REPRESENTATIONS OF COHOMOLOGICAL HALL ALGEBRAS AND DONALDSON-THOMAS THEORY WITH CLASSICAL STRUCTURE GROUPS

MATTHEW B. YOUNG

Abstract. We introduce a new class of representations of the cohomological Hall algebras of Kontsevich and Soibelman, which we call cohomological Hall modules (CoHM). These representations are constructed from self-dual representations of a quiver with contravariant involution and can be seen as a mathematical model for the space of BPS states in orientifold string theory. We use the CoHM to define a generalization of cohomological Donaldson-Thomas theory of quivers from structure group GL_n to O_n and Sp_{2n}. We prove the integrality conjecture for orientifold Donaldson-Thomas invariants of σ-symmetric quivers and formulate precise conjectures regarding the geometric meaning of these invariants and their relationship to the structure of the CoHM. The conjectures are proved for zero and one loop quivers and the affine Dynkin quiver of type $\tilde{A}_1$. We also describe the CoHM of finite type quivers by constructing explicit Poincaré-Birkhoff-Witt type bases of these representations.

Contents

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Introduction

Motivation. Motivated by the Donaldson-Thomas theory of three dimensional Calabi-Yau categories, Kontsevich and Soibelman introduced in [23] the cohomological Hall algebra (CoHA) of a quiver with potential. We briefly recall the connection between the CoHA and Donaldson-Thomas theory, leaving details to Section 2. For simplicity we assume that the potential is zero and that the quiver Q is symmetric. Let Λ_Q^+ be the monoid of dimension vectors of Q. Denote by $\mathsf{Vect}_{\mathbb{Z}}$ the category of $\mathbb{Z}$-graded rational vector spaces and by $D(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$ the category of Λ_Q^+-graded objects of the unbounded derived category $D(\mathsf{Vect}_{\mathbb{Z}})$. The CoHA is defined to be the shifted direct sum of cohomology groups of stacks of representations of Q,

$$\mathcal{H}_Q = \bigoplus_{d \in \Lambda_Q^+} H^\bullet(\mathbf{M}_d)\{\chi(d,d)/2\} \in D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$$

where χ is the Euler form of Q and the $\mathbb{Z}$-grading is the Hodge theoretic weight grading. A natural correspondence diagram of stacks makes $\mathcal{H}_Q$ into an associative algebra object of the full subcategory $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+} \subset D(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$ of objects with finite dimensional $\Lambda_Q^+ \times \mathbb{Z}$-homogeneous summands. There exists an object $V_Q^{\mathsf{prim}} \in D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$ such that

$$[\mathrm{Sym}(V_Q^{\mathsf{prim}} \otimes \mathbb{Q}[u])] = [\mathcal{H}_Q] \in K_0(D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}). \tag{1}$$

Here u is an indeterminant of degree $(0,2) \in \Lambda_Q^+ \times \mathbb{Z}$ and $\mathrm{Sym}(V)$ is the free supercommutative algebra on V, the $\mathbb{Z}_2$-grading induced by the $\mathbb{Z}$-grading. The motivic Donaldson-Thomas invariant of Q is defined to be

$$\Omega_{Q,d} = [V_{Q,d}^{\mathsf{prim}}] \in K_0(D^{lb}(\mathsf{Vect}_{\mathbb{Z}})).$$

The integrality conjecture [22], [19] states that in fact

$$\Omega_{Q,d} \in \mathrm{im}\left(K_0(D^b(\mathsf{Vect}_{\mathbb{Z}})) \to K_0(D^{lb}(\mathsf{Vect}_{\mathbb{Z}}))\right).$$

A proof of this conjecture for quivers with potential was given in [23, Theorem 10]. However, positivity of motivic Donaldson-Thomas invariants was not proven.

While the definition of Ω_Q involves only the graded dimensions of $\mathcal{H}_Q$, it is natural to expect that an understanding of the algebra structure of $\mathcal{H}_Q$ may lead to additional insights. Not unrelated, the algebra $\mathcal{H}_Q$ has physical significance: it is a model for the algebra of closed oriented BPS states of a quantum field theory or string theory with extended supersymmetry [17], [23]. In this direction, Efimov constructed [12] a subobject $V_Q^{\mathsf{prim}} \otimes \mathbb{Q}[u] \subset \mathcal{H}_Q$, with V_Q^{prim} having finite dimensional Λ_Q^+-homogeneous summands, such that the canonical map

$$\mathrm{Sym}(V_Q^{\mathsf{prim}} \otimes \mathbb{Q}[u]) \to \mathcal{H}_Q \tag{2}$$

is an algebra isomorphism. Upon passing to Grothendieck rings this confirms the integrality and positivity conjectures. The subobject V_Q^{prim} is a cohomologically refined Donaldson-Thomas invariant in the sense of [36]. For an arbitrary quiver with potential W and generic stability θ, it was recently proved in [8] that the slope μ cohomological Donaldson-Thomas invariant $V_{Q,W,\mu}^{\mathsf{prim},\theta}$ can again be constructed as a subobject of $\mathcal{H}_{Q,W,\mu}^{\theta\text{-}ss}$ and that the analogue of the map (2) is an isomorphism in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$ (or better, with $\mathsf{Vect}_{\mathbb{Z}}$ replaced by a more refined category). Moreover, the integrality conjecture holds in this more general setting. In this way $\mathcal{H}_{Q,W,\mu}^{\theta\text{-}ss}$ acquires a Poincaré-Birkhoff-Witt type basis. The results of [8] rely on an interpretation of $V_{Q,W,\mu}^{\mathsf{prim},\theta}$ in terms of intersection cohomology of quiver moduli [27].

As an application, the structure of $\mathcal{H}^{\theta\text{-}ss}_{Q,W,\mu}$ was used in [6] to give a new proof of the Kač conjecture.

The representation theory of the CoHA is also relevant to Donaldson-Thomas theory. Physical arguments suggest that the space of open BPS states in a theory with defects forms a representation of the BPS algebra [16]. By the work of [4], such representations are expected to be related to CoHA representations constructed from stable framed objects [34]. See also [35]. In the case of quiver categories, framed CoHA representations have been studied in detail [14], [38], [8]. A similar construction, with framed quiver moduli replaced by Nakajima quiver varieties, was given in [39].

In this paper we introduce a new class of CoHA representations constructed using orthogonal and symplectic analogues of quiver representations. While the framing construction models open BPS states, the constructions used in this paper model unoriented BPS states in orientifold string theory. From another (related) point of view, the formalism we consider provides an extension of Donaldson-Thomas theory from structure group $\mathsf{GL}_n(\mathbb{C})$ to the classical groups $\mathsf{O}_n(\mathbb{C})$ and $\mathsf{Sp}_{2n}(\mathbb{C})$, in the following sense. If G is a reductive group, then the derived moduli stack of G-bundles on a Calabi-Yau threefold X has a canonical (-1)-shifted symplectic structure [28, Corollary 2.6]. The truncation therefore has a symmetric perfect obstruction theory [28, §3.2] which could be used to define the G-Donaldson-Thomas invariants of X. The usual Donaldson-Thomas theory arises when $\mathsf{G} = \mathsf{GL}_n(\mathbb{C})$. For orthogonal or symplectic groups, G-bundles on X are precisely the (frame bundles of) self-dual objects of the category of vector bundles on X. More generally, we expect the correct setting for orientifold Donaldson-Thomas theory to be three dimensional Calabi-Yau categories together with a contravariant duality functor which preserves the Calabi-Yau pairings [40]. The CoHA representations introduced below, and the resulting orientifold Donaldson-Thomas invariants, should be seen as an instance of this theory in the case of quivers.

Main results. Let Q be a quiver with contravariant involution σ. Denote by $\Lambda_Q^{\sigma,+} \subset \Lambda_Q^+$ the submonoid of symmetric dimension vectors. Then $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}$ is naturally a left-module category over $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$. After fixing some combinatorial data, the involution σ induces a contravariant duality on the representation category $\mathsf{Rep}_{\mathbb{C}}(Q)$. Denote by $\mathbf{M}_e^\sigma$ the stack of representations of dimension vector $e \in \Lambda_Q^{\sigma,+}$ which are symmetrically isomorphic to their duals (henceforth, self-dual) and set

$$\mathcal{M}_Q = \bigoplus_{e \in \Lambda_Q^{\sigma,+}} H^\bullet(\mathbf{M}_e^\sigma)\{\mathcal{E}(e)/2\} \in D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}.$$

The function $\mathcal{E} : \Lambda_Q^+ \to \mathbb{Z}$ plays the rôle of the Euler form for self-dual representations. Write $\mathbf{M}_{d,e}^\sigma$ for the stack of flags of representations $U \subset M$ with M self-dual, U isotropic in M and $\mathbf{dim}\, U = d$, $\mathbf{dim}\, M = d + \sigma(d) + e$. The correspondence

$$\begin{array}{ccccc} \mathbf{M}_d \times \mathbf{M}_e^\sigma & \leftarrow & \mathbf{M}_{d,e}^\sigma & \rightarrow & \mathbf{M}^\sigma_{d+\sigma(d)+e} \\ (U, M/\!\!/U) & \leftarrow\!\shortmid & U \subset M & \mapsto & M \end{array}$$

where $/\!\!/$ is a categorical version of symplectic reduction, can be used to give $\mathcal{M}_Q$ the structure of a left $\mathcal{H}_Q$-module object in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}$. See Theorem 3.1. We call $\mathcal{M}_Q$ the cohomological Hall module (CoHM). In Theorem 3.3 we prove that $\mathcal{M}_Q$ has an explicit combinatorial description as a signed shuffle module, analogous to the Feĭgin-Odesskiĭ shuffle algebra structure of $\mathcal{H}_Q$ [23]. This result is crucial for both the computational and theoretical aspects of the paper.

Suppose for simplicity that Q is σ-symmetric. This condition is stronger than symmetry of Q but appears naturally when considering quivers with involution. Let

$W_Q^{\mathsf{prim}} \subset \mathcal{M}_Q$ be a minimal generating subobject with respect to the $\mathcal{H}_Q$-module structure and define the orientifold Donaldson-Thomas invariant by

$$\Omega^{\sigma}_{Q,e} = [W^{\mathsf{prim}}_{Q,e}] \in K_0(D^{lb}(\mathsf{Vect}_{\mathbb{Z}})).$$

Our first main result is the following.

Theorem A (Theorem 3.4)**.** *If Q is σ-symmetric, then the integrality conjecture holds for $\mathcal{M}_Q$. More precisely, for all $e \in \Lambda_Q^{\sigma,+}$ we have*

$$\Omega^{\sigma}_{Q,e} \in \operatorname{im}\big(K_0(D^b(\mathsf{Vect}_{\mathbb{Z}})) \hookrightarrow K_0(D^{lb}(\mathsf{Vect}_{\mathbb{Z}}))\big).$$

The proof is similar to Efimov's proof [12] of the integrality conjecture for $\mathcal{H}_Q$ and relies on the explicit shuffle description of $\mathcal{M}_Q$. Positivity of orientifold Donaldson-Thomas invariants follows immediately from their definition.

We next focus on the analogue of the map (2). The situation is more complicated than that of the CoHA since $\mathcal{M}_Q$ is very far from being a free $\mathcal{H}_Q$-module. Instead, we formulate the following conjecture.

Conjecture A (Conjectures 3.6 and 3.8)**.** *Let Q be σ-symmetric and assume that $\mathcal{H}_Q$ is supercommutative without any twist. There exist $\Lambda_Q^{\sigma,+} \times \mathbb{Z}$-graded subalgebras $\mathcal{H}_Q(e) \subset \mathcal{H}_Q$, $e \in \Lambda_Q^{\sigma,+}$, such that the CoHA action map*

$$\bigoplus_{e \in \Lambda_Q^{\sigma,+}} \mathcal{H}_Q(e) \boxtimes W^{\mathsf{prim}}_{Q,e} \xrightarrow{\star} \mathcal{M}_Q$$

is an isomorphism in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}$. Moreover, the restriction to the summand $\mathcal{H}_Q(e) \boxtimes W^{\mathsf{prim}}_{Q,e}$ is a $\mathcal{H}_Q(e)$-module isomorphism onto its image.

Each subalgebra $\mathcal{H}_Q(e)$ is explicitly defined and is, roughly, a free supercommutative algebra on the pure cohomology of an e-dependent $\mathbb{Z}_2$-quotient of the stack of stable quiver representations. Passing to Grothendieck rings, Conjecture A implies an orientifold analogue of the factorization (1),

$$\sum_{e \in \Lambda_Q^{\sigma,+}} [\mathcal{H}_Q(e)] \cdot \Omega^{\sigma}_{Q,e} = [\mathcal{M}_Q] \in K_0(D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}).$$

In this case of loop quivers this equation can be used to compute Ω^{σ}_Q from Ω_Q. In general, $\mathbb{Z}_2$-equivariant refinements of Ω_Q are needed to compute Ω^{σ}_Q.

We also formulate a conjectural geometric interpretation of orientifold Donaldson-Thomas invariants. Let $\mathfrak{M}_e^{\sigma,st}$ be the moduli space of stable self-dual representations of dimension vector e and let $PH^{\bullet}(\mathfrak{M}_e^{\sigma,st})$ be the pure part of its cohomology.

Conjecture B (Conjecture 3.11)**.** *If Q is σ-symmetric, then there is a canonical isomorphism*

$$W^{\mathsf{prim}}_{Q,e} \simeq PH^{\bullet}(\mathfrak{M}_e^{\sigma,st})\{\mathcal{E}(e)/2\}.$$

The analogue of Conjecture B for Donaldson-Thomas invariants was proved by Chen [3]. As initial evidence for Conjecture B, in Proposition 3.10 we construct a surjection $W^{\mathsf{prim}}_{Q,e} \twoheadrightarrow PH^{\bullet}(\mathfrak{M}_e^{\sigma,st})\{\mathcal{E}(e)/2\}$.

In Section 4 we study in detail a number of examples of $\mathcal{M}_Q$ for σ-symmetric. The main results can be summarized as follows.

Theorem B (Theorems 4.2, 4.5, 4.8 and 4.11)**.** *Conjectures A and B hold for disjoint union quivers, zero and one loop quivers and the symmetric orientation of the affine Dynkin quiver of type $\tilde{A}_1$.*

In each case we explicitly compute all orientifold Donaldson-Thomas invariants and describe the module structure of $\mathcal{M}_Q$. In contrast to the case of Donaldson-Thomas invariants, in some of these examples there are already infinitely many non-zero orientifold Donaldson-Thomas invariants.

In Section 5 we study the CoHM of a finite type quiver with involution. As these quivers are not σ-symmetric, their CoHM have a rather different structure than those of σ-symmetric quivers. The non-trivial task is to describe the CoHM of Dynkin type A quivers.

Theorem C (Theorem 5.8). *Let Q be a Dynkin type A quiver with involution. Then $\mathcal{M}_Q$ admits two Poincaré-Birkhoff-Witt type bases, each of which is determined by a simple/indecomposable Poincaré-Birkhoff-Witt type basis of $\mathcal{H}_Q$ and the set of simple/indecomposable self-dual representations of Q.*

Theorem C categorifies the orientifold quantum dilogarithm identities found in [41]. To prove Theorem C we develop a modification of Rimányi's approach to the study of the CoHA of a finite type quiver [32]. Along the way we prove a number of results that are of independent interest. For example, in Corollary 5.6 we prove that Thom polynomials of orbit closures of self-dual quiver representations appear as structure constants of the CoHM.

In this paper we have made calculations only in the case of zero potential; see however Section 3.5 for the construction of the critical CoHM. There are also a number of expected applications which we have not discussed. Perhaps the most exciting is the connection between the CoHA and the cohomology of character varieties for $\mathsf{GL}_n(\mathbb{C})$ [7]. It is natural to expect a connection between the corresponding CoHM and character varieties associated to the groups $\mathsf{O}_n(\mathbb{C})$ and $\mathsf{Sp}_{2n}(\mathbb{C})$.

Notation. All cohomology groups have $\mathbb{Q}$ coefficients and, unless explicitly mentioned otherwise, all tensor products are over $\mathbb{Q}$.

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1. Background material

1.1. **Classical groups.** We fix some notation regarding the classical groups. Each such group G_n is the automorphism group of a pair $(V_n, \langle \cdot, \cdot \rangle)$ consisting of a finite dimensional complex vector space with nondegenerate bilinear form.

(1) Types B_n and D_n. Let $V_n = \mathbb{C}^{2n+1}$ with basis $x_1, \dots, x_n, w, y_1, \dots, y_n$ in type B_n and $V_n = \mathbb{C}^{2n}$ with basis $x_1, \dots, x_n, y_1, \dots, y_n$ in type D_n. Define a symmetric bilinear form in this basis by $\langle x_i, y_j \rangle = \delta_{i,j}$, and $\langle w, w \rangle = 1$ in type B_n, all other pairings being zero. Then G_n is the orthogonal group $\mathsf{O}_{2n+1}(\mathbb{C})$ or $\mathsf{O}_{2n}(\mathbb{C})$. It is important in what follows that we use the full orthogonal group and not the special orthogonal group.

(2) Type C_n. Let $V_n = \mathbb{C}^{2n}$ with basis $x_1, \dots, x_n, y_1, \dots, y_n$ and skew-symmetric bilinear form determined by $\langle x_i, y_j \rangle = \delta_{i,j}$, all other pairings between basis vectors being zero. Then G_n is the symplectic group $\mathsf{Sp}_{2n}(\mathbb{C})$.

Consider the maximal torus

$$\mathsf{T}_n = \{\mathrm{diag}(t_1, \dots, t_n, (1), t_1^{-1}, \dots, t_n^{-1}) \mid t_i \in \mathbb{C}^{\times}\} \subset \mathsf{G}_n,$$

omitting the middle 1 except in type B_n. For each $1 \leq i \leq n$ define a character $e_i : \mathsf{T}_n \to \mathbb{C}^\times$ by $t \mapsto t_i$. Then the positive roots are

$$\begin{array}{ll} \text{Type } B_n: & \Delta = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \sqcup \{e_i \mid 1 \leq i \leq n\} \\ \text{Type } C_n: & \Delta = \{e_i \pm e_j \mid 1 \leq i < j \leq n\} \sqcup \{2e_i \mid 1 \leq i \leq n\} \\ \text{Type } D_n: & \Delta = \{e_i \pm e_j \mid 1 \leq i < j \leq n\}. \end{array}$$

The Weyl groups $W_{\mathsf{G}_n} = N_{\mathsf{G}_n}(\mathsf{T}_n)/\mathsf{T}_n$ are

$$W_{\mathsf{O}_{2n+1}} \simeq (\mathbb{Z}_2^n \rtimes \mathfrak{S}_n) \times \mathbb{Z}_2, \qquad W_{\mathsf{Sp}_{2n}} \simeq \mathbb{Z}_2^n \rtimes \mathfrak{S}_n, \qquad W_{\mathsf{O}_{2n}} \simeq \mathbb{Z}_2^n \rtimes \mathfrak{S}_n$$

with $\mathfrak{S}_n$ the symmetric group on n letters.

1.2. **Quiver representations.** Let Q be a quiver with finite sets of nodes Q_0 and arrows Q_1. Write $\alpha : i \to j$ for an arrow α with tail i and head j. Let $\mathsf{Rep}_{\mathbb{C}}(Q)$ be the hereditary abelian category of finite dimensional complex representations of Q. Objects of $\mathsf{Rep}_{\mathbb{C}}(Q)$ are pairs (U, u), often denoted by just U, where $U = \bigoplus_{i \in Q_0} U_i$ is a finite dimensional Q_0-graded complex vector space and $u = \{U_i \xrightarrow{u_\alpha} U_j\}_{i \xrightarrow{\alpha} j \in Q_1}$ is a collection of linear maps. Let $\Lambda_Q^+ = \mathbb{Z}_{\geq 0} Q_0$ be the abelian monoid dimension vectors. Set also $\Lambda_Q = \mathbb{Z} Q_0$.

The Euler form of $\mathsf{Rep}_{\mathbb{C}}(Q)$ is

$$\chi(U, V) = \dim_{\mathbb{C}} \operatorname{Hom}(U, V) - \dim_{\mathbb{C}} \operatorname{Ext}^1(U, V).$$

It descends to a bilinear form on Λ_Q which has the explicit expression

$$\chi(d, d') = \sum_{i \in Q_0} d_i d'_i - \sum_{i \xrightarrow{\alpha} j \in Q_1} d_i d'_j.$$

For each $d \in \Lambda_Q^+$ let $R_d = \bigoplus_{i \xrightarrow{\alpha} j} \operatorname{Hom}_{\mathbb{C}}(\mathbb{C}^{d_i}, \mathbb{C}^{d_j})$. The algebraic group $\mathsf{GL}_d = \prod_{i \in Q_0} \mathsf{GL}_{d_i}(\mathbb{C})$ acts linearly on R_d by change of basis. The GL_d-orbits of R_d are in bijection with the isomorphism classes of representations of dimension vector d.

1.3. **Self-dual quiver representations.** For a detailed discussion of self-dual quiver representations see [10], [42, §3.2].

An involution σ of a quiver Q is a pair of involutions

$$\sigma : Q_0 \to Q_0, \qquad \sigma : Q_1 \to Q_1$$

such that

(i) if $i \xrightarrow{\alpha} j \in Q_1$, then $\sigma(j) \xrightarrow{\sigma(\alpha)} \sigma(i) \in Q_1$, and
(ii) if $i \xrightarrow{\alpha} \sigma(i) \in Q_1$, then $\alpha = \sigma(\alpha)$.

Given an involution, let Λ_Q^σ be the subgroup of fixed points of the induced involution $\sigma : \Lambda_Q \to \Lambda_Q$. Set also $\Lambda_Q^{\sigma,+} = \Lambda_Q^+ \cap \Lambda_Q^\sigma$. The group homomorphism

$$H : \Lambda_Q \to \Lambda_Q^\sigma, \qquad d \mapsto d + \sigma(d)$$

makes Λ_Q^σ into a Λ_Q-module.

A duality structure on (Q, σ) is a pair of functions

$$s : Q_0 \to \{\pm 1\}, \qquad \tau : Q_1 \to \{\pm 1\}$$

such that s is σ-invariant and $\tau_\alpha \tau_{\sigma(\alpha)} = s_i s_j$ for every arrow $i \xrightarrow{\alpha} j$. Given a duality structure we define an exact contravariant functor $S : \mathsf{Rep}_{\mathbb{C}}(Q) \to \mathsf{Rep}_{\mathbb{C}}(Q)$ as follows. At the level of objects S is given by

$$S(U)_i = U_{\sigma(i)}^\vee, \qquad S(u)_\alpha = \tau_\alpha u_{\sigma(\alpha)}^\vee.$$

Here $(-)^\vee = \operatorname{Hom}_{\mathbb{C}}(-, \mathbb{C})$ is the linear duality functor on the category of finite dimensional complex vector spaces. If $\phi : U \to U'$ is a morphism, then $S(\phi) : S(U') \to S(U)$ has components $S(\phi)_i = \phi_{\sigma(i)}^\vee$. Setting $\Theta_U = \oplus_{i \in Q_0} s_i \cdot \mathrm{ev}_{U_i}$, with

ev_V the canonical evaluation isomorphism from a finite dimensional vector space V to its double dual $V^{\vee\vee}$, defines an isomorphism of functors $\Theta : \mathbf{1}_{\mathsf{Rep}(Q)} \xrightarrow{\sim} S^2$ which satisfies $S(\Theta_U)\Theta_{S(U)} = \mathbf{1}_{S(U)}$. The triple $(\mathsf{Rep}_{\mathbb{C}}(Q), S, \Theta)$ is therefore an abelian category with duality in the sense of [1].

A self-dual representation is a pair (M, ψ_M) consisting of a representation M and an isomorphism $\psi_M : M \xrightarrow{\sim} S(M)$ which satisfies $S(\psi_M)\Theta_M = \psi_M$. Geometrically, a self-dual representation is a representation M together with a nondegenerate bilinear form $\langle \cdot, \cdot \rangle$ such that

(i) M_i and M_j are orthogonal unless $i = \sigma(j)$,
(ii) the restriction of $\langle \cdot, \cdot \rangle$ to $M_i + M_{\sigma(i)}$ satisfies $\langle x, x' \rangle = s_i \langle x', x \rangle$, and
(iii) for all arrows $i \xrightarrow{\alpha} j$ the structure maps of M satisfy

$$\langle m_\alpha x, x' \rangle - \tau_\alpha \langle x, m_{\sigma(\alpha)} x' \rangle = 0, \qquad x \in M_i, \ x' \in M_{\sigma(j)}. \tag{3}$$

Fix a partition $Q_0 = Q_0^- \sqcup Q_0^\sigma \sqcup Q_0^+$ such that Q_0^σ consists of the nodes fixed by σ and $\sigma(Q_0^-) = Q_0^+$. Similarly, fix a partition $Q_1 = Q_1^- \sqcup Q_1^\sigma \sqcup Q_1^+$.

Let $e \in \Lambda_Q^{\sigma,+}$ with e_i even for all $i \in Q_0^\sigma$ with $s_i = -1$. The trivial representation of dimension vector e admits a self-dual structure $\langle \cdot, \cdot \rangle$ which is unique up to Q_0-graded isometry. Denote by $R_e^\sigma \subset R_e$ the linear subspace of representations whose structure maps satisfy equation (3) with respect to $\langle \cdot, \cdot \rangle$. There is an isomorphism

$$R_e^\sigma \simeq \bigoplus_{i \xrightarrow{\alpha} j \in Q_1^+} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{e_i}, \mathbb{C}^{e_j}) \oplus \bigoplus_{i \xrightarrow{\alpha} \sigma(i) \in Q_1^\sigma} \mathrm{Bil}^{s_i \tau_\alpha}(\mathbb{C}^{e_i})$$

where $\mathrm{Bil}^\epsilon(\mathbb{C}^{e_i})$ denotes the vector space of symmetric ($\epsilon = 1$) or skew-symmetric ($\epsilon = -1$) bilinear forms on $\mathbb{C}^{e_i}$. The subgroup $\mathsf{G}_e^\sigma \subset \mathsf{GL}_e$ which preserves $\langle \cdot, \cdot \rangle$ is

$$\mathsf{G}_e^\sigma \simeq \prod_{i \in Q_0^+} \mathsf{GL}_{e_i}(\mathbb{C}) \times \prod_{i \in Q_0^\sigma} \mathsf{G}_{e_i}^{s_i}$$

where

$$\mathsf{G}_{e_i}^{s_i} = \begin{cases} \mathsf{Sp}_{e_i}(\mathbb{C}), & \text{if } s_i = -1 \\ \mathsf{O}_{e_i}(\mathbb{C}), & \text{if } s_i = 1. \end{cases}$$

The group G_e^σ acts linearly on R_e^σ with orbits in bijection with isometry classes of self-dual representations of dimension vector e.

