CLASSIFICATION PROBLEM OF HOLOMORPHIC ISOMETRIES OF THE UNIT DISK INTO POLYDISKS

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Abstract. We study the classification problem of holomorphic isometric embeddings of the unit disk into polydisks as in [Ng10] and [Ch16]. We can give complete classification when the target is the 4-disks and also some holomorphic isometric embeddings with certain prescribed sheeting numbers (cf. [Ng10]).

1. Introduction

Mok ([Mok11], p. 262-263) has raised a question about the structure of the space $\mathbf{HI}_k(\Delta,\Delta^p)$ of holomorphic isometric embeddings $(\Delta, kds^2_\Delta)\to(\Delta^p, ds^2_{\Delta^p})$. Ng [Ng10] has provided a complete description of $\mathbf{HI}_k(\Delta,\Delta^p)$ for $p=2,3$. Recently, the author [Ch16] has proven that any $f\in\mathbf{HI}_1(\Delta,\Delta^p;p)$ is the p-th root embedding up to reparametrizations, where $p\geq 2$ is an integer. In particular, the 4-th root embedding in $\mathbf{HI}_1(\Delta,\Delta^4;4)$ is globally rigid in the sense of [Mok11], p. 261. The main purpose of this article is to provide a complete description of $\mathbf{HI}_k(\Delta,\Delta^4)$ so that the classification problem of holomorphic isometric embeddings $(\Delta, kds^2_\Delta)\to(\Delta^4, ds^2_{\Delta^4})$ with the isometric constant k shall be solved as follows:

Theorem 1.1. *Let $f\in\mathbf{HI}_k(\Delta,\Delta^4)$ be a holomorphic isometric embedding such that all component functions of f are non-constant.*

(1) *If the isometric constant $k=1$, then f is one of the following up to reparametrizations:*
   (a) *the 4-th root embedding $F_4:\Delta\to\Delta^4$,*
   (b) $(\alpha_1,\, \alpha_2\circ\beta_1,\, \alpha_3\circ(\beta_2\circ\beta_1),\, \beta_3\circ(\beta_2\circ\beta_1))$*, where* $(\alpha_j,\beta_j)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$ *for* $j=1,2,3$,
   (c) $(\alpha_1, h^2\circ\alpha_2, h^3\circ\alpha_2, h^4\circ\alpha_2)$*, where* $(\alpha_1,\alpha_2)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$ *and* $(h^2,h^3,h^4)\in\mathbf{HI}_1(\Delta,\Delta^3;3)$,
   (d) $(\beta_1,\alpha_1\circ\beta_2,\alpha_2\circ\beta_2,\beta_3)$*, where* $(\beta_1,\beta_2,\beta_3)\in\mathbf{HI}_1(\Delta,\Delta^3;3)$ *and* $(\alpha_1,\alpha_2)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$,
   (e) $(\alpha_1\circ\alpha_2,\beta_1\circ\alpha_2,\alpha_3\circ\beta_2,\beta_3\circ\beta_2)$*, where* $(\alpha_j,\beta_j)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$ *for* $j=1,2,3$.

(2) *If the isometric constant $k=2$, then $f(z)$ is one of the following up to reparametrizations:*
   (a) $(\alpha_1(z),\beta_1(z),\alpha_2(z),\beta_2(z))$*, where* $(\alpha_j,\beta_j)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$ *for* $j=1,2$.
   (b) $(z,\alpha_1(z),(\alpha_2\circ\beta_1)(z),(\beta_2\circ\beta_1)(z))$*, where* $(\alpha_j,\beta_j)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$ *for* $j=1,2$.
   (c) $(z,\alpha_1(z),\alpha_2(z),\alpha_3(z))$*, where* $(\alpha_1,\alpha_2,\alpha_3)\in\mathbf{HI}_1(\Delta,\Delta^3;3)$.

(3) *If the isometric constant $k=3$, then*
$$f(z)=(z,z,\alpha(z),\beta(z))$$
up to reparametrizations, where $(\alpha,\beta)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$.

(4) *If the isometric constant $k=4$, then f is the diagonal embedding $f(z)=(z,z,z,z)$ up to reparametrizations.*

Remark. Actually, this theorem says that all holomorphic isometric embeddings $f:(\Delta,kds^2_\Delta)\to(\Delta^4,ds^2_{\Delta^4})$ with the isometric constant k are parametrized by diagonal embeddings, automorphisms of Δ (resp. Δ^4) and p-th root embeddings up to reparametrizations, for $2\leq p\leq 4$. This answers the question for the case $\mathbf{HI}_k(\Delta,\Delta^4)$ in problem 5.1.2. in [Mok11], p. 262-263.

Moreover, we shall provide some generalizations to the study of Ng [Ng10] and the author [Ch16] in certain cases and provided complete description of some holomorphic isometric embeddings with certain prescribed sheeting numbers (cf. [Ng10]).

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1.1. **Preliminary.** Let $\Delta \subset \mathbb{C}$ be the open unit disk with the Poincaré metric $ds^2_\Delta = 2\,\mathrm{Re}(g dz \otimes d\overline{z})$, where $g = -2\frac{\partial^2}{\partial z \partial \overline{z}} \log(1-|z|^2)$. For integer $p \geq 2$, let $\Delta^p = \{(z_1,\ldots,z_p) \in \mathbb{C}^p \mid |z_j| < 1,\ 1 \leq j \leq p\}$ be the polydisk, which is viewed as p copies of Δ. Moreover, Δ^p is equipped with the Kähler metric $ds^2_{\Delta^p}$, which is the product metric induced from the Poincaré metric ds^2_Δ. More precisely, we take the real analytic function $-2\sum_{j=1}^p \log(1-|z_j|^2)$ as Kähler potential for $ds^2_{\Delta^p}$ (cf. [Ng10], p. 2908). Let $\mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\}$ be the Riemann sphere.

Let $f : (\Delta, k ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$ be a holomorphic isometric embedding with the isometric constant k and the sheeting number n. In this article, all holomorphic isometric embeddings

$$f = (f^1,\ldots,f^p) : (\Delta, k ds^2_\Delta) \to (\Delta^p, ds^2_{\Delta^p})$$

will be assumed to be *genuine*, i.e. all component functions of f are non-constant, as mentioned in [Ng08], p. 7. From [Ng08], we have $1 \leq k \leq p$. We can always assume that $f(0) = \mathbf{0}$ after compositing some $\Psi \in \mathrm{Aut}(\Delta^p)$. In [Ng10], we have the following functional equation

$$\prod_{\mu=1}^{p} \left(1 - |f^\mu(z)|^2\right) = (1-|z|^2)^k$$

and also the polarized functional equation

$$\prod_{\mu=1}^{p} \left(1 - f^\mu(z)\overline{f^\mu(w)}\right) = (1 - z\overline{w})^k.$$

Let $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$ be the irreducible projective-algebraic curve such that $\mathrm{Graph}(f) \subset V$ as obtained in [Ng10]. From [Ng10], $V_j := P_j(V)$ is a projective-algebraic curve containing the graph of f^j, where $P_j : V \to \mathbb{P}^1 \times \mathbb{P}^1$ is defined by $P_j(z, w_1,\ldots,w_p) = (z, w_j)$, $1 \leq j \leq p$. Let $\pi : V \to \mathbb{P}^1$ be the finite branched covering $\pi(z, w_1,\ldots,w_p) = z$ and $\pi_j : V_j \to \mathbb{P}^1$ is defined by $\pi_j(z, w_j) = z$, $1 \leq j \leq p$. We refer to [Ng10], p. 2910-2913, for details.

For bounded symmetric domains $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$, Mok [Mok11] has introduced the space $\mathbf{HI}(D,\Omega)$ of holomorphic isometries $(D, \lambda ds^2_D) \to (\Omega, ds^2_\Omega)$ for some real constant $\lambda > 0$, where ds^2_D, ds^2_Ω are Bergman metrics of D, Ω respectively. In particular, in case $D = \Delta$ and $\Omega = \Delta^p$, we also have spaces $\mathbf{HI}_k(\Delta,\Delta^p)$, $\mathbf{HI}_k(\Delta,\Delta^p; n)$ and $\mathbf{HI}_k(\Delta,\Delta^p; n; s_1,\ldots,s_p)$ so as to specify the isometric constant k, the sheeting numbers s_j of each component functions of isometries and the global sheeting number n (cf. [Mok11, p. 263]).

If $\pi' : V' \to Y$ is a finite branched covering, where V' is a smooth irreducible algebraic curve and Y is a compact Riemann surface, then for each point $y \in Y$, denote by $v(\pi', x)$ the ramification index of π' at x and by $b(\pi', y)$ the branching order of π' at y in the sense of [GH78] (p.217), where $x \in \pi'^{-1}(y)$. From [Ng08], [Ng10] and [Ch16], for $f \in \mathbf{HI}_1(\Delta,\Delta^p; n; s_1,\ldots,s_p)$, we denote all branches of f^j over Δ by f^j_l while all branches of f^j over $\mathcal{O} := \mathbb{P}^1 \smallsetminus \overline{\Delta}$ by $f^j_{l,-}$, $1 \leq l \leq s_j$, and $f^j_1 = f^j$, $1 \leq j \leq p$.

Mok [Mok12] has defined the map $\rho_p : \mathcal{H} \to \mathcal{H}^p$ $(p \geq 2)$ by

$$\rho_p(\tau) = \left(\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \ldots, \gamma^{p-1}\tau^{\frac{1}{p}}\right),$$

where $\gamma = e^{\frac{i\pi}{p}}$ and $\tau^{\frac{1}{p}} = r^{\frac{1}{p}} e^{\frac{i\theta}{p}}$ if $\tau = re^{i\theta}$, $0 < \theta < \pi$. From [Mok12], the map ρ_p is a non-totally geodesic holomorphic isometric embedding. Then, the p-th root embedding $F_p : \Delta \to \Delta^p$ can be defined from ρ_p via the Cayley transform $\iota : \mathcal{H} \to \Delta$, $\tau \mapsto \frac{\tau - i}{\tau + i}$ and target automorphisms.

2. General properties of holomorphic isometries in $\mathbf{HI}_1(\Delta,\Delta^p)$

2.1. **Special branching behaviour of certain holomorphic isometries in $\mathbf{HI}_k(\Delta,\Delta^p)$.** For holomorphic isometric embeddings $f \in \mathbf{HI}_k(\Delta,\Delta^p)$ satisfying certain branching behaviour, we shall prove that the classification problem of this kind of isometries can be reduced to that of holomorphic isometric embeddings in $\mathbf{HI}_k(\Delta,\Delta^{p-1})$.

