Hyperbolic Anderson Model with space-time homogeneous Gaussian noise

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Abstract

In this article, we study the stochastic wave equation in arbitrary spatial dimension d, with a non-linear multiplicative term of the form $\sigma(u) = u$, also known in the literature as the Hyperbolic Andreson Model. This equation is perturbed by a general Gaussian noise, which is homogeneous in both space and time. We prove the existence of a solution of this equation (in the Skorohod sense) and the Hölder continuity of its sample paths, under the same respective conditions on the spatial spectral measure of the noise as in the case of the white noise in time, regardless of the temporal covariance function of the noise.

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1 Introduction

In this article, we are interested in studying the stochastic wave equation with multiplicative noise:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + u(t,x)\dot{W}(t,x), & t>0, x \in \mathbb{R}^d \\ u(0,x) = 1, & x \in \mathbb{R}^d \\ \dfrac{\partial u}{\partial t}(0,x) = 0, & x \in \mathbb{R}^d \end{cases} \tag{1}$$

This problem is also known in the literature as the *Hyperbolic Anderson Model*, by analogy with the Parabolic Anderson Model in which the wave operator is replaced by

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the heat operator. We assume that the noise W is Gaussian with covariance structure specified by two locally integrable non-negative definitive functions $\gamma : \mathbb{R} \to [0, \infty]$ in time and $f : \mathbb{R}^d \to [0, \infty]$ in space. Since the noise is not a martingale in time, the stochastic integral with respect to W cannot be defined in the Itô sense. To define the concept of solution we use the divergence operator from Malliavin calculus. We refer the reader to Section 2 below for the precise definitions of the noise and the solution.

The Parabolic Anderson Model with the same noise W as in the present article has been studied extensively in the recent years. These investigations culminated with the recent impressive article [17], in which the authors have obtained a Feynman-Kac formula for the moments of the solution (for general covariance kernels γ and f), as well as exponential bounds for these moments (under some quantitative conditions on γ and f). The exact asymptotics for these moments were obtained in [8]. These extend some earlier results of [18] and [19], in the case when the noise W was fractional in space and time with index $H > 1/2$ in time, and indices $H_1, \ldots, H_d > 1/2$ in space.

In contrast with its parabolic counterpart, the Hyperbolic Anderson Model with noise W as above received less attention in the literature. However, there is a large amount of literature dedicated to the stochastic wave equation with spatially-homogeneous Gaussian noise which is white in time and has spectral covariance measure μ in space. (The covariance kernel f is the Fourier transform of μ.) We describe briefly the most important contributions in this area. In the landmark article [10], Robert Dalang introduced an Itô-type stochastic integral with respect to this noise (building upon the theory of martingale measures developed in [25]), and proved that the solution of the stochastic wave equation with this type of noise (and possibly a Lipschitz non-linear term $\sigma(u)$ multiplying the noise) exists in any dimension $d = 1, 2, 3$, provided that the measure μ satisfies what is now called *Dalang's condition*:

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^2} \mu(d\xi) < \infty. \tag{2}$$

This result was extended to any dimension d in [9]. In [9], it was also proved that the solution of the wave equation with affine term $\sigma(u) = u + b$ is Hölder continuous, provided that μ satisfies:

$$\int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi|^2} \right)^{\beta} \mu(d\xi) < \infty, \quad \text{for some} \quad \beta \in (0, 1). \tag{3}$$

A deeper study of the Hölder continuity of the solution of the wave equation in dimension $d = 3$ (with general Lipschitz function σ) was carried out in [13] and [15]. Exponential bounds for the moments of the solution of the Hyperbolic Anderson Model in dimension $d = 3$ were obtained in [12]. The fact that the solution of the wave equation (with general Lipschitz function σ) has a density was proved in [23] for any dimension d. In [16], it was shown that this density in smooth for dimensions $d = 1, 2, 3$.

The existence and Hölder continuity of the solution of equation (1) with noise W which is fractional in time with index $H > 1/2$ and has a spatial covariance function given by the Riesz kernel $f(x) = |x|^{-\alpha}, 0 < \alpha < d$ was proved in [1] under the conditions $\alpha < 2$, respectively $\alpha/2 < \beta < 1$ (which are restatements of conditions (2) and (3) for the

Riesz kernel). Exponential bounds for the moments of this solution were obtained in [4], for several examples of covariance functions f. Interestingly, the condition $\alpha < 2$ does not depend on H, which is in sharp contrast with the necessary and sufficient condition $\alpha < 2H + 1$ obtained in [6] for the existence of the solution of the wave equation with additive noise: $\frac{\partial^2 u}{\partial t^2} = \Delta u + \dot{W}$. In the case of the Parabolic Anderson Model with the same noise W, it was proved in [3] that $\alpha < 2$ is the necessary and sufficient condition for the existence of the solution. We believe that this is also the case for the Hyperbolic Anderson Model, and more generally that (2) is the necessary and sufficient condition for the existence of a solution of equation of (1), regardless of the temporal covariance function γ. In the present article, we only show the sufficiency part, extending in this way the results of [1] to arbitrary covariance functions γ and f. As far as we know, the question of necessity of (2) is still open even for the white noise in time.

This article is organized as follows. In Section 2, we gather some preliminary results about the space of integrands with respect to the noise W and the existence of solution to equation (1). In Section 3, we show that this solution exists for any temporal covariance function γ and for any spectral measure μ which satisfies (2). In Section 4, we prove that this solution is Hölder continuous in time and space, provided that μ satisfies (3).

We conclude the introduction with a few words about the notation. We let $\mathcal{D}_{\mathbb{C}}(\mathbb{R}^d)$ be the set of complex-valued infinitely differentiable functions on $\mathbb{R}^d$ with compact support. For any $p > 0$, we denote by $L^p_{\mathbb{C}}(\mathbb{R}^d)$ the space of complex-valued functions φ on $\mathbb{R}^d$ such that $|\varphi|^p$ is integrable with respect to the Lebesgue measure. We let $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^d)$ be the set of complex-valued rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$. We denote by $\mathcal{D}'_{\mathbb{C}}(\mathbb{R}^d)$ and $\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^d)$ the space of all complex-valued linear functionals defined on $\mathcal{D}_{\mathbb{C}}(\mathbb{R}^d)$, respectively $\mathcal{S}_{\mathbb{C}}(\mathbb{R}^d)$. Similar notations are used for spaces of real-valued elements, with the subscript $\mathbb{C}$ omitted. We denote by $x \cdot y = \sum_{i=1}^d x_i y_i$ the inner product in $\mathbb{R}^d$ and by $|x| = (x \cdot x)^{1/2}$ the Euclidean norm in $\mathbb{R}^d$. We denote by $\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i\xi \cdot x}\varphi(x)dx$ the Fourier transform of a function $\varphi \in L^1(\mathbb{R}^d)$. We use the same notation $\mathcal{F}$ for the Fourier transform of functions on $\mathbb{R}, \mathbb{R}^d$ or $\mathbb{R}^{d+1}$, but whenever there is a risk of confusion, the notation will be clearly specified.

