{D}] \subset \mathcal{T}$ and hence inductively $\mathcal{D}^\ell + [\mathcal{D}, \mathcal{D}^\ell] \subset \mathcal{T} \subsetneq \mathcal{O}(T(S))$ for $\ell \geq 0$, so that $\mathcal{D}$ is not bracket generating, a plain contradiction. We conclude that $\dim(\mathcal{V}_{j,q}) = s$ for some $j \geq 0$, $1 \leq q \leq m^{j+1}$. From the construction of $\mathcal{W}_{j,q}$ it follows that $Z := \mathcal{V}_{j,q}$ contains a nonempty open subset of S, hence $Z \supset S$, proving Main Theorem 2. □

Remarks

(a) Fixing (j,q) as in the last paragraph in the above, for any $i \geq 0$ and for $1 \leq r \leq m^{i+1}$ we have from the construction clearly $\mathcal{V}_{i,r} \subset \mathcal{V}_{j+i+1,t}$ for some $\mathcal{W}_{j+i+1,t}$ lying over $\mathcal{W}_{j,q}$, hence $\mathcal{V}_{i,r} \subset \mathcal{V}_{j,q} = Z$.

(b) For a uniruled projective manifold $(X, \mathcal{K})$, $\mathscr{C}_x(X) \subset \mathbb{P}T_x(X)$ at a general point may be reducible. There are straightforward generalizations of Theorem 1.4 and Main Theorem 2 to take care of this, where we assume $(\mathscr{C}_x(S), \mathscr{C}_x(X))$ is a proper pair in the sense of Remark (b) after Definition 5.3.

9. On the recognition of uniruled projective spaces by sub-VMRT structures on complete intersections of rational homogeneous spaces

We proceed to examine examples of sub-VMRT structures where the ambient manifold is a smooth complete intersection of a rational homogeneous space of Picard number 1 which remains to be uniruled by projective lines. For this purpose we introduce a quantitative measure of the extent to which nondegeneracy for substructures holds. We adhere to the notation used in Definition 5.2.

Definition 9.1. *Let $(\mathcal{B}, \mathcal{A})$, $\mathcal{B} = \mathcal{A} \cap \mathbb{P}(E)$, be a proper pair of projective subvarieties, and α be a smooth point of both $\widetilde{\mathcal{B}}$ and $\widetilde{\mathcal{A}}$. For $\eta \in T_\alpha(\widetilde{\mathcal{A}})$ write $\delta_\alpha(\eta; \mathcal{B}, \mathcal{A}; E) := \dim\big(\tau_\alpha(\mathbb{C}\eta \otimes T_\alpha(\widetilde{\mathcal{B}}))\big)$ and define $\ell(\alpha)$ to be the minimum of $\delta_\alpha(\eta; \mathcal{B}, \mathcal{A}; E)$ among all $\eta \in T_\alpha(\widetilde{\mathcal{A}}) - E$. Let $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_m$ be the decomposition of $\mathcal{B}$ into irreducible components. Let ℓ_k be the value of $\ell(\chi_k)$ at a general smooth point χ_k of $\mathcal{B}_k$. The minimum $\ell = \ell(\mathcal{B}, \mathcal{A}; E)$ of all ℓ_k, $1 \leq k \leq m$, is called the index of nondegeneracy for substructures of $(\mathcal{A}, \mathcal{B}; E)$. For $p \geq 1$, $(\mathcal{B}, \mathcal{A}; E)$ is said to be p-nondegenerate for substructures if and only if $\ell(\mathcal{B}, \mathcal{A}; E) \geq p$.*

Note that $(\mathcal{B}, \mathcal{A}; E)$ is 1-nondegenerate for substructures if and only if is nondegenerate for substructures. We have

Definition 9.2. *For an admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1, we denote by $\ell(X_0, X)$ the nonnegative integer $\ell(\mathscr{C}_0(X_0), \mathscr{C}_0(X); T_0(X_0))$ at the reference point $0 \in X_0$ and call it the index of nondegeneracy of (X_0, X).*

Noting that sub-VMRTs are obtained as linear sections of VMRTs, the following result gives sufficient conditions guaranteeing nondegeneracy for substructures.