Lemma 2.1. *Let $g : \Delta \to \Delta$ be a component function of a holomorphic isometric embedding $f = (f^1,\ldots,f^p) \in \mathbf{HI}_k(\Delta,\Delta^p)$ satisfying $f(0) = \mathbf{0}$. Suppose that there is $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$ such that*

$\varphi\circ g$ is also a component function of f, where $\varphi(z)=\frac{az+b}{cz+d}$ with $\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\begin{pmatrix}u_3 & 0\\ -\det U & u_1\end{pmatrix}$ for some unitary matrix $U=\begin{pmatrix}u_1 & u_2\\ u_3 & u_4\end{pmatrix}$ satisfying $u_1,u_3\in\mathbb{C}\smallsetminus\{0\}$. Then, we have

$$(1-|g(z)|^2)(1-|\varphi(g(z))|^2)=1-|h(z)|^2,$$

where $h:\Delta\to\mathbb{C}$ is a holomorphic function defined by

$$h(z):=\frac{g(z)-u_4(g(z))^2}{u_1-(\det U)g(z)}.$$

Proof. Without loss of generality, we can assume that $g=f^1$ and $\varphi\circ g=f^2$. Then $R_1(f^1(z))=z=R_2(f^2(z))=R_2(\varphi(f^1(z)))$ so that R_1 and $R_2\circ\varphi$ are meromorphic functions on $\mathbb{P}^1$ such that $R_1|_{U'}=(R_2\circ\varphi)|_{U'}$, where U' is the image of f^1 in $\mathbb{P}^1$, which is an open subset by the open mapping theorem for holomorphic functions. In particular, $R_1=R_2\circ\varphi$ by the identity theorem. We compute

$$u_1h(z)+u_2f^1(z)(\varphi\circ f_1)(z)=\frac{u_1f^1(z)-u_1u_4(f^1(z))^2}{u_1-(\det U)f^1(z)}+u_2\frac{u_3(f^1(z))^2}{u_1-(\det U)f^1(z)}=f^1(z)$$

and

$$\begin{aligned}u_3h(z)+u_4f^1(z)(\varphi\circ f_1)(z)&=\frac{u_3f^1(z)-u_3u_4(f^1(z))^2}{u_1-(\det U)f^1(z)}+u_4\frac{u_3(f^1(z))^2}{u_1-(\det U)f^1(z)}\\&=\frac{u_3f^1(z)}{u_1-(\det U)f^1(z)}=\varphi(f^1(z)).\end{aligned}$$

Thus, we have

$$\begin{pmatrix}f^1(z)\\ \varphi(f^1(z))\end{pmatrix}=U\cdot\begin{pmatrix}h(z)\\ f^1(z)\varphi(f^1(z))\end{pmatrix}.$$

Actually, we also need to show that $f^1(z)\neq\frac{u_1}{\det U}$ for $z\in\overline{\Delta}$ so as to ensure that h is holomorphic. Suppose that $f^1(z_0)=\frac{u_1}{\det U}$ for some $z_0\in\overline{\Delta}$, then $\varphi(f^1(z_0))=\infty$. This would imply that $\infty=R_2(\infty)=R_2(\varphi(f^1(z_0)))=R_1(f^1(z_0))=z_0$ by [Ng10] and $R_2\circ\varphi=R_1$, which is a contradiction. Thus, $f^1(z)\neq\frac{u_1}{\det U}$ for $z\in\overline{\Delta}$ so that the function h is holomorphic on Δ and continuous on $\overline{\Delta}$, i.e. the extension $\widetilde{h}:\overline{\Delta}\to\overline{\Delta}$ of h is continuous. Now, we have

$$|f^1(z)|^2+|\varphi(f^1(z))|^2=|h(z)|^2+|f^1(z)\varphi(f^1(z))|^2$$

for $z\in\Delta$ because U is an unitary matrix and thus U preserves Euclidean norm of the holomorphic mappings. The result follows. □

Theorem 2.2. *Let $f=(f^1,\ldots,f^p)\in\mathbf{HI}_k(\Delta,\Delta^p;n;s_1,\ldots,s_p)$ with $f(0)=\mathbf{0}$, where $p\geq 4$ is an integer. Suppose that there is a point $z_0\in\partial\Delta$ such that $v(R_{\sigma(j)},f^{\sigma(j)}(z_0))\geq 2$ ($j=p-1,p$) and $v(R_{\sigma(\mu)},f^{\sigma(\mu)}(z_0))=1$ ($\mu=1,\ldots,p-2$) for some $\sigma\in S_p$, then $s_{\sigma(p-1)}=s_{\sigma(p)}$ are even integers and $\exists\,\psi\in\mathrm{Aut}(\mathbb{P}^1)$ with $\psi(0)=0$ such that $\psi\circ f_1^{\sigma(p-1)}=f_1^{\sigma(p)}$ so that $R_{\sigma(p)}\circ\psi=R_{\sigma(p-1)}$ and ψ is of the form $\psi(z)=\frac{u_3z}{-(\det U)z+u_1}$ for some unitary matrix $U=\begin{pmatrix}u_1 & u_2\\ u_3 & u_4\end{pmatrix}$ satisfying $u_1,u_3\in\mathbb{C}\smallsetminus\{0\}$. In particular, we have*

$$(1-|f^{\sigma(p-1)}(z)|^2)(1-|f^{\sigma(p)}(z)|^2)=1-|h(z)|^2$$

for some holomorphic function h on Δ and thus

$$(f^{\sigma(1)},\ldots,f^{\sigma(p-2)},h):(\Delta,kds_\Delta^2)\to(\Delta^{p-1},ds^2_{\Delta^{p-1}})$$

is a holomorphic isometric embedding.

Remark. The assumption in the theorem can be replaced by the existence of certain branch of f which is of the form $(f_1^1,\ldots,f_1^{p-2},f_{l_{p-1}}^{p-1},f_{l_p}^p)$ up to permutation of component functions, where $l_j\neq 1$ for $j=p-1,p$. This can be also considered as the existence of a continuous path $\gamma:[0,1]\to\mathbb{P}^1\smallsetminus B_\pi$ such that $\gamma(0)=\gamma(1)=0$ and perform analytic continuation of $f=(f_1^1,\ldots,f_1^p)$ along γ would come up with a branch of f which is of the form $(g_1,\ldots,g_p)$, where $g_{\sigma(j)}:=f_1^{\sigma(j)}$ for $1\leq j\leq p-2$ and $g_{\sigma(\mu)}:=f^{\sigma(\mu)}_{l_{\sigma(\mu)}}$ with $l_{\sigma(\mu)}\neq 1$ for $\mu=p-1,p$, for some $\sigma\in S_p$.

Proof. Without loss of generality, we can assume that $\sigma = \mathrm{Id}$. Starting with the branch $f = (f_1^1, \ldots, f_1^p)$ at 0, we perform (multivalued) analytic continuation along some simple closed loop around z_0 once to obtain $(f_1^1, \ldots, f_1^{p-2}, f_2^{p-1}, f_2^p)$. Note that we label branches of each f^j so that we can obtain f_2^j by performing analytic continuation of f_1^j along some simple closed loop around z_0 once for $j = p-1, p$. By the polarized functional equation, we have

$$\left(1 - f_1^{p-1}(z)\overline{f_2^{p-1}(0)}\right)\left(1 - f_1^p(z)\overline{f_2^p(0)}\right) = 1$$

for $z \in \Delta$ so that $f_1^p(z) = \psi(f_1^{p-1}(z))$, where $\psi(w) = \frac{1}{\overline{f_2^p(0)}} \frac{w}{w - \frac{1}{\overline{f_2^{p-1}(0)}}}$. Note that $f_2^j(0) \in \mathbb{C}^*$ for $j = p-1, p$, thus $\psi \in \mathrm{Aut}(\mathbb{P}^1)$ because $\det \begin{pmatrix} \frac{1}{\overline{f_2^p(0)}} & 0 \\ 1 & -\frac{1}{\overline{f_2^{p-1}(0)}} \end{pmatrix} = -\frac{1}{\overline{f_2^p(0)}\,\overline{f_2^{p-1}(0)}} \neq 0$. In particular, $s_{p-1} = s_p$ and $R_p \circ \psi = R_{p-1}$. From the polarized functional equation, we also have

$$\left(1 - f_2^{p-1}(z)\overline{f_2^{p-1}(0)}\right)\left(1 - f_2^p(z)\overline{f_2^p(0)}\right) = 1$$

so that $\psi(f_2^{p-1}(z)) = f_2^p(z)$ for $z \in \Delta$. Now, we have $f_2^p(0) = \psi(f_2^{p-1}(0)) = \frac{|f_2^{p-1}(0)|^2}{\overline{f_2^p(0)} \cdot (|f_2^{p-1}(0)|^2 - 1)}$ so that

$$\frac{1}{|f_2^p(0)|^2} + \frac{1}{|f_2^{p-1}(0)|^2} = 1.$$

Then we also have $|f_2^j(0)|^2 > 1$ for $j = p-1, p$. Now, one can verify that $\psi(z) = \frac{u_3 z}{-(\det U)z + u_1}$, where

$$U = \begin{pmatrix} u_1 & u_2 \\ u_3 & u_4 \end{pmatrix} = \begin{pmatrix} -\lambda\overline{f_2^p(0)} & \frac{\frac{1}{f_2^p(0)}}{\overline{f_2^{p-1}(0)}\left(1 - \frac{1}{|f_2^p(0)|^2}\right)} \\ \lambda \overline{f_2^{p-1}(0)} & \end{pmatrix}$$

with $\lambda = \sqrt{\left(1 - \frac{1}{|f_2^p(0)|^2}\right)\frac{1}{|f_2^p(0)|^2}}\, e^{i\theta_0}$ for some $\theta_0 \in [0, 2\pi)$. By Lemma 2.1, the holomorphic function h on Δ defined by

$$h(z) := \frac{f^{p-1}(z) - u_4 (f^{p-1}(z))^2}{u_1 - (\det U) f^{p-1}(z)}$$

satisfies

$$(1 - |f^{p-1}(z)|^2)(1 - |f^p(z)|^2) = 1 - |h(z)|^2$$

Then $(f^1, \ldots, f^{p-2}, h) : \Delta \to \Delta^{p-1}$ is clearly a holomorphic isometric embedding. Thus, there is a rational function R_h such that $R_h(h(z)) = z$, and we have $2 \cdot \deg R_h = \deg R_{p-1} = s_{p-1} = s_p$ so that $s_p = s_{p-1}$ is an even integer. □

2.2. **Special sheeting numbers of holomorphic isometries.** In the study of the structure of $\mathbf{HI}_1(\Delta, \Delta^p; n; s_1, \ldots, s_p)$ in [Ng10], if $s_j = 2$ for some j, then the study of holomorphic isometries $f = (f^1, \ldots, f^p) : \Delta \to \Delta^p$ can be reduced to the study of holomorphic isometries $\Delta \to \Delta^{p-1}$. For example, in the proof of Theorem 6.8 in [Ng10], Ng has reduced the study of certain $f \in \mathbf{HI}(\Delta, \Delta^p)$ to the understanding of the space $\mathbf{HI}(\Delta, \Delta^{p-1})$ and so on. For the study of the space $\mathbf{HI}_1(\Delta, \Delta^p; n; s_1, \ldots, s_p)$, one may ask whether $s_j = q$ for some prime number $q \geq 3$ and some j could lead to a similar phenomenon as in the case of $s_j = 2$ for some j. We do not have any general method to handle such problem. However, for some small prime number $q \geq 3$, it may be possible for us to use the method in [Ch16] to deal with the problem. In this section, we shall show that when $q = 3$, then we could show that a similar phenomenon occurs as in the case of $s_j = 2$ for some j.