2 Preliminaries

In this section, we give the rigourous definition of the noise W, we establish a criterion for integrability with respect to W, and we apply this criterion to the fundamental solution of the wave equation on $\mathbb{R}_+ \times \mathbb{R}^d$. Next, we introduce the basic elements of Malliavin calculus, and we define the concept of solution to equation (1). Finally, we give a necessary and sufficient condition for the existence of this solution.

We assume that $W = \{W(\varphi); \varphi \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)\}$ is a zero-mean Gaussian process, defined on a complete probability space $(\Omega, \mathcal{F}, P)$, with covariance

$$E[W(\varphi_1)W(\varphi_2)] = \int_{\mathbb{R}^2 \times \mathbb{R}^{2d}} \gamma(t-s)f(x-y)\varphi_1(t,x)\varphi_2(s,y)dxdydtds =: J(\varphi_1, \varphi_2),$$

where $\gamma : \mathbb{R} \to [0, \infty]$ and $f : \mathbb{R}^d \to [0, \infty]$ are continuous, symmetric, locally integrable functions, such that

$$\gamma(t) < \infty \quad \text{if and only if} \quad t \neq 0$$

$$f(x) < \infty \quad \text{if and only if} \quad x \neq 0.$$

We denote by $\mathcal{H}$ the completion of $\mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ defined by

$$\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}} = J(\varphi_1, \varphi_2).$$

We are mostly interested in variables $W(\varphi)$ with $\varphi \in \mathcal{D}(\mathbb{R}_+ \times \mathbb{R}^d)$.

We assume that the functions γ and f are *non-negative definite* (in the sense of distributions), i.e. for any $\phi \in \mathcal{S}(\mathbb{R})$ and $\varphi \in \mathcal{S}(\mathbb{R}^d)$

$$\int_{\mathbb{R}} (\phi * \widetilde{\phi})(t)\gamma(t)dt \geq 0 \quad \text{and} \quad \int_{\mathbb{R}^d} (\varphi * \widetilde{\varphi})(x) f(x)dx \geq 0,$$

where $\widetilde{\phi}(t) = \phi(-t)$ and $\widetilde{\varphi}(x) = \varphi(-x)$.

By the Bochner-Schwartz Theorem, there exists a tempered measure ν on $\mathbb{R}$ such that γ is the Fourier transform of ν in $\mathcal{S}'_{\mathbb{C}}(\mathbb{R})$, i.e.

$$\int_{\mathbb{R}^d} \phi(t)\gamma(t)dt = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}\phi(\tau)\nu(d\tau) \quad \text{for all} \quad \phi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}).$$

Similarly, there exists a tempered measure μ on $\mathbb{R}^d$ such that f is the Fourier transform of μ in $\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^d)$, i.e.

$$\int_{\mathbb{R}^d} \varphi(x) f(x)dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\mu(d\xi) \quad \text{for all} \quad \varphi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^d). \tag{4}$$

It follows that for any functions $\phi_1, \phi_2 \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$ and $\varphi_1, \varphi_2 \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^d)$

$$\int_{\mathbb{R}} \int_{\mathbb{R}} \gamma(t-s)\phi_1(t)\overline{\phi_2(s)}dtds = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}\phi_1(\tau)\overline{\mathcal{F}\phi_2(\tau)}\nu(d\tau) \tag{5}$$

and

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x-y)\varphi_1(x)\overline{\varphi_2(y)}dxdy = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}\varphi_1(\xi)\overline{\mathcal{F}\varphi_2(\xi)}\mu(d\xi). \tag{6}$$

The next result shows that the functional J has an alternative expression, in terms of Fourier transforms. In particular, this shows that J is non-negative definite.

Lemma 2.1. *For any $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R} \times \mathbb{R}^d)$, we have:*

$$J(\varphi_1, \varphi_2) = \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^{d+1}} \mathcal{F}\varphi_1(\tau, \xi)\overline{\mathcal{F}\varphi_2(\tau, \xi)}\nu(d\tau)\mu(d\xi), \tag{7}$$

where $\mathcal{F}$ denotes the Fourier transform in both variables t and x. Moreover, J is non-negative definite.

Proof: Since $\varphi_k(t,\cdot) \in \mathcal{D}(\mathbb{R}^d)$ for any $t \in \mathbb{R}$ and $k=1,2$, by (6) we have:

$$J(\varphi_1,\varphi_2)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}}\int_{\mathbb{R}}\gamma(t-s)\left(\int_{\mathbb{R}^d}\mathcal{F}\varphi_1(t,\cdot)(\xi)\overline{\mathcal{F}\varphi_2(s,\cdot)(\xi)}\mu(d\xi)\right)dtds.$$

For any $\xi \in \mathbb{R}^d$ fixed, we denote $\phi_\xi^{(k)}(t)=\mathcal{F}\varphi_k(t,\cdot)(\xi)=\int_{\mathbb{R}^d}e^{-i\xi\cdot x}\varphi_k(t,x)dx$. Note that $\phi_\xi^{(k)} \in \mathcal{D}_{\mathbb{C}}(\mathbb{R})$ for any $\xi \in \mathbb{R}^d$ and $k=1,2$. Hence, by Fubini's theorem and (5), we have

$$\begin{aligned}J(\varphi_1,\varphi_2) &= \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\gamma(t-s)\phi_\xi^{(1)}(t)\overline{\phi_\xi^{(2)}(\xi)}dtds\right)\mu(d\xi)\\ &= \frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}}\mathcal{F}\phi_\xi^{(1)}(\tau)\overline{\phi_\xi^{(2)}(\tau)}\nu(d\tau)\mu(d\xi), \qquad (8)\end{aligned}$$

where for any $\tau \in \mathbb{R}$ and $k=1,2$, we denote

$$\mathcal{F}\phi_\xi^{(k)}(\tau)=\int_{\mathbb{R}}e^{-i\tau\cdot t}\phi_\xi^{(k)}(t)dt=\int_{\mathbb{R}}e^{-i\tau\cdot t}\left(\int_{\mathbb{R}^d}e^{-i\xi\cdot x}\varphi_k(t,x)dx\right)dt=\mathcal{F}\varphi_k(\tau,\xi).$$

This proves (7). Consequently, for any $a_1,\ldots,a_n \in \mathbb{C}$ and $\varphi_1,\ldots,\varphi_n \in \mathcal{D}_{\mathbb{C}}(\mathbb{R}_+ \times \mathbb{R}^d)$,

$$\sum_{j,k=1}^n a_j\overline{a}_k J(\varphi_j,\varphi_k)=\int_{\mathbb{R}^d}\int_{\mathbb{R}}\left|\sum_{j=1}^n a_j\mathcal{F}\varphi_j(\tau,\xi)\right|^2\nu(d\tau)\mu(d\xi)\geq 0.$$

This proves that J is non-negative definite. □

The map $\varphi \mapsto W(\varphi)$ is an isometry which can be extended to $\mathcal{H}$. For any $\varphi \in \mathcal{H}$, we say that $W(\varphi)$ is the Wiener integral of φ with respect to W and we denote

$$W(\varphi)=\int_{\mathbb{R}}\int_{\mathbb{R}^d}\varphi(t,x)W(dt,dx).$$

We note that the space $\mathcal{H}$ may contain distributions in $\mathcal{S}'(\mathbb{R}^{d+1})$.

