REMARK ON THE PAPER
"ON PRODUCTS OF FOURIER COEFFICIENTS OF CUSP FORMS"

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ABSTRACT. Let $a(n)$ be the Fourier coefficient of a holomorphic cusp form on some discrete subgroup of $SL_2(\mathbb{R})$. This note is to refine a recent result of Hofmann and Kohnen on the non-positive (and non-negative resp.) product of $a(n)a(n+r)$ for a fixed positive integer r.

1. INTRODUCTION

Let Γ be a subgroup of $SL_2(\mathbb{R})$. Assume as in [2] that

(i) Γ is a finitely generated Fuchsian group of the first kind,
(ii) $-I \in \Gamma$ where I is the identity,
(iii) Γ contains $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ exactly if b is an integer.

Conditions (ii) and (iii) may be formulated as: Γ has a cusp at $i\infty$ and its stabilizer $\Gamma_{i\infty}$ is generated by $\pm\begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix}$.

Let $k > 2$ be any even integer. Write $S_k(\Gamma)$ for the space of all elliptic cusp forms of weight k on Γ (with trivial multiplier system). Throughout, a cusp form is tacitly assumed to be holomorphic. Suppose all the coefficients of f in its Fourier expansion at $i\infty$ are real. In [2], Hofmann and Kohnen showed the infinitude of non-vanishing terms in the sequence $\{a_f(n)a_f(n+r)\}_{n\geq 1}$. Moreover, when Γ is a congruence subgroup, they showed that the sequence has infinitely many non-negative (resp. non-positive) terms, i.e. for $\epsilon = +$ or $-$ respectively,

$$\mathcal{C}^{\epsilon}_{f,r}(x) := \#\{n \in [1,x] : \epsilon\, a_f(n)a_f(n+r) \geq 0\} \to \infty \tag{1.1}$$

as $x \to \infty$, where $\#\{\cdots\}$ denotes the cardinality of the set $\{\cdots\}$. It is also pointed out that (1.1) holds for more general, but still restricted as in [5], subgroups Γ; in particular, it requires the discrete subgroups Γ (considered here) that have a cusp at 0.

This note aims at some refinement: First we show that (1.1) will hold when Γ satisfies merely Conditions (i)-(iii). Second we give a quantitative version for the case of congruence subgroups – a lower bound for $\mathcal{C}^{\pm}_{f,r}(x)$.

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Theorem 1.1. *Suppose Γ satisfies Conditions (i)-(iii) and f is a cusp form of even integral weight $k > 2$ on Γ whose Fourier coefficients are all real. For any $r \in \mathbb{N}$, we have $\mathcal{C}^{\pm}_{f,r}(x) \to \infty$ as $x \to \infty$.*

Theorem 1.2. *Let Γ be a congruence subgroup and f a cusp form of even integral weight $k > 2$ on Γ. Suppose all the Fourier coefficients of f are real. There exist positive constants $c_1 = c_1(f, r)$ and $x_1 = x_1(f, r)$ such that for all $x \geq x_1$,*

$$a_f(n_1)a_f(n_1 + r) \geq 0 \qquad \text{and} \qquad a_f(n_2)a_f(n_2 + r) \leq 0$$

for some integers $n_1, n_2 \in (x, x + c_1x^{1/2}]$. In particular, we have $\mathcal{C}^{\pm}_{f,r}(x) \gg_{f,r} x^{1/2}$ for all $x \geq x_1$.

Remark 1.1. 1. Compared with [2, Theorem 2], the condition on Γ there is relaxed in Theorem 1.1 while an explicit lower bound is given in Theorem 1.2 (for congruence subgroup Γ).

2. Both results are derived, similarly to the argument in [2], via counting the sign changes of $a_f(n)$ in arithmetic progressions $\mathcal{A} = \mathcal{A}_{a,r}$ where

$$\mathcal{A}_{a,r} := \{n \in \mathbb{N} : n \equiv a \bmod r\} \tag{1.2}$$

for $r \in \mathbb{N}$ and $a \in \mathbb{Z}$. However we detect the sign-changes in ways different than the method in [5] (used in [2]).

3. In Theorem 1.2, f is not necessarily a Hecke eigenform or primitive form. The Fourier coefficients of a primitive form are multiplicative. Recently Matomäki and Radziwill [8] obtained very strong results on sign-changes via their theory on multiplicative functions. Hence it is plausible to get a lower bound much better than $x^{1/2}$ for primitive forms.