Lemma 2.3. *Suppose that h is a component function of a holomorphic isometric embedding $f : (\Delta, k ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$ such that* $\deg h = 3$, *then for any branch point $a \in \partial\Delta$ of R_h, we have $|w| = 1 \ \ \forall\ w \in R_h^{-1}(a)$, where $R_h : \mathbb{P}^1 \to \mathbb{P}^1$ is the rational function of degree 3 such that $R_h(h(z)) = z$, $R_h\left(\frac{1}{\overline{w}}\right) = \frac{1}{\overline{R_h(w)}}$ and $R_h(\partial\Delta) \subset \partial\Delta$.*

Proof. Without loss of generality, we can suppose that $f(0) = \mathbf{0}$. Let m be the number of distinct branch points of R_h, $\{a_1, \ldots, a_m\}$ be the set of all distinct branch points of R_h and the branching order of a_j is denoted by b_j for $1 \leq j \leq m$. Since $\deg h = 3$, we have $\sum_{i=1}^m b_i = 4$ so that $2 \leq m \leq 4$.

After reordering branch points of h if necessary, we can assume that $b_1 \le \cdots \le b_m$ without loss of generality. Then, we have the following possibilities:

(1) $m = 2$ and $(b_1, b_2) = (2, 2)$;
(2) $m = 3$ and $(b_1, b_2, b_3) = (1, 1, 2)$;
(3) $m = 4$ and $(b_1, b_2, b_3, b_4) = (1, 1, 1, 1)$.

If $b_i = 1$ for some i, then $|R_h^{-1}(a_i)| = 2$ and thus $R_h^{-1}(a_i) = \{w_1, w_2\}$ such that ramification index of R_h at w_1 (resp. w_2) equals 1 (resp. 2). for some distinct $w_1, w_2 \in \mathbb{P}^1$. Either $|w_1| = |w_2| = 1$ or $w_1 = \frac{1}{\overline{w_2}}$. If $w_1 = \frac{1}{\overline{w_2}}$, then ramification order of R_h at w_1 would be the same as that of R_h at w_2, which contradicts to the assumption when $b_i = 1$. Thus, we must have $|w_1| = |w_2| = 1$.
If $b_i = 2$, then clearly $|R_h^{-1}(a_i)| = 1$ and $w \in R_h^{-1}(a_i)$ would satisfies $|w| = 1$ because $(a_i, w) \in V_h \iff \left(a_i, \frac{1}{\overline{w}}\right) \in V_h$. Thus, we have verified that if h is a component function of a holomorphic isometric embedding $\Delta \to \Delta^p$ with $\deg h = 3$, then we have $|w| = 1 \ \forall\, w \in R_h^{-1}(a_i)$ for $i = 1, \ldots, m$. On the other hand, we have shown that for an arbitrary branch h_l of h, we have $|h_l(a_i)| = 1$ for $i = 1, \ldots, m$. □

Note that Lemma 6.7 in [Ng10, p. 2917] shows that if the sheeting number of some component function g of a holomorphic isometry $\Delta \to \Delta^p$ is equal to 2, then there exists a holomorphic function $h : \Delta \to \Delta$ such that $(g, h) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$. The following proposition provides a similar result in case the sheeting number is equal to 3.

Proposition 2.4. *Let $p \ge 3$ be an integer. If $h^1, h^2 : \Delta \to \Delta$ are two distinct component functions of a holomorphic isometric embedding $f = (f^1, \ldots, f^p) : (\Delta, ds_\Delta^2) \to (\Delta^p, ds_{\Delta^p}^2)$ such that $\deg h^1 = \deg h^2 = 3$, then there is a holomorphic function $h^3 : \Delta \to \Delta$ such that $(h^1, h^2, h^3) : \Delta \to \Delta^3$ is the cube root embedding up to reparametrizations, i.e. $(h^1, h^2, h^3) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$.*

Proof. Without loss of generality, suppose that $f^1 = h^1$, $f^2 = h^2$ and $f(0) = \mathbf{0}$. Let $\{a_1, \ldots, a_m\} \subset \partial\Delta$ be the set of all distinct branch points of f^1. Suppose that $m \ge 3$, then there is a branch point $a = a_i \in \partial\Delta$ such that $b_i = 1$. Therefore, there is a branch f_l^1 of f^1 such that the ramification index of π_1 at $(a, f_l^1(a))$ is equal to 1 and $|f_l^1(a)| = 1$. Then we have a branch $(f_l^1, f_{l_2}^2, f_{l_3}^3, \ldots, f_{l_p}^p)$ of f for some l_j. Consider the functional equation

$$\left(1 - f_l^1(z)\overline{f_l^1(a)}\right) \cdot \prod_{j=2}^{p} \left(1 - f_{l_j}^j(z)\overline{f_{l_j}^j(a)}\right) = 1 - z\overline{a}.$$

By comparing vanishing order of both sides of the above equation at a, we see that $|f_{l_j}^j(a)| \ne 1$ for $2 \le j \le p$. Thus, a is not a branch point of π_2; otherwise we would have $|f_{l_j}^2(a)| = 1$ by the previous lemma because $\deg f^2 = 3$.
Since $\pi_2 : V_2 \to \mathbb{P}^1$ is not branched over $a \in \partial\Delta$, we have $|(\pi_2)^{-1}(a)| = 3$ and the set $(R_2)^{-1}(a)$ contains at least one unimodular value because $(z, w) \in V_2 \iff \left(\frac{1}{\overline{z}}, \frac{1}{\overline{w}}\right) \in V_2$. Then, we can choose l' such that $|f_{l'}^2(a)| = 1$ and we have a branch $(f_{l_1'}^1, f_{l'}^2, f_{l_3'}^3, \ldots, f_{l_p'}^p)$ of f for some l_j'. Consider the functional equation

$$\left(1 - f_{l'}^2(z)\overline{f_{l'}^2(a)}\right) \prod_{1\le j\le p,\ j\ne 2} \left(1 - f_{l_j'}^j(z)\overline{f_{l_j'}^j(a)}\right) = 1 - z\overline{a}.$$

Since $a \in \partial\Delta$ is a branch point of π_1 and $\deg f^1 = 3$, we have $|f_{l_1'}^1(a)| = 1$ by the previous lemma. Now, we have $|f_{l_1'}^1(a)| = |f_{l'}^2(a)| = 1$. Note that we have the Puiseux series

$$f_{l_1'}^1(z) = \varphi_{l_1'}^1\left((z - a)^{\frac{1}{v}}\right)$$

for $z \in B^1(a, \varepsilon)$, where $\varepsilon > 0$ such that $B^1(a, \varepsilon) \smallsetminus \{a\}$ does not contain any branch point of any component function of f and $\varphi_{l_1'}^1$ is some holomorphic function on $B^1\left(0, \varepsilon^{\frac{1}{v}}\right)$. Here $v = 1$ or $v = 2$. Then we have

$$\left(1 - \varphi_{l_1'}^1(\xi)\,\overline{\varphi_{l_1'}^1(0)}\right)\left(1 - f_{l'}^2(\xi^v + a)\overline{f_{l'}^2(a)}\right)\psi(\xi) = -\overline{a}\xi^v, \tag{2.1}$$

where $\psi(\xi) := \prod_{j=3}^{p}\left(1 - f_{l_j'}^j(\xi^v + a)\overline{f_{l_j'}^j(a)}\right)$. Note that $1 - \varphi_{l_1'}^1(\xi)\,\overline{\varphi_{l_1'}^1(0)}$ has a zero of order 1 at $\xi = 0$ and that $1 - f_{l'}^2(\xi^v + a)\overline{f_{l'}^2(a)}$ has a zero of order v at $\xi = 0$ since a is not a branch point of

π_2. Thus, the left hand side of (2.1) has a zero of order at least $v+1$ at $\xi=0$. However, the right hand side of (2.1) has a zero of order v at $\xi=0$, which is a contradiction. Thus, $b_i\neq 1$ for all i, $1\leq i\leq m$. Hence we must have $m=2$, i.e. f^1 has precisely two distinct branch points. Similarly, f^2 can only have two distinct branch points. Then, f^1 and f^2 are component functions of the cube root embedding up to reparametrizations by [Ng10].
We claim that f^1, f^2 has the same set of branch points, say $a_1, a_2\in\partial\Delta$. Assume the contrary that $a=a_j$ for some j such that a is a branch point of R_1 but not a branch point of R_2, then $|f^1_l(a)|=1$ for $l=1,2,3$ by lemma 2.3. But then $\exists\ l'\in\{1,2,3\}$ such that $|f^2_{l'}(a)|=1$ since $|(R_2)^{-1}(a)|=3$ and $(z,w)\in V_2 \iff \left(\frac{1}{\bar z},\frac{1}{\bar w}\right)\in V_2$ (cf. [Ng10]). Then we obtain a contradiction by considering polarized functional equation as before. Thus, if a is a branch point of f^1, then a is a branch point of f^2. Similarly, if a is a branch point of f^2, then a is a branch point of f^1. Thus, branching loci of R_1 and R_2 are the same.
From Lemma 4.9 and the proof of Theorem 6.5 in [Ng10], we see that there is a single reparmetrization such that f^1, f^2 would become one of the component functions of the cube root embedding. Then, $f^1\neq f^2$ since for each branch of $f=(f^1,\dots,f^p)$, there is only one infinite value as $z\to\infty$ (cf. [Ng10], p.2917). Thus f^1, f^2 are precisely two distinct component functions of the cube root embedding. Recall that $h^j=f^j$ for $j=1,2$. Thus, there is a holomorphic function $h^3:\Delta\to\Delta$ such that $h^3(0)=0$ and $(h^1,h^2,h^3):\Delta\to\Delta^3$ is the cube root embedding up to reparametrizations, i.e. $(h^1,h^2,h^3)\in\mathbf{HI}_1(\Delta,\Delta^3;3)$. □

Remark. This proposition can be used for classifying holomorphic isometric embeddings

$$f:(\Delta, ds^2_{\Delta})\to(\Delta^p, ds^2_{\Delta^p})$$

with some special sheeting numbers $s_1,\dots,s_p$. For example, the structure of the space

$$\mathbf{HI}_1(\Delta,\Delta^{2q+1};n;3,3,3^2,3^2,\dots,3^{q-1},3^{q-1},3^q,3^q,3^q)$$

can be completely described by induction, where $q\geq 2$ and n satisfying $3^q|n$, $2q+1<n\leq 2^{2q}$. Roughly speaking, the above space is constructed by composition of q holomorphic isometries in $\mathbf{HI}_1(\Delta,\Delta^3;3)$. Similarly, the structure of the space

$$\mathbf{HI}_1(\Delta,\Delta^{2q'+2};n';3,3,3^2,3^2,\dots,3^{q'},3^{q'},2\cdot 3^{q'},2\cdot 3^{q'})$$

can be completely described by induction, where $q'\geq 1$ and n' satisfying $(2\cdot 3^{q'})|n'$, $2q'+2<n'\leq 2^{2q'+1}$. Roughly speaking, the above space is constructed by composition of q' holomorphic isometries in $\mathbf{HI}_1(\Delta,\Delta^3;3)$ and a holomorphic isometry in $\mathbf{HI}_1(\Delta,\Delta^2)$.