Let M be a self-dual representation with isotropic subrepresentation $U \subset M$. Then the orthogonal complement $U^\perp \subset M$ is a subrepresentation which contains U and the quotient $M /\!\!/ U = U^\perp / U$ inherits a canonical self-dual structure.

Example. Let $U \in \mathsf{Rep}_{\mathbb{C}}(Q)$. The hyperbolic representation $H(U)$ is the self-dual structure on $U \oplus S(U)$ given by $\psi_{H(U)} = \left(\begin{smallmatrix} 0 & \mathbf{1}_{S(U)} \\ \Theta_U & 0 \end{smallmatrix}\right)$. ◁

For any $U \in \mathsf{Rep}_{\mathbb{C}}(Q)$, the pair (S, Θ) determines a linear $\mathbb{Z}_2$-action on $\mathrm{Ext}^i(S(U), U)$. Write $\mathrm{Ext}^i(S(U), U)^{\pm S}$ for the subspace of (anti-)invariants and define

$$\mathcal{E}(U) = \dim_{\mathbb{C}} \mathrm{Hom}(S(U), U)^{-S} - \dim_{\mathbb{C}} \mathrm{Ext}^1(S(U), U)^S.$$

It was proved in [42, Proposition 3.3] that $\mathcal{E}(U)$ depends only on the dimension vector of U and that the resulting function $\mathcal{E} : \Lambda_Q \to \mathbb{Z}$ is given by

$$\mathcal{E}(d) = \sum_{i \in Q_0^\sigma} \frac{d_i(d_i - s_i)}{2} + \sum_{i \in Q_0^+} d_{\sigma(i)} d_i - \\ \sum_{\sigma(i) \xrightarrow{\alpha} i \in Q_1^\sigma} \frac{d_i(d_i + \tau_\alpha s_i)}{2} - \sum_{i \xrightarrow{\alpha} j \in Q_1^+} d_{\sigma(i)} d_j. \tag{4}$$

The function $\mathcal{E}$ satisfies the identity

$$\mathcal{E}(d+d') = \mathcal{E}(d) + \mathcal{E}(d') + \chi(\sigma(d), d'), \qquad d, d' \in \Lambda_Q. \tag{5}$$

Following [22], to each quiver we associate a quantum torus $\hat{\mathbb{T}}_Q = \mathbb{Q}(q^{\frac{1}{2}})[\![\Lambda_Q^+]\!]$, the $\mathbb{Q}(q^{\frac{1}{2}})$-vector space with topological basis $\{t^d \mid d \in \Lambda_Q^+\}$ and multiplication

$$t^d \cdot t^{d'} = q^{\frac{1}{2}(\chi(d,d') - \chi(d',d))} t^{d+d'}.$$

As in [41, §4.1], for a fixed duality structure we will also consider the vector space $\hat{\mathbb{S}}_Q = \mathbb{Q}(q^{\frac{1}{2}})[\![\Lambda_Q^{\sigma,+}]\!]$ with topological basis $\{\xi^e \mid e \in \Lambda_Q^{\sigma,+}\}$. The formula

$$t^d \star \xi^e = q^{\frac{1}{2}(\chi(d,e) - \chi(e,d) + \mathcal{E}(\sigma(d)) - \mathcal{E}(d))} \xi^{H(d)+e}$$

gives $\hat{\mathbb{S}}_Q$ the structure of a left $\hat{\mathbb{T}}_Q$-module.

Finally, we recall how the theory of stability of quiver representations [21] can be adapted to the self-dual setting. For details see [41, §3]. A stability $\theta \in \mathrm{Hom}_{\mathbb{Z}}(\Lambda_Q, \mathbb{Z})$ is called σ-compatible if it satisfies $\sigma^* \theta = -\theta$. Fix a σ-compatible stability θ. A self-dual representation M is called σ-semistable if $\mu(U) \leq \mu(M)$ for all non-zero isotropic subrepresentations $U \subset M$; if this inequality is strict then M is called σ-stable. Here $\mu(U) = \frac{\theta(\mathbf{dim}\, U)}{\dim U}$ is the slope of U. The slope of a self-dual representation is necessarily zero.

The moduli space of σ-semistable self-dual representations of dimension vector e is the θ-linearized geometric invariant theory quotient $\mathfrak{M}_e^{\sigma,\theta} = R_e^{\sigma} /\!\!/_{\theta} \mathsf{G}_e^{\sigma}$. It parameterizes S-equivalence classes of σ-semistable representations. There is an open subvariety $\mathfrak{M}_e^{\sigma,\theta\text{-}st} \subset \mathfrak{M}_e^{\sigma,\theta}$ parameterizing isometry classes of σ-stable representations. In general, $\mathfrak{M}_e^{\sigma,\theta\text{-}st}$ is an orbifold. A σ-stable representation M can be written uniquely as an orthogonal direct sum $M = \bigoplus_{i=1}^k M_i$, where M_i are pairwise non-isometric self-dual representations which are stable as ordinary representations [41, Proposition 3.5]. In this case $\mathrm{Aut}_S(M) \simeq \mathbb{Z}_2^k$. If $k = 1$, then M is called regularly σ-stable and gives a smooth point of $\mathfrak{M}_e^{\sigma,\theta\text{-}st}$. By convention we set $\mathfrak{M}_0^{\sigma,\theta\text{-}st} = pt$.

Remark. The bounded derived category of the Ginzburg dg algebra associated to Q, denoted $D^b_{fd}(\Gamma_Q\text{-}\mathsf{mod})$, is a three dimensional triangulated Calabi-Yau category for which $\mathsf{Rep}_{\mathbb{C}}(Q)$ is the heart of a bounded t-structure [15]. A duality structure on Q induces a triangulated duality structure on $D^b(\Gamma_Q\text{-}\mathsf{mod})$ which, up to a sign, preserves the Calabi-Yau pairing. This gives an abstract version of the three dimensional Calabi-Yau orientifolds considered in the string theory literature.

1.4. **Equivariant cohomology.** Fix an integer $n > 0$. If $N > n$, then the variety $M^*_{N,n}$ of complex $N \times n$ matrices of rank n is $2(N-n)$-connected and carries a free right action of GL_n. The quotients $M^*_{N,n} \to M^*_{N,n}/\mathsf{GL}_n$ form an injective system $\{E_N \to B_N\}_{N>n}$ of finite dimensional approximations by varieties to the universal GL_n-bundle $E\mathsf{GL}_n \to B\mathsf{GL}_n$. More generally, if G is a linear algebraic group with a closed embedding $\mathsf{G} \hookrightarrow \mathsf{GL}_n$, then $\{E_N \to E_N/\mathsf{G}\}_{N>n}$ approximates $E\mathsf{G} \to B\mathsf{G}$. If $\mathsf{H} \subset \mathsf{G}$ is a closed subgroup, then the canonical morphism $B\mathsf{H} \to B\mathsf{G}$ is a fibration with fibre G/H.

Suppose that G acts on a variety X. Then the G-equivariant cohomology of X is defined to be

$$H^\bullet_{\mathsf{G}}(X) = \varprojlim H^\bullet(X \times_{\mathsf{G}} E_N; \mathbb{Q}). \tag{6}$$

Here $H^\bullet(-;\mathbb{Q})$ denotes singular cohomology with rational coefficients.

We write $H^\bullet_{\mathsf{G}}$ for $H^\bullet_{\mathsf{G}}(pt)$. If $\mathsf{T}_{\mathsf{GL}_n} \subset \mathsf{GL}_n$ denotes the diagonal maximal torus, then there are ring isomorphisms

$$H^\bullet_{\mathsf{GL}_n} \simeq H^\bullet(B\mathsf{T}_{\mathsf{GL}_n})^{W_{\mathsf{GL}_n}} \simeq \mathbb{Q}[x_1, \dots, x_n]^{\mathfrak{S}_n}.$$

Similarly, if G_n is a classical group of type B_n, C_n or D_n, then the inclusion $\mathsf{T}_n \hookrightarrow \mathsf{G}_n$ induces ring isomorphisms

$$H^\bullet_{\mathsf{G}_n} \simeq H^\bullet(B\mathsf{T}_n)^{W_{\mathsf{G}_n}} \simeq \mathbb{Q}[z_1^2, \dots, z_n^2]^{\mathfrak{S}_n}. \tag{7}$$

Here it is essential that G_n is the full orthogonal group in type D_n. The generators x_i, z_i have cohomological degree two.

We record the following results for later use.

Lemma 1.1.

(1) *Let $\phi : \mathsf{GL}_n \to \mathsf{GL}_n$ be the automorphism $\phi(g) = (g^{-1})^t$. The induced map $(B\phi)^* : H^\bullet_{\mathsf{GL}_n} \to H^\bullet_{\mathsf{GL}_n}$ is given by*

$$(B\phi)^* f(x_1, \dots, x_n) = f(-x_1, \dots, -x_n).$$

(2) *Let $h : \mathsf{GL}_n \hookrightarrow \mathsf{G}_n$ be the hyperbolic embedding. The induced map $(Bh)^* : H^\bullet_{\mathsf{G}_n} \to H^\bullet_{\mathsf{GL}_n}$ is given by $(Bh)^* z_i = x_i$.*

(3) *Let $\iota : \mathsf{G}_n \hookrightarrow \mathsf{GL}_{2n+\epsilon}$ be the embedding arising from the description of G_n given in Section 1.1, where $\epsilon = 1$ in type B_n and $\epsilon = 0$ otherwise. Under the identification*

$$H^\bullet_{\mathsf{GL}_{2n+\epsilon}} \simeq \mathbb{Q}[x_1, \dots, x_n, y_1, \dots, y_n, (w)]^{\mathfrak{S}_{2n+\epsilon}}$$

the induced map $(B\iota)^* : H^\bullet_{\mathsf{GL}_{2n+\epsilon}} \to H^\bullet_{\mathsf{G}_n}$ is given by

$$(B\iota)^* x_i = z_i, \qquad (B\iota)^* y_i = -z_i, \qquad (B\iota)^* w = 0.$$

Finally, recall that $H^\bullet_{\mathsf{G}}(X)$ (and the compactly supported variant $H^\bullet_{c,\mathsf{G}}(X)$) has a canonical mixed Hodge structure [9]. The pure part of $H^\bullet_{\mathsf{G}}(X)$ is

$$PH^\bullet_{\mathsf{G}}(X) = \bigoplus_{k \geq 0} W_k H^k_{\mathsf{G}}(X)$$

where $0 = W_{-1} \subset W_0 \subset \cdots \subset W_{2k} = H^k_{\mathsf{G}}(X)$ is the weight filtration.

2. Cohomological Hall algebras

2.1. **Definition of the CoHA.** We recall some material from [23, §2].

Fix a quiver Q. Let $\mathsf{Vect}_{\mathbb{Z}}$ be the abelian category of finite dimensional $\mathbb{Z}$-graded rational vector spaces. Write $D^{lb}(\mathsf{Vect}_{\mathbb{Z}}) \subset D(\mathsf{Vect}_{\mathbb{Z}})$ for the full subcategory of objects whose cohomological and $\mathbb{Z}$ degrees are bounded from below. Let also $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$ be the category whose objects are Λ_Q^+-graded objects of $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})$ with finite dimensional $\Lambda_Q^+ \times \mathbb{Z}$-homogeneous summands and whose morphisms preserve the $\Lambda_Q^+ \times \mathbb{Z}$-grading. Define a monoidal product $\boxtimes^{\mathsf{tw}}$ on $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$ by

$$\bigoplus_{d \in \Lambda_Q^+} \mathcal{U}_d \boxtimes^{\mathsf{tw}} \bigoplus_{d \in \Lambda_Q^+} \mathcal{V}_d = \bigoplus_{d \in \Lambda_Q^+} \Big(\bigoplus_{d = d' + d''} \mathcal{U}_{d'} \otimes \mathcal{V}_{d''} \Big) \{(\chi(d', d'') - \chi(d'', d'))/2\}.$$

Here $\{\frac{1}{2}\}$ denotes tensor product with the one dimensional vector space of cohomological and $\mathbb{Z}$ degree -1.

Let $d', d'' \in \Lambda_Q^+$ and put $d = d' + d''$. Write $\mathbb{C}^{d'} \subset \mathbb{C}^d$ for the Q_0-graded subspace spanned by the first d' coordinate directions. Let $R_{d',d''} \subset R_d$ be the subspace of representations which preserve $\mathbb{C}^{d'}$ and let $\mathsf{GL}_{d',d''} \subset \mathsf{GL}_d$ be the parabolic subgroup which preserves $\mathbb{C}^{d'}$. The cohomological Hall algebra (henceforth CoHA) of Q is

$$\mathcal{H}_Q = \bigoplus_{d \in \Lambda_Q^+} H^\bullet_{\mathsf{GL}_d}(R_d)\{\chi(d,d)/2\} \in D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}.$$

The $\mathbb{Z}$-grading is the Hodge theoretic weight grading and coincides with the cohomological degree by purity. Define a multiplication $\mathcal{H}_Q \boxtimes^{\mathsf{tw}} \mathcal{H}_Q \to \mathcal{H}_Q$ by requiring its restriction to $\mathcal{H}_{Q,d'} \boxtimes^{\mathsf{tw}} \mathcal{H}_{Q,d''}$ to be the composition

$$\begin{aligned} &H^{\bullet}_{\mathsf{GL}_{d'}}(R_{d'}) \otimes H^{\bullet}_{\mathsf{GL}_{d''}}(R_{d''}) \xrightarrow{\sim} H^{\bullet}_{\mathsf{GL}_{d'} \times \mathsf{GL}_{d''}}(R_{d'} \times R_{d''}) \xrightarrow{\sim} \\ &\qquad H^{\bullet}_{\mathsf{GL}_{d',d''}}(R_{d',d''}) \to H^{\bullet}_{\mathsf{GL}_{d',d''}}(R_d)\{(2\Delta_1)/2\} \to H^{\bullet}_{\mathsf{GL}_d}(R_d)\{(2\Delta_1 + 2\Delta_2)/2\}, \end{aligned}$$

where for ease of notation the degree shifts in $\mathcal{H}_{Q,d}$ and $\boxtimes^{\mathsf{tw}}$ are omitted. The maps in the composition are defined using the morphisms

$$R_{d'} \times R_{d''} \xleftarrow{\pi} R_{d',d''} \xhookrightarrow{i} R_d, \qquad \mathsf{GL}_{d'} \times \mathsf{GL}_{d''} \xleftarrow{p} \mathsf{GL}_{d',d''} \xhookrightarrow{j} \mathsf{GL}_d. \tag{8}$$

The first map in the CoHA multiplication is the Künneth map, the second is induced by the homotopy equivalences π and p, the third is the pushforward along the $\mathsf{GL}_{d',d''}$-equivariant inclusion i and the last is the pushforward along the fundamental class of $\mathsf{GL}_d/\mathsf{GL}_{d',d''}$. The degree shift is $\Delta_1 + \Delta_2 = -\chi(d', d'')$. It is shown in [23, Theorem 1] that this product gives $\mathcal{H}_Q$ the structure of an associative algebra object in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$.

The CoHA product can be written explicitly using localization in equivariant cohomology. To do so, identify $\mathcal{H}_{Q,d}$ with the vector space of polynomials in variables $\{x_{i,1}, \dots, x_{i,d_i}\}_{i \in Q_0}$ which are invariant under the Weyl group $W_{\mathsf{GL}_d} \simeq \mathfrak{S}_d = \prod_{i \in Q_0} \mathfrak{S}_{d_i}$. The product of $f_1 \in \mathcal{H}_{Q,d'}$ and $f_2 \in \mathcal{H}_{Q,d''}$ will be viewed as a polynomial in $\{x_{i,1}, \dots, x_{i,d_i}\}_{i \in Q_0}$ by identifying $x'_{i,k}$ and $x''_{i,k}$ with $x_{i,k}$ and x_{i,d'_i+k}, respectively. Let $\mathfrak{sh}_{d',d''} \subset \mathfrak{S}_d$ be the set of 2-shuffles of type (d', d''), that is, elements $\{\pi_i\}_{i \in Q_0} \in \mathfrak{S}_d$ which satisfy

$$\pi_i(1) < \dots < \pi_i(d'_i), \qquad \pi_i(d'_i + 1) < \dots < \pi_i(d_i), \quad i \in Q_0.$$

Then $\mathfrak{sh}_{d',d''}$ acts on polynomials in $\{x_{i,1}, \dots, x_{i,d_i}\}_{i \in Q_0}$ via the action of $\mathfrak{S}_d$.

Theorem 2.1 ([23, Theorem 2]). *The CoHA product of $f_1 \in \mathcal{H}_{Q,d'}$ and $f_2 \in \mathcal{H}_{Q,d''}$ is given by*

$$f_1 \cdot f_2 = \sum_{\pi \in \mathfrak{sh}_{d',d''}} \pi \left(f_1(x') f_2(x'') \frac{\prod_{i \xrightarrow{\alpha} j \in Q_1} \prod_{b=1}^{d''_j} \prod_{a=1}^{d'_i} \left(x''_{j,b} - x'_{i,a} \right)}{\prod_{i \in Q_0} \prod_{b=1}^{d''_i} \prod_{a=1}^{d'_i} \left(x''_{i,b} - x'_{i,a} \right)} \right).$$

The motivic DT series of Q is the class of $\mathcal{H}_Q$ in the Grothendieck ring of $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$,

$$A_Q(q^{\frac{1}{2}}, t) = \sum_{(d,k) \in \Lambda_Q^+ \times \mathbb{Z}} \dim_{\mathbb{Q}} \mathcal{H}_{Q,(d,k)} (-q^{\frac{1}{2}})^k t^d \in \mathbb{Z}[\![q^{\frac{1}{2}}, \Lambda_Q^+]\!].$$

It can be written explicitly as

$$A_Q(q^{\frac{1}{2}}, t) = \sum_{d \in \Lambda_Q^+} \frac{(-q^{\frac{1}{2}})^{\chi(d,d)}}{\prod_{i \in Q_0} \prod_{j=1}^{d_i} (1 - q^j)} t^d.$$

The series A_Q is naturally viewed as an element of the quantum torus $\hat{\mathbb{T}}_Q$ since the product in the latter agrees with the product induced by $\boxtimes^{\mathsf{tw}}$. Passing from motivic DT series to motivic DT invariants is most easily explained in the case of symmetric quivers. We do this in the next section.

2.2. **The CoHA of a symmetric quiver.** A quiver is called symmetric if its Euler form is a symmetric bilinear form. Throughout this section we assume that Q is symmetric. In this case $\boxtimes^{\mathsf{tw}}$ reduces to the standard symmetric monoidal product $\boxtimes$ on $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$ and $\mathcal{H}_Q$ can be considered as a $\Lambda_Q^+ \times \mathbb{Z}$-graded algebra.

Define a $\mathbb{Z}_2$-grading on $\mathcal{H}_Q$ by the reduction modulo two of the shifted cohomological degree. If the Euler form satisfies

$$\chi(d, d') \equiv \chi(d, d)\chi(d', d') \mod 2 \tag{9}$$

for all $d, d' \in \Lambda_Q^+$, then $\mathcal{H}_Q$ is a supercommutative algebra. Writing a_{ij} for the number of arrows from i to j, equation (9) holds if and only if

$$a_{ij} \equiv (1 + a_{ii})(1 + a_{jj}) \mod 2$$

for all $i, j \in Q_0$ with $i \neq j$. If the Euler form does not satisfy equation (9), then the CoHA multiplication can be twisted by a sign so as to make $\mathcal{H}_Q$ supercommutative [23, §2.6]. Since all (connected) symmetric quivers studied in this paper satisfy equation (9) we do not recall this twist here.

Write $\mathrm{Sym}(V)$ for the free supercommutative algebra generated by a $\Lambda_Q^+ \times \mathbb{Z}$-graded vector space V. The following result was conjectured by Kontsevich and Soibelman [23, Conjecture 1] and proved by Efimov.

Theorem 2.2 ([12, Theorem 1.1])**.** *Let Q be a symmetric quiver and let u be a formal variable of degree $(0, 2)$. Then there exists a $\Lambda_Q^+ \times \mathbb{Z}$-graded rational vector space of the form $V_Q = V_Q^{\mathsf{prim}} \otimes \mathbb{Q}[u]$ such that, with its supercommutative structure, $\mathcal{H}_Q \simeq \mathrm{Sym}(V_Q)$. Moreover, each Λ_Q^+-homogeneous summand*

$$V_{Q,d}^{\mathsf{prim}} \subset V_Q^{\mathsf{prim}}, \qquad d \in \Lambda_Q^+$$

is finite dimensional.

If we do not use the supercommutative twist, then instead $\mathcal{H}_Q \simeq \mathrm{Sym}(V_Q)$ only as objects of $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$. The second part of Theorem 2.2, known as the integrality conjecture [22], asserts that V_Q^{prim} defines an element of $D^b(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$, the full subcategory of $D(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$ consisting of objects whose Λ_Q^+-homogeneous components lie in $D^b(\mathsf{Vect}_{\mathbb{Z}})$.

Definition. *The motivic Donaldson-Thomas invariant of a symmetric quiver Q is the class of V_Q^{prim} in the Grothendieck ring of $D^b(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$,*

$$\Omega_Q(q^{\frac{1}{2}}, t) = \sum_{(d,k) \in \Lambda_Q^+ \times \mathbb{Z}} \dim_{\mathbb{Q}} V_{Q,(d,k)}^{\mathsf{prim}} (-q^{\frac{1}{2}})^k t^d \in \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}][\![\Lambda_Q^+]\!].$$

For symmetric quivers the parity-twisted Hilbert-Poincaré series of $\mathcal{H}_Q$ coincides with A_Q. Using this observation, Theorem 2.2 implies that A_Q can be written as a product of q-Pochhammer symbols $(t; q)_\infty = \prod_{i \geq 0}(1 - q^i t)$.

Corollary 2.3 ([12, Corollary 4.1])**.** *Let Q be a symmetric quiver. Then*

$$A_Q(q^{\frac{1}{2}}, t) = \prod_{(d,k) \in \Lambda_Q^+ \times \mathbb{Z}} (q^{\frac{k}{2}} t^d; q)_\infty^{-\Omega_{Q,(d,k)}}$$

where $\Omega_{Q,(d,k)}$ is the coefficient of $q^{\frac{k}{2}} t^d$ in Ω_Q.

The factorization of Corollary 2.3 is often used as the definition of Ω_Q, in which case *a priori* $\Omega_Q \in \mathbb{Q}(q^{\frac{1}{2}})[\![\Lambda_Q^+]\!]$. Theorem 2.2 provides a conceptual reason for the existence of such factorizations and proves integrality as well as positivity, $\Omega_Q(-q^{\frac{1}{2}}, t) \in \mathbb{Z}_{\geq 0}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}][\![\Lambda_Q^+]\!]$.

Finally, we recall a geometric interpretation of Ω_Q. Let $\mathbf{M}_d^{st}$ be the stack of stable representations of dimension vector d with respect to the trivial stability. The map to the coarse moduli space $\mathbf{M}_d^{st} \to \mathfrak{M}_d^{st}$ is a $\mathbb{C}^\times$-gerbe and induces an isomorphism of mixed Hodge structures $H^\bullet(\mathbf{M}_d^{st}) \simeq H^\bullet(\mathfrak{M}_d^{st}) \otimes \mathbb{Q}[u]$.

Theorem 2.4 ([3, Theorem 2.2])**.** *Let Q be the double of a quiver. For each $d \in \Lambda_Q^+$, the restriction $H^\bullet_{\mathsf{GL}_d}(R_d) \to H^\bullet(\mathbf{M}_d^{st})$ induces an isomorphism of $\mathbb{Z}$-graded vector spaces $V_{Q,d}^{\mathsf{prim}} \xrightarrow{\sim} PH^{\bullet - \chi(d,d)}(\mathfrak{M}_d^{st})$.*

For more general geometric interpretations of Ω_Q see [18], [27].

3. Cohomological Hall modules

We introduce the cohomological Hall module of a quiver with duality structure, describe some of its basic properties and formulate the main conjectures regarding its structure.

3.1. **Definition of the CoHM.** Fix a quiver with involution (Q,σ) and duality structure (s,τ). Let $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}} \subset D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$ be the full subcategory of $\Lambda_Q^{\sigma,+}$-graded objects. Equation (5) shows that $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}$ becomes a left module category for $(D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}, \boxtimes^{\mathsf{tw}})$ via

$$\bigoplus_{d \in \Lambda_Q^+} \mathcal{U}_d \boxtimes^{S\text{-}\mathsf{tw}} \bigoplus_{e \in \Lambda_Q^{\sigma,+}} \mathcal{X}_e = \bigoplus_{e \in \Lambda_Q^{\sigma,+}} \Big(\bigoplus_{e = H(d') + e''} \mathcal{U}_{d'} \otimes \mathcal{X}_{e''} \Big) \{\epsilon(d', e'')/2\}$$

where

$$\epsilon(d,e) = \chi(d,e) - \chi(e,d) + \mathcal{E}(\sigma(d)) - \mathcal{E}(d).$$

Let $d \in \Lambda_Q^+$ and $e \in \Lambda_Q^{\sigma,+}$ with e_i even for all $i \in Q_0^\sigma$ with $s_i = -1$. The subspace $R_{d,e}^\sigma \subset R_{H(d)+e}^\sigma$ of self-dual structure maps on the orthogonal direct sum $H(\mathbb{C}^d) \oplus \mathbb{C}^e$ which preserve the canonical Q_0-graded isotropic subspace $\mathbb{C}^d$ can be identified with the subspace of

$$R_d \oplus R_e^\sigma \oplus \bigoplus_{i \xrightarrow{\alpha} j} \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}^{e_i}, \mathbb{C}^{d_j}) \oplus \bigoplus_{i \xrightarrow{\alpha} j} \mathrm{Hom}_{\mathbb{C}}((\mathbb{C}^{d_{\sigma(i)}})^\vee, \mathbb{C}^{d_j})$$

whose final component $\{m_\alpha\} \in \bigoplus_{i \xrightarrow{\alpha} j} \mathrm{Hom}_{\mathbb{C}}((\mathbb{C}^{d_{\sigma(i)}})^\vee, \mathbb{C}^{d_j})$ satisfies $\Theta_{\mathbb{C}^{d_j}} m_\alpha = -\tau_\alpha m_{\sigma(\alpha)}^\vee$. Let also $\mathsf{G}_{d,e}^\sigma \subset \mathsf{G}_{H(d)+e}^\sigma$ be the parabolic subgroup which preserves $\mathbb{C}^d$.