Proposition 9.1. *Let $(\mathcal{B}, \mathcal{A})$ be a proper pair of projective subvarieties, $\mathcal{A} \subset \mathbb{P}(V)$, $\mathcal{B} := \mathcal{A} \cap \mathbb{P}(E)$, $E \subsetneq V$, $\mathcal{B} \subsetneq \mathcal{A}$, and suppose $(\mathcal{B}, \mathcal{A}; E)$ is p-nondegenerate for substructures, $p \geq 1$. Let $\mathcal{D} \subset \mathbb{P}(V)$ be a subvariety of codimension $\leq r$ and suppose $(\mathcal{B} \cap \mathcal{D}, \mathcal{A} \cap \mathcal{D})$ is a proper pair of projective subvarieties. If $0 \leq r \leq p - 1$, then $(\mathcal{B} \cap \mathcal{D}, \mathcal{A} \cap \mathcal{D})$ is $(p - r)$-nondegenerate for substructures, $p - r \geq 1$. In particular, it is nondegenerate for substructures.*

Proof. Let $\Gamma \not\subset \mathrm{Sing}(\mathcal{A})$ be any one of the irreducible components $\mathcal{B}_k$ of $\mathcal{B}$, $1 \leq k \leq m$. For a general point $\gamma \in \widetilde{\Gamma}$, we have $\dim\big(\tau_\gamma(\mathbb{C}\eta \otimes T_\gamma(\widetilde{\mathcal{B}}))\big) \geq p$. Let χ be a general smooth point of $\mathcal{B} \cap \mathcal{D}$ such that χ is also a smooth point of $\mathcal{A} \cap \mathcal{D}$. Since $\mathcal{D} \subset \mathbb{P}(V)$ is of codimension r, we have $\dim(T_\chi(\mathcal{B} \cap \mathcal{D})) \geq \dim(T_\chi(\mathcal{B})) - r$. Hence, for any $\eta \in T_\chi(\mathcal{B} \cap D)$ we have $\dim\big(\tau_\chi(\mathbb{C}\eta \otimes T_\chi(\mathcal{B} \cap \mathcal{D}))\big) \geq p - r \geq 1$ by assumption, and Proposition 9.1 follows. $\square$

For the prototypical case of the Grassmannian we have the following lemma resulting from a straightforward computation.

Lemma 9.1. *Suppose* $1 \le p' \le p, 1 \le q' \le q$, $(p', q') \ne (p, q)$. *For the admissible pair* $(G(p', q'), G(p, q))$ *we have* $\ell((G(p', q'), G(p, q)) = \min(p'-1, q'-1)$ *if* $p' < p$ *and* $q' < q$. *Moreover* $\ell((G(p, q'), G(p, q)) = p-1$ *for* $q' < q$.

In what follows we examine cases where Theorem 1.4 and Main Theorem 2 apply to yield structural results on germs of submanifolds inheriting sub-VMRT structures on certain complete intersections on rational homogeneous spaces. For a rational homogeneous space X of Picard number 1, we denote by δ the positive generator of $H^2(X, \mathbb{Z}) \cong \mathbb{Z}$. Denoting by $\mathcal{O}(1)$ the positive generator of $\mathrm{Pic}(X) \cong \mathbb{Z}$, $c_1(\mathcal{O}(1)) = \delta$, for a minimal rational curve ℓ on X we have $T(X)|_\ell \cong \mathcal{O}(2) \oplus (\mathcal{O}(1))^p \oplus \mathcal{O}^q$, thus we have $c_1(X) = (2+p)\delta$, noting that $p = \dim(\mathscr{C}_0(X)$ for any reference point $0 \in X$ (cf. Lemma 5.1 and the reference there.) We will identify $H^2(X, \mathbb{Z})$ with $\mathbb{Z}$ and write $c_1(X) = 2 + p$, etc. X will be identified as a projective submanifold by means of the first canonical embedding $\mu : X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{O}(1))^*) := \mathbb{P}(V)$. Let $k \ge 1$, $H_i \subset \mathbb{P}(V)$, $1 \le i \le m$, be hypersurfaces of degree k_i, $k := k_1 + \cdots + k_m$, such that $Z := X \cap (H_1 \cap \cdots \cap H_m) \subset X$ is smooth and of codimension m. We have the following lemma concerning VMRTs on Z (cf. Kollár [Ko96, Chapter 5, §4, esp. Exercises 4.6, p.270]).