3. Proof of the Theorem 1.1

From [Ng10], if $f\in\mathbf{HI}_k(\Delta,\Delta^4)$ is a holomorphic isometric embedding such that all component functions of f are non-constant, then we have $f\in\mathbf{HI}_k(\Delta,\Delta^4;n;s_1,s_2,s_3,s_4)$ for some positive integers n,s_1,s_2,s_3,s_4 satisfying $\frac{4}{k}\leq n\leq 8$, $\sum_{l=1}^4\frac{1}{s_l}=k$ and $s_j|n$ for $j=1,2,3,4$. Note that $1\leq k\leq 4$ from [Ng08]. It turns out that given some positive integers n,s_1,s_2,s_3,s_4 satisfying $\frac{4}{k}\leq n\leq 8$, $\sum_{l=1}^4\frac{1}{s_l}=k$ and $s_j|n$ for $j=1,2,3,4$, it is possible that the space $\mathbf{HI}_k(\Delta,\Delta^4;n;s_1,s_2,s_3,s_4)$ is empty due to the structure of the irreducible projective-algebraic curve V and the branching behaviour of each component functions of f.

3.1. Classification of $\mathbf{HI}_1(\Delta,\Delta^4)$.

Lemma 3.1. *Let $p\geq 2$ be an integer and n be a prime number satisfying $p<n\leq 2^{p-1}$, then the space $\mathbf{HI}_1(\Delta,\Delta^p;n)$ is empty.*

Remark. Note that such prime n does not exist when $p=2,3$, thus the condition $p\geq 2$ could be replaced by $p\geq 4$.

Proof. Assume the contrary that the space $\mathbf{HI}_1(\Delta,\Delta^p;n)$ is non-empty, then there is a holomorphic isometric embedding $f=(f^1,\dots,f^p):(\Delta,ds^2_\Delta)\to(\Delta^p,ds^2_{\Delta^p})$ such that the sheeting number of f^j equals s_j, $s_j|n$ for $1\leq j\leq p$ and $\sum_{j=1}^p\frac{1}{s_j}=1$ (cf. [Ng10]). Then, we have $s_j=n$ for $1\leq j\leq p$ because $\sum_{j=1}^p\frac{1}{s_j}=1$ so that $s_j\neq 1$ for any j. This would imply that $1=\sum_{j=1}^p\frac{1}{s_j}=\frac{p}{n}$ so that $n=p$, contradicts to $n>p$. Hence, we have $\mathbf{HI}_1(\Delta,\Delta^p;n)=\varnothing$. □

By the Lemma 3.1, we have $\mathbf{HI}_1(\Delta, \Delta^4; n) = \varnothing$ for $n = 5, 7$. Thus, we only need to consider the cases $n = 4$, 6 or 8. The following are all possibilities of global sheeting number n and sheeting numbers $s_1, \ldots, s_4$:

(1) $(n, s_1, s_2, s_3, s_4) = (4, 4, 4, 4, 4)$.
(2) $(n, s_1, s_2, s_3, s_4) = (6, 3, 6, 6, 3)$ or $(n, s_1, s_2, s_3, s_4) = (6, 2, 6, 6, 6)$.
(3) $(n, s_1, s_2, s_3, s_4) = (8, 4, 4, 4, 4)$ or $(n, s_1, s_2, s_3, s_4) = (8, 2, 4, 8, 8)$.

In case $(n, s_1, s_2, s_3, s_4) = (4, 4, 4, 4, 4)$, we can apply the global rigidity of the p-th root embedding for $p \geq 2$ (cf. [Ch16]). More precisely, any $f \in \mathbf{HI}_1(\Delta, \Delta^4; 4)$ is the 4-th root embedding up to reparametrizations.

Proposition 3.2. *Let* $f \in \mathbf{HI}_1(\Delta, \Delta^4; 8; 2, 4, 8, 8)$, *then*

$$f = (\alpha_1,\, \alpha_2 \circ \beta_1,\, \alpha_3 \circ (\beta_2 \circ \beta_1),\, \beta_3 \circ (\beta_2 \circ \beta_1))$$

up to reparametrizations, where $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *for* $j = 1, 2, 3$.

Proof. Actually, the result follows directly from Theorem 6.8 in [Ng10]. More precisely, $\forall\ f \in \mathbf{HI}_1(\Delta, \Delta^4; 8; 2, 4, 8, 8)$, we have

$$f(z) = (\alpha_1(z), g(\beta_1(z))),$$

where $g \in \mathbf{HI}_1(\Delta, \Delta^3; 4; 2, 4, 4)$ and $(\alpha_1, \beta_1) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$. Moreover, from [Ng10], we have

$$g(z) = (\alpha_2(z), \alpha_3(\beta_2(z)), \beta_3(\beta_2(z))),$$

for some $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ for $j = 2, 3$. Hence, we have

$$f(z) = (\alpha_1(z), \alpha_2(\beta_1(z)), \alpha_3(\beta_2(\beta_1(z))), \beta_3(\beta_2(\beta_1(z))))$$

for some $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$, $j = 1, 2, 3$. □

Proposition 3.3. *Let* $f \in \mathbf{HI}_1(\Delta, \Delta^4; 6; 2, 6, 6, 6)$, *then*

$$f = (\alpha_1, h^2 \circ \alpha_2, h^3 \circ \alpha_2, h^4 \circ \alpha_2)$$

up to reparametrizations, where $(\alpha_1, \alpha_2) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *and* $(h^2, h^3, h^4) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$.

Proof. From [Ng10], we have $f^1 = \alpha_1$ for some holomorphic isometric embedding $(\alpha_1, \alpha_2) : \Delta \to \Delta^2$ with isometric constant 1. Then since $(1 - |\alpha_1(z)|^2)(1 - |\alpha_2(z)|^2) = 1 - |z|^2$, we have

$$(1 - |f^2(z)|^2)(1 - |f^3(z)|^2)(1 - |f^4(z)|^2) = 1 - |\alpha_2(z)|^2.$$

Since 0 is not a branch point, locally there is an inverse $\alpha_2^{-1} : U \subset \Delta \to \Delta$ of α_2. Then

$$(1 - |f^2(\alpha_2^{-1}(z))|^2)(1 - |f^3(\alpha_2^{-1}(z))|^2)(1 - |f^4(\alpha_2^{-1}(z))|^2) = 1 - |z|^2,$$

i.e. $(f^2 \circ \alpha_2^{-1}, f^3 \circ \alpha_2^{-1}, f^4 \circ \alpha_2^{-1}) : U \to \Delta^3$ is a holomorphic isometric embedding with isometric constant 1. From [Mok12], we know that $(f^2 \circ \alpha_2^{-1}, f^3 \circ \alpha_2^{-1}, f^4 \circ \alpha_2^{-1})$ can be extended to the whole Δ, and we let $(h^2, h^3, h^4) : \Delta \to \Delta^3$ be the extension. Then $f^j \circ \alpha_2^{-1} = h^j$ for $j = 2, 3, 4$ and thus $f^j = h^j \circ \alpha_2$ on some open subset. Now, we have local inverse $(f^j)^{-1} = \alpha_2^{-1} \circ (h^j)^{-1}$. Since the degree of $(f^j)^{-1}$ equals 6 while the degree of α_2^{-1} equals 2, so the degree of $(h^j)^{-1}$ should be equal to 3. Thus $(h^2, h^3, h^4) : \Delta \to \Delta^3$ is the cube-root embedding up to reparametrizations by Theorem 8.1 in [Ng10]. Hence f is of the form

$$f = (f^1, f^2, f^3, f^4) = (\alpha_1, h^2 \circ \alpha_2, h^3 \circ \alpha_2, h^4 \circ \alpha_2)$$

up to reparametrizations. □

Proposition 3.4. *Let* $f \in \mathbf{HI}_1(\Delta, \Delta^4; 6; 3, 6, 6, 3)$, *then*

$$f = (\beta_1, \alpha_1 \circ \beta_2, \alpha_2 \circ \beta_2, \beta_3)$$

up to reparametrizations, where $(\beta_1, \beta_2, \beta_3) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$ *and* $(\alpha_1, \alpha_2) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$.

Proof. Without loss of generality, we can assume that $f = (f^1, f^2, f^3, f^4) \in \mathbf{HI}_1(\Delta, \Delta^4; 6; 3, 6, 6, 3)$ satisfying $f(0) = \mathbf{0}$. Then, there is a holomorphic function $g : \Delta \to \Delta$ with $g(0) = 0$ such that $(f^1, f^4, g) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$ by Proposition 2.4. From the functional equation, we have

$$(1 - |f^2(z)|^2)(1 - |f^3(z)|^2) = 1 - |g(z)|^2.$$

Since g is a component function of some holomorphic isometry in $\mathbf{HI}_1(\Delta,\Delta^3;3)$, from [Ng10], we have a local inverse g^{-1} of g around $0\in\Delta$ so that

$$(1-|f^2\circ g^{-1}(z)|^2)(1-|f^3\circ g^{-1}(z)|^2)=1-|z|^2$$

on some open neighborhood of 0 in Δ. Thus $(f^2\circ g^{-1},f^3\circ g^{-1}):\Delta\to\Delta^2$ is a germ of holomorphic isometric embedding. In particular, $(f^2\circ g^{-1},f^3\circ g^{-1})$ is the germ of the square root embedding at 0 up to reparametrizations. From [Mok12], such germ of holomorphic isometric embedding can be extended to a holomorphic isometric embedding $\Delta\to\Delta^2$. Thus we have $f^2\circ g^{-1}=\alpha_1|_U,\ f^3\circ g^{-1}=\alpha_2|_U$ for some neighborhood U of 0 in Δ, where $(\alpha_1,\alpha_2)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$. Thus $f^2=\alpha_1\circ g,\ f^3=\alpha_2\circ g$ on Δ. Hence

$$f=(f^1,f^2,f^3,f^4)=(\beta_1,\alpha_1\circ\beta_2,\alpha_2\circ\beta_2,\beta_3),$$

where $(\beta_1,\beta_2,\beta_3)\in\mathbf{HI}_1(\Delta,\Delta^3;3)$ and $(\alpha_1,\alpha_2)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$. □

Let $f=(f^1,f^2,f^3,f^4)\in\mathbf{HI}_1(\Delta,\Delta^4;8;4,4,4,4)$ and $\nu:X\to V$ be the normalization, where X is a compact Riemann surface of genus $g(X)$. Without loss of generality, we can assume that $f(0)=\mathbf{0}$. The universal cover of X is either $\mathbb{P}^1$, $\mathbb{C}$ or Δ by the Uniformization Theorem. In any cases, we can use global holomorphic coordinate ζ on $\mathbb{P}^1=\mathbb{C}\sqcup\{\infty\}$, $\mathbb{C}$ or Δ to represent a point in X. Given a non-constant meromorphic function $\hat{S}$ on X, denote by $\mathrm{Zeros}(\hat{S}(\zeta))$ (resp. $\mathrm{Poles}(\hat{S}(\zeta))$) the set of all zeros (resp. poles) of $\hat{S}$ not counting multiplicities.