The cohomological Hall module (henceforth CoHM) is

$$\mathcal{M}_Q = \bigoplus_{e \in \Lambda_Q^{\sigma,+}} H^\bullet_{\mathsf{G}_e^\sigma}(R_e^\sigma)\{\mathcal{E}(e)/2\} \in D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}.$$

Define $\star : \mathcal{H}_Q \boxtimes^{S\text{-}\mathsf{tw}} \mathcal{M}_Q \to \mathcal{M}_Q$ so that its restriction to $\mathcal{H}_{Q,d} \boxtimes^{S\text{-}\mathsf{tw}} \mathcal{M}_{Q,e}$ is

$$\begin{aligned} H^\bullet_{\mathsf{GL}_d}(R_d) \otimes H^\bullet_{\mathsf{G}_e^\sigma}(R_e^\sigma) \xrightarrow{\sim} H^\bullet_{\mathsf{GL}_d \times \mathsf{G}_e^\sigma}(R_d \times R_e^\sigma) \to H^\bullet_{\mathsf{G}_{d,e}^\sigma}(R_{d,e}^\sigma) \to \\ H^\bullet_{\mathsf{G}_{d,e}^\sigma}(R_{H(d)+e}^\sigma)\{2\delta_1/2\} \to H^\bullet_{\mathsf{G}_{H(d)+e}^\sigma}(R_{H(d)+e}^\sigma)\{(2\delta_1 + 2\delta_2)/2\}, \end{aligned}$$

where again the degree shifts in $\mathcal{H}_{Q,d}$, $\mathcal{M}_{Q,e}$ and $\boxtimes^{S\text{-}\mathsf{tw}}$ are omitted. The maps in the composition are defined analogously to those appearing in the CoHA multiplication, where the maps (8) are replaced by

$$R_d \times R_e^\sigma \xleftarrow{\pi} R_{d,e}^\sigma \xhookrightarrow{i} R_{H(d)+e}^\sigma, \qquad \mathsf{GL}_d \times \mathsf{G}_e^\sigma \xleftarrow{p} \mathsf{G}_{d,e}^\sigma \xhookrightarrow{j} \mathsf{G}_{H(d)+e}^\sigma.$$

The degree shifts are

$$\delta_1 = \dim_{\mathbb{C}} R_{H(d)+e}^\sigma - \dim_{\mathbb{C}} R_{d,e}^\sigma, \qquad \delta_2 = -\dim_{\mathbb{C}} \mathsf{G}_{H(d)+e}^\sigma - \dim_{\mathbb{C}} \mathsf{G}_{d,e}^\sigma.$$

A direct calculation shows that $\delta_1 + \delta_2 = -\chi(d,e) - \mathcal{E}(\sigma(d))$.

Theorem 3.1. *The $\star$ action gives $\mathcal{M}_Q$ the structure of a left $\mathcal{H}_Q$-module object in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}$.*

Proof. The commutative diagram used to prove associativity of the CoHA multiplication in [23, §2.3] has a natural modification in the self-dual setting, obtained by requiring that the structure maps and isometry groups preserve multi-step isotropic flags. This modified commutative diagram establishes the $\mathcal{H}_Q$-module structure of $\mathcal{M}_Q$. □

Remark. While there are versions of $\mathcal{H}_Q$ and $\mathcal{M}_Q$ defined using cohomology with integer coefficients, the results of this paper require rational coefficients.

Let $\mathsf{W}(Q)$ be the abelian group defined by the exact sequence

$$\Lambda_Q^+ \xrightarrow{H} \Lambda_Q^{\sigma,+} \xrightarrow{\nu} \mathsf{W}(Q) \to 0.$$

Explicitly, $\mathsf{W}(Q) \simeq \prod_{i \in Q_0^\sigma} \mathbb{Z}_2$ with ν sending a dimension vector to its parities at Q_0^σ. The following result is immediate.

Proposition 3.2. *For each $w \in \mathsf{W}(Q)$ the subspace*

$$\mathcal{M}_Q^{(w)} = \bigoplus_{\{e \in \Lambda_Q^{\sigma,+} \mid \nu(e) = w\}} \mathcal{M}_{Q,e} \subset \mathcal{M}_Q$$

is a $\mathcal{H}_Q$-submodule. Moreover, $\mathcal{M}_Q = \bigoplus_{w \in \mathsf{W}(Q)} \mathcal{M}_Q^{(w)}$ as $\mathcal{H}_Q$-modules.

Remark. The module $\mathcal{M}_Q^{(w)}$ is zero unless $s_i = 1$ for all $i \in Q_0^\sigma$ with $w_i \neq 0$.

The motivic orientifold DT series of Q is the class of $\mathcal{M}_Q$ in the Grothendieck ring of $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}$,

$$A_Q^\sigma(q^{\frac{1}{2}}, \xi) = \sum_{(e,l) \in \Lambda_Q^{\sigma,+} \times \mathbb{Z}} \dim_{\mathbb{Q}} \mathcal{M}_{Q,(e,l)} (-q^{\frac{1}{2}})^l \xi^e \in \mathbb{Z}[\![q^{\frac{1}{2}}, \Lambda_Q^{\sigma,+}]\!].$$

Using the G_e^σ-equivariant contractibility of R_e^σ and the isomorphisms(7) we compute

$$A_Q^\sigma = \sum_{e \in \Lambda_Q^{\sigma,+}} \frac{(-q^{\frac{1}{2}})^{\mathcal{E}(e)}}{\prod_{i \in Q_0^+} \prod_{j=1}^{e_i} (1-q^j) \prod_{i \in Q_0^\sigma} \prod_{j=1}^{\lfloor \frac{e_i}{2} \rfloor} (1-q^{2j})} \xi^e. \tag{10}$$

We will view A_Q^σ as an element of the $\hat{\mathbb{T}}_Q$-module $\hat{\mathbb{S}}_Q$.

Also inspired by orientifold DT theory, in [41] a different generating series was attached to a quiver with duality structure. Given a finite field $\mathbb{F}_q$ of odd characteristic, the $\mathcal{E}$-weighted generating series[1] of the number of $\mathbb{F}_q$-rational points of stacks of self-dual representations is

$$\mathfrak{A}_{Q,\mathbb{F}_q}^\sigma(\xi) = \sum_M \frac{(-q^{\frac{1}{2}})^{\mathcal{E}(\mathbf{dim}\, M)}}{\#\mathrm{Aut}_S(M)} \xi^{\mathbf{dim}\, M}.$$

The sum runs over isometry classes of self-dual representations and $\mathrm{Aut}_S(-)$ denotes the isometry group. Comparing equation (10) and [41, Proposition 4.2] shows that $A_Q^\sigma(q^{-\frac{1}{2}}, \xi) = \mathfrak{A}_{Q,\mathbb{F}_q}(\xi)$. Hence the cohomological and finite field approaches to orientifold DT theory are consistent.

[1]We have renormalized the integration map from [41] to match the conventions of this paper. The series $\mathfrak{A}_{Q,\mathbb{F}_q}^\sigma$ was denoted by A_Q^σ in [41].

3.2. **The CoHM as a signed shuffle module.** In this section we derive an explicit combinatorial expression for the action of $\mathcal{H}_Q$ on $\mathcal{M}_Q$.

Using the isomorphism (7), for each $e \in \Lambda_Q^{\sigma,+}$ we identify $\mathcal{M}_{Q,e}$ with the vector space of polynomials in the variables

$$\{z_{i,1}, \dots, z_{i,e_i}\}_{i \in Q_0^+}, \qquad \{z_{i,1}^2, \dots, z^2_{i, \lfloor \frac{e_i}{2} \rfloor}\}_{i \in Q_0^\sigma}$$

which are invariant under the group $\prod_{i \in Q_0^+} \mathfrak{S}_{e_i} \times \prod_{i \in Q_0^\sigma} \mathfrak{S}_{\lfloor \frac{e_i}{2} \rfloor}$. We also identify polynomials in the variables

$$\{x'_{i,1}, \dots, x'_{i,d_i}\}_{i \in Q_0}, \quad \text{and} \quad \{z''_{i,1}, \dots, z''_{i,e_i}\}_{i \in Q_0^+}, \ \{z''_{i,1}, \dots, z''_{i,\lfloor \frac{e_i}{2} \rfloor}\}_{i \in Q_0^\sigma}$$

with polynomials in the variables

$$\{z_{i,1}, \dots, z_{i,d_i + e_i + d_{\sigma(i)}}\}_{i \in Q_0^+}, \qquad \{z_{i,1}, \dots, z_{i,d_i + \lfloor \frac{e_i}{2} \rfloor}\}_{i \in Q_0^\sigma} \tag{11}$$

via

$$x'_{i,j} \mapsto z_{i,j}, \qquad z''_{i,j} \mapsto z_{i,d_i+j}, \qquad x'_{\sigma(i),j} \mapsto -z_{i,d_i+e_i+j}, \qquad i \in Q_0^+$$

and

$$x'_{i,j} \mapsto z_{i,j}, \qquad z''_{i,j} \mapsto z_{i,d_i+j}, \qquad i \in Q_0^\sigma.$$

The minus sign arises from the minus sign in the first part of Lemma 1.1.

Given $m,n,p \in \mathbb{Z}_{\geq 0}$ let $\mathfrak{sh}_{m,n,p} \subset \mathfrak{S}_{m+n+p}$ be the set of 3-shuffles of type (m,n,p). Define the set of σ-shuffles of type $(d,e) \in \Lambda_Q^+ \times \Lambda_Q^{\sigma,+}$ by

$$\mathfrak{sh}^\sigma_{d,e} = \prod_{i \in Q_0^+} \mathfrak{sh}_{d_i, e_i, d_{\sigma(i)}} \times \prod_{i \in Q_0^\sigma} \left(\mathbb{Z}_2^{d_i} \times \mathfrak{sh}_{d_i, d_i + \lfloor \frac{e_i}{2} \rfloor} \right).$$

There is a natural action of $\mathfrak{sh}^\sigma_{d,e}$ on the vector space of polynomials in the variables (11), the shuffle factors acting as usual and the $\mathbb{Z}_2$ factors acting by multiplication by -1 on the first d_i elements of $\{z_{i,1}, \dots, z_{i,d_i+\lfloor \frac{e_i}{2} \rfloor}\}_{i \in Q_0^\sigma}$.

For each $i \in Q_0$ define $\varepsilon_i : \Lambda_Q \to \{0,1\}$ by $e \mapsto e_i \mod 2$. Write $\leq_t$ for $<$ if $t=-1$ and $\leq$ if $t=+1$.

Theorem 3.3. *Let $f \in \mathcal{H}_{Q,d}$ and $g \in \mathcal{M}_{Q,e}$. Then*

$$f \star g = \sum_{\pi \in \mathfrak{sh}^\sigma_{d,e}} \pi \left(f(x') g(z'') \frac{\prod_{\alpha \in Q_1^+ \sqcup Q_1^\sigma} V_\alpha(x',z'')}{\prod_{i \in Q_0^+ \sqcup Q_0^\sigma} D_i(x',z'')} \right)$$

where the factors of the denominator are

$$D_i = \prod_{k=1}^{e_i} \prod_{l=1}^{d_i} (z''_{i,k} - x'_{i,l}) \prod_{m=1}^{d_{\sigma(i)}} \prod_{l=1}^{d_i} (-x'_{\sigma(i),m} - x'_{i,l}) \prod_{m=1}^{d_{\sigma(i)}} \prod_{k=1}^{e_i} (-x'_{\sigma(i),m} - z''_{i,k})$$

if $i \in Q_0^+$ and

$$D_i = g(x'_{i,1}, \dots, x'_{i,d_i}) \prod_{1 \leq k < l \leq d_i} (x'_{i,k} + x'_{i,l}) \prod_{l=1}^{d_i} \prod_{k=1}^{\lfloor \frac{e_i}{2} \rfloor} (x'^2_{i,l} - z''^2_{i,k})$$

with

$$g(x'_{i,1}, \dots, x'_{i,d_i}) = \begin{cases} \prod_{l=1}^{d_i} x'_{i,l}, & \text{if } \mathsf{G}^{s_i}_{2d_i+e_i} \text{ is type } B_{d_i + \lfloor \frac{e_i}{2} \rfloor} \\ \prod_{l=1}^{d_i} 2x'_{i,l}, & \text{if } \mathsf{G}^{s_i}_{2d_i+e_i} \text{ is type } C_{d_i + \lfloor \frac{e_i}{2} \rfloor} \\ 1, & \text{if } \mathsf{G}^{s_i}_{2d_i+e_i} \text{ is type } D_{d_i + \lfloor \frac{e_i}{2} \rfloor} \end{cases}$$

if $i \in Q_0^\sigma$ and the factors of the numerator are defined as follows:

- *If* $i \xrightarrow{\alpha} j \in Q_1^+$, *then* $V_\alpha = \widetilde{V}_\alpha^{(i)} \widetilde{V}_\alpha^{(j)} \prod_{m=1}^{d_{\sigma(j)}} \prod_{l=1}^{d_i} (-x'_{\sigma(j),m} - x'_{i,l})$ *where*

$$\widetilde{V}_\alpha^{(i)} = \begin{cases} \prod_{m=1}^{d_{\sigma(j)}} \prod_{k=1}^{e_i} (-x'_{\sigma(j),m} - z''_{i,k}), & \text{if } i \notin Q_0^\sigma \\ \prod_{m=1}^{d_{\sigma(j)}} \prod_{k=1}^{\lfloor \frac{e_i}{2} \rfloor} (x'^2_{\sigma(j),m} - z''^2_{i,k}) \prod_{m=1}^{d_{\sigma(j)}} (-x'_{\sigma(j),m})^{\varepsilon_i(e)}, & \text{if } i \in Q_0^\sigma \end{cases}$$

and

$$\widetilde{V}_\alpha^{(j)} = \begin{cases} \prod_{k=1}^{e_j} \prod_{l=1}^{d_i} (z''_{j,k} - x'_{i,l}), & \text{if } j \notin Q_0^\sigma \\ \prod_{l=1}^{d_i} \prod_{k=1}^{\lfloor \frac{e_j}{2} \rfloor} (x'^2_{i,l} - z''^2_{j,k}) \prod_{l=1}^{d_i} (-x'_{i,l})^{\varepsilon_j(e)}, & \text{if } j \in Q_0^\sigma. \end{cases}$$

- *If* $\sigma(i) \xrightarrow{\alpha} i \in Q_1^\sigma$, *then* $V_\alpha = \widetilde{V}_\alpha \prod_{1 \le j \le_{s_i \tau_\alpha} k \le d_{\sigma(i)}} (-x'_{\sigma(i),j} - x'_{\sigma(i),k})$ *where*

$$\widetilde{V}_\alpha = \begin{cases} \prod_{k=1}^{e_i} \prod_{l=1}^{d_{\sigma(i)}} (z''_{i,k} - x'_{\sigma(i),l}), & \text{if } i \notin Q_0^\sigma \\ \prod_{l=1}^{d_{\sigma(i)}} \prod_{k=1}^{\lfloor \frac{e_i}{2} \rfloor} (x'^2_{\sigma(i),l} - z''^2_{i,k}) \prod_{l=1}^{d_{\sigma(i)}} (-x'_{\sigma(i),l})^{\varepsilon_i(e)}, & \text{if } i \in Q_0^\sigma. \end{cases}$$

Proof. Similar to [23, §2.4], we regard f and g as classes in $H^\bullet(B\mathsf{GL}_d \times B\mathsf{G}_e^\sigma)$ and let $\mathsf{Eu}_{\mathsf{G}_{d,e}^\sigma}(N_{R_{H(d)+e}^\sigma / R_{d,e}^\sigma})$ be the $\mathsf{G}_{d,e}^\sigma$-equivariant Euler class of the fibre of the normal bundle to $R_{d,e}^\sigma \subset R_{H(d)+e}^\sigma$ at the origin. Then

$$f \star g = \int_{[\mathsf{G}_{H(d)+e}^\sigma / \mathsf{G}_{d,e}^\sigma]} f \cdot g \cdot \mathsf{Eu}_{\mathsf{G}_{d,e}^\sigma}(N_{R_{H(d)+e}^\sigma / R_{d,e}^\sigma})$$

where $[\mathsf{G}_{H(d)+e}^\sigma / \mathsf{G}_{d,e}^\sigma]$ is the $\mathsf{G}_{H(d)+e}^\sigma$-equivariant fundamental class of $\mathsf{G}_{H(d)+e}^\sigma / \mathsf{G}_{d,e}^\sigma$, the fibre of $B\mathsf{G}_{d,e}^\sigma \to B\mathsf{G}_{H(d)+e}^\sigma$. We will compute this integral by equivariant localization with respect to the action of the maximal torus $\mathsf{T} = \mathsf{T}_{H(d)+e} \subset \mathsf{G}_{H(d)+e}^\sigma$.

Let $U \in R_d$ and $N \in R_{H(d)+e}^\sigma$. An inclusion $U \hookrightarrow N$ is isotropic if and only if we have an commutative diagram of the form

$$\begin{tikzcd} U_i \arrow[r, hook] \arrow[d, "u_\alpha"'] & (U^\perp)_i \arrow[r, hook] \arrow[d] & N_i \arrow[d, "n_\alpha"] \\ U_j \arrow[r, hook] & (U^\perp)_j \arrow[r, hook] & N_j \end{tikzcd} \qquad i \xrightarrow{\alpha} j \in Q_1. \tag{12}$$

We first compute the equivariant Euler class of the tangent space at a T-fixed point of $\mathsf{G}_{H(d)+e}^\sigma / \mathsf{G}_{d,e}^\sigma$. The inclusions of diagram (12) lead to the identification

$$\mathsf{G}_{H(d)+e}^\sigma / \mathsf{G}_{d,e}^\sigma \simeq \prod_{i \in Q_0^+} \mathsf{Fl}(d_i, e_i, d_{\sigma(i)}) \times \prod_{i \in Q_0^\sigma} \mathsf{IGr}^{s_i}(d_i, 2d_i + e_i)$$

where $\mathsf{Fl}(a,b,c)$ is the variety of flags of the form $\mathbb{C}^a \subset \mathbb{C}^{a+b} \subset \mathbb{C}^{a+b+c}$ and $\mathsf{IGr}^s(a,b)$ is the variety of a-dimensional isotropic subspaces of a b-dimensional orthogonal ($s=1$) or symplectic ($s=-1$) vector space. The T-fixed points of $\mathsf{Fl}(d_i, e_i, d_{\sigma(i)})$ are two-step coordinate flags and are labelled by disjoint pairs of increasing sequences in $\{1, \dots, d_i + e_i + d_{\sigma(i)}\}$ of the form

$$\pi = \{a_1 < \cdots < a_{d_i};\ b_1 < \cdots < b_{e_i}\}.$$

Such pairs are in bijection with $\mathfrak{sh}_{d_i,e_i,d_{\sigma(i)}}$. The T-character of the tangent space to a flag $U_i \subset (U^\perp)_i \subset N_i$ is the product of the following factors:

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{C}}(U_i,(N/\!\!/U)_i) &\rightsquigarrow \prod_{k=1}^{e_i}\prod_{l=1}^{d_i}(z''_{i,k}-x'_{i,l}) \\
\mathrm{Hom}_{\mathbb{C}}(U_i,U^\vee_{\sigma(i)}) &\rightsquigarrow \prod_{m=1}^{d_{\sigma(i)}}\prod_{l=1}^{d_i}(-x'_{\sigma(i),m}-x'_{i,l}) \\
\mathrm{Hom}_{\mathbb{C}}((N/\!\!/U)_i,U^\vee_{\sigma(i)}) &\rightsquigarrow \prod_{m=1}^{d_{\sigma(i)}}\prod_{k=1}^{e_i}(-x'_{\sigma(i),m}-z''_{i,k}).
\end{aligned}$$

The T-fixed points of $\mathsf{IGr}^{s_i}(d_i,2d_i+e_i)$ are isotropic coordinate planes and are in bijection with $\mathbb{Z}_2^{d_i}\times\mathfrak{sh}_{d_i,d_i+\lfloor\frac{e_i}{2}\rfloor}$ via

$$\mathbb{Z}_2^{d_i}\times\mathfrak{sh}_{d_i,d_i+\lfloor\frac{e_i}{2}\rfloor}\ni(p,\pi)\mapsto\mathrm{span}_{\mathbb{C}}\{v_{\pi(1),p(1)},\dots,v_{\pi(d_i),p(d_i)}\}$$

where, in the notation of Section 1.1,

$$v_{i,p}=\begin{cases} x_i, & \text{if } p=1\\ y_i, & \text{if } p=-1.\end{cases}$$

The T-character of the tangent space at a fixed point is the product of the positive roots of $\mathsf{G}^{s_i}_{2d_i+e_i}$ are not in the corresponding parabolic Lie subalgebra; see Section 1.1 for conventions. These calculations gives the denominators D_i as stated.

Next we compute the restriction of $\mathsf{Eu}_{\mathsf{G}^\sigma_{d,e}}(N_{R^\sigma_{H(d)+e}/R^\sigma_{d,e}})$ to a T-fixed point. From the vertical arrows of diagram (12) we see that the contribution V_α of $\alpha\in Q_1^+$ to $\mathsf{Eu}_{\mathsf{G}^\sigma_{d,e}}(N_{R^\sigma_{H(d)+e}/R^\sigma_{d,e}})$ is the product of the following T-weights:

$$\mathrm{Hom}_{\mathbb{C}}(U_i,(N/\!\!/U)_j)\rightsquigarrow\begin{cases}\displaystyle\prod_{k=1}^{e_j}\prod_{l=1}^{d_i}(z''_{j,k}-x'_{i,l}), & \text{if } j\notin Q_0^\sigma\\ \displaystyle\prod_{k=1}^{\lfloor\frac{e_j}{2}\rfloor}\prod_{l=1}^{d_i}(-z''^2_{j,k}+x'^2_{i,l})\prod_{l=1}^{d_i}(-x'_{i,l})^{\varepsilon_j(e)}, & \text{if } j\in Q_0^\sigma\end{cases}$$

and

$$\mathrm{Hom}_{\mathbb{C}}(U_i,U^\vee_{\sigma(j)})\rightsquigarrow\prod_{m=1}^{d_{\sigma(j)}}\prod_{l=1}^{d_i}(-x'_{\sigma(j),m}-x'_{i,l})$$

and

$$\mathrm{Hom}_{\mathbb{C}}((N/\!\!/U)_i,U^\vee_{\sigma(j)})\rightsquigarrow\begin{cases}\displaystyle\prod_{m=1}^{d_{\sigma(j)}}\prod_{k=1}^{e_i}(-x'_{\sigma(j),m}-z''_{i,k}), & \text{if } i\notin Q_0^\sigma\\ \displaystyle\prod_{m=1}^{d_{\sigma(j)}}\prod_{k=1}^{\lfloor\frac{e_i}{2}\rfloor}(x'^2_{\sigma(j),m}-z''^2_{i,k})\prod_{m=1}^{d_{\sigma(j)}}(-x'_{\sigma(j),m})^{\varepsilon_i(e)}, & \text{if } i\in Q_0^\sigma.\end{cases}$$

Similarly, the contribution V_α of $\sigma(j)\xrightarrow{\alpha}j\in Q_1^\sigma$ is the product of the T-weights

$$\mathrm{Hom}_{\mathbb{C}}(U_{\sigma(j)},(N/\!\!/U)_j)\rightsquigarrow\begin{cases}\displaystyle\prod_{k=1}^{e_j}\prod_{m=1}^{d_{\sigma(j)}}(z''_{j,k}-x'_{\sigma(j),m}), & \text{if } j\notin Q_0^\sigma\\ \displaystyle\prod_{k=1}^{\lfloor\frac{e_j}{2}\rfloor}\prod_{m=1}^{d_{\sigma(j)}}(-z''^2_{j,k}+x'^2_{\sigma(j),m})\prod_{m=1}^{d_{\sigma(j)}}(-x'_{\sigma(j),m})^{\varepsilon_j(e)}, & \text{if } j\in Q_0^\sigma.\end{cases}$$

and

$$\mathrm{Hom}_{\mathbb{C}}(U_{\sigma(j)}, U^{\vee}_{\sigma(j)}) \rightsquigarrow \prod_{1 \leq m \leq_{\pm} k \leq d_{\sigma(j)}} (-x'_{\sigma(j),m} - x'_{\sigma(j),k}).$$

There is no separate contribution from $\mathrm{Hom}_{\mathbb{C}}((N/\!\!/U)_j, U^{\vee}_{\sigma(j)})$; the symmetry of n_α requires that these elements be dual to those of $\mathrm{Hom}_{\mathbb{C}}(U_{\sigma(j)}, (N/\!\!/U)_j)$. □

3.3. **The CoHM of a σ-symmetric quiver.** In Section 2.2 we saw that the abstract structure of the cohomological Hall algebra of a symmetric quiver is relatively simple. In general, we do not know if the supercommutative twist of the multiplication in $\mathcal{H}_Q$ can be lifted to $\mathcal{M}_Q$. Hence we will consider $\mathcal{H}_Q$ with its standard (possibly non-supercommutative) multiplication. In the self-dual setting it is natural to impose the following stronger notion of symmetry.

Definition. *A quiver with involution and duality structure is called σ-symmetric if it is symmetric and $\mathcal{E}(d) = \mathcal{E}(\sigma(d))$ for all $d \in \Lambda_Q$.*

Using equation (4) we find that a symmetric quiver is σ-symmetric if and only if

$$\sum_{\sigma(i) \xrightarrow{\alpha} i \in Q_1^\sigma} \tau_\alpha = \sum_{i \xrightarrow{\alpha} \sigma(i) \in Q_1^\sigma} \tau_\alpha, \qquad \forall\, i \in Q_0. \tag{13}$$

Here, in contrast to all other places in the paper, the sums run over arrows with fixed initial and final vertices.

If Q is σ-symmetric, then $\boxtimes^{S\text{-}\mathsf{tw}}$ reduces to the $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$-module structure defined using only the Λ_Q-module structure of Λ_Q^σ. Somewhat abusively, we denote this by $\boxtimes$. In particular, $\mathcal{M}_Q$ is a $\Lambda_Q^{\sigma,+} \times \mathbb{Z}$-graded $\mathcal{H}_Q$-module.