4. Our results hold for maass cusp forms (of weight 0 and trivial multiplier system) with real coefficients. In view of the proof (in Section 4), the analogue of Theorem 1.2 is clear while for Theorem 1.1, one may apply [9, Theorem 5.1], the pointwise bound $a(n) \ll |n|^{2/5+\varepsilon}$ in [10, Corollary 1] and [4, (8.23)] instead (cf. Section 2).

2. Proof of Theorem 1.1

Let $f \neq 0$ be given as in Theorem 1.1. By [1, Theorem 2] and its Corollary, we have $a_f(n) \ll n^{(k-1)/2+1/3}$ and

$$\sum_{n \leq x} a_f(n)^2 \sim C_f x^k \qquad \text{as } x \to \infty$$

where $C_f > 0$ is a constant depending on f. Thus, there exists $1 \leq a \leq r$ such that

$$\sum_{\substack{x/2 < n \leq x \\ n \equiv a \bmod r}} |a_f(n)| \gg_{f,r} x^{k/2+1/6}. \tag{2.1}$$

On the other hand, from [3, Corollary 5.4], it follows that for any $1 \le b \le r$,

$$\sum_{\substack{n\le x\\ n\equiv b \bmod r}} a_f(n) \ll x^{k/2}\log 2x.$$

Hence for all large x, there are $u, v \in (x, 2x] \cap \mathcal{A}_{a,r}$ for some a such that $a_f(u)a_f(v) < 0$, and consequently $a_f(n)a_f(n+r) \le 0$ for some $n \in (x, 2x] \cap \mathcal{A}_{a,r}$, implying the infinitude of non-positive $a_f(n)a_f(n+r)$.

Suppose $a_f(n)a_f(n+r) < 0$ for all n. Then $a_f(n)a_f(n+2r) > 0$ for all n which contradicts to the last assertion with $2r$ in place of r. This completes the proof.

3. Preliminaries for Theorem 1.2

Define for $M, N \in \mathbb{N}$,

$$\begin{aligned}\Gamma_0(M,N) &= \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in SL_2(\mathbb{Z}):\ M|c,\ N|b\right\},\\ \Gamma(M,N) &= \left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_0(M,N):\ a\equiv d\equiv 1 \ (\mathrm{mod}\ [M,N])\right\}\end{aligned}$$

where $[M, N]$ denotes the least common multiple of M and N.[†] Then $\Gamma_0(M, 1) = \Gamma_0(M)$, $\Gamma(M, 1) = \Gamma_1(M)$ and $\Gamma(M, M) = \Gamma(M)$ of the usual notation. Recall a subgroup Γ of $SL_2(\mathbb{Z})$ is a congruence subgroup of level N if $\Gamma(N) \subseteq \Gamma$ for some $N \in \mathbb{N}$.

Also we write

$$n(x) = \begin{pmatrix} 1 & x\\ & 1\end{pmatrix} \quad \text{and} \quad W = \begin{pmatrix} & -1\\ 1 & \end{pmatrix}.$$

For any $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in GL_2(\mathbb{R})$ and any function g, define

$$g\big|_\gamma(z) = \left(\frac{cz+d}{\sqrt{\det\gamma}}\right)^{-k} g(\gamma z) \quad \text{where } \ \gamma z = \frac{az+b}{cz+d}.$$

Lemma 3.1. *(i) Suppose $f \in S_k(G(M, N))$ where $G(M, N) = \Gamma_0(M, N)$ or $\Gamma(M, N)$. We have $f\big|_W \in S_k(G(N, M))$.*

(ii) If $f \in S_k(\Gamma(M, N))$ and $r \in \mathbb{N}$, then $f\big|_{n(\frac{Nu}{r})}$ is a cusp form on $\Gamma(Mr^2, N)$ for any u (mod r).

(iii) Let $\mathbb{1}_{\mathcal{A}}$ be the characteristic function on $\mathcal{A} = \mathcal{A}_{a,r}$ (cf. (1.2)). The twist

$$f \otimes \mathbb{1}_{\mathcal{A}}(z) := \sum_{n\ge 1} a_f(n)\mathbb{1}_{\mathcal{A}}(n)e(nz/N)$$

is a cusp form on $\Gamma(Mr^2, N)$.

Proof. (i) Let $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix}$. Then $W\gamma W^{-1} = \begin{pmatrix} d & -c\\ -b & a\end{pmatrix}$. So $W\gamma W^{-1} \in G(M, N)$ if γ is belonged to $G(N, M)$.

[†] In [7], the product MN is used instead of the least common multiple.