Recall that $\pi:V\to\mathbb{P}^1$ is the finite branched covering defined by $(z,w_1,w_2,w_3,w_4)\mapsto z$. Then, $\pi\circ\nu(\zeta)=R(\zeta)$ is a non-constant meromorphic function on X with precisely 8 distinct poles and 8 distinct zeros. Denote by $S_j(\zeta)=(\mathrm{Pr}_2\circ(P_j\circ\nu))(\zeta)$ for $1\le j\le 4$, where $\mathrm{Pr}_2:\mathbb{P}^1\times\mathbb{P}^1\to\mathbb{P}^1$ is the projection onto the second factor and $P_j:V\to\mathbb{P}^1\times\mathbb{P}^1$ is defined by $(z,w_1,w_2,w_3,w_4)\mapsto(z,w_j)$ and $V_j=P_j(V)$ for $1\le j\le 4$. Then, S_j is a non-constant meromorphic function on X with precisely two distinct poles and two distinct zeros. Moreover, we have $R(\zeta)=R_j(S_j(\zeta))$ for $1\le j\le 4$.

Let $(f^1_{l_1},f^2_{l_2},f^3_{l_3},f^4_{l_4})$ be a branch of f over Δ for some $l_j\in\{1,2,3,4\}$. For $\zeta\in U':=\nu^{-1}(\mathrm{Graph}(f))$, we have $f^j(R(\zeta))=S_j(\zeta)$ for $1\le j\le 4$. Note that for any branch f^j_l of f^j, $1\le l,j\le 4$, there is precisely two distinct branches of f over Δ with the j-th component function equal to f^j_l because $S_j:X\to\mathbb{P}^1$ is a degree 2 branched covering and the graph of each branch of f over Δ (resp. $\mathbb{P}^1\smallsetminus\overline{\Delta}$) lies in the regular part of the variety V. From the polarized functional equation, for $\zeta\in U':=\nu^{-1}(\mathrm{Graph}(f))$ and $w\in\Delta$, we have

$$\prod_{j=1}^4\left(1-S_j(\zeta)\overline{f^j_{l_j}(w)}\right)=1-R(\zeta)\overline{w}.$$

Fix $w\in\Delta$, then both sides of the above equality are meromorphic functions on X. Thus, by identity theorem of meromorphic functions on compact Riemann surfaces, the above equality holds for $\zeta\in X$ and $w\in\Delta$. Putting $w=0$ in the above equality gives

$$\prod_{j=1}^4\left(1-S_j(\zeta)\overline{f^j_{l_j}(0)}\right)=1\quad\forall\,\zeta\in X.$$

Lemma 3.5. *Let $f=(f^1,\ldots,f^4)\in\mathbf{HI}_1(\Delta,\Delta^4;8;4,4,4,4)$, then there is a branch of f over Δ which is of the form $(g_1,\ldots,g_4)$, where $g_{\sigma(j)}:=f_1^{\sigma(j)}$ $(j=1,2)$ and $g_{\sigma(\mu)}:=f^{\sigma(\mu)}_{l_{\sigma(\mu)}}$ with $l_{\sigma(\mu)}\neq 1$ $(\mu=3,4)$ for some $\sigma\in S_4$.*

Proof. Without loss of generality, we can assume that $f(0)=\mathbf{0}$. Let $\nu:X\to V$ be the normalization. Assume the contrary that f does not have a branch of the required form. From the functional equation, it is known that f cannot have a branch of the form $\left(f^{\sigma(1)},f^{\sigma(2)},f^{\sigma(3)},f^{\sigma(4)}_{j_{\sigma(4)}}\right)$ over Δ up to permutation of component functions of f, where $\sigma\in S_4$ and $j_{\sigma(4)}\neq 1$. Otherwise, we would have $\left|f^{\sigma(4)}_{j_{\sigma(4)}}(z)\right|^2=\left|f^{\sigma(4)}(z)\right|^2$ so that $f^{\sigma(4)}_{j_{\sigma(4)}}(0)=f^{\sigma(4)}(0)=0$, which contradicts to $f^{\sigma(4)}_{j_{\sigma(4)}}$ and $f^{\sigma(4)}$ being distinct branches and 0 is not a branch point of R_4. Then, we have branches of f over Δ of the form

$$\left(f^1,f^2_{l_2^{(1)}},f^3_{l_3^{(1)}},f^4_{l_4^{(1)}}\right),\ \left(f^1_{l_1^{(2)}},f^2,f^3_{l_3^{(2)}},f^4_{l_4^{(2)}}\right),\ \left(f^1_{l_1^{(3)}},f^2_{l_2^{(3)}},f^3,f^4_{l_4^{(3)}}\right),\ \left(f^1_{l_1^{(4)}},f^2_{l_2^{(4)}},f^3_{l_3^{(4)}},f^4\right),\tag{3.1}$$

where $l_j^{(k)} \neq 1$ for each j,k. Note that performing (multivalued) analytic continuation of (f^1, f^2, f^3, f^4) along some simple closed loop around each branch point of R_j in $\mathbb{C}$, $1 \le j \le 4$, would produce all branches of f over Δ because $\mathrm{Reg}(V)$ is connected (cf. Proposition 1 in [MN10], p.2634-2635, for the structure of V and properties for the branches of f). From the polarized functional equation, we have

$$\prod_{j=1}^{3}\left(1 - S_{\sigma(j)}(\zeta)\overline{\beta_{\sigma(j)}^{(\sigma(4))}}\right) = 1$$

for each $\sigma \in S_4$, where for each $k \in \{1,2,3,4\}$, $\beta_j^{(k)} = f^j_{l_j^{(k)}}(0) \in \mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$ for $j \in \{1,2,3,4\} \smallsetminus \{k\}$. Note that the poles of $1 - S_j(\zeta)\overline{\beta_j^{(l)}}$ are precisely the poles of $S_j(\zeta)$ for $j \in \{1,2,3,4\} \smallsetminus \{l\}$ and $l = 1,2,3,4$. Moreover, $1 - S_j(\zeta)\overline{\beta_j^{(l)}}$ has precisely two distinct zeros and two distinct poles for $j \in \{1,2,3,4\} \smallsetminus \{l\}$ and $l = 1,2,3,4$.

Consider the branch $\left(f^1_{l_1^{(4)}}, f^2_{l_2^{(4)}}, f^3_{l_3^{(4)}}, f^4\right)$, then there is a unique branch of f over Δ which is of the form $\left(f^1_{k_1}, f^2_{k_2}, f^4_{l_3^{(4)}}, f^4_{k_4}\right)$ with $k_4 \neq 1$ because we already have the branch (f^1, f^2, f^3, f^4), S_j is a degree 2 branched covering and all points in $\nu^{-1}(\pi^{-1}(\infty))$ are not ramification points of S_l, $1 \le l \le 4$. We claim that $k_j \neq l_j^{(4)}$ for $j = 1,2$.
If $k_j = l_j^{(4)}$ for $j = 1,2$, then we would have $|f^4(z)|^2 = |f^4_{k_4}(z)|^2$ for $z \in \Delta$, which leads to a contradiction by the arguments above. If $k_1 = l_1^{(4)}$ and $k_2 \neq l_2^{(4)}$, then we have

$$\left(1 - S_2(\zeta)\overline{\beta_2^{(4)}}\right) = \left(1 - S_2(\zeta)\overline{f^2_{k_2}(0)}\right)\left(1 - S_4(\zeta)\overline{f^2_{k_4}(0)}\right)$$

from the functional equation so that

$$S_4(\zeta) = \frac{1}{\overline{f^4_{k_4}(0)}}\frac{\left(\overline{\beta_2^{(4)}} - \overline{f^2_{k_2}(0)}\right)\cdot S_2(\zeta)}{1 - S_2(\zeta)\overline{f^2_{k_2}(0)}}.$$

Thus, $S_4 = \varphi \circ S_2$ for some $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$. But then this implies that all branches of f are of the form $(f^1_{l_1}, f^2_l, f^3_{l_3}, f^4_l)$ for some $l_1, l_3, l \in \{1,2,3,4\}$ by performing (multivalued) analytic continuation, which contradicts to the existence of the branch $\left(f^1_{l_1^{(4)}}, f^2_{l_2^{(4)}}, f^3_{l_3^{(4)}}, f^4\right)$. Similarly, if $k_2 = l_2^{(4)}$ and $k_1 \neq l_1^{(4)}$, then this also leads to a contradiction. Hence, $k_j \neq l_j^{(4)}$ for $j = 1,2$.
From the functional equation, we have

$$1 - S_4(\zeta)\overline{f^4_{k_4}(0)} = \frac{1 - S_1(\zeta)\overline{\beta_1^{(4)}}}{1 - S_1(\zeta)\overline{f^1_{k_1}(0)}}\frac{1 - S_2(\zeta)\overline{\beta_2^{(4)}}}{1 - S_2(\zeta)\overline{f^2_{k_2}(0)}}$$

and $\prod_{j=1}^{3}\left(1 - S_j(\zeta)\overline{\beta_j^{(4)}}\right) = 1$. Thus, we have

$$\begin{aligned}\mathrm{Zeros}\left(1 - S_4(\zeta)\overline{f^4_{k_4}(0)}\right) &\subseteq \mathrm{Zeros}\left(\left(1 - S_1(\zeta)\overline{\beta_1^{(4)}}\right)\left(1 - S_2(\zeta)\overline{\beta_2^{(4)}}\right)\right)\\ &= \mathrm{Zeros}\left(\frac{1}{1 - S_3(\zeta)\overline{\beta_3^{(4)}}}\right) = \mathrm{Poles}(S_3(\zeta))\end{aligned}$$