Let $\mathcal{H}_{Q,+}$ be the augmentation ideal of $\mathcal{H}_Q$.

Definition. *The cohomological orientifold Donaldson-Thomas invariant of a σ-symmetric quiver Q is the $\Lambda_Q^{\sigma,+} \times \mathbb{Z}$-graded vector space*

$$W_Q^{\mathsf{prim}} = \mathcal{M}_Q / (\mathcal{H}_{Q,+} \star \mathcal{M}_Q).$$

By picking a vector space splitting we will view W_Q^{prim} as a subspace of $\mathcal{M}_Q$.

The next result asserts that the orientifold analogue of the integrality conjecture holds. For its proof we choose the partition $Q_1 = Q_1^+ \sqcup Q_1^\sigma \sqcup Q_1^-$ such that a configuration

$$i \xrightarrow{\alpha} j = \sigma(j) \xrightarrow{\sigma(\alpha)} \sigma(i)$$

in Q implies that $i \in Q_0^+$ if and only if $\alpha \in Q_1^+$. This can always be achieved by permuting elements of $Q_1^+ \sqcup Q_1^-$.

Theorem 3.4. *Let Q be a σ-symmetric quiver. Then each $\Lambda_Q^{\sigma,+}$-homogeneous summand*

$$W_{Q,e}^{\mathsf{prim}} \subset W_Q^{\mathsf{prim}}, \qquad e \in \Lambda_Q^{\sigma,+}$$

is finite dimensional.

Proof. We modify the argument of [12, §3]. Define

$$X_{Q,d} = \mathbb{Q}[x_{i,j} \mid i \in Q_0,\ 1 \leq j \leq d_i], \qquad d \in \Lambda_Q^+$$

and

$$Z_{Q,e} = \mathbb{Q}[z_{i,j} \mid i \in Q_0^+,\ 1 \leq j \leq e_i] \otimes \mathbb{Q}[z_{i,j} \mid i \in Q_0^\sigma,\ 1 \leq j \leq \lfloor \frac{e_i}{2} \rfloor], \qquad e \in \Lambda_Q^{\sigma,+}$$

both of which we consider as $\mathbb{Z}$-graded polynomial algebras with generators in degree two. The Weyl groups W_{GL_d} and $W_{\mathsf{G}_e^\sigma}$ act on $X_{Q,d}$ and $Z_{Q,e}$, respectively, and up to constant degree shifts we obtain $\mathbb{Z}$-graded vector space isomorphisms $\mathcal{H}_{Q,d} \simeq X_{Q,d}^{W_{\mathsf{GL}_d}}$ and $\mathcal{M}_{Q,e} \simeq Z_{Q,e}^{W_{\mathsf{G}_e^\sigma}}$. Denote by

$$\mathcal{K}^{\sigma}_{d',e''}(x',z'') = \frac{\prod_{\alpha \in Q_1^+ \sqcup Q_1^\sigma} V_\alpha(x',z'')}{\prod_{i \in Q_0^+ \sqcup Q_0^\sigma} D_i(x',z'')}$$

the kernel from Theorem 3.3 and let $Z_{Q,e}^{loc}$ be the localization of $Z_{Q,e}$ at the denominators of $\mathcal{K}^{\sigma}_{d',e''}$, for all $(d',e'') \in \Lambda_Q^+ \times \Lambda_Q^{\sigma,+}$ satisfying $H(d') + e'' = e$ and $d' \neq 0$.

Let $L_{Q,e} \subset Z_{Q,e}^{loc}$ be the smallest $W_{\mathsf{G}_e^\sigma}$-stable $Z_{Q,e}$-submodule such that $\mathcal{K}^{\sigma}_{d',e''} \in L_{Q,e}$ for all $(d',e'') \in \Lambda_Q^+ \times \Lambda_Q^{\sigma,+}$ as above. We claim that $\mathcal{M}_{Q,e} = W_{Q,e}^{\mathsf{prim}} \oplus L_{Q,e}^{W_{\mathsf{G}_e^\sigma}}$ or, equivalently, that $L_{Q,e}^{W_{\mathsf{G}_e^\sigma}}$ is the image of the CoHA action map

$$\bigoplus_{\substack{(d',e'') \in \Lambda_Q^+ \times \Lambda_Q^{\sigma,+} \\ H(d')+e''=e, \ d' \neq 0}} \mathcal{H}_{Q,d'} \boxtimes \mathcal{M}_{Q,e''} \xrightarrow{\star} \mathcal{M}_{Q,e}. \tag{14}$$

To see this, first note that $L_{Q,e}^{W_{\mathsf{G}_e^\sigma}}$ is $\mathbb{Q}$-linearly spanned by $W_{\mathsf{G}_e^\sigma}$-symmetrizations of functions of the form

$$f(x')g(z'')\mathcal{K}^{\sigma}_{d',e''}(x',z''), \qquad f \in X_{Q,d'}, \ \ g \in Z_{Q,e''}. \tag{15}$$

It follows that the image of the map (14) is contained in $L_{Q,e}^{W_{\mathsf{G}_e^\sigma}}$. For the reverse inclusion, suppose we are given an element of the form (15). By symmetrizing with respect to $W_{\mathsf{GL}_{d'}}$ and $W_{\mathsf{G}_{e''}^\sigma}$, both of which are subgroups of $W_{\mathsf{G}_e^\sigma}$, we may assume that $f \in \mathcal{H}_{Q,d'}$ and $g \in \mathcal{M}_{Q,e''}$. Then, up to a non-zero constant, the $W_{\mathsf{G}_e^\sigma}$-symmetrization of $fg\mathcal{K}^{\sigma}_{d',e''}$ is $f \star g$.

Hence, we must show that $L_{Q,e}^{W_{\mathsf{G}_e^\sigma}} \subset \mathcal{M}_{Q,e}$ has finite codimension. Adding a loop at each node, with duality structure $\tau = -1$ for nodes in Q_0^σ, does not decrease the ideal $L_{Q,e}$. By adding loops we can therefore avoid localizing $Z_{Q,e}$. In this case $\mathcal{M}_{Q,e}/L_{Q,e}^{W_{\mathsf{G}_e^\sigma}} \hookrightarrow Z_{Q,e}/L_{Q,e}$ and it suffices to show that $L_{Q,e} \subset Z_{Q,e}$ has finite codimension. Interpret $Z_{Q,e}$ as the algebra of functions on the affine space $\mathbb{Q}^D$, where

$$D = \sum_{i \in Q_0^+} e_i + \sum_{i \in Q_0^\sigma} \lfloor \frac{e_i}{2} \rfloor,$$

and suppose that $z \in \overline{\mathbb{Q}}^D$ satisfies $h(z) = 0$ for all $h \in L_{Q,e}$. We claim that $z = 0$. Suppose to the contrary that $z \neq 0$. By using the action of $W_{\mathsf{G}_e^\sigma}$ we will write $z = \{\vec{z}_i\}_{i \in Q_0^+ \sqcup Q_0^\sigma}$ as

$$\vec{z}_i = (x'_{i,1}, \dots, x'_{i,d'_i}, z''_{i,1}, \dots, z''_{i,e''_i}, -x'_{\sigma(i),1}, \dots, -x'_{\sigma(i),d'_{\sigma(i)}}), \qquad i \in Q_0^+$$

and

$$\vec{z}_i = (x'_{i,1}, \dots, x'_{i,d'_i}, z''_{i,1}, \dots, z''_{i,\lfloor \frac{e''_i}{2} \rfloor}), \qquad i \in Q_0^\sigma$$

for some $d' \neq 0$ so that $\mathcal{K}^{\sigma}_{d',e''}(x',z'') \neq 0$, giving a contradiction.

Define z'' to be the collection of vanishing coordinates of z and let x be what remains. By assumption $x \neq 0$. Up to the action of $W_{\mathsf{G}_e^\sigma}$, we need to write x as $\{(\vec{x}'_i, -\vec{x}'_{\sigma(i)})\}_{i \in Q_0^+} \sqcup \{\vec{x}'_i\}_{i \in Q_0^\sigma}$ so that $\mathcal{K}^{\sigma}_{d',e''}(x',z'') \neq 0$, which by Theorem 3.3 is equivalent to the following conditions[2]:

(i) $\prod_{m=1}^{d_{\sigma(i)}} \prod_{l=1}^{d_i} (-x'_{\sigma(i),m} - x'_{i,l}) \neq 0$ if $i \in Q_0^+$.

[2]Because signs are included in the definition of x' we do not need to make additional sign substitutions in these equations.

(ii) $\prod_{l\leq k<l\leq d_i}(x'_{i,k}+x'_{i,l})\neq 0$ if $i\in Q_0^\sigma$.

(iii) $\prod_{m=1}^{d_{\sigma(j)}}\prod_{l=1}^{d_i}(-x'_{\sigma(j),m}-x'_{i,l})\neq 0$ if $i\xrightarrow{\alpha} j\in Q_1^+$.

(iv) $\prod_{1\leq j\leq k\leq d_{\sigma(i)}}(-x'_{\sigma(i),j}-x'_{\sigma(i),k})\neq 0$ if $\sigma(i)\xrightarrow{\sigma} i\in Q_1^\sigma$.

These conditions can be satisfied as follows. For each $i\in Q_0^+$ by using permutations ensure that the x'_i and $-x'_{\sigma(i)}$ coordinates have no common values. Then (i) holds. For each $i\in Q_0^\sigma$ act by the sign change subgroup at i to ensure that the x'_i coordinates contain no $\pm$ pairs, that is, pairs $(a,-a)$ for some $a\in\overline{\mathbb{Q}}$. Then (ii) holds. It is easy to see that (i) and (ii) imply (iv). By our choice of partition $Q_1=Q_1^+\sqcup Q_1^\sigma\sqcup Q_1^-$, condition (iii) can be broken into three cases:

(1) Both i,j are in Q_0^σ. Use the sign change subgroups to ensure that there are no $\pm$ pairs among all Q_0^σ variables.
(2) Neither i nor j is in Q_0^σ. Use the symmetric groups to ensure that there are no $\pm$ pairs among all Q_0^+ (and hence Q_0^-) variables and no common values among the Q_0^+ (and hence Q_0^-) variables.
(3) One of i,j is in Q_0^+ and one is in Q_0^σ. Use the sign change subgroups to ensure that there are no $\pm$ pairs among all Q_0^σ and Q_0^+ variables.

This completes the proof. $\square$

Definition. *The motivic orientifold Donaldson-Thomas invariant of a σ-symmetric quiver Q is*

$$\Omega_Q^\sigma(q^{\frac{1}{2}},\xi)=\sum_{(e,l)\in\Lambda_Q^{\sigma,+}\times\mathbb{Z}}\dim_{\mathbb{Q}}W^{\mathsf{prim}}_{Q,(e,l)}(-q^{\frac{1}{2}})^l\xi^e\in\mathbb{Z}[q^{\frac{1}{2}},q^{-\frac{1}{2}}][\![\Lambda_Q^{\sigma,+}]\!].$$

More precisely, the invariant Ω_Q^σ, like Ω_Q of Section 2.2, is defined for the trivial stability condition. Theorem 3.4 implies that numerical orientifold DT invariants can be defined as the $q^{\frac{1}{2}}\mapsto 1$ specialization of $\Omega_Q^\sigma(q^{\frac{1}{2}},\xi)$. In the orientifold setting there is no need to remove from W_Q^{prim} an infinite factor of the form $\mathbb{Q}[u]$. This reflects the isomorphism between the rational cohomologies of the moduli stack and moduli space of σ-stable representations; see Lemma 3.9 below.

We now turn to a more detailed study of the module structure of $\mathcal{M}_Q$. Our goal is to formulate a conjectural analogue for $\mathcal{M}_Q$ of the freeness of the CoHA of a symmetric quiver. To begin, observe that a duality structure on an arbitrary quiver induces linear isomorphisms $R_d\to R_{\sigma(d)}$ which are equivariant with respect to the isomorphisms

$$\mathsf{GL}_d\to\mathsf{GL}_{\sigma(d)},\qquad \{g_i\}_{i\in Q_0}\mapsto\{(g_{\sigma(i)}^{-1})^t\}_{i\in Q_0}. \tag{16}$$

Contravariance of the functor $S:\mathsf{Rep}_{\mathbb{C}}(Q)\to\mathsf{Rep}_{\mathbb{C}}(Q)$ implies that these maps define an algebra anti-involution $S_{\mathcal{H}}:\mathcal{H}_Q\to\mathcal{H}_Q$. Explicitly, using equation (16) and the first part of Lemma 1.1 we have

$$S_{\mathcal{H}}(f)(\{x_{i,j}\}_{i\in Q_0,\,1\leq j\leq d_{\sigma(i)}})=f(\{\tilde{x}_{i,j}\}_{i\in Q_0,\,1\leq j\leq d_i})_{|\tilde{x}_{i,j}=-x_{\sigma(i),j}} \tag{17}$$

for all $f\in\mathcal{H}_{Q,d}$.

Proposition 3.5. *Let Q be a σ-symmetric quiver. For all $f\in\mathcal{H}_{Q,d}$ and $g\in\mathcal{M}_{Q,e}$ the equality*

$$S_{\mathcal{H}}(f)\star g=(-1)^{\chi(e,d)+\mathcal{E}(d)}f\star g$$

holds.

Proof. Let $\varpi\in\mathfrak{sh}_{d,e}^\sigma$ be the signed shuffle defined by the maps of ordered sets

$$[d_i]\sqcup[e_i]\sqcup[d_{\sigma(i)}]\mapsto[d_{\sigma(i)}]\sqcup[e_i]\sqcup[d_i],\qquad i\in Q_0^+$$

and

$$[d_i] \sqcup \big[\lfloor \frac{e_i}{2} \rfloor\big] \mapsto [-d_i] \sqcup \big[\lfloor \frac{e_i}{2} \rfloor\big], \qquad i \in Q_0^\sigma.$$

Here $[n] = \{z_1, \dots, z_n\}$. Precomposition with ϖ defines a bijection $\mathfrak{sh}^\sigma_{\sigma(d),e} \to \mathfrak{sh}^\sigma_{d,e}$. Moreover, using equation (17) we see that, after identifying variables as in Section 3.2, the polynomials f and $S_{\mathcal{H}}(f)$ differ exactly by ϖ. It is clear that ϖ fixes g.

We claim that

$$\varpi(\mathcal{K}^\sigma_{\sigma(d),e}) = (-1)^{\chi(e,d)+\mathcal{E}(d)} \mathcal{K}^\sigma_{d,e}, \qquad d \in \Lambda_Q^+, \ e \in \Lambda_Q^{\sigma,+}. \tag{18}$$

To prove this we use the explicit form of $\mathcal{K}^\sigma_{d,e}$ from Theorem 3.3. Applying ϖ to a factor D_i, $i \in Q_0^\sigma$, results in multiplication by $(-1)^{d_i + \frac{d_i(d_i-1)}{2}}$ in types B and C and $(-1)^{\frac{d_i(d_i-1)}{2}}$ in type D. If instead $i \in Q_0^+$, then the result is multiplication by $(-1)^{e_i d_i + d_i d_{\sigma(i)} + e_i d_{\sigma(i)}}$. The action of ϖ on the denominator of $\mathcal{K}^\sigma_{d,e}$ therefore results in multiplication by $(-1)^{\chi_{Q_0}(e,d)+\mathcal{E}_{Q_0}(d)}$, the subscripts indicating that only summands of χ and $\mathcal{E}$ associated to nodes are included. The action of ϖ on a factor V_α is multiplication by $(-1)^{d_i d_{\sigma(j)} + e_i d_{\sigma(j)} + d_i e_j}$ for $i \xrightarrow{\alpha} j \in Q_1^+$ and by $(-1)^{e_i d_{\sigma(i)} + \frac{d_{\sigma(i)}(d_{\sigma(i)} + \tau_\alpha s_i)}{2}}$ for $\sigma(i) \xrightarrow{\alpha} i \in Q_1^\sigma$. Using equation (13) we conclude that the sign change of the numerator is $(-1)^{\chi_{Q_1}(e,d)+\mathcal{E}_{Q_1}(d)}$. Equation (18) follows.

We now compute

$$\begin{aligned}
S_{\mathcal{H}}(f) \star g &= \sum_{\pi \in \mathfrak{sh}^\sigma_{\sigma(d),e}} \pi(S(f) g \mathcal{K}^\sigma_{\sigma(d),e}) \\
&= \sum_{\pi \in \mathfrak{sh}^\sigma_{\sigma(d),e}} \pi(\varpi(f) g \mathcal{K}^\sigma_{\sigma(d),e}) \\
&= (-1)^{\chi(e,d)+\mathcal{E}(d)} \sum_{\pi \in \mathfrak{sh}^\sigma_{\sigma(d),e}} \pi \circ \varpi(f g \mathcal{K}^\sigma_{d,e}) \\
&= (-1)^{\chi(e,d)+\mathcal{E}(d)} \sum_{\pi' \in \mathfrak{sh}^\sigma_{d,e}} \pi'(f g \mathcal{K}^\sigma_{d,e}) \\
&= (-1)^{\chi(e,d)+\mathcal{E}(d)} f \star g,
\end{aligned}$$

finishing the proof. □

Since $S_{\mathcal{H}}$ is an anti-involution the image of the CoHA multiplication map

$$\mathcal{H}_{Q,+} \boxtimes \mathcal{H}_{Q,+} \to \mathcal{H}_Q$$

is stable under $S_{\mathcal{H}}$. It follows that V_Q inherits the structure of a $\mathbb{Z}_2$-representation. In fact, $V_Q = V_Q^{\mathsf{prim}} \otimes \mathbb{Q}[u]$ as $\mathbb{Z}_2$-representations with $S_{\mathcal{H}}$ sending u to $-u$ as follows from the first part of Lemma 1.1. Interpreting V_Q^{prim} geometrically as in Theorem 2.4 or [27], the $\mathbb{Z}_2$-representation agrees with that induced by the $\mathbb{Z}_2$-action on $\bigsqcup_{d \in \Lambda_Q^+} \mathfrak{M}_d^{st}$.

Motivated by Proposition 3.5, for fixed $e \in \Lambda_Q^{\sigma,+}$ define a twisted $\mathbb{Z}_2$-representation on $\mathcal{H}_Q$ by

$$f \mapsto (-1)^{\chi(e,d)+\mathcal{E}(d)} S_{\mathcal{H}}(f), \qquad f \in \mathcal{H}_{Q,d}.$$

As representations $V_Q = V_{Q,\mathsf{perm}} \oplus V_{Q,\mathsf{fix}}$ where

$$V_{Q,\mathsf{perm}} = \bigoplus_{\substack{d \in \Lambda_Q^+ \\ d \neq \sigma(d)}} V_{Q,d}, \qquad V_{Q,\mathsf{fix}} = \bigoplus_{d \in \Lambda_Q^{\sigma,+}} V_{Q,d}.$$

The subrepresentation $V_{Q,\mathsf{perm}}$ is a direct sum of permutation representations and so can be written (non-canonically) as

$$V_{Q,\mathsf{perm}} = V_{Q,+} \oplus V_{Q,-} \tag{19}$$

for some $\Lambda_Q^+ \times \mathbb{Z}$-graded subspaces $V_{Q,+}$, $V_{Q,-}$ which are permuted by the $\mathbb{Z}_2$-action. Define a $\Lambda_Q^+ \times \mathbb{Z}$-graded vector space by

$$V_Q(e) = V_{Q,+} \oplus (V_{Q,\mathsf{fix}})_{(\mathbb{Z}_2,e)}$$

where $(-)_{(\mathbb{Z}_2,e)}$ denotes $\mathbb{Z}_2$-coinvariants. By identifying invariants and coinvariants we regard $V_Q(e)$ as a subspace of V_Q.

Conjecture 3.6. *Let Q be a σ-symmetric quiver. Then the CoHA action map*

$$\bigoplus_{e \in \Lambda_Q^{\sigma,+}} \mathrm{Sym}(V_Q(e)) \boxtimes W_{Q,e}^{\mathsf{prim}} \xrightarrow{\star} \mathcal{M}_Q$$

is an isomorphism in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}$.

When $\mathcal{H}_Q$ is supercommutative without any twist (see Section 2.2) there is a refinement of Conjecture 3.6 which partially describes the module structure of $\mathcal{M}_Q$. We require the following basic result.

Lemma 3.7. *Let Q be a σ-symmetric quiver. If $\mathcal{H}_Q$ is supercommutative, then the $\mathbb{Z}_2$-grading of $\mathcal{M}_Q$ defined by the reduction modulo two of the shifted cohomological degree makes $\mathcal{M}_Q$ is a super $\mathcal{H}_Q$-module.*

Proof. First observe that for an arbitrary quiver with involution the equality

$$\chi(d,d') = \chi(\sigma(d'),\sigma(d)), \qquad d,d' \in \Lambda_Q \tag{20}$$

holds. In the σ-symmetric case, the parity of elements of $\mathcal{H}_{Q,(d,k)} \star \mathcal{M}_{Q,(e,l)}$ is $\mathcal{E}(H(d)+e)$. Modulo two we have

$$\begin{aligned}
\mathcal{E}(H(d)+e) &\equiv \mathcal{E}(d) + \mathcal{E}(\sigma(d)) + \chi(d,d) + \mathcal{E}(e) + \chi(d,e) + \chi(\sigma(d),e) \\
&\equiv \mathcal{E}(d) + \mathcal{E}(\sigma(d)) + \chi(d,d) + \mathcal{E}(e) + \chi(d,e) + \chi(d,e) \\
&\equiv \mathcal{E}(d) + \mathcal{E}(\sigma(d)) + \chi(d,d) + \mathcal{E}(e) \\
&\equiv \chi(d,d) + \mathcal{E}(e).
\end{aligned}$$

The first equality follows by using equation (5) twice, the second from equation (20), the third from symmetry of Q and the last from σ-symmetry of Q. Since $\chi(d,d) + \mathcal{E}(e)$ is the sum of the parities of $\mathcal{H}_{Q,(d,k)}$ and $\mathcal{M}_{Q,(e,l)}$ the lemma follows. $\square$

Consider V_Q^{prim} as a $\Lambda_Q^{\sigma,+} \times \mathbb{Z}$-graded $\mathbb{Z}_2$-representation by setting

$$\tilde{V}_{Q,e}^{\mathsf{prim}} = \bigoplus_{\substack{d \in \Lambda_Q^+ \\ H(d)=e}} V_{Q,d}^{\mathsf{prim}}, \qquad e \in \Lambda_Q^{\sigma,+}.$$

Using Proposition 3.5 we see that if $g \in \mathcal{M}_{Q,e}$, then $\mathcal{H}_Q \star g \subset \mathcal{H}_Q$ is naturally a module over the $\Lambda_Q^{\sigma,+} \times \mathbb{Z}$-graded supercommutative algebra

$$\mathcal{H}_Q(e) = \mathrm{Sym}((\tilde{V}_Q)_{(\mathbb{Z}_2,e)}). \tag{21}$$

The strengthened form of Conjecture 3.6 reads as follows.

Conjecture 3.8. *Let Q be a σ-symmetric quiver and assume that $\mathcal{H}_Q$ is supercommutative. Then the CoHA action map*

$$\bigoplus_{e \in \Lambda_Q^{\sigma,+}} \mathcal{H}_Q(e) \boxtimes W_{Q,e}^{\mathsf{prim}} \xrightarrow{\star} \mathcal{M}_Q$$

is an isomorphism in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}$. *Moreover, the restriction to the summand* $\mathcal{H}_Q(e) \boxtimes W^{\mathsf{prim}}_{Q,e}$ *is a* $\mathcal{H}_Q(e)$*-module isomorphism onto its image.*

We will verify some instances of Conjecture 3.8 in Section 4.

Remark. A duality structure induces an involution of the stack $\mathbf{M}^{st} = \bigsqcup_{d \in \Lambda_Q^+} \mathbf{M}^{st}_d$ and $H^\bullet(\mathbf{M}^{st}/\mathbb{Z}_2) \simeq H^\bullet(\mathbf{M}^{st})^{\mathbb{Z}_2}$ as mixed Hodge structures. The algebra $\mathcal{H}_Q(e)$ is not $\mathrm{Sym}(PH^\bullet(\mathbf{M}^{st}/\mathbb{Z}_2))$, but is instead $\mathrm{Sym}(PH^\bullet(\mathbf{M}^{st})^{(\mathbb{Z}_2,e)})$ where we use the non-geometric e-twisted $\mathbb{Z}_2$-action.

Conjectures 3.6 and 3.8 lead to factorizations of orientifold DT series in terms of orientifold DT invariants and equivariant refinements of DT invariants, analogous to the factorization of Corollary 2.3. To explain this, we first work in the setting of Conjecture 3.8.

Definition. *Let* $e' \in \Lambda_Q^{\sigma,+}$. *The* $\mathbb{Z}_2$*-equivariant motivic Donaldson-Thomas invariant is the class of* $\tilde{V}_Q^{\mathsf{prim}}$ *in the* $\Lambda_Q^{\sigma,+} \times \mathbb{Z}$*-graded representation ring of* $\mathbb{Z}_2$*:*

$$\tilde{\Omega}_Q = \sum_{(e,k) \in \Lambda_Q^{\sigma,+} \times \mathbb{Z}} \Big(\dim_{\mathbb{Q}} (\tilde{V}^{\mathsf{prim}}_{Q,(e,k)})^+ + \dim_{\mathbb{Q}} (\tilde{V}^{\mathsf{prim}}_{Q,(e,k)})^- \eta \Big) (-q^{\frac{1}{2}})^k \xi^e$$
$$\in \mathbb{Q}(q^{\frac{1}{2}})[\![\Lambda_Q^{\sigma,+}]\!][\eta]/(\eta^2-1).$$

Here $(-)^{\pm}$ *denotes the subspace of (anti-)invariants for the* e'*-twisted* $\mathbb{Z}_2$*-action.*

For ease of notation we do not indicate the e'-dependence of $\tilde{\Omega}_Q$. Since the character of $\mathbb{Q}[u]$ is $\frac{1+q\eta}{1-q^2}$ we find that the Grothendieck class of $(\tilde{V}_Q)_{(\mathbb{Z}_2,e)}$ is

$$\frac{1}{1-q^2} \sum_{(e,k) \in \Lambda_Q^{\sigma,+} \times \mathbb{Z}} (\tilde{\Omega}^+_{Q,(e,k)} + \tilde{\Omega}^-_{Q,(e,k)} q) q^{\frac{k}{2}} \xi^e.$$

It follows that the parity-twisted Hilbert-Poincaré series of $\mathcal{H}_Q(e')$ is

$$A_Q(e') = \prod_{\substack{(e,k) \in \Lambda_Q^{\sigma,+} \times \mathbb{Z} \\ \lambda \in \{\pm\}}} (q^{\frac{k}{2} + \delta_{-1,\lambda}} \xi^e ; q^2)_\infty^{-\tilde{\Omega}^\lambda_{Q,(e,k)}} \in \mathbb{Q}(q^{\frac{1}{2}})[\![\Lambda_Q^{\sigma,+}]\!].$$

Assuming Conjecture 3.8 holds, we see that

$$A_Q^\sigma = \sum_{e \in \Lambda_Q^{\sigma,+}} A_Q(e) \cdot \Omega^\sigma_{Q,e} \xi^e, \tag{22}$$

interpreted as an equality in $\hat{\mathbb{S}}_Q$ with its commutative multiplication. Equation (22) uniquely determines Ω_Q^σ from A_Q^σ and the $\mathbb{Z}_2$-equivariant motivic DT invariants.