To obtain a criterion for integrability, we need the following approximation result.

Lemma 2.2. *If μ is a tempered measure on $\mathbb{R}^d$, then $\mathcal{F}(\mathcal{D}(\mathbb{R}^d))$ is dense in $\widetilde{L}^2_{\mathbb{C}}(\mathbb{R}^d,\mu)$, where*

$$\widetilde{L}^2_{\mathbb{C}}(\mathbb{R}^d,\mu)=\{\varphi \in L^2_{\mathbb{C}}(\mathbb{R}^d,\mu);\varphi(\xi)=\overline{\varphi(-\xi)} \text{ for all } \xi \in \mathbb{R}^d\}.$$

Proof: We refer the reader to the proof of Theorem 3.2 of [20] for the case $d=1$. The same argument can be used for higher dimensions d. □

We also need the following result on the "energy" of a complex-valued function φ with respect to a kernel κ.

Lemma 2.3. *Let m be a tempered measure on $\mathbb{R}^d$ whose Fourier transform in $\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^d)$ is a locally integrable function $\kappa:\mathbb{R}^d \to [0,\infty]$ such that $\kappa(x)<\infty$ if and only if $x \neq 0$. Then for any bounded function $\varphi:\mathbb{R}^d \to \mathbb{C}$ with bounded support, which is continuous almost everywhere, we have:*

$$\mathcal{E}_\kappa(\varphi):=\int_{\mathbb{R}^d}\kappa(x-y)\varphi(x)\overline{\varphi(y)}dxdy=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\mathcal{F}\varphi(\xi)|^2 m(d\xi). \qquad (9)$$

Remark 2.4. *If we assume that κ is a kernel of positive type (i.e. the measure m is absolutely continuous with respect to the Lebesgue measure), relation* (9) *can be deduced from Lemma 5.6 of [21] for any function $\varphi \in L^1_{\mathbb{C}}(\mathbb{R}^d)$ with $\mathcal{E}_\kappa(|\varphi|) < \infty$. In the proof of Theorem 2.5 below, we will use relation* (9) *for the kernel $\kappa = \gamma$ on $\mathbb{R}$ and the measure $m = \nu$. We do not use the result of [21] since we do not assume that ν is absolutely continuous with respect to the Lebesque measure. (Relation* (9) *will also be used in the proof of Theorem 2.10 below for the kernel $\kappa = \gamma_n$ on $\mathbb{R}^n$, with $\gamma_n(t_1, \ldots, t_n) = \prod_{i=1}^n \gamma(t_i)$.)*

Proof of Lemma 2.3: Suppose that $\varphi = \varphi_1 + i\varphi_2$, $|\varphi(x)| \leq K$ for all $x \in \mathbb{R}^n$ and the support of φ is contained in the set $\{x \in \mathbb{R}^n; |x| \leq M\}$. We proceed by approximation. Let $p \in \mathcal{D}(\mathbb{R}^n)$ be such that $p \geq 0$, $\int_{\mathbb{R}^n} p(x)dx = 1$ and the support of p is contained in $\{x \in \mathbb{R}^n; |x| \leq 1\}$. For any $\varepsilon > 0$, we define $p_\varepsilon(x) = \varepsilon^{-d} p(x/\varepsilon)$ for all $x \in \mathbb{R}^d$. Let

$$\varphi_\varepsilon = \varphi * p_\varepsilon = \varphi_{\varepsilon,1} + i\varphi_{\varepsilon,2},$$

where $\varphi_{\varepsilon,1} = \varphi_1 * p_\varepsilon$ and $\varphi_{\varepsilon,2} = \varphi_2 * p_\varepsilon$. Then $\varphi_\varepsilon \in \mathcal{D}_{\mathbb{C}}(\mathbb{R}^d)$, $|\varphi_\varepsilon(x)| \leq K$ for all $x \in \mathbb{R}^d$, $\varphi_\varepsilon(x) \to \varphi(x)$ for any continuity point x of φ, and the support of φ_ε is contained in the set $\{x \in \mathbb{R}^d; |x| \leq M+1\}$, for any $\varepsilon \in (0,1)$. Moreover, $\mathcal{F}\varphi_\varepsilon = \mathcal{F}\varphi \mathcal{F}p_\varepsilon \to \mathcal{F}\varphi$ as $\varepsilon \downarrow 0$ and $|\mathcal{F}\varphi_\varepsilon| \leq |\mathcal{F}\varphi|$. By the definition of the Fourier transform in $\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^d)$, for any $\varepsilon > 0$,

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \kappa(x-y)\varphi_\varepsilon(x)\overline{\varphi_\varepsilon(y)}dxdy = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\mathcal{F}\varphi_\varepsilon(\xi)|^2 m(d\xi). \tag{10}$$

Note that

$$\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \kappa(x-y)\varphi_\varepsilon(x)\overline{\varphi_\varepsilon(y)}dxdy = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \kappa(x-y)\varphi(x)\overline{\varphi(y)}dxdy. \tag{11}$$

(This follows by applying the dominated convergence theorem to the real and imaginary part of the integrals above. In fact, since the integral on the right-hand side of (10) is real-valued, the term on the left-hand side has to be real-valued for any $\varepsilon > 0$, and hence its limit as $\varepsilon \downarrow 0$ is real-valued.) On the other hand, by Fatou's lemma,

$$\int_{\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|^2 m(d\xi) \leq \liminf_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} |\mathcal{F}\varphi_\varepsilon(\xi)|^2 m(d\xi). \tag{12}$$

From (10), (11) and (12), we obtain that

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|^2 m(d\xi) \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \kappa(x-y)\varphi(x)\overline{\varphi(y)}dxdy.$$

Hence, if the right-hand side of (9) is infinite, so must be the left-hand side. If the right-hand side of (9) is finite, then by the dominated convergence theorem, we have:

$$\int_{\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|^2 m(d\xi) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} |\mathcal{F}\varphi_\varepsilon(\xi)|^2 m(d\xi). \tag{13}$$

In this case, relation (9) follows by (10), (11) and (13). □

Recall that the Fourier transform $\mathcal{F}S$ of a distribution $S \in \mathcal{S}'(\mathbb{R}^d)$ is defined by $\mathcal{F}S(\varphi) = S(\mathcal{F}\varphi)$ for all $\varphi \in \mathcal{S}(\mathbb{R}^d)$. When S is a genuine distribution and $\mathcal{F}S = g$ is a function, this means that

$$\int_{\mathbb{R}^d} g(\xi)\varphi(\xi)d\xi = S(\mathcal{F}\varphi) \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^d). \tag{14}$$

In this case, $\mathcal{F}S$ is understood as the equivalence class of all functions g which satisfy (14). If g is an element of this class, we say that g is a version of $\mathcal{F}S$. If g_1 and g_2 are two versions of $\mathcal{F}S$, then $g_1 = g_2$ a.e. This leads us to the following hypothesis.