Since S_3 has two distinct simple poles and $1 - S_4(\zeta)\overline{f^4_{k_4}(0)}$ has two distinct simple zeros, we have $\mathrm{Zeros}\left(1 - S_4(\zeta)\overline{f^4_{k_4}(0)}\right) = \mathrm{Poles}(S_3(\zeta))$. Therefore, there are two distinct points $y_1, y_2 \in V$ (resp. $x_1, x_2 \in X$) such that $\nu(x_j) = y_j$,

$$y_j = \left(\infty, \alpha_1^j, \alpha_2^j, \infty, \frac{1}{\overline{f^4_{k_4}(0)}}\right)$$

for $j=1,2$, and $\{x_1,x_2\} = \mathrm{Zeros}\left(1-S_4(\zeta)\overline{f^4_{k_4}(0)}\right) = \mathrm{Poles}(S_3(\zeta))$, where $\alpha^j_1,\alpha^j_2\in\mathbb{C}^*$, $j=1,2$. Note that $x_1,x_2\in X$ are two distinct unramified points of $\pi\circ\nu: X\to\mathbb{P}^1$ and $y_1,y_2\in V$ are smooth points on V. Then, we have two distinct branches of f over $\mathbb{P}^1\smallsetminus\overline{\Delta}$ which are of the form $\left(f^1_{l_1,-},f^2_{l_2,-},f^3_{l_3,-},f^4_{l_4,-}\right)$, $\left(f^1_{n_1,-},f^2_{n_2,-},f^3_{l_3,-},f^4_{l_4,-}\right)$ such that

$$y_1 = \left(\infty, f^1_{l_1,-}(\infty), f^2_{l_2,-}(\infty), f^3_{l_3,-}(\infty), f^4_{l_4,-}(\infty)\right),$$
$$y_2 = \left(\infty, f^1_{n_1,-}(\infty), f^2_{n_2,-}(\infty), f^3_{l_3,-}(\infty), f^4_{l_4,-}(\infty)\right).$$

If $n_j=l_j$ and $n_i\neq l_i$ for distinct $i,j\in\{1,2\}$, then we have

$$1-f^i_{l_i,-}(z)\overline{f^i_{l_i,-}(w)} = 1-f^i_{n_i,-}(z)\overline{f^i_{l_i,-}(w)}$$

for $z,w\in\mathbb{P}^1\smallsetminus\overline{\Delta}$ from the functional equation, which implies that $f^i_{l_i,-}=f^i_{n_i,-}$ so that $l_i=n_i$, a contradiction. Thus, $n_j\neq l_j$ for $j=1,2$. Now, we have $\alpha^1_l\neq\alpha^2_l$ for $l=1,2$. From the functional equation, we have

$$\left(1-f^1_{l_1,-}(z)\overline{f^1_{n_1,-}(w)}\right)\left(1-f^2_{l_2,-}(z)\overline{f^2_{n_2,-}(w)}\right) = \left(1-f^1_{l_1,-}(z)\overline{f^1_{l_1,-}(w)}\right)\left(1-f^2_{l_2,-}(z)\overline{f^2_{l_2,-}(w)}\right)$$

so that

$$\frac{1-f^1_{l_1,-}(z)\overline{\alpha^2_1}}{1-f^1_{l_1,-}(z)\overline{\alpha^1_1}} = \frac{1-f^2_{l_2,-}(z)\overline{\alpha^1_2}}{1-f^2_{l_2,-}(z)\overline{\alpha^2_2}}$$

which implies that $f^1_{l_1,-}(z)=\varphi(f^2_{l_2,-}(z))$ for some $\varphi\in\mathrm{Aut}(\mathbb{P}^1)$ satisfying $\varphi(0)=0$. Denote by $\mathcal{O}=\mathbb{P}^1\smallsetminus\overline{\Delta}$. Thus, $R_1\circ\varphi|_{f^2_{l_2,-}(\mathcal{O})} = R_2|_{f^2_{l_2,-}(\mathcal{O})}$. Since $f^2_{l_2,-}(\mathcal{O})\subset\mathbb{P}^1$ is open, we have $R_1\circ\varphi=R_2$ by the Identity Theorem for meromorphic functions on irreducible holomorphic varieties ([Gun90], p.177). We claim that $R_j(h(z))=z$ for some holomorphic function h on Δ implies $h=f^j_l$ for some l and $h(0)=f^j_l(0)$. Actually, $\exists$ an open neighborhood B_0 of 0 in Δ such that $R_j|_{U_l}:U_l\to B_0$ is biholomorphic and $h(0)=f^j_l(0)$ for some l since 0 is not a branch point of R_j, where U_l is some open neighborhood of $f^j_l(0)$ in $\mathbb{P}^1$. Then $(R_j|_{U_l})^{-1}|_{B_0} = h|_{B_0} = f^j_l|_{B_0}$ and thus $h=f^j_l$ by the Identity Theorem.

Therefore, this implies that $\varphi\circ f^2$ is one of the branches of f^1 over Δ. Since $(\varphi\circ f^2)(0)=0$, we have $\varphi\circ f^2=f^1$ because 0 is not a branch point of any R_j, $1\le j\le 4$. But then performing (multivalued) analytic continuation of (f^1,f^2,f^3,f^4) could only produce branches of f over Δ of the form $(f^1_l,f^2_l,f^3_{l_3},f^4_{l_3})$ for some $l,l_3,l_4\in\{1,2,3,4\}$, which contradicts to the assumption 3.1. Hence, there is a branch of f over Δ which is of the required form. □

Proposition 3.6. *Let $f\in\mathbf{HI}_1(\Delta,\Delta^4;8;4,4,4,4)$, then*

$$f=(\alpha_1\circ\alpha_2,\beta_1\circ\alpha_2,\alpha_3\circ\beta_2,\beta_3\circ\beta_2)$$

up to reparametrizations, where $(\alpha_j,\beta_j)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$, $j=1,2,3$.

Proof. Without loss of generality, we can assume that $f(0)=\mathbf{0}$. By the Lemma 3.5, there is a branch of f over Δ which is of the form $(g_1,\ldots,g_4)$, where $g_{\sigma(j)}:=f^{\sigma(j)}_1$ for $1\le j\le 2$ and $g_{\sigma(\mu)}:=f^{\sigma(\mu)}_{l_{\sigma(\mu)}}$ with $l_{\sigma(\mu)}\neq 1$ for $\mu=3,4$, for some $\sigma\in S_4$. By Theorem 2.2,

$$(1-|f^{\sigma(3)}(z)|^2)(1-|f^{\sigma(4)}(z)|^2) = 1-|h(z)|^2$$

for some holomorphic function $h:\Delta\to\mathbb{C}$. Thus, from [Ng10], $(f^{\sigma(1)},f^{\sigma(2)},h)\in\mathbf{HI}_1(\Delta,\Delta^3)$ so that sheeting number of h equals 2 and h is a component function of some isometry in $\mathbf{HI}_1(\Delta,\Delta^2;2)$ (cf. [Ng10]). This shows that $(f^{\sigma(1)},f^{\sigma(2)},h)\in\mathbf{HI}_1(\Delta,\Delta^3;4;4,4,2)$. From [Ng10], we have

$$(f^{\sigma(1)},f^{\sigma(2)},h)=(\alpha_5\circ g,\beta_5\circ g,h)$$

up to reparametrizations for some $(\alpha_5,\beta_5)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$ and $(g,h)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$ for some holomorphic function $g:\Delta\to\Delta$. Moreover, $(1-|f^{\sigma(3)}(h^{-1}(z))|^2)(1-|f^{\sigma(4)}(h^{-1}(z))|^2)=1-|z|^2$ for $z\in B^1(0,\varepsilon)\subset\Delta$ for some $\varepsilon>0$. Thus, $(f^{\sigma(3)}\circ h^{-1},f^{\sigma(4)}\circ h^{-1}):B^1(0,\varepsilon)\to\Delta^2$ is a local holomorphic isometric embedding which can be extended to the whole unit disk Δ (cf. [Mok12]), so $f^{\sigma(3)}=\alpha_4\circ h$, $f^{\sigma(4)}=\beta_4\circ h$ for some $(\alpha_4,\beta_4)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$. Hence, $(f^{\sigma(1)},f^{\sigma(2)},f^{\sigma(3)},f^{\sigma(4)})=(\alpha_5\circ g,\beta_5\circ g,\alpha_4\circ h,\beta_4\circ h)$ up to reparametrizations so that $f=(\alpha_1\circ\alpha_2,\beta_1\circ\alpha_2,\alpha_3\circ\beta_2,\beta_3\circ\beta_2)$ up to reparametrizations, where $(\alpha_j,\beta_j)\in\mathbf{HI}_1(\Delta,\Delta^2;2)$, $j=1,2,3$. □

Combining the above results, part (1) of the Theorem 1.1 is proved.

3.2. **Classification of $\mathbf{HI}_k(\Delta, \Delta^4)$ for $2 \leq k \leq 4$.** Now, we consider the case $k = 2, 3$ or 4. The following is part (2) of the Theorem 1.1.

Proposition 3.7. *Let $f : (\Delta, 2ds^2_\Delta) \to (\Delta^4, ds^2_{\Delta^4})$ be a holomorphic isometric embedding, then $f(z)$ is of one of the following form up to reparametrizations:*

(1) $(\alpha_1(z), \beta_1(z), \alpha_2(z), \beta_2(z))$*, where* $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *for* $j = 1, 2$.
(2) $(z, \alpha_1(z), (\alpha_2 \circ \beta_1)(z), (\beta_2 \circ \beta_1)(z))$*, where* $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ *for* $j = 1, 2$.
(3) $(z, \alpha_1(z), \alpha_2(z), \alpha_3(z))$*, where* $(\alpha_1, \alpha_2, \alpha_3) \in \mathbf{HI}_1(\Delta, \Delta^3; 3)$.

Moreover, the space $\mathbf{HI}_2(\Delta, \Delta^4; n; 2, 2, 2, 2)$ is non-empty only when $n = 2$ or $n = 4$.