In the setting of Conjecture 3.6, note that as operators on $\hat{\mathbb{S}}_Q$ we have $A_Q(e) \cdot = [\mathrm{Sym}(V_Q(e))]\star$. In particular, the right hand side is independent of the splitting (19). Hence Conjecture 3.6 also implies equation (22).

3.4. **Orientifold Donaldson-Thomas invariants and Hodge theory.** We continue to assume that Q is σ-symmetric. In this section we describe a connection between W_Q^{prim} and the Hodge theory of $\bigsqcup_{e \in \Lambda_Q^{\sigma,+}} \mathfrak{M}_e^{\sigma,st}$. We use the trivial stability, $\theta = 0$, for which a self-dual representation is σ-stable if and only if it has no non-trivial isotropic subrepresentations.

We begin with a simple lemma.

Lemma 3.9. *Let* $e \in \Lambda_Q^{\sigma,+}$.

(1) The canonical map

$$H^{\bullet}(\mathfrak{M}_e^{\sigma,st}) \to H^{\bullet}_{\mathsf{G}_e^{\sigma}}(R_e^{\sigma,st}) \tag{23}$$

is an isomorphism of mixed Hodge structures.

(2) For each $k \geq 0$ the subspace $W_{k-1}H^k(\mathfrak{M}_e^{\sigma\text{-}st})$ is trivial.

Proof. Since $H^{\bullet}_{\mathsf{G}_e^{\sigma}}(R_e^{\sigma,st})$ is isomorphic to the cohomology of the Deligne-Mumford stack $[R_e^{\sigma,st}/\mathsf{G}_e^{\sigma}]$ and $[R_e^{\sigma,st}/\mathsf{G}_e^{\sigma}] \to \mathfrak{M}_e^{\sigma,st}$ is a coarse moduli space, the map (23) is a graded vector space isomorphism [11, Theorem 4.40]. To prove that (23) is a morphism of mixed Hodge structures, observe that the morphisms

$$R_e^{\sigma,st} \times_{\mathsf{G}_e^{\sigma}} E_N \to \mathfrak{M}_e^{\sigma,st}, \tag{24}$$

in the notation of Section 1.4, approximate the morphism

$$R_e^{\sigma,st} \times_{\mathsf{G}_e^{\sigma}} E\mathsf{G}_e^{\sigma} \to R_e^{\sigma,st}/\mathsf{G}_e^{\sigma} = \mathfrak{M}_e^{\sigma,st}.$$

The maps in cohomology induced by (24) are morphisms of mixed Hodge structures. Passing to the limit finishes the proof of the first part of the lemma.

Since $\mathfrak{M}_e^{\sigma,st}$ is an orbifold the second part follows from [9, Théorèm 8.2.4 (iv)]. $\square$

The next result gives a partial analogue of Theorem 2.4.

Proposition 3.10. *Let Q be a σ-symmetric quiver. For each $e \in \Lambda_Q^{\sigma,+}$ the composition $H^{\bullet}_{\mathsf{G}_e^{\sigma}}(R_e^{\sigma}) \to H^{\bullet}_{\mathsf{G}_e^{\sigma}}(R_e^{\sigma,st}) \simeq H^{\bullet}(\mathfrak{M}_e^{\sigma,st})$ factors through a surjective morphism*

$$W_{Q,e}^{\mathsf{prim}} \to PH^{\bullet-\mathcal{E}(e)}(\mathfrak{M}_e^{\sigma,st}).$$

Proof. As the argument is similar to [3], we will be brief. Poincaré duality for smooth Artin stacks gives a perfect pairing

$$H^{\bullet}_{\mathsf{G}_e^{\sigma}}(R_e^{\sigma}) \otimes H^{-2\mathcal{E}(e)-\bullet}_{c,\mathsf{G}_e^{\sigma}}(R_e^{\sigma}) \to \mathbb{Q}(-\mathcal{E}(e)).$$

Here we have used that $\dim_{\mathbb{C}}[R_e^{\sigma}/\mathsf{G}_e^{\sigma}] = -\mathcal{E}(e)$. By [9, Théorème 9.1.1] the mixed Hodge structure on $H^i_{\mathsf{G}_e^{\sigma}}(R_e^{\sigma}) \simeq H^i(B\mathsf{G}_e^{\sigma})$ is pure of weight i. Hence $H^i_{c,\mathsf{G}_e^{\sigma}}(R_e^{\sigma})$ is pure of weight i.

Consider the long exact sequence associated to the pair $(R_e^{\sigma,st}, R_e^{\sigma}\backslash R_e^{\sigma,st})$:

$$\cdots \to H^{i-1}_{c,\mathsf{G}_e^{\sigma}}(R_e^{\sigma}\backslash R_e^{\sigma,st}) \to H^i_{c,\mathsf{G}_e^{\sigma}}(R_e^{\sigma,st}) \to H^i_{c,\mathsf{G}_e^{\sigma}}(R_e^{\sigma}) \to H^i_{c,\mathsf{G}_e^{\sigma}}(R_e^{\sigma}\backslash R_e^{\sigma,st}) \to \cdots$$

Since the weights of $H^{i-1}_{c,\mathsf{G}_e^{\sigma}}(R_e^{\sigma}\backslash R_e^{\sigma,st})$ are bounded above by $i-1$, the restriction $PH^i_{c,\mathsf{G}_e^{\sigma}}(R_e^{\sigma,st}) \to H^i_{c,\mathsf{G}_e^{\sigma}}(R_e^{\sigma})$ is an injection. By duality, $H^i_{\mathsf{G}_e^{\sigma}}(R_e^{\sigma}) \to PH^i_{\mathsf{G}_e^{\sigma}}(R_e^{\sigma,st})$ is a surjection.

Next, a straightforward modification of the proof of [3, Lemma 2.1] shows that for each $e \in \Lambda_Q^{\sigma,+}$ the composition of the CoHA action map

$$\bigoplus_{\substack{(d',e') \in \Lambda_Q^+ \times \Lambda_Q^{\sigma,+} \\ H(d')+e'=e,\ d' \neq 0}} \mathcal{H}_{Q,d'} \boxtimes \mathcal{M}_{Q,e'} \xrightarrow{\star} \mathcal{M}_{Q,e} = H^{\bullet-\mathcal{E}(e)}_{\mathsf{G}_e^{\sigma}}(R_e^{\sigma})$$

with the restriction

$$H^{\bullet}_{\mathsf{G}_e^{\sigma}}(R_e^{\sigma}) \to H^{\bullet}_{\mathsf{G}_e^{\sigma}}(R_e^{\sigma,st}) \simeq H^{\bullet}(\mathfrak{M}_e^{\sigma,st})$$

is zero. The last isomorphism follows from the first part of Lemma 3.9. Combined with the previous paragraph, this implies that the restriction $W_{Q,e}^{\mathsf{prim}} \to PH^{\bullet-\mathcal{E}(e)}(\mathfrak{M}_e^{\sigma,st})$ is surjective. $\square$

The proof of injectivity in Theorem 2.4 uses a cohomological interpretation of Ω_Q due to Hausel, Letellier and Rodriguez-Villegas [18] which relies on the smoothness of Nakajima quiver varieties. As there are no smooth analogues of Nakajima varieties for self-dual representations, it is not clear how to adapt the proof from [3]. In any case, it is natural to make the following conjecture.

Conjecture 3.11. *The surjection $W_{Q,e}^{\mathsf{prim}} \twoheadrightarrow PH^{\bullet-\mathcal{E}(e)}(\mathfrak{M}_e^{\sigma,st})$ is an isomorphism.*

We will confirm some instances of Conjecture 3.11 in Section 4.

In view of results of [27] it is also natural to conjecture that W_Q^{prim} computes the intersection cohomology of the closure of $\mathfrak{M}_e^{\sigma,st} \subset \mathfrak{M}_e^{\sigma,ss}$:

$$W_{Q,e}^{\mathsf{prim}} \simeq IC^{\bullet-\mathcal{E}(e)}(\overline{\mathfrak{M}_e^{\sigma,st}}).$$

This can be verified in all examples in which Conjecture 3.11 is verified below.

3.5. **The critical semistable CoHM.** We explain how to generalize Section 3.1 to define the CoHM in the presence of a stability and a potential.

Fix a stability θ and a potential $W \in \mathbb{C}Q/[\mathbb{C}Q,\mathbb{C}Q]$. Let $d', d'' \in \Lambda_Q^+$ and set $d = d' + d''$. Let $R_d^{\theta\text{-}ss} \subset R_d$ be the open subvariety of semistable representations and define $R_{d',d''}^{\theta\text{-}ss} = R_{d',d''} \cap R_d^{\theta\text{-}ss}$. The canonically defined trace functions $\mathrm{tr}(W)_d : R_d^{\theta\text{-}ss} \to \mathbb{C}$ and $\mathrm{tr}(W)_{d',d''} : R_{d',d''}^{\theta\text{-}ss} \to \mathbb{C}$ are invariant under the actions of GL_d and $\mathsf{GL}_{d',d''}$, respectively. Recall that the full subcategory of $\mathsf{Rep}_{\mathbb{C}}(Q)$ consisting of the zero object and all semistable representations of fixed slope is abelian. Using this fact, if $\mu(d') = \mu(d'')$, then upon restriction of the maps (8) we get

$$R_{d'}^{\theta\text{-}ss} \times R_{d''}^{\theta\text{-}ss} \xleftarrow{\pi} R_{d',d''}^{\theta\text{-}ss} \xhookrightarrow{i} R_d^{\theta\text{-}ss}.$$

The trace functions pull back along these maps according to

$$\pi^*\left(\mathrm{tr}(W)_{d'} \boxplus \mathrm{tr}(W)_{d''}\right) = \mathrm{tr}(W)_{d',d''} = i^*\mathrm{tr}(W)_d.$$

Let $\varphi_{\mathrm{tr}(W)_d}\mathbb{Q}_{R_d^{\theta\text{-}ss}} \in D_c^b(R_d^{\theta\text{-}ss})$ be the sheaf of vanishing cycles of $\mathrm{tr}(W)_d$. See [20] for background. We abbreviate $\varphi_{\mathrm{tr}(W)_d}\mathbb{Q}_{R_d^{\theta\text{-}ss}}$ to $\varphi_{\mathrm{tr}(W)_d}$. The slope μ semistable critical CoHA [23, §7] has underlying $\mathbb{Q}$-vector space[3] the direct sum of the duals of compactly supported equivariant cohomology with coefficients in the sheaf of vanishing cycles,

$$\mathcal{H}_{Q,W,\mu}^{\theta\text{-}ss} = \bigoplus_{\{d \in \Lambda_Q^+ \mid \mu(d) = \mu\}} H_{c,\mathsf{GL}_d}^{\bullet}(R_d^{\theta\text{-}ss}, \varphi_{\mathrm{tr}(W)_d})^{\vee}\{\chi(d,d)/2\}.$$

As in Section 1.4, these cohomology groups are defined by a limiting procedure. An associative product on $\mathcal{H}_{Q,W,\mu}^{\theta\text{-}ss}$ is defined via a pull-push procedure as in Section 2.1; see [23, §7], [5, §3.2] for details. The GL_d-equivariant open inclusions $R_d^{\theta\text{-}ss} \hookrightarrow R_d$ induce an algebra homomorphism $\mathcal{H}_{Q,W,\mu}^{\theta} \to \mathcal{H}_{Q,W,\mu}^{\theta\text{-}ss}$. Here $\mathcal{H}_{Q,W,\mu}^{\theta} \subset \mathcal{H}_{Q,W}$ is the subalgebra associated to the submonoid of dimension vectors of slope μ.

Suppose now that Q has an involution and duality structure. Assume that θ is σ-compatible. We say that a potential W is S-compatible if its associated trace functions are invariant under the isomorphisms $R_d \xrightarrow{\sim} R_{\sigma(d)}$. The self-dual trace functions $\mathrm{tr}(W)_e^{\sigma} : R_e^{\sigma,\theta\text{-}ss} \to \mathbb{C}$ and $\mathrm{tr}(W)_{d',e'}^{\sigma} : R_{d',e'}^{\sigma,\theta\text{-}ss} \to \mathbb{C}$ are invariant under G_e^{σ} and $\mathsf{G}_{d',e'}^{\sigma}$, respectively.

We need the following simple observation.

[3]In fact, the underlying object of $\mathcal{H}_{Q,W,\mu}^{\theta\text{-}ss}$ has the structure of a monodromic mixed Hodge module, but we will not use this in this paper.

Lemma 3.12. *Let X be a complex manifold and $f : X \to \mathbb{C}$ a holomorphic function. For any $c \in \mathbb{R}_{>0}$ there is a canonical isomorphism of vanishing cycle functors $\varphi_f \simeq \varphi_{cf}$. In particular, $\varphi_f \mathbb{Q}_X \simeq \varphi_{cf} \mathbb{Q}_X$.*

The next result defines the critical semistable CoHM.

Proposition 3.13. *Let θ be a σ-compatible stability and W a S-compatible potential. Then*

$$\mathcal{M}_{Q,W}^{\theta\text{-}ss} = \bigoplus_{e \in \Lambda_Q^{\sigma,+}} H^\bullet_{c,\mathsf{G}_e^\sigma}(R_e^{\sigma,\theta\text{-}ss}, \varphi_{\operatorname{tr}(W)_e^\sigma})^\vee \{\mathcal{E}(e)/2\}$$

has a natural $\mathcal{H}_{Q,W,0}^{\theta\text{-}ss}$-module structure defined via a pull-push procedure. Moreover, the map $\mathcal{M}_{Q,W} \to \mathcal{M}_{Q,W}^{\theta\text{-}ss}$ induced by the G_e^σ-equivariant open inclusions $R_e^{\sigma,\theta\text{-}ss} \hookrightarrow R_e^\sigma$ is a module homomorphism over $\mathcal{H}_{Q,W,0}^\theta \to \mathcal{H}_{Q,W,0}^{\theta\text{-}ss}$.

Proof. We need the following simple result. Let $U \subset N$ be an isotropic subrepresentation and assume that U is semistable of slope zero and $N /\!\!/ U$ is σ-semistable. Then N is also σ-semistable. Indeed, we have short exact sequences in $\mathsf{Rep}_{\mathbb{C}}(Q)$:

$$0 \to U \to U^\perp \to N /\!\!/ U \to 0, \qquad\qquad 0 \to U^\perp \to N \to S(U) \to 0.$$

Since $N /\!\!/ U$ is σ-semistable it is semistable [41, Proposition 3.2]. Then $U^\perp$ is semistable of slope zero, implying that that N is semistable and hence σ-semistable.

Using this observation, for each $d \in \Lambda_Q^+$ of slope zero and $e \in \Lambda_Q^{\sigma,+}$ we obtain well-defined morphisms

$$R_d^{\theta\text{-}ss} \times R_e^{\sigma,\theta\text{-}ss} \overset{\pi}{\twoheadleftarrow} R_{d,e}^{\sigma,\theta\text{-}ss} \overset{i}{\hookrightarrow} R_{H(d)+e}^{\sigma,\theta\text{-}ss}$$

for which

$$i^* \operatorname{tr}(W)_{H(d)+e}^\sigma = \operatorname{tr}(W)_{d,e}^\sigma = \pi^* \left(2 \operatorname{tr}(W)_d \boxplus \operatorname{tr}(W)_e^\sigma\right).$$

Lemma 3.12 followed by the Thom-Sebastiani isomorphism [26] gives

$$H^\bullet_{c,\mathsf{GL}_d}(R_d^{\theta\text{-}ss}, \varphi_{\operatorname{tr}(W)_d})^\vee \otimes H^\bullet_{c,\mathsf{G}_e^\sigma}(R_e^{\sigma,\theta\text{-}ss}, \varphi_{\operatorname{tr}(W)_e^\sigma})^\vee \xrightarrow{\sim} H^\bullet_{c,\mathsf{GL}_d \times \mathsf{G}_e^\sigma}(R_d^{\theta\text{-}ss} \times R_e^{\sigma,\theta\text{-}ss}, \varphi_{2\operatorname{tr}(W)_d \boxplus \operatorname{tr}(W)_e^\sigma})^\vee.$$

From this point on the construction of the $\mathcal{H}_{Q,W,0}^{\theta\text{-}ss}$-module structure of $\mathcal{M}_{Q,W}^{\theta\text{-}ss}$ is the natural common generalization of [23, §7] and Section 3.1.

The second statement follows from the fact that the diagram

$$\begin{tikzcd}
R_{d,e}^{\sigma,\theta\text{-}ss} \arrow[r, hook] \arrow[d] & R_{d,e}^\sigma \arrow[d] \\
R_d^{\theta\text{-}ss} \times R_e^{\sigma,\theta\text{-}ss} \arrow[r, hook] & R_d \times R_e^\sigma
\end{tikzcd}$$

is Cartesian which in turn follows from the first paragraph of the proof. $\square$

When $W = 0$ and Q is σ-symmetric set

$$W_Q^{\mathsf{prim},\theta} = \mathcal{M}_Q^{\theta\text{-}ss} / (\mathcal{H}_{Q,\mu=0,+}^{\theta\text{-}ss} \star \mathcal{M}_Q^{\theta\text{-}ss}).$$

As in Section 3.3 we expect that $\mathcal{M}_Q^{\theta\text{-}ss}$ is a direct sum of free modules over subalgebras of $\mathcal{H}_{Q,\mu 0}^{\theta\text{-}ss}$, leading to an identity in $\hat{\mathbb{S}}_Q$ of form

$$A_Q^{\sigma,\theta\text{-}ss} = \sum_{e \in \Lambda_Q^{\sigma,+}} A_{Q,0}^{\theta\text{-}ss}(e) \cdot \Omega_{Q,e}^{\sigma,\theta} \xi^e. \tag{25}$$

If equation (25) indeed holds, then orientifold DT invariants are independent of θ. Indeed, this follows from the wall-crossing formula [41, Theorem 4.5]

$$A_Q^{\sigma} = \prod_{\mu \in \mathbb{Q}_{>0}} A_{Q,\mu}^{\theta\text{-}ss} \star A_Q^{\sigma,\theta\text{-}ss}. \tag{26}$$

This should be compared with the fact that DT invariants of symmetric quivers are independent of stability.

To end this section we briefly describe the expected general structure of $\mathcal{M}_{Q,W}$. Let (Q,W) be an arbitrary quiver with potential and generic stability θ. Motivated by the existence and uniqueness of Harder-Narasimhan filtrations, in [23, §5.2] (see also [4, §8.1]) it was asked if there exist algebra embeddings $\mathcal{H}_{Q,W,\mu}^{\theta\text{-}ss} \hookrightarrow \mathcal{H}_{Q,W}$ such that slope ordered CoHA multiplication

$$\overleftarrow{\boxtimes}^{\mathsf{tw}}_{\mu \in \mathbb{Q}} \mathcal{H}_{Q,W,\mu}^{\theta\text{-}ss} \to \mathcal{H}_{Q,W}$$

is an isomorphism in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$. Moreover, each factor $\mathcal{H}_{Q,W,\mu}^{\theta\text{-}ss}$ is expected to be the universal enveloping algebra of a Lie superalgebra structure on $V_{Q,W,\mu}^{\mathsf{prim},\theta} \otimes \mathbb{Q}[u]$ whose definition involves only the stack of semistable representations of slope μ. In this way $\mathcal{H}_{Q,W}$ obtains a Poincaré-Birkhoff-Witt (PBW) type basis. See [8] for results in this direction. Conjecturally, $V_{Q,W}^{\mathsf{prim},\theta}$ can be interpreted as the space of closed oriented single-particle BPS states.

Consider now the orientifold setting and assume that θ is σ-compatible. Every self-dual representation M has a unique self-dual Harder-Narasimhan filtration [41, Proposition 3.3], that is, an isotropic filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_r \subset M$$

such that $U_1/U_0, \dots, U_r/U_{r-1}$ are semistable with strictly decreasing positive slopes and $M /\!\!/ U_r$ is zero or σ-semistable. It is therefore natural to ask for a $\mathcal{H}_{Q,W,0}^{\theta\text{-}ss}$-module[4] embedding $\mathcal{M}_{Q,W}^{\theta\text{-}ss} \hookrightarrow \mathcal{M}_{Q,W}$ such that the CoHA action

$$\overleftarrow{\boxtimes}^{\mathsf{tw}}_{\mu \in \mathbb{Q}_{>0}} \mathcal{H}_{Q,W,\mu}^{\theta\text{-}ss} \boxtimes^{S\text{-}\mathsf{tw}} \mathcal{M}_{Q,W}^{\theta\text{-}ss} \to \mathcal{M}_{Q,W} \tag{27}$$

deifnes an isomorphism in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}$. Together with the natural extension of Conjecture 3.6 to $\mathcal{M}_{Q,W}^{\theta\text{-}ss}$, an isomorphism of the form (27) would determine a PBW type basis of $\mathcal{M}_{Q,W}$ in terms of $W_{Q,W}^{\mathsf{prim},\theta}$ and the PBW bases of $\mathcal{H}_{Q,W,\mu>0}^{\theta\text{-}ss}$ and $\mathcal{H}_{Q,W,0}^{\theta\text{-}ss}$. Conjecturally, $W_{Q,W}^{\mathsf{prim},\theta}$ can be interpreted as the space of single-particle BPS states of the orientifolded theory. Decompositions similar to (27) occur in physical definitions of unoriented BPS invariants [33], [37].

4. Symmetric examples

We study a number of examples and illustrate some instances of the conjectures from Section 3.

4.1. **Disjoint union quivers.** Let Q and Q' be quivers. Their disjoint union $Q \sqcup Q'$ is the quiver with nodes $Q_0 \sqcup Q_0'$ and arrows $Q_1 \sqcup Q_1'$. The opposite Q^{op} is the quiver with nodes Q_0 and an arrow $j \xrightarrow{\alpha^{op}} i$ for each arrow $i \xrightarrow{\alpha} j \in Q_1$.

Lemma 4.1. *There are canonical algebra isomorphisms*

$$\mathcal{H}_{Q \sqcup Q'} \simeq \mathcal{H}_Q \otimes \mathcal{H}_{Q'}, \qquad \mathcal{H}_{Q^{op}} \simeq \mathcal{H}_Q^{op}$$

where $\mathcal{H}_Q^{op}$ is the algebra opposite to $\mathcal{H}_Q$.

[4]More precisely, we should restrict to subalgebras of $\mathcal{H}_{Q,W,0}^{\theta\text{-}ss}$ as above.

Proof. The isomorphism $\mathcal{H}_{Q \sqcup Q'} \xrightarrow{\sim} \mathcal{H}_Q \otimes \mathcal{H}_{Q'}$ is the pullback along the isomorphisms

$$R_d(Q) \times R_{d'}(Q') \xrightarrow{\sim} R_{(d,d')}(Q \sqcup Q')$$

while $\mathcal{H}_{Q^{op}} \xrightarrow{\sim} \mathcal{H}_Q^{op}$ is the pullback along the isomorphisms $R_d(Q) \xrightarrow{\sim} R_d(Q^{op})$ sending a representation to its transpose. □

The quiver $Q^{\sqcup} = Q \sqcup Q^{op}$ has a canonical involution σ which swaps the nodes and arrows of Q and Q^{op}. Fix a compatible duality structure. Representations of $Q^{\sqcup}$ are of the form $U_1 \oplus S(U_2)$ for unique $U_1, U_2 \in \mathsf{Rep}_{\mathbb{C}}(Q)$. Self-dual representations have $U_1 = U_2$. The resulting isomorphism $R_d \xrightarrow{\sim} R^{\sigma}_{H(d)}$, $d \in \Lambda_Q^+$, is equivariant with respect to $\mathsf{GL}_d \xrightarrow{\sim} \mathsf{G}^{\sigma}_{H(d)}$. Let $\mathcal{M}_{Q^{\sqcup}} \xrightarrow{\sim} \mathcal{H}_Q$ be the associated vector space isomorphism. Lemma 4.1 implies that $\mathcal{M}_{Q^{\sqcup}}$ is a $\mathcal{H}_Q \otimes \mathcal{H}_Q^{op}$-module. Similarly, $\mathcal{H}_Q$ is a $\mathcal{H}_Q \otimes \mathcal{H}_Q^{op}$-module, the regular left $\mathcal{H}_Q$-bimodule.