Hypothesis A. μ is absolutely continuous with respect to the Lebesgue measure.

Using the alternative expression given by (8) for the inner product $\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}}$ and the previous lemmas, we obtain the following criterion for integrability with respect to W.

Theorem 2.5. *Let $\mathbb{R} \ni t \mapsto S(t) \in \mathcal{S}'(\mathbb{R}^d)$ be a deterministic function such that $\mathcal{F}S(t,\cdot)$ is a function for all $t \in \mathbb{R}$. If $\mathcal{F}S(t,\cdot)$ is uniquely determined only up to a set of Lebesgue measure zero, we assume that μ satisfies Hypothesis A. Suppose that:*
(i) for each $t \in \mathbb{R}$, there exists a version of $\mathcal{F}S(t,\cdot)$ such that $(t,\xi) \mapsto \mathcal{F}S(t,\cdot)(\xi) =: \phi_\xi(t)$ is measurable on $\mathbb{R} \times \mathbb{R}^d$;
(ii) for all $\xi \in \mathbb{R}^d$, $\int_{\mathbb{R}} |\phi_\xi(t)|dt < \infty$.

Then the following statements hold:

a) The function $(\tau, \xi) \mapsto \mathcal{F}\phi_\xi(\tau)$ is measurable on $\mathbb{R} \times \mathbb{R}^d$, where $\mathcal{F}\phi_\xi$ denotes the Fourier transform of ϕ_ξ, i.e. $\mathcal{F}\phi_\xi(\tau) = \int_{\mathbb{R}} e^{-i\tau t}\phi_\xi(t)dt$, $\tau \in \mathbb{R}$.

b) If

$$\|S\|_0^2 := \frac{1}{(2\pi)^{d+1}} \int_{\mathbb{R}^d} \int_{\mathbb{R}} |\mathcal{F}\phi_\xi(\tau)|^2 \nu(d\tau)\mu(d\xi) < \infty \tag{15}$$

then $S \in \mathcal{H}$ and $\|S\|_{\mathcal{H}}^2 = \|S\|_0^2$.

c) Assume in addition that $S(t,\cdot) = 0$ for all $t \notin [0,T]$, for some $T > 0$. If for every $\xi \in \mathbb{R}^d$, the function $t \mapsto \mathcal{F}S(t,\cdot)(\xi)$ is bounded and continuous almost everywhere on $[0,T]$, and

$$I_T := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_0^T \int_0^T \gamma(t-s)\mathcal{F}S(t,\cdot)(\xi)\overline{\mathcal{F}S(t,\cdot)(\xi)}dtds\mu(d\xi) < \infty,$$

then $S \in \mathcal{H}$ and $\|S\|_{\mathcal{H}}^2 = I_T$.

Proof: a) This follows by Fubini's theorem, using the fact that $(t,\tau,\xi) \mapsto e^{-i\tau t}\phi_\xi(t)$ is measurable on $\mathbb{R} \times \mathbb{R} \times \mathbb{R}^d$, by hypothesis (i).

b) By Theorem 3.9 of [2], we know that $S \in \mathcal{H}_{\mathbb{C}}$, where $\mathcal{H}_{\mathbb{C}}$ is the completion of $\mathcal{D}_{\mathbb{C}}(\mathbb{R}^{d+1})$ with respect to the inner product

$$\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}_{\mathbb{C}}} = \int_{\mathbb{R}^2 \times \mathbb{R}^{2d}} \gamma(t-s) f(x-y) \varphi_1(t,x)\overline{\varphi_2(s,y)}dxdydtds.$$

We will now prove that S lies in the smaller space $\mathcal{H}$ (of real-valued elements).

Let $a(\tau,\xi)=\mathcal{F}\phi_{\xi}(\tau)$. By (15) and part a), the function a lies in $L^2_{\mathbb{C}}(\mathbb{R}^{d+1},\Pi)$, where

$$\Pi(d\tau,d\xi)=\frac{1}{(2\pi)^{d+1}}\nu(d\tau)\mu(d\xi).$$

We denote $\widetilde{L}^2_{\mathbb{C}}(\mathbb{R}^{d+1},\Pi)=\{\varphi\in L^2_{\mathbb{C}}(\mathbb{R}^{d+1},\Pi);\varphi(\tau,\xi)=\overline{\varphi(-\tau,-\xi)}$ for all $\tau\in\mathbb{R},\xi\in\mathbb{R}^d\}$.

We observe that $a\in\widetilde{L}^2_{\mathbb{C}}(\mathbb{R}^{d+1},\Pi)$, since by Lemma 3.3 of [7],

$$\phi_{-\xi}(t)=\mathcal{F}S(t,\cdot)(-\xi)=\overline{\mathcal{F}S(t,\cdot)(\xi)}=\overline{\phi_{\xi}(t)}\quad\text{for all }\xi\in\mathbb{R}^d,$$

and hence

$$a(-\tau,-\xi)=\int_{\mathbb{R}}e^{i\tau t}\phi_{-\xi}(t)dt=\int_{\mathbb{R}}\overline{e^{-i\tau t}\phi_{\xi}(t)}dt=\overline{a(\tau,\xi)}\quad\text{for all }\tau\in\mathbb{R},\xi\in\mathbb{R}^d.$$

By Lemma 2.2, $\mathcal{F}(\mathcal{D}(\mathbb{R}^{d+1}))$ is dense in $\widetilde{L}^2_{\mathbb{C}}(\mathbb{R}^{d+1},\Pi)$. Hence, for any $\varepsilon>0$, there exists a function $l=l(\varepsilon)\in\mathcal{D}(\mathbb{R}^{d+1})$ such that

$$\int_{\mathbb{R}^{d+1}}|a(\tau,\xi)-\mathcal{F}l(\tau,\xi)|^2\Pi(d\tau,d\xi)<\varepsilon^2.$$

Note that the previous integral is $\int_{\mathbb{R}^{d+1}}|\mathcal{F}\phi_{\xi}(\tau)-\mathcal{F}\psi_{\xi}(\tau)|^2\Pi(d\tau,d\xi)=:\|S-l\|_0^2$, where $\mathcal{F}\psi_{\xi}$ is the Fourier transform of the function $t\mapsto\psi_{\xi}(t)=\mathcal{F}l(t,\cdot)(\xi)$. The conclusion follows using expression (8) for the inner product in $\mathcal{H}$.