Proof. Without loss of generality, we can assume that $f(0) = \mathbf{0}$. Let s_j be the sheeting number of f^j and n be the global sheeting number (cf. [Ng10]). In case $k = 2$, we have $2 \leq n \leq 8$. If $n = 5$, then we have $\sum_{j=1}^4 \frac{1}{s_j} = 2$ with $s_j|5$ for $1 \leq j \leq 4$. Thus, $l + \frac{4-l}{5} = 2$ for some integer $l \geq 0$, but this would imply that $4l = 6$, a contradiction. If $n = 7$, then we have $\sum_{j=1}^4 \frac{1}{s_j} = 2$ with $s_j|7$ for $1 \leq j \leq 4$. Thus, $l + \frac{4-l}{7} = 2$ for some integer $l \geq 0$, but this would imply that $6l = 10$, a contradiction. Then, $n \notin \{5, 7\}$. Therefore, we have $n = 2, 3, 4, 6$ or 8.
In priori for $n = 6$ or $n = 8$, we would have $(n, s_1, s_2, s_3, s_4) = (6, 2, 2, 2, 2)$, $(6, 1, 3, 3, 3)$, $(6, 1, 2, 3, 6)$, $(8, 2, 2, 2, 2)$ or $(8, 1, 2, 4, 4)$.
If $s_1 = 1$, then $f^1(z) = z$ up to reparametrizations so that the problem reduces to the study of $\mathbf{HI}_1(\Delta, \Delta^3)$, which is completely described by Ng [Ng10]. If $(n, s_1, s_2, s_3, s_4) = (6, 1, 3, 3, 3)$, then (f^2, f^3, f^4) is the cube root embedding up to reparametrizations by [Ng10] and this implies that $n = 3$, which is a contradiction. If $(n, s_1, s_2, s_3, s_4) = (6, 1, 2, 3, 6)$, then we would have a holomorphic isometry in $\mathbf{HI}_1(\Delta, \Delta^3; n'; 2, 3, 6)$ so that $n' \geq 6$, which contradicts to $n' \leq 4$ (cf. [Ng10]). If $(n, s_1, s_2, s_3, s_4) = (8, 1, 2, 4, 4)$, then (f^2, f^3, f^4) is of the form $(\alpha_1, \alpha_2 \circ \beta_1, \beta_2 \circ \beta_2)$ for $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2)$ by Ng [Ng10] and thus $n = 4$, a contradiction. This rules out the cases $(n, s_1, s_2, s_3, s_4) = (6, 1, 3, 3, 3)$, $(n, s_1, s_2, s_3, s_4) = (6, 1, 2, 3, 6)$, $(n, s_1, s_2, s_3, s_4) = (8, 1, 2, 4, 4)$.
Therefore, the only possible global sheeting numbers n and sheeting numbers $s_1, \ldots, s_4$ are the following:

(1) $(n, s_1, s_2, s_3, s_4) = (n, 2, 2, 2, 2)$, $n = 2, 4, 6$ or 8,
(2) $(n, s_1, s_2, s_3, s_4) = (4, 1, 2, 4, 4)$,
(3) $(n, s_1, s_2, s_3, s_4) = (3, 1, 3, 3, 3)$.

Now, we deal with these cases:

(1) Let $f = (f^1, f^2, f^3, f^4) \in \mathbf{HI}_2(\Delta, \Delta^4; n; 2, 2, 2, 2)$, then each f^j becomes one of the component functions of the square root embedding from [Ng10]. From [Ng10], for each branch point $a \in \partial\Delta$ of some component function f^j of f, we have $|f^j(a)|^2 = 1$. From the use of Puiseux series of each component function f^j of f around a branch point $a \in \partial\Delta$ of f^j, we see that either a is a branch point of all component functions of f or a is a branch point of another component f^l of f ($l \neq j$) and a is not a branch point of other component functions f^μ of f ($\mu \notin \{l, j\}$).
Then either (i) branching loci of all component functions of f are the same or (ii) for any branch point $a \in \partial\Delta$ of each component function f^j of f, a is only a branch point of f^l for some $l \neq j$ and not a branch point of f^μ for $\mu \notin \{l, j\}$.
(i) If branching loci of all component functions of f are the same, then there is a single reparametrization of f so that each f^j is one of the α_1, β_1, where $(\alpha_1, \beta_1) \in \mathbf{HI}_1(\Delta, \Delta^2)$ is the square root embedding. From [Ng10], since for every branch of f, there is precisely two component functions of f which takes value ∞ at ∞, so only two of the f^j's is α_1 and the other two are β_1 up to reparametrizations. In particular, f is $(\alpha_1, \beta_1, \alpha_1, \beta_1)$ up to reparametrizations for some $(\alpha_1, \beta_1) \in \mathbf{HI}_1(\Delta, \Delta^2)$.
(ii) Suppose that for any branch point $a \in \partial\Delta$ of each component function f^j of f, a is only a branch point of f^l for some $l \neq j$ and not a branch point of f^μ for $\mu \notin \{l, j\}$. We can assume that f^1 and f^2 have a common branch point $a \in \partial\Delta$ and a is not a branch point of f^3, f^4, then after performing (multivalued) analytic continuation around $a \in \partial\Delta$ along a simple continuous closed loop around a once, we have another branch (f^1_l, f^2_l, f^3, f^4) of

f for some $l \neq 1$. Then from the proof of Theorem 2.2, we actually have

$$(1-|f^1(z)|^2)(1-|f^2(z)|^2) = 1-|h(z)|^2$$

for some holomorphic function $h : \Delta \to \Delta$. Then $(h, f^3, f^4) \in \mathbf{HI}_2(\Delta, \Delta^3)$ and actually the sheeting number of h has to be 1, i.e. $h(z) = z$ up to reparametrization. In particular, $(f^1, f^2) \in \mathbf{HI}_1(\Delta, \Delta^2)$ and thus $(f^3, f^4) \in \mathbf{HI}_1(\Delta, \Delta^2)$. Hence, f is $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ up to reparametrizations for some $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2)$, $j = 1, 2$.

In particular, any $f \in \mathbf{HI}_2(\Delta, \Delta^4; n; 2, 2, 2, 2)$ is $(\alpha_1, \beta_1, \alpha_2, \beta_2)$ up to reparametrizations for some $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2)$, $j = 1, 2$. Note that branching loci of α_j and β_j are the same for each $j = 1, 2$. By performing (multivalued) analytic continuation, the global sheeting number is at most 4, i.e. either $n = 2$ or $n = 4$.

If $f = (f^1, f^2, f^3, f^4) \in \mathbf{HI}_2(\Delta, \Delta^4; 2; 2, 2, 2, 2)$, then branching loci of all f^j are the same so that there is a single parametrization of f to make f^j to be either α_1 or β_1, where $(\alpha_1, \beta_1) : \Delta \to \Delta^2$ is the square root embedding. Moreover, since for each branch of f, there are only two component functions takes value ∞ at ∞, so $f = (\alpha_1, \beta_1, \alpha_1, \beta_1)$ up to reparametrizations.

If $f \in \mathbf{HI}_2(\Delta, \Delta^4; 4; 2, 2, 2, 2)$, then $f = (\alpha_1, \beta_1, \alpha_2, \beta_2)$ up to reparametrizations, where $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ for $j = 1, 2$ such that branching loci of (α_1, β_1) is different from that of (α_2, β_2).

(2) Let $f = (f^1, f^2, f^3, f^4) \in \mathbf{HI}_2(\Delta, \Delta^4; 4; 1, 2, 4, 4)$, then $f^1(z) = z$ up to reparametrizations, so we have $(f^2, f^3, f^4) \in \mathbf{HI}_1(\Delta, \Delta^3; 4; 2, 4, 4)$. From [Ng10], we have

$$(f^2, f^3, f^4) = (\alpha_1, \alpha_2 \circ \beta_1, \beta_2 \circ \beta_1)$$

up to reparametrizations, where $(\alpha_j, \beta_j) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$ for $j = 1, 2$.

(3) Now, we consider the case $n = 3$, then the only possibility is that $(s_1, s_2, s_3, s_4) = (1, 3, 3, 3)$. Then, $f^1(z) = z$ up to reparametrizations, then

$$\left(1-|f^2(z)|^2\right)\left(1-|f^3(z)|^2\right)\left(1-|f^4(z)|^2\right) = 1-|z|^2$$

so that $(f^2, f^3, f^4) : \Delta \to \Delta^3$ is a holomorphic isometric embedding with isometric constant $k = 1$. From [Ng10], (f^2, f^3, f^4) has to be the cube-root embedding up to reparametrizations. Thus $f(z) = (z, \alpha_1(z), \alpha_2(z), \alpha_3(z))$, where $(\alpha_1, \alpha_2, \alpha_3) : \Delta \to \Delta^3$ is the cube-root embedding with the isometric constant 1 up to reparametrizations.

□

The following is part (3) of the Theorem 1.1.

Proposition 3.8. *Let $f : (\Delta, 3ds^2_\Delta) \to (\Delta^4, ds^2_{\Delta^4})$ be a holomorphic isometric embedding with the isometric constant $k = 3$, then*

$$f(z) = (z, z, \alpha(z), \beta(z))$$

up to reparametrizations, where $(\alpha, \beta) \in \mathbf{HI}_1(\Delta, \Delta^2; 2)$.

Proof. Without loss of generality, we can assume that $f(0) = \mathbf{0}$. Note that $\sum_{j=1}^4 \frac{1}{s_j} = 3$, so $\exists\, j$ such that $\frac{1}{s_j} \geq \frac{3}{4}$, but then $s_j \leq \frac{4}{3} < 2 \implies s_j = 1$, which implies $f^j(z) = z$ up to reparametrizations, say $f^1(z) = z$ without loss of generality. Then

$$\left(1-|f^2(z)|^2\right)\left(1-|f^3(z)|^2\right)\left(1-|f^4(z)|^2\right) = (1-|z|^2)^2$$

so that from [Ng10], $(f^2, f^3, f^4) : \Delta \to \Delta^3$ is a holomorphic isometric embedding with isometric constant 2 and thus $(f^2(z), f^3(z), f^4(z)) = (z, \alpha(z), \beta(z))$ up to reparametrizations, where $(\alpha, \beta) : \Delta \to \Delta^2$ is a holomorphic isometric embedding with isometric constant 1. Thus, $f(z) = (z, z, \alpha(z), \beta(z))$ up to reparametrizations. □

Combining the results in the previous section, Proposition 3.7 and Proposition 3.8, the Theorem 1.1 is proved when $k = 1, 2, 3$. For the case of isometric constant $k = 4$, it is known from [Ng08] that $f(z) = (z, z, z, z)$ is the diagonal embedding up to reparametrizations, i.e. the space $\mathbf{HI}_4(\Delta, \Delta^4)$ consists of only the diagonal embedding up to reparametrizations. Hence, the Theorem 1.1 is proven completely.

4. Generalizations of the global rigidity of the p-th root embedding

In [Ch16], we have obtained that all holomorphic isometric embeddings in $\mathbf{HI}_1(\Delta,\Delta^p;p)$ is the p-th root embedding F_p up to reparametrizations, which means that F_p is globally rigid in $\mathbf{HI}_1(\Delta,\Delta^p;p)$ in the sense of [Mok11]. This phenomenon also occurs for the space $\mathbf{HI}_k\left(\Delta,\Delta^p;\frac{p}{k}\right)$, where k,p are positive integers satisfying $p\ge 2$, $k|p$ and $\frac{p}{k}\ge 2$. Note that the case of $\mathbf{HI}_k\left(\Delta,\Delta^p;\frac{p}{k}\right)$ is precisely the minimal case of $\mathbf{HI}_k(\Delta,\Delta^p)$ in terms of the global sheeting number. More precisely, we shall show that all holomorphic isometries in $\mathbf{HI}_k\left(\Delta,\Delta^{qk};q\right)$ are globally rigid for positive integers q,k satisfying $q\ge 2$ and $k\ge 1$. The following can be regarded as an analogue of the Theorem 1.1. in [Ch16] because the techniques of proving Theorem 1.1. in [Ch16] are still valid for a more general situation with slight modifications.