Theorem 4.2. *The map $\mathcal{M}_{Q^{\sqcup}} \to \mathcal{H}_Q$ is an isomorphism of $\mathcal{H}_Q \otimes \mathcal{H}_Q^{op}$-modules.*

Proof. The action of $f_1 \otimes f_3 \in \mathcal{H}_Q \otimes \mathcal{H}_Q^{op}$ on $f_2 \in \mathcal{H}_Q$ is $f_1 \cdot f_2 \cdot f_3 \in \mathcal{H}_Q$, which is in turn the image of $f_1 \otimes f_2 \otimes f_3$ under the composition (degree shifts are omitted)

$$\begin{aligned} H^{\bullet}_{\mathsf{GL}_{d_1}}(R_{d_1}) \otimes H^{\bullet}_{\mathsf{GL}_{d_2}}(R_{d_2}) \otimes H^{\bullet}_{\mathsf{GL}_{d_3}}(R_{d_3}) \to H^{\bullet}_{\mathsf{GL}_{d_1,d_2,d_3}}(R_{d_1,d_2,d_3}) \\ \to H^{\bullet}_{\mathsf{GL}_{d_1+d_2+d_3}}(R_{d_1+d_2+d_3}). \end{aligned}$$

The isomorphism $R_d \simeq R^{\sigma}_{H(d)}$ identifies $R^{\sigma}_{d_1+\sigma(d_2),H(d_3)} \subset R^{\sigma}_{H(d_1+d_2+d_3)}$ with the subspace $R_{d_1,d_3,d_2} \subset R_{d_1+d_2+d_3}$ preserving the Q_0-graded flag

$$\mathbb{C}^{d_1} \subset (\mathbb{C}^{\sigma(d_2)})^{\perp} \cap \mathbb{C}^{d_1+d_2+d_3} \subset \mathbb{C}^{d_1+d_2+d_3}$$

and identifies $\mathsf{G}^{\sigma}_{d_1+\sigma(d_2),H(d_3)} \subset \mathsf{G}^{\sigma}_{H(d_1+d_2+d_3)}$ with $\mathsf{GL}_{d_1,d_3,d_2} \subset \mathsf{GL}_{d_1+d_2+d_3}$. Using these identifications we find that $(f_1 \otimes f_3) \star f_2$ is equal to $f_1 \cdot f_2 \cdot f_3$. That the isomorphism $\mathcal{M}_{Q^{\sqcup}} \xrightarrow{\sim} \mathcal{H}_Q$ respects the gradings follows from the equality

$$\mathcal{E}_{Q^{\sqcup}}(U_1 \oplus S(U_2)) = \chi_Q(U_2, U_1), \qquad U_1, U_2 \in \mathsf{Rep}_{\mathbb{C}}(Q), \tag{28}$$

which is easily verified. □

Remark. The natural generalization of Theorem 4.2 to the critical semistable CoHM holds as well. The proof is similar.

Corollary 4.3. *Conjectures 3.8 and 3.11 hold for $Q^{\sqcup}$.*

Proof. Equation (28) implies that $Q^{\sqcup}$ is σ-symmetric if Q is symmetric. Consider $\mathcal{H}_Q$ with its twisted supercommutative product. Theorems 2.2 and 4.2 give

$$\mathcal{H}_{Q^{\sqcup}} \simeq \mathcal{H}_Q \otimes \mathcal{H}_Q^{op} \simeq \mathrm{Sym}\big(\big(V_Q^{\mathsf{prim}} \oplus S(V_Q^{\mathsf{prim}})\big) \otimes \mathbb{Q}[u]\big).$$

Lift the supercommutative twist of $\mathcal{H}_Q$ by taking $\mathcal{M}_{Q^{\sqcup}}$ to be the regular super $\mathcal{H}_Q$-bimodule. Then $\mathcal{M}_{Q^{\sqcup}}$ is a rank one free module with basis $\mathbf{1}_0^{\sigma} \in \mathcal{M}_{Q^{\sqcup},0}$ over the subalgebra of $\mathcal{H}_{Q^{\sqcup}}$ generated by $v + S(v)$ with $v \in V_Q^{\mathsf{prim}} \otimes \mathbb{Q}[u]$. Hence Conjecture 3.8 holds. Since $Q^{\sqcup}$ has no σ-stable representations Conjecture 3.11 also holds. □

Similarly, $\mathcal{M}_{Q^{\sqcup}}$ is a rank one free module over $\mathcal{H}_Q \subset \mathcal{H}_{Q^{\sqcup}}$. This module structure is the PBW factorization (27) associated to a σ-compatible stability on $Q^{\sqcup}$ whose restriction to $\Lambda_Q^+ \subset \Lambda_{Q^{\sqcup}}^+$ is positive.

4.2. **Zero and one loop quivers.** Let L_m be the quiver with one node and $m \geq 0$ loops. It is symmetric and its CoHA is supercommutative without any twist. If $f_1 \in \mathcal{H}_{L_m,d'}$ and $f_2 \in \mathcal{H}_{L_m,d''}$, then

$$f_1 \cdot f_2 = \sum_{\pi \in \mathfrak{sh}_{d',d''}} \pi\Big(f_1(x'_1,\dots,x'_{d'})f_2(x''_1,\dots,x''_{d''})\prod_{l=1}^{d''}\prod_{k=1}^{d'}(x''_l - x'_k)^{m-1}\Big).$$

The (unique) involution of L_m fixes the node and arrows. Hence L_m is σ-symmetric. A duality structure is determined by signs s and $\tau_1,\dots,\tau_m$. Suppose that τ_+ of the latter are positive and $\tau_- = m - \tau_+$ are negative. When $s=1$ Proposition 3.2 gives $\mathcal{M}_{L_m} = \mathcal{M}^B_{L_m} \oplus \mathcal{M}^D_{L_m}$ with summands spanned by odd and even dimensional self-dual representations, respectively. When $s=-1$ write $\mathcal{M}^C_{L_m}$ for $\mathcal{M}_{L_m}$. Applying Theorem 3.3 to $f \in \mathcal{H}_{L_m,d}$ and $g \in \mathcal{M}_{L_m,e}$ gives

$$f \star g = 2^{(\tau_s - \frac{1-s}{2})d} \sum_{\pi \in \mathfrak{sh}^{\sigma}_{d,e}} \pi\Bigg[f(x_1,\dots,x_d)g(z_1,\dots,z_{\lfloor \frac{e}{e} \rfloor})\times$$

$$\prod_{i=1}^{d} x_i^{N(s,\tau)}\Big(\prod_{1\leq i<j\leq d}(x_i+x_j)\prod_{i=1}^{d}\prod_{j=1}^{\lfloor \frac{e}{2}\rfloor}(x_i^2 - z_j^2)\Big)^{m-1}\Bigg]$$

where

$$N(s,\tau) = \begin{cases} 2\tau_+ + \tau_- - 1, & \text{in type } B \\ \tau_- - 1, & \text{in type } C \\ \tau_+, & \text{in type } D. \end{cases}$$

Using this we obtain the following degree $(0,0)$ or $(1,0)$ isomorphisms:

$$\begin{aligned} &\text{(i) If } \tau_+ = 0, \text{ then } \mathcal{M}^B_{L_m} \simeq \mathcal{M}^C_{L_m}. \\ &\text{(ii) If } \tau_+ = \tau_- - 1, \text{ then } \mathcal{M}^C_{L_m} \simeq \mathcal{M}^D_{L_m}. \\ &\text{(iii) If } m=1 \text{ then } \mathcal{M}^D_{L_1} \simeq \mathcal{M}^B_{L_1}. \end{aligned} \tag{29}$$

As the cases $m \leq 1$ serve as building blocks for more complicated examples, we now study these in detail.

4.2.1. *Zero loops.* Let $m=0$. The CoHA $\mathcal{H}_{L_0}$ is a free supercommutative algebra generated by the odd variables $x^i \in \mathcal{H}_{L_0,1}$, $i \geq 0$, of degree $(1, 2i+1)$ [23, §2.5]. Explicitly, if $\mathbf{i} = (i_d,\dots,i_1)$ is a strictly decreasing partition, then

$$x^{i_1}\cdots x^{i_d} = s_{\mathbf{i}-\delta_d}.$$

Here s_λ is the Schur polynomial associated to a partition λ and $\delta_r = (r-1,\dots,1,0)$. In particular, $V^{\mathsf{prim}}_{L_0} = \mathbb{Q}\cdot \mathbf{1}_1 = \mathbb{Q}_{(1,1)}$.

The first isomorphism of (29) implies $\mathcal{M}^B_{L_0} \simeq \mathcal{M}^C_{L_0}$, so we consider only $\mathcal{M}^B_{L_0}$ and $\mathcal{M}^D_{L_0}$. Given $f \in \mathbb{Q}[x_1,\dots,x_d]$ let $\tilde{f}(x_1,\dots,x_d) = f(x_1^2,\dots,x_d^2)$.

Lemma 4.4. *Let* $\mathbf{i}$ *be a strictly decreasing partition of length* d.

(1) Type B: If all i_j *are odd, then* $s_{\mathbf{i}-\delta_d} \star \mathbf{1}^{\sigma}_0 = 2^d \tilde{s}_{\frac{\mathbf{i}-\mathbf{1}}{2}-\delta_d}$.

(2) Type D: If all i_j *are even, then* $s_{\mathbf{i}-\delta_d} \star \mathbf{1}^{\sigma}_0 = 2^d \tilde{s}_{\frac{\mathbf{i}}{2}-\delta_d}$.

Proof. Consider type B and proceed by induction on d. If $i \geq 1$ is odd, then

$$(x^i \star \mathbf{1}^{\sigma}_0)(z) = z^i(z)^{-1} + (-z)^i(-z)^{-1} = 2(z^2)^{\frac{i-1}{2}} = 2\tilde{s}_{(\frac{i-1}{2})}(z).$$

This confirms the case $d=1$. Assuming the lemma holds for partitions of length $d-1$, we find that $x^{i_1}\cdots x^{i_d} \star \mathbf{1}^{\sigma}_0 = x^{i_1} \star (x^{i_2}\cdots x^{i_d} \star \mathbf{1}^{\sigma}_0)$ is equal to

$$2^d \sum_{p=1}^{d} \frac{z_p^{i_1-1}}{\prod_{\substack{j=1 \\ j\neq p}}^{d}(z_j^2 - z_p^2)} \tilde{s}_{\frac{\mathbf{i}'-\mathbf{1}}{2}-\delta_{d-1}}$$

where $\mathbf{i}' = (i_d, \ldots, i_2)$. A direct calculation shows that this coincides with $2^d \tilde{s}_{\frac{\mathbf{i}-\mathbf{1}}{2}-\delta_d}$.

The proof in type D is similar and is omitted. ☐

Remark. By Proposition 3.5, if $\mathbf{i}$ is not purely odd/even, then $s_{\mathbf{i}-\delta_r}$ annihilates $\mathcal{M}_{L_0}^{B/D}$. A similar statement holds for $\mathcal{M}_{L_1}$ below.

Let $\mathcal{H}_{L_0}^{even}$, $\mathcal{H}_{L_0}^{odd}$ be the subalgebras generated by $\{x^{2i}\}_{i\geq 0}$, $\{x^{2i+1}\}_{i\geq 0}$, respectively. Equivalently, $\mathcal{H}_{L_0}^{even} = \mathrm{Sym}(\mathbb{Q}_{(1,1)} \otimes \mathbb{Q}[u^2])$ and $\mathcal{H}_{L_0}^{odd} = \mathrm{Sym}(\mathbb{Q}_{(1,1)} \otimes u\mathbb{Q}[u^2])$. These are the subalgebras defined in equation (21); they are independent of e.

Theorem 4.5.
(1) $\mathcal{M}_{L_0}^B$ is a free $\mathcal{H}_{L_0}^{odd}$-module with basis $\mathbf{1}_1^\sigma \in \mathcal{M}_{L_0,1}^B$.
(2) $\mathcal{M}_{L_0}^D$ is a free $\mathcal{H}_{L_0}^{even}$-module with basis $\mathbf{1}_0^\sigma \in \mathcal{M}_{L_0,0}^D$.

Proof. The map $\mathbf{i} \mapsto \frac{\mathbf{i}-\mathbf{1}}{2}$ is a bijection between the set of strictly decreasing purely odd partitions of length d and the set of strictly decreasing partitions of length d. Since the Schur functions $\tilde{s}_{\mathbf{i}'-\delta_d}$ parameterized by the former set are an additive basis of $\mathcal{M}_{L_0,2d+1}^B \simeq \mathbb{Q}[z_1^2, \ldots, z_d^2]^{\mathfrak{S}_d}$, the statement in type B follows from Lemma 4.4.

In type D use instead the bijection $\mathbf{i} \mapsto \frac{\mathbf{i}}{2}$ between the set of strictly decreasing purely even and the set of strictly decreasing partitions. ☐

Corollary 4.6. *The motivic orientifold DT invariants of L_0 are*
$$\Omega_{L_0}^B = \xi, \qquad \Omega_{L_0}^C = 1, \qquad \Omega_{L_0}^D = 1.$$
Conjectures 3.8 and 3.11 hold for L_0.

Proof. The calculation of the orientifold DT invariants and the validity of Conjecture 3.8 follow from Theorem 4.5. Conjecture 3.11 follows from the isomorphisms
$$\mathfrak{M}_{2e}^{\mathfrak{sp},st} = \varnothing, \ \ e \geq 1, \qquad \mathfrak{M}_e^{\mathfrak{o},st} = \begin{cases} pt, & \text{if } e = 1, \\ \varnothing, & \text{if } e \geq 2, \end{cases}$$
the superscripts $\mathfrak{sp}$ and $\mathfrak{o}$ indicating type C or types B or D, respectively. ☐

4.2.2. *One loop.* Let $m = 1$. The CoHA $\mathcal{H}_{L_1}$ is a free supercommutative algebra generated by even variables $x^i \in \mathcal{H}_{L_1,1}$, $i \geq 0$, of degree $(1, 2i)$ [23, §2.5]. Explicitly,
$$x^{i_1} \cdots x^{i_d} = N(\mathbf{i}) m_{\mathbf{i}}$$
where $m_{\mathbf{i}}$ is the monomial symmetric polynomial and $N(\mathbf{i}) = \prod_{k\geq 0} \#\{j \geq 1 \mid i_j = k\}!$. Hence $V_{L_1}^{\mathsf{prim}} = \mathbb{Q} \cdot \mathbf{1}_1 = \mathbb{Q}_{(1,0)}$.

The isomorphisms (29) give $\mathcal{M}_{L_1}^B \simeq \mathcal{M}_{L_1}^C \simeq \mathcal{M}_{L_1}^D$ if $\tau = -1$ and $\mathcal{M}_{L_1}^B \simeq \mathcal{M}_{L_1}^D$ if $\tau = 1$. So we consider only $\mathcal{M}_{L_1}^B$ if $\tau = -1$ and $\mathcal{M}_{L_1}^{C,D}$ if $\tau = 1$.

Lemma 4.7. *Let $\mathbf{i}$ be a partition of length d.*

1. *Type B, $\tau = -1$: If $\mathbf{i}$ is purely even, then $m_{\mathbf{i}} \star \mathbf{1}_0^\sigma = 2^d \tilde{m}_{\frac{\mathbf{i}}{2}}$.*
2. *Type C, $\tau = 1$: If $\mathbf{i}$ is purely odd, then $m_{\mathbf{i}} \star \mathbf{1}_0^\sigma = 2^d \tilde{m}_{\frac{\mathbf{i}-\mathbf{1}}{2}}$.*
3. *Type D, $\tau = 1$: If $\mathbf{i}$ is purely odd, then $m_{\mathbf{i}} \star \mathbf{1}_{2e}^\sigma = 2^d \tilde{m}_{(\frac{\mathbf{i}+\mathbf{1}}{2}, \mathbf{0}^e)}$, where $(\mathbf{i}, \mathbf{0}^e)$ denotes the length $d+e$ partition obtained by appending e zeros to $\mathbf{i}$.*

Proof. The proof is similar to that of Lemma 4.4 and so is omitted. ☐

Let $\mathcal{H}_{L_1}^{even} = \mathrm{Sym}(\mathbb{Q}_{(1,0)} \otimes \mathbb{Q}[u^2])$ and $\mathcal{H}_{L_1}^{odd} = \mathrm{Sym}(\mathbb{Q}_{(1,0)} \otimes u\mathbb{Q}[u^2])$.

Theorem 4.8.

1. *If $\tau = -1$, then $\mathcal{M}_{L_1}^B$ is a free $\mathcal{H}_{L_1}^{even}$-module with basis $\mathbf{1}_0^\sigma \in \mathcal{M}_{L_1,1}^B$.*
2. *If $\tau = 1$, then $\mathcal{M}_{L_1}^C$ is a free $\mathcal{H}_{L_1}^{odd}$-module with basis $\mathbf{1}_0^\sigma \in \mathcal{M}_{L_1,0}^C$.*

(3) If $\tau = 1$, then $\mathcal{M}^D_{L_1}$ is a free $\mathcal{H}^{odd}_{L_1}$-module with basis $\mathbf{1}^\sigma_{2e} \in \mathcal{M}^D_{L_1,2e}$, $e \geq 0$.

Proof. The proof is similar to that of Theorem 4.5, using Lemma 4.7 instead of Lemma 4.4. □

Corollary 4.9. *The motivic orientifold DT invariants of L_1 are*

$$\tau = -1: \qquad \Omega^B_{L_1} = \xi, \qquad \Omega^C_{L_1} = 1, \qquad \Omega^D_{L_1} = 1$$

and

$$\tau = 1: \qquad \Omega^B_{L_1} = \frac{q^{-\frac{1}{2}}\xi}{1 - q^{-1}\xi^2}, \qquad \Omega^C_{L_1} = 1, \qquad \Omega^D_{L_1} = \frac{1}{1 - q^{-1}\xi^2}.$$

Conjectures 3.8 and 3.11 hold for L_1.

Proof. The calculation of the orientifold DT invariants and the validity of Conjecture 3.8 follow from Theorem 4.8. For $\tau = -1$ we find

$$\mathfrak{M}^{\mathfrak{sp},st}_{2e} = \varnothing, \ \ e \geq 1, \qquad \mathfrak{M}^{\mathfrak{o},st}_{e} = \begin{cases} pt, & \text{if } e = 1, \\ \varnothing, & \text{if } e \geq 2. \end{cases}$$

while for $\tau = -1$ we find $\mathfrak{M}^{\mathfrak{sp},st}_{2e} = \varnothing$ and

$$\mathfrak{M}^{\mathfrak{o}}_{e} = \mathsf{Symm}_{e \times e} /\!\!/ \mathsf{O}_e \simeq \mathrm{Sym}^e \mathbb{C}, \qquad \mathfrak{M}^{\mathfrak{o},st}_{e} = \mathrm{Sym}^e \mathbb{C} \backslash \Delta, \ \ e \geq 1.$$

Here $\mathsf{Symm}_{e\times e}$ is the variety of symmetric $e \times e$ matrices and Δ is the big diagonal consisting of unordered n-tuples of points of $\mathbb{C}$ not all of which are distinct. Conjecture 3.11 is now immediate except in the last case, where it reads

$$PH^0(\mathfrak{M}^{\mathfrak{o},st}_{e}) \simeq \mathbb{Q}(0), \quad PH^k(\mathfrak{M}^{\mathfrak{o},st}_{e}) = 0 \quad \text{ if } e, \, k \geq 1.$$

In this case the claim follows from the isomorphism of mixed Hodge structures $H^\bullet(\mathrm{Sym}^e \mathbb{C} \backslash \Delta) \simeq H^\bullet(\mathbb{C} \backslash \{0\})$. □

4.2.3. *Higher loops.* When $m \geq 2$ the situation is more complicated as neither $\mathcal{H}_{L_m}$ nor $\mathcal{M}_{L_m}$ is finitely generated. However, Conjecture 3.8 can be made quite explicit and can be used to give a numerical method to compute orientifold DT invariants. We have

$$\chi(e,d) + \mathcal{E}(d) \equiv d\delta_B + \mathcal{E}(d) \mod 2$$

where δ_B is one in type B zero otherwise. Then $\mathcal{H}_Q(e)$ depends only on the type and not e. Write $\mathcal{H}^\circ_Q$ for $\mathcal{H}_Q(e)$. Each CoHA summand $\mathcal{H}_{L_m,(d,k)}$ is isotypical as a $\mathbb{Z}_2$-representation and the $\mathbb{Z}_2$-equivariant DT invariants are

$$\tilde{\Omega}^+_{2d,k} = \begin{cases} \Omega_{d,k}, & \text{if } d\delta_B + \mathcal{E}(d) + \frac{k-(1-m)d^2}{2} \equiv 0 \mod 2 \\ 0, & \text{if } d\delta_B + \mathcal{E}(d) + \frac{k-(1-m)d^2}{2} \equiv 1 \mod 2 \end{cases}$$

and

$$\tilde{\Omega}^-_{2d,k} = \begin{cases} 0, & \text{if } d\delta_B + \mathcal{E}(d) + \frac{k-(1-m)d^2}{2} \equiv 0 \mod 2 \\ \Omega_{d,k}, & \text{if } d\delta_B + \mathcal{E}(d) + \frac{k-(1-m)d^2}{2} \equiv 1 \mod 2. \end{cases}$$

Conjecture 3.8 states that $\mathcal{M}_{L_m}$ is a free module over $\mathcal{H}^\circ_Q$. Equation (22) becomes

$$A^\sigma_{L_m} = A^\circ_{L_m} \cdot \Omega^\sigma_{L_m}.$$

Since Ω_{L_m} have been computed by Reineke [31, Theorem 6.8] and $A^\sigma_{L_m}$ is given explicitly by equation (10), this gives a way to compute $\Omega^\sigma_{L_m}$.

Example. For $m = 2$ we have

$$\Omega_{L_2} = -q^{-\frac{1}{2}}t + q^{-2}t^2 - q^{-\frac{9}{2}}t^3 + (q^{-6} + q^{-8})t^4 + O(t^5).$$

When $\tau = -1$ the $\mathbb{Z}_2$-equivariant DT invariants are

$$\tilde{\Omega}^+_{L_2} = -q^{-\frac{9}{2}}\xi^6 + (q^{-6} + q^{-8})\xi^8 + O(\xi^{10}), \qquad \tilde{\Omega}^-_{L_2} = -q^{-\frac{1}{2}}\xi^2 + q^{-2}\xi^4 + O(\xi^{10})$$

and equation (22) predicts

$$\begin{aligned}\Omega^B_{L_2} &= \xi - q^{-\frac{3}{2}}\xi^3 + (q^{-5}+q^{-3})\xi^5 - (q^{-\frac{21}{2}} + q^{-\frac{17}{2}} + 2q^{-\frac{13}{2}} + q^{-\frac{9}{2}})\xi^7 + \\ &\quad (q^{-18}+q^{-16}+2q^{-14}+3q^{-12}+4q^{-10}+3q^{-8}+q^{-6})\xi^9 + O(\xi^{11}).\end{aligned}$$

Up to $\Lambda^{\sigma,+}_Q$-degree five the generators of $\mathcal{M}^B_{L_2}$ can be taken to be $\mathbf{1}^\sigma_1, \mathbf{1}^\sigma_3$, $\mathbf{1}^\sigma_5$ and $z_1^2 + z_2^2$. ◁

Example. For $m = 3$ we have

$$\Omega_{L_2} = -q^{-\frac{1}{2}}t + q^{-2}t^2 - q^{-\frac{9}{2}}t^3 + (q^{-6}+q^{-8})t^4 + O(t^5).$$

When $\tau = 1$ the $\mathbb{Z}_2$-equivariant DT invariants are

$$\tilde{\Omega}^+_{L_3} = q^{-4}\xi^4 + q^{-6}\xi^6 + (q^{-8}+2q^{-10}+2q^{-12}+q^{-14}+q^{-16})\xi^8 + O(\xi^{10})$$

and

$$\tilde{\Omega}^-_{L_3} = q^{-1}\xi^2 + (q^{-7}+q^{-9})\xi^6 + (q^{-9}+q^{-11}+q^{-13})\xi^8 + O(\xi^{10})$$

and equation (22) predicts

$$\Omega^D_{L_3,0} = 1, \qquad \Omega^D_{L_3,2} = q^{-4}+q^{-2}, \qquad \Omega^D_{L_3,4} = q^{-12}+q^{-10}+2q^{-8}+2q^{-6}+q^{-4}$$

and

$$\Omega^D_{L_3,6} = q^{-24}+q^{-22}+2q^{-20}+3q^{-18}+4q^{-16}+5q^{-14}+6q^{-12}+6q^{-10}+4q^{-8}+q^{-6}$$

and

$$\begin{aligned}\Omega^D_{L_3,8} &= q^{-40}+q^{-38}+2q^{-36}+3q^{-34}+5q^{-32}+6q^{-30}+9q^{-28}+11q^{-26}+14q^{-24}+ \\ &\quad 16q^{-22}+19q^{-20}+20q^{-18}+21q^{-16}+19q^{-14}+14q^{-12}+6q^{-10}+q^{-8}.\end{aligned}$$

◁

4.3. **Symmetric $\tilde{A}_1$ quiver.** Let Q be the following affine Dynkin quiver,

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left, "\alpha"] & \underset{2}{\bullet} \arrow[l, bend left, "\beta"] \end{tikzcd}$$

The CoHA $\mathcal{H}_Q$ is supercommutative without any twist. The product of $f_1 \in \mathcal{H}_{Q,d'}$ and $f_2 \in \mathcal{H}_{Q,d''}$ is

$$f_1 \cdot f_2 = \sum_{\pi \in \mathfrak{sh}_{d',d''}} \pi\Big(f_1(x'_1,\dots,x'_{d'_1},y'_1,\dots,y'_{d'_2}) f_2(x''_1,\dots,x''_{d''_1},y''_1,\dots,y''_{d''_2}) \times \\ \frac{\prod_{j=1}^{d''_2}\prod_{i=1}^{d'_1}(y''_j - x'_i)\prod_{j=1}^{d''_1}\prod_{i=1}^{d'_2}(x''_j - y'_i)}{\prod_{i=1}^{d''_1}\prod_{j=1}^{d'_1}(x''_i - x'_j)\prod_{i=1}^{d''_2}\prod_{j=1}^{d'_2}(y''_i - y'_j)}\Big).$$

A representation of Q of dimension vector (d_1, d_2) consists of a pair of matrices

$$A \in \mathsf{Mat}_{d_2 \times d_1}, \quad B \in \mathsf{Mat}_{d_1 \times d_2}.$$

For stability $\theta = (1, -1)$ the semistable representations are

(i) the direct sums of simples $S_1^{\oplus k}$, $k \geq 1$, having slope 1,
(ii) the direct sums of simples $S_2^{\oplus k}$, $k \geq 1$, having slope -1, and
(iii) the pairs $(A, B) \in \mathsf{GL}_d(\mathbb{C}) \times \mathsf{Mat}_{d\times d}$, $d \geq 1$, having slope 0.