c) For every $\xi\in\mathbb{R}^d$ fixed, we apply Lemma 2.3 to the bounded function $\phi_{\xi}:\mathbb{R}\to\mathbb{C}$ which is continuous a.e and has support contained in $[0,T]$. We apply this lemma for the measure $m=\nu$ and the kernel $\kappa=\gamma$ on $\mathbb{R}$. We obtain that, for any $\xi\in\mathbb{R}^d$,

$$\int_0^T\int_0^T\gamma(t-s)\phi_{\xi}(t)\overline{\phi_{\xi}(s)}dtds=\frac{1}{2\pi}\int_{\mathbb{R}}|\mathcal{F}\phi_{\xi}(\tau)|^2\nu(d\tau).$$

We integrate with respect to $\mu(d\xi)$ and we multiply by $(2\pi)^{-d}$. We obtain that

$$I_T=\frac{1}{(2\pi)^{d+1}}\int_{\mathbb{R}^d}\int_{\mathbb{R}}|\mathcal{F}\phi_{\xi}(\tau)|^2\nu(d\tau)\mu(d\xi)=:\|S\|_0^2.$$

Since $I_T<\infty$, it follows that $\|S\|_0^2<\infty$. The conclusion follows by part b). $\Box$

We are interested in applying Theorem 2.5 to the case when φ is related to the fundamental solution G of the wave equation on $\mathbb{R}_+\times\mathbb{R}^d$. We recall that:

$$\begin{aligned}G(t,x)&=\frac{1}{2}1_{\{|x|<t\}}\quad\text{if }d=1\\G(t,x)&=\frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}1_{\{|x|<t\}}\quad\text{if }d=2\\G(t,\cdot)&=\frac{1}{4\pi t}\sigma_t,\quad\text{if }d=3,\end{aligned}$$

where σ_t is the surface measure on the sphere $\{x \in \mathbb{R}^3; |x| = t\}$. If $d = 1$ or $d = 2$, $G(t, \cdot)$ is a non-negative function in $L^1(\mathbb{R}^d)$, and if $d = 3$, $G(t, \cdot)$ is a finite measure in $\mathbb{R}^3$.

If $d \geq 4$ is even, $G(t, \cdot)$ is a distribution with compact support in $\mathbb{R}^d$ given by:

$$G(t, \cdot) = \frac{1}{1 \cdot 3 \cdot \ldots \cdot (d-1)} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{(d-2)/2} (t^{d-1} \Upsilon_t), \quad \Upsilon_t(\varphi) = \frac{1}{\omega_{d+1}} \int_{B(0,1)} \frac{\varphi(ty)}{\sqrt{1 - |x|^2}} dx,$$

and if $d \geq 5$ is odd, $G(t, \cdot)$ is a distribution with compact support in $\mathbb{R}^d$ given by:

$$G(t, \cdot) = \frac{1}{1 \cdot 3 \cdot \ldots \cdot (d-2)} \left(\frac{1}{t} \frac{\partial}{\partial t} \right)^{(d-3)/2} (t^{d-2} \Sigma_t), \quad \Sigma_t(\varphi) = \frac{1}{\omega_d} \int_{\partial B(0,1)} \varphi(tz) d\sigma(z),$$

where ω_d is the surface area of the unit sphere $\partial B(0, 1)$ in $\mathbb{R}^d$, and σ is the surface measure on $\partial B(0, 1)$ (see e.g. Theorem (5.28), page 176 of [14]).

It is known that for any $d \geq 1$, the Fourier transform of $G(t, \cdot)$ is given by:

$$\mathcal{F}G(t, \cdot)(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad \xi \in \mathbb{R}^d. \tag{16}$$

Note that when $d = 1, 2, 3$, the previous formula uniquely determines $\mathcal{F}G(t, \cdot)$ as the Fourier transform of a function in $L^1(\mathbb{R}^d)$ for $d = 1, 2$, or the Fourier transform of a finite measure for $d = 3$. But when $d \geq 4$, (16) is interpreted in the sense of distributions, and the definition of $\mathcal{F}G(t, \cdot)$ is unique only up to a set of Lebesgue measure zero.

We have the following result about the integrability of G.

Theorem 2.6. *For any $t > 0$ and $x \in \mathbb{R}^d$, we define $g_{t,x}(s, \cdot) = G(t - s, x - \cdot)1_{[0,t]}(s)$ for any $s \in \mathbb{R}$. If $d \geq 4$, we assume that μ satisfies Hypothesis A. Suppose that*

$$I_t := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_0^t \int_0^t \gamma(r - s) \frac{\sin((t - r)|\xi|) \sin((t - s)|\xi|)}{|\xi|^2} dr ds \mu(d\xi) < \infty \tag{17}$$

for any $t > 0$. Then, for any $t > 0$ and $x \in \mathbb{R}^d$, $g_{t,x} \in \mathcal{H}$, the stochastic integral

$$v(t, x) := \int_0^t \int_{\mathbb{R}} G(t - s, x - y) W(ds, dy)$$

is well-defined and $E|v(t, x)|^2 = I_t$. In particular, (17) holds for any $t > 0$ if the measure μ satisfies (2). (Note that v is the solution of the linear wave equation $\frac{\partial^2 v}{\partial x^2}(t, x) = \Delta v(t, x) + \dot{W}(t, x), t > 0, x \in \mathbb{R}^d$ with zero initial conditions.)

Proof: By applying Theorem 2.5.c) to the function $S = g_{t,x}$ we infer that $g_{t,x} \in \mathcal{H}$. To see that $g_{t,x}$ satisfies the conditions of this theorem, we note that, due to (16), for all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}^d$,

$$\phi_\xi(s) := \mathcal{F}g_{t,x}(s, \cdot)(\xi) = e^{-i\xi \cdot x} \frac{\sin((t - s)|\xi|)}{|\xi|} 1_{[0,t]}(s).$$

Then $|\phi_\xi(s)| \leq (t - s)1_{[0,t]}(s) \leq t1_{[0,t]}(s)$ for all $s \in \mathbb{R}$ and $\xi \in \mathbb{R}$. It follows that $g_{t,x}$ satisfies conditions (i) and (ii) of Theorem 2.5.

By the construction of the stochastic integral, $E|v(t,x)|^2 = \|g_{t,x}\|_0^2 = I_t$.

Finally, we note that I_t coincides with the term $\alpha_1(t)$ which appears in the series representation (25) of the second moment of the solution $u(t,x)$ to equation (1). (See definition (34) of $\alpha_n(t)$ below.) In Section 3 below we will prove that the series $\sum_{n\geq 1} \alpha_n(t)/n!$ converges under condition (2). In particular, this implies that $\alpha_1(t) < \infty$ under (2). $\square$

Remark 2.7. Theorem 2.5.c) can also be applied to the function $S = g_{t,x}$ where $g_{t,x}(s,\cdot) = G(t-s, x-\cdot)1_{[0,t]}(s)$ and

$$G(t,x) = \frac{1}{(2\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{2t}\right) \tag{18}$$

is the fundamental solution of the heat equation $\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u$ on $\mathbb{R}_+ \times \mathbb{R}^d$. Since $g_{t,x}(s,\cdot) \in L^1(\mathbb{R}^d)$, its Fourier transform is uniquely determined and we do not need to assume that μ is absolutely continuous with respect to the Lebesgue measure. Note that $g_{t,x} \in \mathcal{H}$ provided that, for any $t > 0$,

$$I_t := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_0^t \int_0^t \gamma(r-s) \exp\left(-\frac{(t-r)|\xi|^2}{2}\right) \exp\left(-\frac{(t-s)|\xi|^2}{2}\right) drds\mu(d\xi) < \infty.$$

In this case, $v(t,x) = W(g_{t,x})$ is the solution of $\frac{\partial v}{\partial t} = \frac{1}{2}\Delta v + \dot{W}$ and $E|v(t,x)|^2 = I_t$.