Lemma 9.2. *Suppose* $c_1(X) - k \ge 2$. *Then,* $Z \subset X \subset \mathbb{P}(V)$ *is uniruled by projective lines. Moreover, denoting by* $\mathcal{H}$ *the minimal rational component of projective lines on* Z, *writing* $E \subset Z$ *for the bad locus of* $(Z, \mathcal{H})$, *and* $\pi_Z : \mathscr{C}(Z) \to Z$ *for the VMRT structure defined by* $(Z, \mathcal{H})$, *for* $x \in Z - E$, $\mathscr{C}_x(Z)$ *is smooth and given by* $\mathscr{C}_x(Z) = \mathscr{C}_x(X) \cap \mathcal{J}_x$ *for some projective subvariety* $\mathcal{J}_x \subset \mathbb{P}T_x(X)$ *of codimension* k.

Proof. For $1 \le i \le m$ denote by F_i a defining homogeneous polynomial of degree k_i of the reduced hypersurface H_i. Let $x \in Z \subset X$. In terms of homogeneous coordinates $[z_0, \cdots, z_N]$ on $\mathbb{P}(V)$, writing $x = [a] =: [a_0, \cdots, a_N]$, $\eta = (\eta_0, \cdots, \eta_N) \in V$ for a nonzero vector, let $\ell = \ell(x, \eta) \subset \mathbb{P}(V)$ be the projective line given by $\ell := \big\{[a_0 + t\eta_0, a_N + t\eta_N] : t \in \mathbb{C}\big\} \cup \big\{[\eta_0, \cdots, \eta_N]\big\}$. Then, $\ell \subset Z$ if and only if $[\eta] \in \mathscr{C}_x(X) \subset \mathbb{P}T_x(\mathbb{P}V) = V/\mathbb{C}a$ and $F_i(a_0 + t\eta_0, a_N + t\eta_N) = 0$ for every i, $1 \le i \le m$, and for every complex number t. For each i, expanding in t we have $F_i(a_0 + t\eta_0, a_N + t\eta_N) = F_i(a_0, \cdots, a_N) + tF_i^1(\eta_0, \cdots, \eta_N) + \cdots + t^{k_i} F_i^{k_i}(\eta_0, \cdots, \eta_N)$, where F_i^j is a homogeneous polynomial of degree $k_i - j$ for $1 \le j \le k_i$,. Since $x = [a] \in Z$, we have $F_i(a_0, \cdots, a_N) = 0$, and $\ell \subset Z$ if and only if $[\eta] \in \mathscr{C}_x(X)$ and $F_i^j(\eta_0, \cdots, \eta_N) = 0$ for $1 \le i \le m$ and for $1 \le j \le k_i$. Hence $\dim(\mathscr{C}_x(Z)) \ge \dim(\mathscr{C}_x(X)) - (k_1 + \cdots + k_m) = c_1(X) - 2 - k \ge 0$ by the hypothesis of the lemma. Define now $\mathcal{J}_x \subset$

$\mathbb{P}T_x(X)$ to be the common zero set of the k homogeneous polynomials $\{F_1^1, \cdots, F_1^{k_1}; \cdots; F_m^1, \cdots, F_m^{k_m}\}$. Then, $\dim(\mathscr{C}_x(Z)) \geq c_1(X)-2-k \geq 0$, hence Z is uniruled by projective lines. Now the normal bundle $N_{Z|X}$ is isomorphic to the direct sum $\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_n)$ on Z, and by the adjunction formula we have $c_1(Z) = c_1(X) - (k_1 + \cdots k_m) = c_1(X) - k$. For a point $x \in Z-E$, every projective line ℓ on Z passing through x is a free rational curve, hence a standard rational curve (since $N_{\ell|Z}$ is semi-positive and $N_{\ell|Z} \subset N_{\ell|\mathbb{P}V} \cong \mathcal{O}(1)^{N-1}$). It follows that the tangent map $\tau_x : \mathcal{U}_x \to \mathscr{C}_x(Z) \subset \mathbb{P}T_x(Z)$ is an immersion (cf. Lemma 5.1) and hence a biholomorphism since no two distinct projective lines passing through x can share the same tangent space at x, hence $\mathscr{C}_x(Z)$ is smooth for $x \in Z-E$. We have $T(X)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^a \oplus \mathcal{O}^b$, $c_1(Z) = 2+a$, so that each component of $\mathscr{C}_x(Z)$ is of dimension $a = c_1(Z)-2 = c_1(X)-k-2$ and hence $\mathcal{J}_x$ must be of codimension k. The proof of Lemma 9.2 is complete. $\square$