The semistable algebras $\mathcal{H}^{\theta\text{-}ss}_{Q,\mu=1}$ and $\mathcal{H}^{\theta\text{-}ss}_{Q,\mu=-1}$ are isomorphic to $\mathcal{H}_{L_0}$ and embed canonically as subalgebras of $\mathcal{H}_Q$. On the other hand, the inclusion

$$\mathsf{Mat}_{d\times d} \to \mathsf{GL}_d(\mathbb{C}) \times \mathsf{Mat}_{d\times d}, \quad B \mapsto (\mathbb{I}_{d\times d}, B)$$

descends to an isomorphism from the stack of d-dimensional representations of the one loop quiver L_1 to the stack of (d,d)-dimensional semistable representations of Q. This induces a graded algebra isomorphism $\mathcal{H}^{\theta\text{-}ss}_{Q,\mu=0} \simeq \mathcal{H}_{L_1}$ and the map

$$\Psi_0 : \mathcal{H}_{L_1} \to \mathcal{H}_Q, \quad x^i \mapsto x^i y^0$$

extends to an algebra embedding. In [14, Proposition 2.4] Franzen proved that the slope ordered CoHA multiplication

$$\Psi : \mathcal{H}^{\theta\text{-}ss}_{Q,\mu=1} \boxtimes \mathcal{H}^{\theta\text{-}ss}_{Q,\mu=0} \boxtimes \mathcal{H}^{\theta\text{-}ss}_{Q,\mu=-1} \to \mathcal{H}_Q, \quad a \otimes b \otimes c \mapsto a\Psi_0(b)c \tag{30}$$

is an isomorphism of $\Lambda_Q^+ \times \mathbb{Z}$-graded supercommutative algebras. In particular,

$$V_Q^{\mathsf{prim}} = \mathbb{Q} \cdot \mathbf{1}_{(1,0)} \oplus \mathbb{Q} \cdot \mathbf{1}_{(1,1)} \oplus \mathbb{Q} \cdot \mathbf{1}_{(0,1)}.$$

Let σ be the involution of Q that swaps the nodes and fixes the arrows. Then

$$\mathcal{E}(d_1,d_2) = d_1 d_2 - \frac{d_1(d_1 + s\tau_\alpha)}{2} - \frac{d_2(d_2 + s\tau_\beta)}{2}.$$

This shows that there are two inequivalent σ-symmetric duality structures on (Q,σ), say $s = 1$ and $\tau = \pm 1$. The structure maps (A,B) of a self-dual representation are symmetric if $\tau = 1$ and skew-symmetric if $\tau = -1$. If $f \in \mathcal{H}_{Q,(d_1,d_2)}$ and $g \in \mathcal{M}_{Q,(e,e)}$, then $f \star g$ is equal to

$$\sum_{\pi \in \mathfrak{sh}^\sigma_{d,e}} \pi\Big(f(x_1,\dots,x_{d_1},y_1,\dots,y_{d_2}) g(z_1,\dots,z_e) \times$$

$$\frac{\prod_{1 \le j \le_\tau l \le d_1} (-x_j - x_l) \prod_{l=1}^{d_1} \prod_{k=1}^{e} (-z_k - x_l) \prod_{1 \le j \le_\tau m \le d_2} (-y_j - y_m) \prod_{m=1}^{d_2} \prod_{k=1}^{e} (z_k - y_m)}{\prod_{l=1}^{d_1} \prod_{k=1}^{e} (z_k - x_l) \prod_{k=1}^{e} \prod_{m=1}^{d_2} (-y_m - z_k) \prod_{l=1}^{d_1} \prod_{m=1}^{d_2} (-y_m - x_l)} \Big).$$

The non-empty subvarieties of semistable self-dual representations are

$$\tau = 1 : \quad R^{\sigma,\theta\text{-}ss}_{(e,e)} = (\mathsf{Symm}_{e \times e} \cap \mathsf{GL}_e(\mathbb{C})) \times \mathsf{Symm}_{e \times e}$$

and

$$\tau = -1 : \quad R^{\sigma,\theta\text{-}ss}_{(2e,2e)} = (\mathsf{Skew}_{2e \times 2e} \cap \mathsf{GL}_{2e}(\mathbb{C})) \times \mathsf{Skew}_{2e \times 2e}.$$

From this we see that the stack of semistable self-dual representations of Q is isomorphic to the stack of self-dual representations of L_1 with duality structure

$$(s_{L_1} = \tau, \tau_{L_1} = +1).$$

The induced map $\mathcal{M}_Q^{\theta\text{-}ss} \xrightarrow{\sim} \mathcal{M}_{L_1}$ is a module isomorphism over $\mathcal{H}^{\theta\text{-}ss}_{Q,\mu=0} \xrightarrow{\sim} \mathcal{H}_{L_1}$.

Lemma 4.10. *In dimension vector $(e,e) \in \Lambda_Q^{\sigma,+}$ the kernel of the restriction morphism $\mathcal{M}_Q \to \mathcal{M}_Q^{\theta\text{-}ss}$ is the image of the CoHA action map*

$$\bigoplus_{d=1}^{e} \mathcal{H}_{Q,(d,0)} \boxtimes \mathcal{M}_{Q,(e-d,e-d)} \xrightarrow{\star} \mathcal{M}_{Q,(e,e)}.$$

Proof. Let M be a self-dual representation determined by matrices (A,B). Then $0 \subset \ker A \subset M$ is the self-dual Harder-Narasimhan filtration of M. The Harder-Narasimhan strata of R_e^σ are therefore the locally closed subsets consisting of self-dual representations with fixed $\dim_{\mathbb{C}} \ker A$. The closure of a stratum is thus a union of strata. Using this observation, [14, Lemma 2.1] can be applied with only obvious modifications to complete the proof. In slightly more detail, the methods of [14] can be used to prove the present lemma for the Chow theoretic Hall module, defined similarly to $\mathcal{M}_Q$ but using equivariant Chow groups instead of equivariant cohomology. In the case at hand the (semistable) cohomological and Chow theoretic Hall modules are isomorphic, as can be verified directly. Hence the lemma also follows in the cohomological case. □

We can now describe $\mathcal{M}_Q$. Let $\mathcal{H}_Q^{\circ} \subset \mathcal{H}_Q$ be the subalgebra generated by

$$V_Q^{\circ} = \big(\mathbb{Q} \cdot \mathbf{1}_{(1,0)} \otimes \mathbb{Q}[u]\big) \oplus \big(\mathbb{Q} \cdot \mathbf{1}_{(1,1)} \otimes u\mathbb{Q}[u^2]\big) \subset V_Q.$$

There is an isomorphism of algebras $\mathcal{H}_Q^{\circ} \simeq \mathcal{H}_{Q,\mu=1}^{\theta\text{-}ss} \otimes \mathcal{H}_{Q,\mu=0}^{\theta\text{-}ss,odd}$, the second factor being an isomorphic image of $\mathcal{H}_{L_1}^{odd}$. The map sending $\mathbf{1}_0^{\sigma} \in \mathcal{M}_{L_1,0}$ to $\mathbf{1}_{(0,0)}^{\sigma} \in \mathcal{M}_{Q,(0,0)}$ extends to a $\mathcal{H}_{Q,\mu=0}^{\theta\text{-}ss,odd}$-module embedding $\mathcal{M}_Q^{\theta\text{-}ss} \hookrightarrow \mathcal{M}_Q$.

Theorem 4.11. *The semistable CoHM $\mathcal{M}_Q^{\theta\text{-}ss}$ is a free $\mathcal{H}_{Q,\mu=0}^{\theta\text{-}ss,odd}$-module with basis*

(1) $\mathbf{1}_0^{\sigma} \in \mathcal{M}_{Q,(0,0)}$ if $\tau = -1$, and
(2) $\mathbf{1}_{(e,e)}^{\sigma} \in \mathcal{M}_{Q,(e,e)}$, $e \geq 0$, if $\tau = 1$.

Moreover, the CoHA action

$$\mathcal{H}_{Q,\mu=1}^{\theta\text{-}ss} \boxtimes \mathcal{M}_Q^{\theta\text{-}ss} \xrightarrow{\star} \mathcal{M}_Q \tag{31}$$

is an isomorphism of $\Lambda_Q^{\sigma,+} \times \mathbb{Z}$-graded $\mathcal{H}_Q^{\circ}$-modules. In particular, $\mathcal{M}_Q$ is a free $\mathcal{H}_Q^{\circ}$-module and Conjecture 3.8 holds for Q.

Proof. The first statement follows from Theorem 4.8 and the $\mathcal{H}_{L_1}$-module isomorphism $\mathcal{M}_Q^{\theta\text{-}ss} \simeq \mathcal{M}_{L_1}$.

Turning to the second statement, direct calculation shows that in this case the restriction map $\mathcal{M}_Q \to \mathcal{M}_Q^{\theta\text{-}ss}$ is surjective. From this and Lemma 4.10 we conclude that the map (31) is also surjective. The wall-crossing formula (26) for Q reads $A_{Q,\mu=1}^{\theta\text{-}ss} \star A_Q^{\sigma,\theta\text{-}ss} = A_Q^{\sigma}$ and implies that the domain and codomain of the map (31) have the same Hilbert-Poincaré series. Hence the map (31) is an isomorphism. □

Corollary 4.12. *The motivic orientifold DT invariants of Q are*

$$\tau = -1 : \qquad \Omega_Q^{\sigma} = 1$$

and

$$\tau = 1 : \qquad \Omega_Q^{\sigma} = \frac{1}{1 - q^{-\frac{1}{2}}\xi^{(1,1)}}.$$

Conjecture 3.11 holds for $\tilde{A}_1$.

Proof. When $\tau = -1$ the corollary follows immediately from Theorem 4.11.

When $\tau = 1$ Theorem 4.11 shows that $(1 - q^{-\frac{1}{2}}\xi^{(1,1)})^{-1}$ is an upper bound for Ω_Q^{σ}. To prove that it is also a lower bound, observe that the cohomological degree shift of the action of $\mathcal{H}_{Q,d}$ on $\mathcal{M}_{Q,e}$ is

$$(d_1 - d_2)^2 + d_1 + d_2 \geq 0.$$

From this it follows that $\mathbf{1}_{(e,e)}^{\sigma}$, $e \geq 0$, must be included as generators of $\mathcal{M}_Q$. Hence the orientifold DT invariants are as stated.

To verify Conjecture 3.11 when $\tau = 1$ we must prove that $PH^{\bullet}(\mathfrak{M}_{(e,e)}^{\sigma,st}) \simeq \mathbb{Q}(0)$. Note that we take the trivial stability. It is clear that $\mathfrak{M}_{(1,1)}^{\sigma,st} \simeq \mathbb{C}^{\times}$ and that there are no other regularly σ-stable representations. It follows that $\mathfrak{M}_{(e,e)}^{\sigma,st} \simeq \operatorname{Sym}^e \mathbb{C}^{\times} \backslash \Delta$. Consider the open inclusions

$$\operatorname{Sym}^e \mathbb{C}^{\times} \backslash \Delta \xhookrightarrow{i} \operatorname{Sym}^e \mathbb{C} \backslash \Delta \hookrightarrow \operatorname{Sym}^e \mathbb{P}^1.$$

As $\mathrm{Sym}^e \mathbb{P}^1$ is a smooth compactification of both $\mathrm{Sym}^e \mathbb{C}^\times$ and $\mathrm{Sym}^e \mathbb{C}$, we obtain a commutative diagram

$$\begin{tikzcd}
H^\bullet(\mathrm{Sym}^e \mathbb{P}^1) \arrow[r] \arrow[dr, two heads] \arrow[drr, bend right, two heads] & H^\bullet(\mathrm{Sym}^e \mathbb{C} \backslash \Delta) \arrow[r] & H^\bullet(\mathrm{Sym}^e \mathbb{C}^\times \backslash \Delta) \\
& PH^\bullet(\mathrm{Sym}^e \mathbb{C} \backslash \Delta) \arrow[u, hook] \arrow[r, "i^*"] & PH^\bullet(\mathrm{Sym}^e \mathbb{C}^\times \backslash \Delta) \arrow[u, hook]
\end{tikzcd}$$

The surjections follow from [29, Proposition 6.29]. Hence i^* is also surjective. Since $PH^\bullet(\mathrm{Sym}^e \mathbb{C} \backslash \Delta) \simeq \mathbb{Q}(0)$ we also have $PH^\bullet(\mathrm{Sym}^e \mathbb{C}^\times \backslash \Delta) \simeq \mathbb{Q}(0)$. ☐

Remarks.

(1) The isomorphism (31) is an instance of the PBW factorization (27).
(2) Let $\theta = (1, -1)$. If $\tau = 1$, then $\mathfrak{M}_e^{\sigma, \theta\text{-}st} \simeq \mathrm{Sym}^e \mathbb{C} \backslash \Delta$ and the proof of Corollary 4.12 gives $i^* : PH^\bullet(\mathfrak{M}_e^{\sigma,\theta\text{-}st}) \xrightarrow{\sim} PH^\bullet(\mathfrak{M}_e^{\sigma,st})$. This is an example of the lack of wall-crossing for σ-symmetric quivers.

5. Cohomological Hall modules of finite type quivers

A quiver is called finite type if it has only finitely many indecomposable representations up to isomorphism. Gabriel proved that a quiver is finite type if and only if it is a disjoint union of quivers whose underlying graphs are Dynkin diagrams of ADE type. The only connected finite type quivers with involution are of type A; all other finite type quivers with involution are disjoint unions of these and quivers of the form $ADE^{\sqcup}$. By Theorem 4.2 the CoHM of a quiver of the latter type reduces to the CoHA of a connected finite type quiver, whose structure will be recalled in Section 5.1. The problem is therefore to describe the CoHM of a type A quiver.

5.1. **Finite type CoHA following Rimányi.** Let Q be a connected finite type quiver. For simplicity we assume that Q is not of type E_8; for E_8 see [32, Remark 11.3]. The sets $\Pi \subset \Delta$ of positive simple and positive roots of Q are in bijection with the sets of isomorphism classes of simple and indecomposable representations of Q, respectively. Identify Δ with a subset of Λ_Q^+ using the dimension vector map and write I_β for the indecomposable representation with dimension vector $\beta \in \Delta$. Fix a total order $\beta_1 < \cdots < \beta_N$ on Δ such that $\mathrm{Hom}(I_{\beta_i}, I_{\beta_j}) = 0 = \mathrm{Ext}^1(I_{\beta_j}, I_{\beta_i})$ if $i < j$. Such an order exists because the Auslander-Reiten quiver Γ_Q is acyclic.

For each $\beta \in \Delta$ consider

$$\mathcal{H}_Q^{\langle \beta \rangle} = \bigoplus_{n \geq 0} H^\bullet_{\mathsf{GL}_{n\beta}}(R_{n\beta})\{\chi(n\beta, n\beta)/2\}$$

and

$$\mathcal{H}_Q^{\langle \beta \rangle, \simeq} = \bigoplus_{n \geq 0} H^\bullet_{\mathsf{GL}_{n\beta}}(R^{\simeq}_{n\beta})\{\chi(n\beta, n\beta)/2\}$$

where $R^{\simeq}_{n\beta} \subset R_{n\beta}$ is the $\mathsf{GL}_{n\beta}$-orbit consisting of representations which are isomorphic to $I_\beta^{\oplus n}$. Then $\mathcal{H}_Q^{\langle \beta \rangle}$ is a subalgebra of $\mathcal{H}_Q$ and the natural Hall product on $\mathcal{H}_Q^{\langle \beta \rangle, \simeq}$ is such that the restriction map $\rho : \mathcal{H}_Q^{\langle \beta \rangle} \to \mathcal{H}_Q^{\langle \beta \rangle, \simeq}$ is a surjective algebra homomorphism. Moreover, $\mathcal{H}_Q^{\langle \beta \rangle, \simeq} \simeq \mathcal{H}_{L_0}$ as algebras. Let $\{\tilde{x}^j\}_{j \geq 0}$ be the associated generators of $\mathcal{H}_Q^{\langle \beta \rangle, \simeq}$, as defined in Section 4.2.1. Choose[5] a node $i(\beta) \in Q_0$

[5]This cannot be done in type E_8.

such that $\dim_{\mathbb{C}}(I_\beta)_{i(\beta)} = 1$ and define a section ψ of ρ by $\psi(\tilde{x}^j) = x^j_{i(\beta)}$. Write $\mathcal{H}_Q^{(\beta)} \subset \mathcal{H}_Q$ for the isomorphic image of ψ.

The following result is due to Rimányi. It was stated for Q of type A_2 by Kontsevich and Soibelman [23, Proposition 2.1].

Theorem 5.1 ([32, Theorem 11.2])**.** *The $<$-ordered CoHA multiplication maps*

$$\overleftarrow{\boxtimes}^{\mathsf{tw}}_{\alpha\in\Pi} \mathcal{H}_Q^{(\alpha)} \to \mathcal{H}_Q, \qquad \overrightarrow{\boxtimes}^{\mathsf{tw}}_{\beta\in\Delta} \mathcal{H}_Q^{(\beta)} \to \mathcal{H}_Q$$

define isomorphisms in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^+}$.

5.2. **Preliminary results for the self-dual case.** Let (Q,σ) be of Dynkin type A. Then Q has two inequivalent duality structures: fixing $\tau = -1$, either $s = +1$ or $s = -1$ corresponding to orthogonal or symplectic representations in the language of [10], respectively. In type A_{2n} (respectively, A_{2n+1}) all orthogonal (symplectic) representations are hyperbolic. In the remaining two cases, henceforth referred to as non-hyperbolic, the positive roots of Q which are fixed by the involution admit unique self-dual structures.

To describe $\mathcal{M}_Q$ we will modify Rimányi's approach to the study of $\mathcal{H}_Q$. Fix $d^\bullet = (d^1,\dots,d^r) \in (\Lambda_Q^+)^r$, $e' \in \Lambda_Q^{\sigma,+}$ and put $e = \sum_{i=1}^r H(d^i) + e'$. Let $\mathsf{G}^\sigma_{d^\bullet,e'} \subset \mathsf{G}^\sigma_e$ be the stabilizer of a Q_0-graded isotropic flag of $\mathbb{C}^e$ of the form

$$0 = U_0 \subset U_1 \subset \cdots \subset U_r \subset \mathbb{C}^e, \qquad \mathbf{dim}\, U_k/U_{k-1} = d^k, \;\; \mathbf{dim}\, \mathbb{C}^e /\!\!/ U_r = e'.$$

Extend $U_\bullet$ to a flag of length $2r+1$ by setting $U_{2r-k+1} = U_k^\perp$ for $k = 0,\dots,r$. Let $\mathsf{Fl}^\sigma_{d^\bullet,e'} \simeq \mathsf{G}^\sigma_e/\mathsf{G}^\sigma_{d^\bullet,e'}$ be the corresponding isotropic flag variety.

For each $k = 1,\dots,2r+1$ let $\mathcal{V}_{i,k}$ be the tautological vector bundle over $\mathsf{Fl}^\sigma_{d^\bullet,e'}$ parameterizing the kth subspace of $\mathbb{C}^e$ at the node i. The quotient bundle $\mathcal{F}_{i,k} = \mathcal{V}_{i,k}/\mathcal{V}_{i,k-1}$ has rank d_i^k. The bilinear form on $\mathbb{C}^e$ induces isomorphisms $\mathcal{F}_{i,k} \simeq \mathcal{F}^\vee_{\sigma(i),2r+1-k}$. By duality this gives a chain of vector bundle isomorphisms

$$\mathrm{Hom}(\mathcal{F}_{i,k},\mathcal{F}_{j,l}) \simeq \mathrm{Hom}(\mathcal{F}^\vee_{j,l},\mathcal{F}^\vee_{i,k}) \simeq \mathrm{Hom}(\mathcal{F}_{\sigma(j),2r+1-l},\mathcal{F}_{\sigma(i),2r+1-k})$$

which induce a linear $\mathbb{Z}_2$-action on

$$\mathcal{G} = \bigoplus_{i \xrightarrow{\alpha} j \in Q_1} \bigoplus_{1\le k<l\le 2r+1} \mathrm{Hom}(\mathcal{F}_{i,k},\mathcal{F}_{j,l}).$$

Denote by $\mathcal{G}^\sigma$ the subbundle of anti-fixed points.

The following result is motivated by [32, Lemmas 8.1 and 8.2].

Lemma 5.2. *Let $f_k \in \mathcal{H}_{Q,d^k}$, $k = 1,\dots,r$, and $g \in \mathcal{M}_{Q,e'}$. Then*

$$(f_1\cdots f_r)\star g = \pi^\sigma_*\left[\left(\prod_{k=1}^r f_k(\mathcal{F}_{\bullet,k})\right) g(\mathcal{F}_{\bullet,0})\mathsf{Eu}_{\mathsf{G}^\sigma_e}(\mathcal{G}^\sigma)\right]$$

where $\pi^\sigma_* : H^\bullet_{\mathsf{G}^\sigma_e}(\mathsf{Fl}^\sigma_{d^\bullet,e'}) \to H^\bullet_{\mathsf{G}^\sigma_e}(pt)$ is the pushforward to a point and $\mathsf{Eu}_{\mathsf{G}^\sigma_e}(\mathcal{G}^\sigma)$ is the G^σ_e-equivariant Euler class of $\mathcal{G}^\sigma \to \mathsf{Fl}^\sigma_{d^\bullet,e'}$.

Proof. The right-hand side of the desired equality can be computed by localization with respect to the maximal torus $\mathsf{T}_e \subset \mathsf{G}^\sigma_e$. The T_e-fixed points are those appearing in the proof of (the r-fold iteration of) Theorem 3.3. As the weights of $\mathsf{Eu}_{\mathsf{G}^\sigma_e}(\mathcal{G}^\sigma)$ and $\mathsf{Eu}_{\mathsf{G}_{d^\bullet,e'}}(N_{R^\sigma_e/R^\sigma_{d^\bullet,e'}})$ at a T_e-fixed point agree, the lemma follows. □

Define a G^σ_e-stable subvariety of $\mathsf{Fl}^\sigma_{d^\bullet,e'} \times R^\sigma_e$ by

$$\Sigma^\sigma = \{(U_\bullet,m) \in \mathsf{Fl}^\sigma_{d^\bullet,e'} \times R^\sigma_e \mid m_\alpha(U_{i,k}) \subset U_{j,k}, \quad \forall\, i \xrightarrow{\alpha} j \in Q_1, \;\; k = 1,\dots,r\}.$$

It has a G^σ_e-equivariant fundamental class

$$[\Sigma^\sigma] \in H^\bullet_{\mathsf{G}^\sigma_e}(\mathsf{Fl}^\sigma_{d^\bullet,e'} \times R^\sigma_e) \simeq H^\bullet_{\mathsf{G}^\sigma_e}(\mathsf{Fl}^\sigma_{d^\bullet,e'}).$$

Lemma 5.3. *The equality* $\mathsf{Eu}_{\mathsf{G}_e^\sigma}(\mathcal{G}^\sigma) = [\Sigma^\sigma]$ *holds in* $H^\bullet_{\mathsf{G}_e^\sigma}(\mathsf{Fl}^\sigma_{d^\bullet, e'})$.

Proof. This can be proved in the same way as [32, Lemma 8.3]. □

The duality structure on $\mathsf{Rep}_{\mathbb{C}}(Q)$ defines an involution of the Auslander-Reiten quiver Γ_Q, sending an indecomposable I to $S(I)$. This involution preserves the levels of Γ_Q which, being in type A, are exactly the orbits of Auslander-Reiten translation. It follows that each level contains at most one fixed point of the duality.

Fix a partition $\Delta = \Delta^- \sqcup \Delta^\sigma \sqcup \Delta^+$ such that Δ^σ is fixed pointwise by S and $S(\Delta^-) = \Delta^+$. Without loss of generality we assume that $\beta_u < S(\beta_u)$ for all $\beta_u \in \Delta^-$. Write $\Delta^- = \{\beta_{u_1} < \cdots < \beta_{u_r}\}$.

Lemma 5.4. *Every self-dual representation* M *has a unique isotropic filtration*

$$0 = U_0 \subset U_1 \subset \cdots \subset U_r \subset M$$

such that $U_j/U_{j-1} \simeq I_{\beta_{u_j}}^{\oplus m_{u_j}}$, $j = 1, \ldots, r$, *and* $M /\!\!/ U_r \simeq \bigoplus_{\beta_u \in \Delta^\sigma} I_{\beta_u}^{\oplus m_u}$.

Proof. Any self-dual representation M can be written as an orthogonal direct sum of indecomposable self-dual representations, that is, self-dual representations which cannot be written as the orthogonal direct sum of two non-trivial self-dual representations. In type A this means that M can be written uniquely as

$$M = \bigoplus_{l=1}^{r} H(I_{\beta_{u_l}})^{\oplus m_{u_l}} \oplus \bigoplus_{\beta_u \in \Delta^\sigma} I_{\beta_u}^{\oplus m_u}. \tag{32}$$

Setting $U_j = \bigoplus_{l=1}^{j} I_{\beta_{u_l}}^{\oplus m_{u_l}}$ gives a filtration with the desired properties.

Suppose that $U'_\bullet \subset M$ is another filtration with the desired properties. By a standard argument the assumption $\beta_{u_1} < \cdots < \beta_{u_r}$ implies that $U'_\bullet = U_\bullet$. So it suffices to show that there is a unique isotropic embedding $U_r \hookrightarrow M$. To do so, first note that $\mathrm{Hom}(I_\beta, I_{\beta'}) = 0$ for all $\beta \in \Delta^-$ and $\beta' \in \Delta^\sigma$. Indeed, if $\mathrm{Hom}(I_\beta, I_{\beta'}) \neq 0$ then $\mathrm{Hom}(I_{\beta'}, S(I_\beta)) \neq 0$. Hence $\beta > \beta'$ and $\beta' > S(\beta)$, whence $S(\beta) < \beta$, a contradiction. Using this, it follows that the summand $U_1 \subset U_r$ must map isomorphically onto $I_{\beta_1}^{\oplus m_1}$. While $U_2 \subset U_r$ could potentially map non-trivially to $S(I_{\beta_{u_1}})$, this would contradict the condition that U_2 be isotropic. Hence U_2 must map isomorphically onto $I_{\beta_{u_1}}^{\oplus m_{u_1}} \oplus I_{\beta_{u_2}}^{\oplus m_{u_2}}$. Continuing in this way we see that $U_r \hookrightarrow M$ is indeed the canonical isotropic embedding. □

We derive two results using Lemma 5.4. The first is an extension to the self-dual setting of a theorem of Reineke [30, Theorem 2.2] and appears in the unpublished thesis of Lovett [24]. For $M \in R_e^\sigma$ let $\eta_M^\sigma \subset \overline{\eta}_M^\sigma \subset R_e^\sigma$ be the G_e^σ-orbit and G_e^σ-orbit closure of M. Elements of $\overline{\eta}_M^\sigma$ are called self-dual degenerations of M.