(ii) For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(Mr^2, N)$,

$$\begin{pmatrix} 1 & Nu/r \\ & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -Nu/r \\ & 1 \end{pmatrix} = \begin{pmatrix} a + N\frac{cu}{r} & b + Nu\frac{d-a}{r} - N\frac{cNu^2}{r^2} \\ c & d - \frac{cNu}{r} \end{pmatrix}$$

which is in $\Gamma(M, N)$ as $a \equiv d \equiv 1 \bmod [Mr^2, N]$ and $c \equiv 0 \bmod Mr^2$. Thus, $f|_{n(Nu/r)\gamma} = f|_{n(Nu/r)}$. Our assertion follows readily.

(iii) It is a direct consequence from (ii) and the fact

$$f \otimes \mathbb{1}_{\mathcal{A}}(z) = \frac{1}{r} \sum_{u \bmod r} e\left(\frac{-au}{r}\right) f\Big|_{n\left(\frac{Nu}{r}\right)}.$$

□

Any cusp form f on $\Gamma(M, N)$ has a Fourier expansion at $i\infty$,

$$f(z) = \sum_{n \geq 1} a_f(n) e(nz/N)$$

where $e(x) = e^{2\pi i x}$. Moreover its associated L-function

$$L(s, f) := \sum_{n \geq 1} \frac{a_f(n)}{n^{(k-1)/2}} n^{-s}$$

is entire and satisfies a functional equation.

Lemma 3.2. *Suppose $f \in S_k(\Gamma(M, N))$ and let $g = f\big|_W$. Then*

$$\left(\frac{N}{2\pi}\right)^s \Gamma(s + \frac{k-1}{2}) L(s, f) = i^k \left(\frac{M}{N}\right)^{\frac{k-1}{2}} \left(\frac{M}{2\pi}\right)^{1-s} \Gamma(1 - s + \frac{k-1}{2}) L(1 - s, g).$$

Proof. For sufficiently large $\Re e\, s$, we have

$$\int_0^\infty f(it) t^{s+(k-1)/2} \frac{dt}{t} = \left(\frac{N}{2\pi}\right)^{s+(k-1)/2} \Gamma(s + \frac{k-1}{2}) L(s, f).$$

The cuspidality of f ensures the absolute convergence of the integral for all $s \in \mathbb{C}$.

Write $g(z) = \sum_{n \geq 1} a_g(n) e(nz/M)$ for $g = f|_W \in S_k(\Gamma(N, M))$ by Lemma 3.1 (i). With a change of variable t into $1/t$, it is apparent that

$$\begin{aligned} \int_0^\infty f(\frac{-1}{it}) t^{-(s+(k-1)/2)} \frac{dt}{t} &= \int_0^\infty (it)^k f\big|_W(it) t^{-(s+(k-1)/2)} \frac{dt}{t} \\ &= i^k \sum_{n \geq 1} a_g(n) \int_0^\infty e^{-2\pi nt/M} t^{1-s+(k-1)/2} \frac{dt}{t} \\ &= i^k \left(\frac{M}{2\pi}\right)^{1-s+(k-1)/2} \Gamma(1 - s + \frac{k-1}{2}) L(1 - s, g), \end{aligned}$$

yielding the functional equation. □

4. Proof of Theorem 1.2

Let us assume more generally $f \in S_k(\Gamma(M, N))$. Write $\mathcal{A} = \mathcal{A}_{a,r}$ for any given $0 \le a<r$. Lemma 3.1 (iii) implies $F := f \otimes \mathbb{1}_{\mathcal{A}} \in S_k(\Gamma(Mr^2, N))$, thus its L-function $L(s, F)$ is entire and satisfies a functional equation with gamma factors. Separating into the two cases of $F = 0$ or not, the following proposition follows from [6, Remark 2 (iii)] with $a_n = a_F(n)/n^{(k-1)/2}$.

Proposition 4.1. *There exist positive constants $c_0 = c_0(F)$ and $x_0 = x_0(F)$ such that $a_F(u)a_F(v) \le 0$ for some $u, v \in (x, x + c_0x^{1/2}]$ for all $x \ge x_0$.*

Arguing as before, we apply the proposition with $a = 0$. There exist constants c_0 and x_0 such that for all $x \ge x_0$, $a_f(n)a_f(n + \ell) \le 0$ for some $n \in (x, x + c_0x^{1/2}] \cap \mathcal{A}_{0,\ell}$ ($\ell = r$ or $2r$). If $a_f(n)a_f(n + r) < 0$ for all $n \in \mathcal{A}_{0,r} \cap (x, x + c_0x^{1/2}]$, then all $a_f(n)$ with $n \in \mathcal{A}_{0,2r} \cap (x, x + c_0x^{1/2}]$ are nonzero and have the same sign, contradicting to Proposition 4.1 for $f \otimes \mathbb{1}_{\mathcal{A}_{0,2r}}$. Now we are done.

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