Remark Since $\tau_x : \mathcal{U}_x \xrightarrow{\cong} \mathscr{C}_x(Z)$, for $x \in Z - E$, the enhanced bad locus $E' \subset Z$ of Z is the same as the bad locus $E \subset Z$.

When an admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1 is nondegenerate for substructures with $\ell(X_0, X) \geq 2$, Proposition 9.1 says that the results in §5 on rational saturation and algebraicity of germs of sub-VMRT structures still hold when one replaces $\big(\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X)\big)$ by smooth slices by projective subvarieties of sufficiently small degrees.

We restrict now to a nonlinear admissible pair (X_0, X) of sub-diagram type marked at a long simple root so that $\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$ are homogeneous submanifolds, and the index of nondegeneracy for substructures $\ell(X_0, X) \geq 1$ can be computed at a single point $[\alpha] \in \mathscr{C}_0(X_0)$. Let $Z := X \cap (H_1 \cap \cdots \cap H_m) \subset X$ be a smooth complete intersection of X with hypersurfaces $H_1, \cdots, H_m$ of degree $k_1, \cdots, k_m$, $k_1 + \cdots + k_m := k$. Since $X_0 \subset X \subset \mathbb{P}(V)$ is a linear section, for general hypersurfaces H_i of degrees k_i, the intersection $M = Z \cap X_0$ is uniruled by projective lines when $k \leq c_1(X_0)-2$. In this case, at a general point $x \in M$, $\big(\mathscr{C}_x(M) \subset \mathscr{C}_x(Z)\big)$ is projectively equivalent to $\big(\mathscr{C}_0(X_0) \cap \mathcal{J}(x) \subset \mathscr{C}_0(X) \cap \mathcal{J}(x)\big)$ for some projective subvariety $\mathcal{J}(x) \subset \mathbb{P}T(X)$ of codimension k. For a sub-VMRT structure on a locally closed complex submanifold $S \subset Z-E$ with $\big(\mathscr{C}_x(S) \subset \mathscr{C}_x(Z)\big)$ modeled on the latter pairs, we prove

Theorem 9.1. *Let S be a complex submanifold of some connected open subset $W \subset Z-E$ such that at a general point x of S, writing $\mathscr{C}_x(S) := \mathscr{C}_x(Z) \cap \mathbb{P}T_0(S)$, $\big(\mathscr{C}_x(S) \subset \mathscr{C}_x(Z)\big)$ is projectively equivalent to $\big(\mathscr{C}_0(X_0) \cap \mathcal{J}(x) \subset \mathscr{C}_0(X) \cap \mathcal{J}(x)\big)$ for some projective subvariety $\mathcal{J}_x \subset \mathscr{C}_0(X)$ of pure codimension k. Assume that $\varpi : \mathscr{C}(S) \to S$ satisfies* Condition (T) *and $\ell(X_0, X) \geq k+1$. Then, $S \subset Z$ is linearly* *saturated with respect to* $(Z, \mathcal{H})$. *If* $\mathscr{C}_x(S) \subset \mathbb{P}T_x(S)$ *is moreover linearly nondegenerate at a general point, then there exists an irreducible subvariety* $Y \subset Z$ *such that* $Y \supset S$ *and* $\dim(Y) = \dim(S)$.

Proof. By Proposition 9.1, the pair $(\mathscr{C}_0(X_0) \cap \mathcal{J}(x), \mathscr{C}_0(X) \cap \mathcal{J}(x))$ is p-nondegenerate for substructures with $p = \ell(X_0, X) - k \geq 1$. By Theorem 1.4, S is rationally saturated with respect to $(Z, \mathcal{H})$. When $\mathscr{C}_x(S) \subset \mathbb{P}T_x(Z)$ is linearly nondegenerate at a general point, the bracket generating property of the distribution $\mathcal{D}$ spanned by $\mathscr{C}(S)$ is automatic, and Main Theorem 2 applies to give an irreducible subvariety $Y \supset S$ such that $\dim(Y) = \dim(S)$. □

Remark In some cases, under dimension restrictions it is possible to adapt the method of parallel transport of VMRTs of Hong-Mok [HoM11] to "recognize" Y as $\gamma(X_0) \cap Z$ for some $\gamma \in \mathrm{Aut}(X)$. For a formulation of the "Recognition Problem" for uniruled projective subvarieties, cf. Mok [Mk16].