We will now extend the previous considerations to multiple Wiener integrals with respect to W. This will allow us to give a rigorous definition of the solution to equation (1), using an approach based on Malliavin calculus with respect to the isonormal Gaussian process $W = \{W(\varphi); \varphi \in \mathcal{H}\}$.

We first recall very briefly some basic elements of Malliavin calculus (see [22] for more details). It is known that every square-integrable random variable F which is measurable with respect to W, has the Wiener chaos expansion:

$$F = E(F) + \sum_{n\geq 1} F_n \quad \text{with} \quad F_n \in \mathcal{H}_n,$$

where $\mathcal{H}_n$ is the n-th Wiener chaos space associated to W. Moreover, each F_n can be represented as $F_n = I_n(f_n)$ for some $f_n \in \mathcal{H}^{\otimes n}$, where $\mathcal{H}^{\otimes n}$ is the n-th tensor product of $\mathcal{H}$ and $I_n : \mathcal{H}^{\otimes n} \to \mathcal{H}_n$ is the multiple Wiener integral with respect to W. By the orthogonality of the Wiener chaos spaces and an isometry-type property of I_n, we obtain that

$$E|F|^2 = (EF)^2 + \sum_{n\geq 1} E|I_n(f_n)|^2 = (EF)^2 + \sum_{n\geq 1} n! \|\widetilde{f}_n\|^2_{\mathcal{H}^{\otimes n}},$$

where $\widetilde{f}_n$ is the symmetrization of f_n in all n variables:

$$\widetilde{f}_n(t_1, x_1, \ldots, t_n, x_n) = \frac{1}{n!} \sum_{\rho \in S_n} f_n(t_{\rho(1)}, x_{\rho(1)}, \ldots, t_{\rho(n)}, x_{\rho(n)}).$$

Here S_n is the set of all permutations of $\{1, \ldots, n\}$. We note that the space $\mathcal{H}^{\otimes n}$ may contain distributions in $\mathcal{S}'(\mathbb{R}^{n(d+1)})$.

We denote by $\delta : \mathrm{Dom}(\delta) \subset L^2(\Omega;\mathcal{H}) \to L^2(\Omega)$ the divergence operator with respect to W, defined as the adjoint of the Malliavin derivative D with respect to W. If $u \in \mathrm{Dom}\,\delta$, we use the notation

$$\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u(t,x) W(\delta t, \delta x),$$

and we say that $\delta(u)$ is the *Skorohod integral* of u with respect to W. In particular, $E[\delta(u)] = 0$.

We let $w = \{w(t,x); t \geq 0, x \in \mathbb{R}^d\}$ be the solution of the homogenous wave equation. Since the initial conditions are $u_0 = 1$ and $v_0 = 0$, we have

$$w(t,x) = 1 \quad \text{for all } t > 0, x \in \mathbb{R}^d.$$

We consider the filtration $\mathcal{F}_t = \sigma(\{W(1_{[0,s]}\varphi); s \in [0,t], \varphi \in \mathcal{D}(\mathbb{R}^d)\}) \vee \mathcal{N}, t \geq 0$, where $\mathcal{N}$ is the σ-field of P-negligible sets.

We are now ready to give the rigorous definition of the solution to equation (1).

Definition 2.8. We say that a process $u = \{u(t,x); t \geq 0, x \in \mathbb{R}^d\}$ is a **(mild) solution** of equation (1) if for any $t > 0$ and $x \in \mathbb{R}^d$, $u(t,x)$ is $\mathcal{F}_t$-measurable, $E|u(t,x)|^2 < \infty$ and the following integral equation holds:

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(\delta s, \delta y), \tag{19}$$

i.e. $v^{(t,x)} \in \mathrm{Dom}\,\delta$ and $u(t,x) = 1 + \delta(v^{(t,x)})$, where

$$v^{(t,x)}(s,\cdot) = 1_{[0,t]}(s) G(t-s, x-\cdot) u(s,\cdot), \quad s \geq 0 \tag{20}$$

and $\cdot$ denotes the missing y-variable. (When $d \geq 3$, $G(t-s, x-\cdot)u(s,\cdot)$ is the multiplication of the distribution $G(t-s, x-\cdot)$ with the function $u(s,\cdot)$.)

The existence of the solution u can be proved exactly as in [1] (in the case $\gamma(t) = H(2H-1)|t|^{2H-2}$ with $\frac{1}{2} < H < 1$). The key idea is to show that the variable $u(t,x)$ has a Wiener chaos expansion in which the kernels $f_n(\cdot,t,x)$ can be written down explicitly. These kernels are defined as follows. If $d = 1$ or $d = 2$,

$$f_n(t_1,x_1,\ldots,t_n,x_n,t,x) = G(t-t_n, x-x_n) \ldots G(t_2-t_1, x_2-x_1) 1_{\{0<t_1<\ldots<t_n<t\}}. \tag{21}$$

If $d = 3$, $f_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)$ is a finite measure on $\mathbb{R}^{3n}$ given by:

$$f_n(t_1,\cdot,\ldots,t_n,\cdot,t,x) = G(t-t_n, x-dx_n) \ldots G(t_2-t_1, x_2-dx_1) 1_{\{0<t_1<\ldots<t_n<t\}}, \tag{22}$$

where $G(t, a-\cdot)$ is the measure defined by $G(t,a-\cdot)(A) = G(t, a-A)$ for all $A \in \mathcal{B}(\mathbb{R}^3)$.

If $d \geq 4$, for every $0 < t_1 < \ldots < t_n < t$, $f_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)$ is the element of $\mathcal{D}'(\mathbb{R}^{nd})$ whose action on a test function $\phi = \phi_1 \otimes \ldots \otimes \phi_n$ with $\phi_i \in \mathcal{D}(\mathbb{R}^d)$ is given by:

$$(f_n(t_1,\cdot,\ldots,t_n,\cdot,t,x), \phi) = \varphi_n(t_2-t_1, t_3-t_2, \ldots, t-t_n, x), \tag{23}$$

where the pairs (ψ_k, φ_k) are defined recursively for $k = 1, \ldots, n$ by the following relations:

$$\begin{aligned}\psi_k(s_1, \ldots, s_{k-1}, \cdot) &= \phi_k(\cdot)\varphi_{k-1}(s_1, \ldots, s_{k-1}, \cdot) \\ \varphi_k(s_1, \ldots, s_k) &= \psi_k(s_1, \ldots, s_{k-1}, \cdot) * G(s_k, \cdot)\end{aligned}$$

with $\varphi_0 = 1$. The function $f_n(t_1, \cdot, \ldots, t_n, \cdot, t, x)$ is defined to be 0 if the relation $0 < t_1 < \ldots < t_n < t$ is not satisfied.