Proposition 4.1.
Let $p\ge 2$ be an integer and $k\in\mathbb{Z}$ satisfying $1\le k\le p$, $\frac{p}{k}\in\mathbb{Z}$ and $\frac{p}{k}\ge 2$. Let $f=(f^1,\dots,f^p):(\Delta,kds^2_\Delta)\to(\Delta^p,ds^2_{\Delta^p})$ be a holomorphic isometric embedding with the sheeting number $q=\frac{p}{k}$ and the isometric constant k. Then $f=(g_1,\dots,g_k)$ up to reparametrizations, where $g_j=F_q$ up to reparametrizations for $1\le j\le k$ such that branching loci of all g_j's are the same and $F_q=(F_q^1,\dots,F_q^q):\Delta\to\Delta^q$ is the q-th root embedding.

Lemma 4.2 (Analogue of Lemma 4.9. in [Ch16])**.** *Suppose the same assumptions as in proposition 4.1, let $q\ge 4$ be an even integer, and suppose that π has 3 distinct branch points $a_1,a_2,a_3\in\partial\Delta$. Then, there is a component function f^j of f such that $\widetilde{f^j}(\overline{\Delta})\subset\Delta$, where $\widetilde{f}=(\widetilde{f^1},\dots,\widetilde{f^{qk}}):\overline{\Delta}\to\overline{\Delta^{qk}}$ is the continuous mapping such that $\widetilde{f}|_\Delta=f$.*

Proof. Let the ramification index of π at a_i be v_i for $i=1,2,3$, then all possible (v_1,v_2,v_3) are listed in table 1 in [Ch16], p. 355. We can write $a_j=e^{\theta_j}$ for $j=1,2,3$ and assume that $0\le\theta_1<\theta_2<\theta_3<2\pi$ without loss of generality. Let $A_{3,1}=\{e^{i\theta}\in\partial\Delta\mid\theta\in(\theta_3,\theta_1+2\pi)\}$, $A_{1,2}=\{e^{i\theta}\in\partial\Delta\mid\theta\in(\theta_1,\theta_2)\}$ and $A_{2,3}=\{e^{i\theta}\in\partial\Delta\mid\theta\in(\theta_2,\theta_3)\}$. Since $m=3$, each component function of f can only map precisely one connected component $A\subset\partial\Delta\smallsetminus\{a_1,a_1,a_3\}$ into $\partial\Delta$. Then, by properness of the holomorphic isometric embedding f (from [Mok12]), we can suppose that $\widetilde{f^\mu}(A_{3,1})\subset\partial\Delta$ for $1\le\mu\le k$ and $\widetilde{f^j}(A_{3,1})\not\subset\partial\Delta$ for $k+1\le j\le qk$; $\widetilde{f^\mu}(A_{1,2})\subset\partial\Delta$ for $k+1\le\mu\le 2k$ and $\widetilde{f^j}(A_{1,2})\not\subset\partial\Delta$ for $1\le j\le k$ or $2k+1\le j\le qk$; $\widetilde{f^\mu}(A_{2,3})\subset\partial\Delta$ for $2k+1\le\mu\le 3k$ and $\widetilde{f^j}(A_{2,3})\not\subset\partial\Delta$ for $1\le j\le 2k$ or $3k+1\le j\le qk$.
For all cases listed in table 1 in [Ch16, p. 355], we have $v_3=2$. In order to be consistent to above settings, by continuity of the map $\widetilde{f}$, we would have $|\widetilde{f^\mu}(a_3)|=1$ for $1\le\mu\le k$ or $2k+1\le\mu\le 3k$, $|\widetilde{f^j}(a_3)|<1$ for $k+1\le j\le 2k$ or $3k+1\le j\le qk$ by arguments in the proof of Lemma 4.3. in [Ch16]; $|\widetilde{f^{\mu'}}(a_2)|=1$ for $2k+1\le\mu'\le 3k$ or $k+1\le\mu'\le 2k$ and $|\widetilde{f^{\mu''}}(a_1)|=1$ for $k+1\le\mu''\le 2k$ or $1\le\mu''\le k$. Actually, arguments in the proof of Lemma 4.3. in [Ch16] would implies that if ramification index of π at $(a_i,f_l^1(a_i),\dots,f_l^{qk}(a_i))$ equals s, then $\exists$ distinct $j_1,\dots,j_{sk}\in\{1,\dots,qk\}$ such that $|f_l^{j_\mu}(a_i)|=1$ for $1\le\mu\le sk$. If $2\le s<q$, then $|f_l^j(a_i)|\ne 1$ for $j\notin\{j_1,\dots,j_{sk}\}$. The only difference is that in the proof of Lemma 4.3. in [Ch16, p. 352], we replace the term $1-|z|^2$ by $(1-|z|^2)^k$ in the functional equation, replace the term $-\overline{a_i}\xi^s$ by $(-\overline{a_i})^k\xi^{ks}$ in the polarized functional equation and also replace p by q. The argument of comparing vanishing order of holomorphic functions at $\xi=0$ is still valid. Now, we assume that contrary that

$$\nexists\, j\in\{1,\dots,kq\}\text{ such that }\widetilde{f^j}(\overline{\Delta})\subset\Delta. \tag{4.1}$$

Then, for $3k+1\le\mu\le qk$, we should have $|\widetilde{f^\mu}(a_2)|=1$ or $|\widetilde{f^\mu}(a_1)|=1$.
In any cases listed in table 1 in [Ch16], p. 355, the number of elements in the set

$$I_2:=\{\mu\in\mathbb{Z}\mid 3k+1\le\mu\le qk,\ |\widetilde{f^\mu}(a_2)|=1\text{ or }|\widetilde{f^\mu}(a_1)|=1\}$$

is at most $2\left(\frac{q}{2}\cdot k-2k\right)=(q-4)k$ because we already have $|\widetilde{f^{\mu'}}(a_2)|=1$ for $2k+1\le\mu'\le 3k$ or $k+1\le\mu'\le 2k$, $|\widetilde{f^{\mu''}}(a_1)|=1$ for $k+1\le\mu''\le 2k$ or $1\le\mu''\le k$ and $v_1,v_2\le\frac{q}{2}$. Note that $|\widetilde{f^\mu}(a_3)|<1$ for $k+1\le j\le 2k$ or $3k+1\le j\le qk$, by the assumption 4.1, the set I_2 must have precisely $(q-3)k$ elements. This leads to a contradiction. Hence, we conclude that $\exists\, j\in\{1,\dots,qk\}$ such that $\widetilde{f^j}(\overline{\Delta})\subset\Delta$. □

Proof of Proposition 4.1. Without loss of generality, assume that $f(0) = \mathbf{0}$. Note that $\sum_{j=1}^{kq} \frac{1}{s_j} = k$ and $s_j|q$ so that $s_j \leq q$, then $k = \sum_{j=1}^{kq} \frac{1}{q} \leq \sum_{j=1}^{kq} \frac{1}{s_j} = k$ implies that $s_j = q$ for $1 \leq j \leq p$. The method used in the proof of global rigidity of p-th root embedding can be applied to the study of $\mathbf{HI}_k(\Delta, \Delta^{kq}; q)$ since $s_j = q$ for $1 \leq j \leq kq$ so that all rational functions R_j are equivalent, i.e. $R_i = R_j \circ \varphi_{ji}$ for some $\varphi_{ji} \in \mathrm{Aut}(\mathbb{P}^1)$. From arguments in the study of minimal case in [Ng10], branching loci of all component functions of f are the same and for each point $(z, w_1, \ldots, w_p) \in V$, ramification index of π_j at (z, w_j) is the ramification index of π_i at (z, w_i) for distinct i, j, $1 \leq i, j \leq p$. Let $\{a_1, \ldots, a_m\} \subset \partial\Delta$ be the set of distinct branch points of $\pi : V \to \mathbb{P}^1$. Then for each connected component $A \subset \partial\Delta \smallsetminus \{a_1, \ldots, a_m\}$, there are precisely k component functions of f which maps A into $\partial\Delta$. From arguments in the proof of Proposition 4.4. in [Ch16], we have $2 \leq m \leq 3$ and the table 1 in [Ch16, p. 355], still provide all possible cases when $q \geq 4$ is even and $m = 3$. Actually, we only need to modify arguments in the proof of proposition 4.4. in [Ch16], namely replacing the term $1 - |z|^2$ (resp. $-\overline{a_i}\xi^s$) by $(1 - |z|^2)^k$ (resp. $(-\overline{a_i})^k \xi^{ks}$) in the functional equation (resp. polarized functional equation) and also replacing p by q. The argument of comparing vanishing order of holomorphic functions at $\xi = 0$ is still valid.

If $q = 2$ or $q \geq 3$ is odd, then from arguments in the proof of Proposition 4.4. and Corollary 4.6. in [Ch16], f has precisely two distinct branch points. If $q \geq 4$ is an even integer and $m = 3$, then by Lemma 4.2, $\widetilde{f^j}(\overline{\Delta}) \subset \Delta$ for some j, and this contradicts to the maximum principle as in the proof of Proposition 4.8. in [Ch16]. Thus $m \neq 3$ so that $m = 2$.

Therefore, all component functions of f are some component functions of the q-th root embedding up to reparametrization (cf. Lemma 4.9 in [Ng10, p. 2913]). Note that $\pi : V \to \mathbb{P}^1$ is also q-sheeted. From the polarized functional equation

$$\prod_{j=1}^{qk} (1 - f^j(z)\overline{f^j(w)}) = (1 - z\overline{w})^k$$

for some fixed $w \in \Delta \smallsetminus \{0\}$, then for each branch of f, there are precisely k of the component functions take the value ∞ at infinity by the proof of Theorem 6.5 in [Ng10]. Thus, these k component functions of f would be the same component function of the q-th root embedding up to reparametrizations. Without loss of generality, we can suppose that $f^{\mu k+1}, \ldots, f^{\mu k+k}$ are the same component function of F_q up to reparametrizations for each $\mu = 0, \ldots, q-1$, and for $1 \leq j, i \leq k$, $f^{\mu k+j}$ and $f^{\mu' k+i}$ are not congruent to the same component function of F_q provided that $\mu \neq \mu'$. Moreover, for $1 \leq j \leq k$, $(f^j, f^{j+k}, \ldots, f^{j+(q-1)k})$ is the q-th root embedding F_q up to reparametrizations. Thus, f is of the required form up to reparametrizations and the result follows. □

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