Theorem 1.4 leads to a characterization of maximal linear subspaces for a general smooth linear section Z of X of sufficiently small codimension, depending on the index of nondegeneracy for substructures, as follows. By a maximal linear subspace Π on a projective subvariety $A \subset \mathbb{P}^m$ we mean a maximal element of the partially ordered set of linear subspaces of $\mathbb{P}^m$ lying on A. We use this notion for linear subspaces containing a smooth point of A, for $Z \subset \mathbb{P}^N$, $\mathscr{C}_x(Z) \subset \mathbb{P}T_x(Z)$ and $\mathscr{C}_x(S) \subset \mathbb{P}T_x(S)$. On a rational homogeneous space X of Picard number 1 the maximal linear subspaces break up into a finite number of equivalence classes modulo the action of $\mathrm{Aut}(X)$, cf. Landsberg-Manivel [LM03]. We have

Theorem 9.2. *Let* $X \subset \mathbb{P}(\Gamma(X, \mathcal{O}(1))^*)$ *be a rational homogeneous space of Picard number 1. Let* $X_i \subset X$, $1 \leq i \leq h$, *be an enumeration of representatives of equivalence classes of maximal linear subspaces modulo the action of* $\mathrm{Aut}(X)$, *exactly one from each equivalence class, and let* ℓ_i *be the index of nondegeneracy of* (X_i, X). *Suppose* $1 \leq k < \ell := \min\{\ell_i, 1 \leq i \leq h\}$. *Let* $(Z, \mathcal{H})$ *be a smooth linear section of codimension* k *endowed with a uniruling by projective lines, and* $E \subset Z$ *be its bad locus. Let* $W \subset Z - E$ *be a connected open subset and let* $S \subset W$ *be a complex submanifold such that* $\mathbb{P}T(S) \subset \mathscr{C}(Z)$ *and* $\mathbb{P}T(S) \cap \mathrm{Reg}(\mathscr{C}(Z)|_S) \neq \emptyset$. *Suppose* $\mathbb{P}T_x(S) \subset \mathscr{C}_x(Z)$ *is a maximal linear subspace for a general point* $x \in S$. *Then,* $S \subset Z$ *is an open subset of a maximal linear subspace.*

As an illustration we have the following special case for Grassmannians.

Corollary 9.1. *When $X = G(p,q)$, $0 \le 3 \le p \le q$, Theorem 9.2 holds for a general smooth linear section $Z \subset G(p,q)$ of codimension $k \le p-2$.*

Proof of Theorem 9.2. Let $x \in S$ be a general point, and $\Pi \subset X$ be a maximal linear subspace passing through x such that $T_x(S) \subset T_x(\Pi)$. There exists $\gamma \in \mathrm{Aut}(X)$ and a maximal linear subspace $X_i \subset X$, $1 \le i \le h$, such that $\Pi = \gamma(X_i)$. Let r be the codimension of $\mathbb{P}T_x(S)$ in $\mathbb{P}T_x(\Pi)$. We have $0 \le r \le k$. By Proposition 9.1, $(\mathbb{P}T_x(S), \mathscr{C}_x(Z))$ is $(\ell_i - r)$-nondegenerate for substructures, where $\ell_i - r \ge \ell_i - k \ge 1$ by hypothesis, hence nondegenerate for substructures. Since $\mathscr{C}(S) := \mathscr{C}(Z) \cap \mathbb{P}T(S) = \mathbb{P}T(S)$, $\mathscr{C}(S)$ trivially satisfies Condition (T). By Theorem 1.4, the tautological foliation on $\mathscr{C}(Z)$ restricts to $\mathbb{P}T(S)$, hence $S \subset Z$ must be a maximal linear subspace, as desired. $\square$

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