The following result is an extension of Theorem 2.8 of [1] to the case of an arbitrary covariance function γ.

Theorem 2.9. *Suppose that $f_n(\cdot, t, x) \in \mathcal{H}^{\otimes n}$ for any $t > 0, x \in \mathbb{R}^d$ and $n \geq 1$. Then equation* (1) *has a solution if and only if for any $t > 0$ and $x \in \mathbb{R}^d$,*

the series $\sum_{n \geq 0} I_n(f_n(\cdot, t, x))$ converges in $L^2(\Omega)$,

where I_n is the n-th order multiple Winer integral with respect to W. In this case, the solution is given by:

$$u(t,x) = \sum_{n \geq 0} J_n(t,x), \quad \text{with} \quad J_n(t,x) = I_n(f_n(\cdot, t, x)).$$

Proof: The proof is identical to the one used in the proof of Theorem 2.8 of [1], replacing $|t-s|^{2H-2}$ by $\gamma(t-s)$. We omit the details. $\square$

From Theorem 2.9, it follows that if we assume that $f_n(\cdot, t, x) \in \mathcal{H}^{\otimes n}$ for any $t > 0, x \in \mathbb{R}^d$ and $n \geq 1$, then a necessary and sufficient condition for the existence of a solution u to equation (1) is:

$$\sum_{n \geq 0} n! \|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}} < \infty, \tag{24}$$

for any $t > 0$ and $x \in \mathbb{R}^d$, and in this case,

$$E|u(t,x)|^2 = \sum_{n \geq 0} \frac{1}{n!} \alpha_n(t), \tag{25}$$

where $\alpha_n(t) = (n!)^2 \|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}}$. Here, we denote by $\widetilde{f}_n(\cdot, t, x)$ the symmetrization of $f_n(\cdot, t, x)$ with respect to the variables $(t_1, x_1), \ldots, (t_n, x_n)$.

To check that the kernel $f_n(\cdot, t, x)$ lies in $\mathcal{H}^{\otimes n}$, we need the following result, which is the counterpart of Theorem 2.5 for multiple Wiener integrals of order n. (See also Theorem 2.2 of [1] for a related result in the case $\gamma(t) = H(2H-1)|t|^{2H-2}$, with $\frac{1}{2} < H < 1$.)

Theorem 2.10. *Let $\mathbb{R}^n \ni (t_1, \ldots, t_n) \mapsto S(t_1, \cdot, \ldots, t_n, \cdot) \in \mathcal{S}'(\mathbb{R}^{nd})$ be a deterministic function such that $\mathcal{F}S(t_1, \cdot, \ldots, t_n, \cdot)$ is a function for all $(t_1, \ldots, t_n) \in \mathbb{R}^n$. If $\mathcal{F}S(t_1, \cdot, \ldots, t_n, \cdot)$ is uniquely determined only up to a set of Lebesgue measure zero, we assume that μ satisfies Hypothesis A. Suppose that:*
(i) for each $(t_1, \ldots, t_n) \in \mathbb{R}^n$, there exists a version of $\mathcal{F}S(t_1, \cdot, \ldots, t_n, \cdot)$ such that the function $(t_1, \ldots, t_n, \xi_1, \ldots, \xi_n) \mapsto \mathcal{F}S(t_1, \cdot, \ldots, t_n, \cdot)(\xi_1, \ldots, \xi_n) =: \phi_{\xi_1, \ldots, \xi_n}(t_1, \ldots, t_n)$ is

measurable on $\mathbb{R}^n \times \mathbb{R}^{nd}$*;*
(ii) for all $\xi_1, \ldots, \xi_n \in \mathbb{R}^d$, $\int_{\mathbb{R}^n} |\phi_{\xi_1,\ldots,\xi_n}(t_1, \ldots, t_n)| dt_1 \ldots dt_n < \infty$.

Then the following statements hold:

a) The function $(\tau_1, \ldots, \tau_n, \xi_1, \ldots, \xi_n) \mapsto \mathcal{F}\phi_{\xi_1,\ldots,\xi_n}(\tau_1, \ldots, \tau_n)$ *is measurable on* $\mathbb{R}^n \times \mathbb{R}^{nd}$*, where* $\mathcal{F}\phi_{\xi_1,\ldots,\xi_n}$ *denotes the Fourier transform of* $\phi_{\xi_1,\ldots,\xi_n}$*, i.e.*

$$\mathcal{F}\phi_{\xi_1,\ldots,\xi_n}(\tau_1, \ldots, \tau_n) = \int_{\mathbb{R}^n} e^{-i(\tau_1 t_1 + \ldots + \tau_n t_n)} \phi_{\xi_1,\ldots,\xi_n}(t_1, \ldots, t_n) dt_1 \ldots dt_n.$$

b) If

$$\|S\|_{0,n}^2 := \frac{1}{(2\pi)^{n(d+1)}} \int_{\mathbb{R}^{nd}} \int_{\mathbb{R}^n} |\mathcal{F}\phi_{\xi_1,\ldots,\xi_n}(\tau_1, \ldots, \tau_n)|^2 \nu(d\tau_1) \ldots \nu(d\tau_n) \mu(d\xi_1) \ldots \mu(d\xi_n) < \infty,$$

then $S \in \mathcal{H}^{\otimes n}$ *and* $\|S\|_{\mathcal{H}^{\otimes n}}^2 = \|S\|_{0,n}^2$.

c) Assume in addition that $S(t_1, \cdot, \ldots, t_n, \cdot) = 0$ *for all* $(t_1, \ldots, t_n) \notin [0, T]^n$*, for some* $T > 0$*. If for every* $\xi_1, \ldots, \xi_n \in \mathbb{R}^d$*, the function* $(t_1, \ldots, t_n) \mapsto \mathcal{F}S(t_1, \cdot, \ldots, t_n, \cdot)(\xi_1, \ldots, \xi_n)$ *is bounded and continuous almost everywhere on* $[0, T]^n$*, and*

$$I_T(n) := \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} \int_{[0,T]^{2n}} \prod_{j=1}^n \gamma(t_j - s_j) \mathcal{F}S(t_1, \cdot, \ldots, t_n, \cdot)(\xi) \overline{\mathcal{F}S(s_1, \cdot, \ldots, s_n, \cdot)(\xi)} d\mathbf{t} d\mathbf{s} \mu_n(d\xi) < \infty,$$

then $S \in \mathcal{H}$ *and* $\|S\|_{\mathcal{H}}^2 = I_T$*. In the integral* $I_T(n)$ *above,* $\mathbf{t} = (t_1, \ldots, t_n)$*,* $\mathbf{s} = (s_1, \ldots, s_n)$ *and* $\mu_n(d\xi_1, \ldots, d\xi_n) = \prod_{j=1}^n \mu(d\xi_j)$ *is a measure on* $\mathbb{R}^{nd}$*.*