Theorem 5.5 ([24])**.** *Let* M *be a self-dual representation. In the notation of Lemma 5.4 set* $d^j = m_j \beta_j$, $j = 1, \ldots, r$, *and* $e' = \mathbf{dim}\, M /\!\!/ U_r$. *Then the canonical morphism* $\pi_M^\sigma : \Sigma^\sigma \to R_e^\sigma$ *is a* G_e^σ*-equivariant resolution of* $\overline{\eta}_M^\sigma$.

Proof. When Q is of type A_3 the statement is proved in [25, Proposition 2.3]. For the general case we use a modification of Reineke's resolution.

It is clear that Σ^σ is smooth and that π_M^σ is proper and equivariant. We prove that $\pi_M^\sigma(\Sigma^\sigma) = \overline{\eta}_M^\sigma$. If $N \in \pi_M^\sigma(\Sigma^\sigma)$, then there is an isotropic filtration

$$0 = V_0 \subset V_1 \subset \cdots \subset V_r \subset N, \qquad \mathbf{dim}\, V_i/V_{i-1} = d^i.$$

Since $\mathrm{Ext}^1(I_\beta, I_\beta) = 0$ for all $\beta \in \Delta$, Voigt's lemma implies that V_i/V_{i-1} is a degeneration of $I_{\beta_{u_i}}^{\oplus m_{u_i}}$. Similarly, $\mathrm{Ext}^1(I_\beta, I_{\beta'}) = 0$ for all $\beta, \beta' \in \Delta^\sigma$ and $N /\!\!/ V_r$ is a degeneration of $\oplus_{\beta_u \in \Delta^\sigma} I_{\beta_u}^{\oplus m_u}$. Applying [30, Lemma 2.3] we conclude that

N is a degeneration of M. It is proved in [10, Theorem 2.6] that two self-dual representations are isometric if and only if they are isomorphic. From this it follows that in fact N is a self-dual degeneration of M. Hence $\eta^{\sigma}_M \subset \pi^{\sigma}_M(\Sigma^{\sigma}) \subset \overline{\eta}^{\sigma}_M$, implying $\pi^{\sigma}_M(\Sigma^{\sigma}) = \overline{\eta}^{\sigma}_M$.

To prove that π^{σ}_M is a resolution it remains to show that it restricts to a bijection over η^{σ}_M. Consider an arbitrary isotropic filtration

$$0 = U_0 \subset U_1 \subset \cdots \subset U_r \subset M, \qquad \mathbf{dim}\, U_i/U_{i-1} = d^i.$$

As above, U_i/U_{i-1} and $M/\!\!/U_r$ are degenerations of $I^{\oplus m_{u_i}}_{\beta_{u_i}}$ and $\oplus_{\beta_u \in \Delta^{\sigma}} I^{\oplus m_u}_{\beta_u}$, respectively. Since $\mathrm{Hom}(I_{\beta_i}, I_{\beta_j}) = 0$ if $i < j$ we can apply the second part of [30, Lemma 2.3] to conclude that $U_i/U_{i-1} \simeq I^{\oplus m_{u_i}}_{\beta_{u_i}}$ and $M/\!\!/U_r \simeq \oplus_{\beta_u \in \Delta^{\sigma}} I^{\oplus m_u}_{\beta_u}$. Lemma 5.4 now implies that $U_{\bullet} \subset M$ is the canonical filtration. $\square$

We can now prove an analogue of [32, Theorem 10.1].

Corollary 5.6. *Let M be a self-dual representation. Then, in the notation of Lemma 5.4, the equality*

$$[\overline{\eta}^{\sigma}_M] = (\mathbf{1}_{m_{u_1}\beta_{u_1}} \cdots \mathbf{1}_{m_{u_r}\beta_{u_r}}) \star \mathbf{1}^{\sigma}_{\sum_{\beta_u \in \Delta^{\sigma}} m_u \beta_u}$$

holds in $\mathcal{M}_Q$.

Proof. Theorem 5.5 implies that $\pi^{\sigma}_*[\Sigma^{\sigma}] = [\overline{\eta}^{\sigma}_M]$. The desired equality then follows from Lemmas 5.2 and 5.3. $\square$

Remark. The class $[\overline{\eta}^{\sigma}_M] \in H^{\bullet}_{\mathsf{G}^{\sigma}_e}(R^{\sigma}_e)$ is the Thom polynomial of the orbit $\eta^{\sigma}_M \subset R^{\sigma}_e$. These classes play the rôle of the quiver polynomials of [2] in the self-dual setting.

Turning to the second application of Lemma 5.4, define putative orientifold DT invariants Ω^{σ}_e to be one if $e \in \Lambda^{\sigma,+}_Q$ is a sum of pairwise distinct positive roots, each of which is the dimension vector of an indecomposable representation which admits a self-dual structure. Otherwise, set $\Omega^{\sigma}_e = 0$. By convention $\Omega^{\sigma}_0 = 1$. Set also $\Pi^+ = \Pi \cap \mathcal{I}^+$ and $\Pi^{\sigma} = \Pi \cap \Delta^{\sigma}$. Let $h = 0$ in the hyperbolic case and $h = 1$ otherwise. Recall that $A_{L_0}(q^{\frac{1}{2}}, t) = (q^{\frac{1}{2}} t; q)_{\infty} = \mathbb{E}_q(t)$ is the quantum dilogarithm.

Theorem 5.7. *The identity*

$$\overleftarrow{\prod_{\alpha \in \Pi^+}} \mathbb{E}_q(t^{\alpha}) \star \sum_{\pi \subset \Pi^{\sigma}} \prod_{\alpha \in \pi} \mathbb{E}_{q^2}(q^{-\frac{1}{2}+h} t^{\alpha}) \star \Omega^{\sigma}_{\sum_{\beta \in \pi} \beta} \xi^{\sum_{\beta \in \pi} \beta} =$$

$$\overrightarrow{\prod_{\beta \in \Delta^-}} \mathbb{E}_q(t^{\beta}) \star \sum_{\pi \subset \Delta^{\sigma}} \Big(\prod_{\beta \in \pi} \mathbb{E}_{q^2}(q^{-\frac{1}{2}+h} t^{\beta}) \cdot \prod_{\beta \notin \pi} \mathbb{E}_{q^2}(q^{-\frac{1}{2}} t^{\beta}) \Big) \star \Omega^{\sigma}_{\sum_{\beta \in \pi} \beta} \xi^{\sum_{\beta \in \pi} \beta}$$

holds in $\hat{\mathbb{S}}_Q$.

Proof. It is straightforward to construct a σ-compatible stability θ_{simp} whose stable representations are the simple representations and whose order by increasing slope agrees with $<$. The existence of unique self-dual Harder-Narasimhan filtrations gives a factorization of the identity characteristic function in the finite field Hall module of Q. Applying the Hall module integration map [41] to this factorization gives the left-hand side of the equality. Lemma 5.4 gives a second factorization of identity characteristic function, the integral of which is the right-hand side. $\square$

Theorem 5.7 can also be proved using Kazarian spectral sequences, as in [32, §6]. The new ingredient is a self-dual version of Voigt's lemma, stating that the codimension of $\eta^{\sigma}_M \subset R^{\sigma}_{\mathbf{dim} M}$ is $\dim_{\mathbb{C}} \mathrm{Ext}^1(M, M)^S$. This can be proved using the cochain description of $\mathrm{Ext}^1(M, M)^S$ given in [42, Proposition 3.3].

5.3. **Type A CoHM.** Let Q be of type A. We begin with an example of rank two.

Example. Consider orthogonal representations of the A_2 quiver

$$\underset{1}{\bullet} \longrightarrow \underset{2}{\bullet}$$

Set $Q_0^+ = \{1\}$. For $f \in \mathcal{H}_{Q,(d_1,d_2)}$ and $g \in \mathcal{M}_{Q,(e,e)}$ we have

$$f \star g = \sum_{\pi \in \mathfrak{sh}_{d_1,e,d_2}} \pi \cdot \Big(f(x_1, \dots, x_{d_1}, y_1, \dots, y_{d_2}) g(z_1, \dots, z_e) \times \\ \frac{\prod_{1 \leq i < j \leq d_1} (-x_i - x_i) \prod_{k=1}^{e} \prod_{i=1}^{d_1} (z_k - x_i)}{\prod_{k=1}^{e} \prod_{i=1}^{d_1} (z_k - x_i) \prod_{l=1}^{d_2} \prod_{i=1}^{d_1} (-y_l - x_i) \prod_{m=1}^{d_2} \prod_{k=1}^{e} (-y_l - z_k)} \Big).$$

Since $x^i \star \mathbf{1}_0^\sigma = z^i$ the set $\{x^i \star \mathbf{1}_0^\sigma\}_{i \geq 0}$ spans $\mathcal{M}_{Q,(1,1)}$. Let β_2 be the non-simple indecomposable and let $\nu_i = y^i \in \mathcal{H}_{Q,(1,1)}$ be a generator of $\mathcal{H}_Q^{(\beta_2)}$. Then

$$(x^i \cdot x^j) \star \mathbf{1}_0^\sigma = -(z_1 + z_2) \frac{z_1^i z_2^j - z_1^j z_2^i}{z_1 - z_2}, \qquad \nu_i \star \mathbf{1}_0^\sigma = (-1)^i \frac{z_1^i - z_2^i}{z_1 - z_2}.$$

Hence $\{(x^i \cdot x^j) \star \mathbf{1}_0^\sigma\}_{i > j}$ spans $(z_1 + z_2)\mathbb{Q}[z_1, z_2]^{\mathfrak{S}_2}$. To generate the remainder of $\mathcal{M}_{Q,(2,2)} = \mathbb{Q}[z_1, z_2]^{\mathfrak{S}_2}$ it suffices to include $\{\nu_{2i+1} \star \mathbf{1}_0^\sigma\}_{i \geq 0}$. In three variables

$$(x^i \cdot x^j \cdot x^k) \star \mathbf{1}_0^\sigma = -(z_1 + z_2)(z_1 + z_3)(z_2 + z_3) s_{(i,j,k) - \delta_3},$$

which freely generate $(z_1 + z_2)(z_1 + z_3)(z_2 + z_3)\mathbb{Q}[z_1, z_2, z_3]^{\mathfrak{S}_3}$. We also have

$$(x^i \cdot \nu_j) \star \mathbf{1}_0^\sigma = \frac{(-1)^j}{(z_1 - z_2)(z_1 - z_3)(z_2 - z_3)} \Big[z_1^i (z_2^j - z_3^j)(z_1 + z_2)(z_1 + z_3) - \\ z_2^i (z_1^j - z_3^j)(z_1 + z_2)(z_3 + z_2) + z_3^i (z_1^j - z_1^j)(z_1 + z_3)(z_2 + z_3) \Big].$$

Using these calculations, direct verification shows that up to $\Lambda_Q^{\sigma,+}$-degree $(3,3)$ the $\star$-action $\mathcal{H}_Q^{(\beta_1)} \boxtimes^{\mathsf{tw}} \mathcal{H}_Q^{(\beta_2),odd} \boxtimes^{S\text{-}\mathsf{tw}} \mathbf{1}_0^\sigma \to \mathcal{M}_Q$ is an isomorphism in $D^{lb}(\mathsf{Vect}_{\mathbb{Z}})_{\Lambda_Q^{\sigma,+}}$, where β_1 is the simple root associated to $1 \in Q_0$. $\triangleleft$

These calculations can be generalized as follows. For each $\beta \in \Delta^\sigma$ consider

$$\mathcal{M}_Q^{\langle \beta \rangle} = \bigoplus_{n \geq 0} H^\bullet_{\mathsf{G}^\sigma_{n\beta}}(R_{n\beta})\{\mathcal{E}(n\beta)/2\}, \qquad \mathcal{M}_Q^{\langle \beta \rangle, \simeq} = \bigoplus_{n \geq 0} H^\bullet_{\mathsf{G}^\sigma_{n\beta}}(R^{\sigma,\simeq}_{n\beta})\{\mathcal{E}(n\beta)/2\}$$

as modules over $\mathcal{H}_Q^{\langle \beta \rangle}$ or $\mathcal{H}_Q^{\langle \beta \rangle, \simeq}$. If I_β does not admit a self-dual structure, then n is necessarily even. We have $\mathcal{M}_Q^{\langle \beta \rangle, \simeq} \simeq \mathcal{M}_{L_0}$ compatibly with $\mathcal{H}_Q^{\langle \beta \rangle, \simeq} \simeq \mathcal{H}_{L_0}$ where the duality structure on L_0 is $s_{L_0} = -1$ in the hyperbolic case and $s_{L_0} = 1$ in the non-hyperbolic case. The structure of $\mathcal{M}_Q^{\langle \beta \rangle, \simeq}$ is therefore determined by Theorem 4.5. There is also a surjective restriction map $\rho^\sigma : \mathcal{M}_Q^{\langle \beta \rangle} \twoheadrightarrow \mathcal{M}_Q^{\langle \beta \rangle, \simeq}$ that is a module homomorphism over $\rho : \mathcal{H}_Q^{\langle \beta \rangle} \to \mathcal{H}_Q^{\langle \beta \rangle, \simeq}$. Define a section of ρ^σ by

$$\psi^\sigma : \begin{cases} \tilde{x}^{2i_1+1} \cdots \tilde{x}^{2i_l+1} \star \mathbf{1}_1^\sigma \mapsto \psi(\tilde{x}^{2i_1+1} \cdots \tilde{x}^{2i_l+1}) \star \mathbf{1}_\beta^\sigma, & \text{in type } B \\ \tilde{x}^{2i_1+1} \cdots \tilde{x}^{2i_l+1} \star \mathbf{1}_0^\sigma \mapsto \psi(\tilde{x}^{2i_1+1} \cdots \tilde{x}^{2i_l+1}) \star \mathbf{1}_0^\sigma, & \text{in type } C \\ \tilde{x}^{2i_1} \cdots \tilde{x}^{2i_l} \star \mathbf{1}_0^\sigma \mapsto \psi(\tilde{x}^{2i_1} \cdots \tilde{x}^{2i_l}) \star \mathbf{1}_0^\sigma, & \text{in type } D. \end{cases}$$

The map ψ^σ is a module embedding over the restriction of ψ to the appropriate even/odd subalgebra of $\mathcal{H}_Q^{\langle \beta \rangle, \simeq}$. Write $\mathcal{M}_Q^{(\beta)}$ for the image of ψ^σ in types C or D

and write $\mathcal{M}_Q^{(\beta),+}$ for the image in type B. In summary, we have a commutative diagram of module homomorphisms over the corresponding algebra morphisms

$$\begin{tikzcd}
\mathcal{M}_Q & \mathcal{M}_Q^{\langle\beta\rangle} \arrow[l, hook'] \arrow[r, two heads, "\rho^\sigma"] \arrow[dr, "\mathsf{res}"'] & \mathcal{M}_Q^{\langle\beta\rangle,\simeq} \arrow[d, "\wr"] \arrow[l, dashed, bend right, "\psi^\sigma"'] \\
 & & \mathcal{M}_{L_0}
\end{tikzcd} \tag{33}$$

The map res is a combination of the restrictions from Lemma 1.1.

In the non-hyperbolic case, for each subset $\varnothing \subseteq \pi \subseteq \Delta^\sigma$ let

$$\mathcal{M}_Q^{(\pi)} = \bigotimes_{\beta\notin\pi} \mathcal{M}_Q^{(\beta)} \otimes \bigotimes_{\beta\in\pi} \mathcal{M}_Q^{(\beta),+}.$$

This is a rank one free module over

$$\mathcal{H}_Q^{(\pi)} = \bigotimes_{\beta\notin\pi} \mathcal{H}_Q^{(\beta),odd} \otimes \bigotimes_{\beta\in\pi} \mathcal{H}_Q^{(\beta),even} \subset \mathcal{H}_Q$$

with generator $\otimes_{\beta\in\pi}\mathbf{1}_\beta^\sigma$. In the hyperbolic case the free $\mathcal{H}_Q^{(\varnothing)}$-module $\mathcal{M}_Q^{(\varnothing)}$ is still defined.

Theorem 5.8. *Let (Q,σ) be a Dynkin quiver of type A. Then the $<$-ordered CoHA action maps*

$$\overleftarrow{\boxtimes}^{\mathsf{tw}}_{\alpha\in\Pi^+} \mathcal{H}_Q^{(\alpha)} \boxtimes^{S\text{-}\mathsf{tw}} \bigoplus_{\varnothing\subseteq\pi\subseteq\Pi^\sigma} \Omega_\pi^\sigma \cdot \mathcal{M}_Q^{(\pi)} \longrightarrow \mathcal{M}_Q \tag{34}$$

and

$$\overrightarrow{\boxtimes}^{\mathsf{tw}}_{\beta\in\Delta^-} \mathcal{H}_Q^{(\beta)} \boxtimes^{S\text{-}\mathsf{tw}} \bigoplus_{\varnothing\subseteq\pi\subseteq\Delta^\sigma} \Omega_\pi^\sigma \cdot \mathcal{M}_Q^{(\pi)} \longrightarrow \mathcal{M}_Q \tag{35}$$

are isomorphisms in $D^{lb}(\mathsf{Vect}_\mathbb{Z})_{\Lambda_Q^{\sigma,+}}$

Proof. Consider the map (34). Let $f_j \in \mathcal{H}_Q^{(\alpha_j)}$ for $\alpha_j \in \Pi^+$. Taking into account the ordering Theorem 3.3 gives $(f_1\cdots f_r)\star\mathbf{1}_0^\sigma = \prod_{j=1}^r f_j$, the multiplication on the right-hand side being polynomial multiplication. Hence the image of $\boxtimes^{\overleftarrow{\mathsf{tw}}}_{\alpha\in\Pi^+}\mathcal{H}_Q^{(\alpha)} \boxtimes^{S\text{-}\mathsf{tw}} \mathbf{1}_0^\sigma$ under (34) is the symmetric polynomials in Q_0^+ variables. In particular, in type A_{2n} (where $Q_0^\sigma=\varnothing$) the map (34) is an isomorphism. In type A_{2n+1} the direct sum $\bigoplus_{\pi\subset\Pi^\sigma}\Omega_\pi^\sigma\cdot\mathcal{M}_Q^{(\pi)}$ consists of symmetric polynomials in Q_0^σ variables. Again by Theorem 3.3, acting on this subspace by $\boxtimes^{\overleftarrow{\mathsf{tw}}}_{\alpha\in\Pi^+}\mathcal{H}_Q^{(\alpha)}$ gives the remainder of $\mathcal{M}_Q$.

To show that (35) is an isomorphism we proceed as in the proof of [32, Theorem 11.2]. To prove injectivity, fix non-negative integers $\{m_{u_j}\}_{\beta_{u_j}\in\Delta^-}$ and $\{m_{\beta_u}\}_{\beta_u\in\Delta^\sigma}$. This determines a self-dual representation M via equation (32). Define $m_{S(u_j)} = m_{u_j}$ for each $\beta_{u_j}\in\Delta^-$ and let $e=\mathbf{dim}\,M$. The isometry group of M is homotopy equivalent to

$$\prod_{j=1}^r \mathsf{GL}_{m_{u_j}} \times \prod_{\beta_u\in\Delta^\sigma} \mathsf{G}_{m_u}^{s_u}$$

where $\mathsf{G}_{m_u}^{s_u}$ is a symplectic group in the hyperbolic case and an orthogonal group otherwise.

Define sets

$$\mathcal{T}_{i,k,v}, \qquad i\in Q_0,\ k=1,\dots,|\Delta|,\ v=1,\dots,m_{u_k}$$

by requiring $|\mathcal{T}_{i,k,v}| = 1$ if $\dim(I_{\beta_{u_k}})_i = 1$ and $\mathcal{T}_{i,k,v} = \varnothing$ otherwise, and

$$\mathcal{T}_{i,1,1} \sqcup \cdots \sqcup \mathcal{T}_{i,|\Delta|,m_{u_{|\Delta|}}} = \{1, \dots, e_i\}$$

as ordered sets. Let $\{\epsilon_{i,1}, \dots, \epsilon_{i,e_i}\}$ be a standard linear basis of $\mathbb{C}^{e_i}$ and let $A_{k,v}$ be the indecomposable representation of type β_{u_k} spanned by $\{\epsilon_{i,j}\}_{i \in Q_0, j \in \mathcal{T}_{i,k,v}}$. Set

$$\Phi^\sigma = \bigoplus_{k=1}^{|\Delta|} \bigoplus_{v=1}^{m_{u_k}} A_{k,v}.$$

Define a self-dual structure on Φ^σ by requiring that

(i) $A_{k,v} \oplus A_{S(k),v}$ be hyperbolic if $u_k \in \Delta^-$,

(ii) $A_{k,v} \oplus A_{k,\lfloor \frac{m_{u_k}}{2} \rfloor + v}$ be hyperbolic if $u_k \in \Delta^\sigma$, $v = 1, \dots, \lfloor \frac{m_{u_k}}{2} \rfloor$, and

(iii) $A_{k,m_{u_k}}$ have its canonical self-dual structure if m_{u_k} is odd.

Then Φ^σ and M are isometric self-dual representations. The restriction homomorphism

$$H^\bullet_{\mathsf{G}^\sigma_e}(R^\sigma_e) \to H^\bullet_{\mathsf{G}^\sigma_e}(\eta^\sigma_M) \simeq H^\bullet(B\mathrm{Aut}_S(\Phi^\sigma))$$

can be computed explicitly using Lemma 1.1. Identifying the groups $H^\bullet_{\mathsf{G}^\sigma_e}(R^\sigma_e)$ and $H^\bullet(B\mathrm{Aut}_S(\Phi^\sigma))$ with appropriately symmetric polynomials in variables $\{z_{i,j}\}$ and $\{\theta_{u,v}\}$, respectively, we have

(i) if $i \in Q_0^+$ and $j \in \mathcal{T}_{i,u,v}$, then

$$z_{i,j} \mapsto \begin{cases} \theta_{u,v}, & \text{if } u \in \Delta^+, \\ -\theta_{u,v}, & \text{if } u \in \Delta^-, \\ \theta_{u,v}, & \text{if } u \in \Delta^\sigma \text{ and } j = 1, \dots, \lfloor \frac{m_u}{2} \rfloor, \\ -\theta_{u,v}, & \text{if } u \in \Delta^\sigma \text{ and } j = \lfloor \frac{m_u}{2} \rfloor + 1, \dots, 2\lfloor \frac{m_u}{2} \rfloor, \\ 0, & \text{if } u \in \Delta^\sigma \text{ and } j = m_u \text{ is odd} \end{cases}$$

and

(ii) if $i \in Q_0^\sigma$ and $j \in \mathcal{T}_{i,u,v}$, then

$$z_{i,j} \mapsto \begin{cases} \theta_{u,v}, & \text{if } u \in \Delta^+, \\ -\theta_{u,v}, & \text{if } u \in \Delta^-, \\ \theta_{u,v}, & \text{if } u \in \Delta^\sigma. \end{cases}$$

Let $f_j \in \mathcal{H}_Q^{(\beta_{u_j})} \cap \mathcal{H}_{Q, m_{u_j} \beta_{u_j}}$ and $g_u \in \mathcal{M}_Q^{(\beta_u)}$. We claim that the image of

$$(f_1 \boxtimes^{\mathsf{tw}} \cdots \boxtimes^{\mathsf{tw}} f_r) \boxtimes^{S\text{-}\mathsf{tw}} \bigotimes_{u \in \Delta^\sigma} g_u \tag{36}$$

under the map (35) is non-zero. It is enough to verify that its image under the restriction $H^\bullet_{\mathsf{G}^\sigma_e}(R^\sigma_e) \to H^\bullet(B\mathrm{Aut}_S(\Phi^\sigma))$ is non-zero. Since $\pi^\sigma : \Sigma^\sigma \to R^\sigma_e$ is a resolution of $\overline{\eta}^\sigma_M$ (Theorem 5.5) there is a single T_e-fixed point above $\Phi^\sigma \in R^\sigma_e$. Hence the restriction of the image of (36) consists of a single term and is equal to

$$\prod_{j=1}^r f_j(\theta_{i(\beta_{u_j}),1}, \dots, \theta_{i(\beta_{u_j}),m_{u_j}}) \prod_{u \in \Delta^\sigma} g_u(\theta_{i(\beta_u),1}, \dots, \theta_{i(\beta_u),m_u}) \mathcal{K}^{(r),\sigma}(z)_{|z \mapsto \theta}. \tag{37}$$

Here $\mathcal{K}^{(r),\sigma}(z)_{|z\mapsto\theta}$ is the r-fold iterated kernel of the CoHM with the above substitutions made. Corollary 5.6 implies that $\mathcal{K}^{(r),\sigma}(z)_{|z\mapsto\theta}$ is equal to the image of $[\overline{\eta}^\sigma_M]$ in $H^\bullet(B\mathrm{Aut}_S(\Phi^\sigma))$, which in turn is equal to $\mathsf{Eu}_{\mathrm{Aut}_S(M)}(N_{R^\sigma_e/\eta^\sigma_M})$. That the latter class is non-zero can be seen by a modification of the proof of [13, Corollary 3.15], which deals with the ordinary case. Hence (37) is non-zero. This proves that the restriction of the map (35) to the summand spanned by elements of the form (36) is injective. This is enough to show that the map (35) itself is injective, since if the image of two or more elements of the form (36), with different $\{m_{u_j}\}$ leading to the same total dimension vector, were linearly dependent, we can restrict to various orbits η_M to derive a contradiction.

We can now complete the proof. Together with the first part of the theorem, Theorem 5.7 implies the equality of the Hilbert-Poincaré series of the domain and codomain of the map (35). Since we have already shown that (35) is injective, it follows that it is in fact an isomorphism of graded vector spaces. $\square$

The isomorphism (34) is the PBW factorization (27) associated to the stability θ_{simp} from the proof of Theorem 5.7. We expect a similar statement for the isomorphism (35), with θ_{simp} replaced by a σ-compatible stability θ_{indec} whose stable objects are the indecomposables and whose order by increasing slope is opposite to $<$. Without requiring σ-compatibility, such a stability is known to exist. In many cases (e.g. the equioriented case) we can check that it may be chosen σ-compatibly. When θ_{indec} indeed exists, the (stability dependent) numbers Ω_π^σ appearing in Theorem 5.8 are consistent with the natural generalization of Conjecture 3.11 to Dynkin quivers.

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Department of Mathematics, The University of Hong Kong, Pokfulam, Hong Kong
E-mail address: `myoung@maths.hku.hk`