Proof: We argue as in the proof of Theorem 2.5.

a) This follows by Fubini's theorem and hypothesis (i).

b) Note that $a \in \widetilde{L}^2_{\mathbb{C}}(\mathbb{R}^{n(d+1)}, \Pi_n)$, where $a(\tau_1, \ldots, \tau_n, \xi_1, \ldots, \xi_n) = \mathcal{F}\phi_{\xi_1,\ldots,\xi_n}(\tau_1, \ldots, \tau_n)$ and

$$\Pi_n(d\tau_1, \ldots, d\tau_n, d\xi_1, \ldots, d\xi_n) = \frac{1}{(2\pi)^{n(d+1)}} \nu(d\tau_1) \ldots \nu(d\tau_n) \mu(d\xi_1) \ldots \mu(d\xi_n).$$

By Lemma 2.2, $\mathcal{F}(\mathcal{D}(\mathbb{R}^{n(d+1)}))$ is dense in $\widetilde{L}^2_{\mathbb{C}}(\mathbb{R}^{n(d+1)}, \Pi_n)$. Hence, for any $\varepsilon > 0$, there exists a function $l = l(\varepsilon) \in \mathcal{D}(\mathbb{R}^{n(d+1)})$ such that

$$\|\varphi - l\|_{0,n} := \int_{\mathbb{R}^{n(d+1)}} |a - \mathcal{F}l|^2 d\Pi_n < \varepsilon^2.$$

The conclusion follows since $\mathcal{H}^{\otimes n}$ is the completion of $\mathcal{D}(\mathbb{R}^{n(d+1)})$ with respect to the inner product $\langle \cdot, \cdot \rangle_{\mathcal{H}^{\otimes n}}$ defined by

$$\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}^{\otimes n}} = \int_{\mathbb{R}^{n(d+1)}} \mathcal{F}\phi^{(1)}_{\xi_1,\ldots,\xi_n}(\tau_1, \ldots, \tau_n) \overline{\mathcal{F}\phi^{(2)}_{\xi_1,\ldots,\xi_n}(\tau_1, \ldots, \tau_n)} \Pi_n(d\tau_1, \ldots, d\tau_n, d\xi_1, \ldots, d\xi_n)$$

where $\phi^{(k)}_{\xi_1,\ldots,\xi_n}(t_1, \ldots, t_n) = \mathcal{F}\varphi(t_1, \cdot \ldots, t_n, \cdot)(\xi_1, \ldots, \xi_n)$ for $k = 1, 2$.

c) For every $\xi_1, \ldots, \xi_n \in \mathbb{R}^d$ fixed, we apply Lemma 2.3 to the bounded function $\phi_{\xi_1,\ldots,\xi_n} : \mathbb{R}^n \to \mathbb{C}$ which is continuous a.e. and has support contained in $[0, T]^n$. We apply this lemma for the measure $m=\nu_n$ and the kernel $\kappa=\gamma_n$ on $\mathbb{R}^n$, where $\nu_n(d\tau_1,\ldots,d\tau_n)=\prod_{j=1}^n \nu(d\tau_j)$ and $\gamma_n(t_1,\ldots,t_n)=\prod_{j=1}^n \gamma(t_j)$. We obtain that, for any $\xi_1,\ldots,\xi_n\in\mathbb{R}^d$,

$$\int_{[0,T]^{2n}}\prod_{j=1}^n \gamma(t_j-s_j)\phi_{\xi_1,\ldots,\xi_n}(\mathbf{t})\overline{\phi_{\xi_1,\ldots,\xi_n}(\mathbf{s})}d\mathbf{t}d\mathbf{s}=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}|\mathcal{F}\phi_{\xi_1,\ldots,\xi_n}(\tau_1,\ldots,\tau_n)|^2\nu_n(d\tau),$$

where $\mathbf{t}=(t_1,\ldots,t_n)$ and $\mathbf{s}=(s_1,\ldots,s_n)$. We integrate with respect to $\mu_n(d\xi_1,\ldots,d\xi_n)$ and we multiply by $(2\pi)^{-nd}$. We obtain that

$$I_T(n)=\frac{1}{(2\pi)^{n(d+1)}}\int_{\mathbb{R}^{nd}}\int_{\mathbb{R}^n}|\mathcal{F}\phi_{\xi_1,\ldots,\xi_n}(\tau_1,\ldots,\tau_n)|^2\nu_n(d\tau)\mu_n(d\xi)=:\|S\|_{0,n}^2.$$

Since $I_T(n)<\infty$, it follows that $\|S\|_{0,n}^2<\infty$. The conclusion follows by part b).
$\square$

As a consequence of the previous theorem, we obtain the following result.

Theorem 2.11. *For any $t>0$, $x\in\mathbb{R}^d$ and $n\geq 1$, let $f_n(\cdot,t,x)$ be defined by* (21) *if $d=1$ or $d=2$,* (22) *if $d=3$ or* (23) *if $d\geq 4$. Suppose that μ satisfies* (2)*. If $d\geq 4$, suppose in addition that Hypothesis A holds. Then for any $t>0$, $x\in\mathbb{R}^d$ and $n\geq 1$,*

$$f_n(\cdot,t,x)\in\mathcal{H}^{\otimes n}\quad \textit{and}\quad \|f_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes n}}^2=I_t(n),$$

where

$$I_t(n)=\frac{1}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}}\int_{[0,t]^{2n}}\frac{\sin((t_2-t_1)|\xi_1|)}{|\xi_1|}\cdot\frac{\sin((t_3-t_2)|\xi_1+\xi_2|)}{|\xi_1+\xi_2|}\cdots\frac{\sin((t-t_n)|\xi_1+\ldots+\xi_n|)}{|\xi_1+\ldots+\xi_n|}$$
$$\frac{\sin((s_2-s_1)|\xi_1|)}{|\xi_1|}\cdot\frac{\sin((s_3-s_2)|\xi_1+\xi_2|)}{|\xi_1+\xi_2|}\cdots\frac{\sin((t-s_n)|\xi_1+\ldots+\xi_n|)}{|\xi_1+\ldots+\xi_n|}$$
$$\prod_{j=1}^n\gamma(t_j-s_j)dt_1\ldots dt_n ds_1\ldots ds_n\mu(d\xi_1)\ldots\mu(d\xi_n).$$

Proof: We apply Theorem 2.10.c) to the function $S=f_n(\cdot,t,x)$ for fixed $t>0$ and $x\in\mathbb{R}^d$, i.e. $S(t_1,\ldots,t_n)=f_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)$. To see that $f_n(\cdot,t,x)$ satisfies the conditions of this theorem, we note that by relation (9) of [1], for any $(t_1,\ldots,t_n)\in\mathbb{R}$ and $\xi_1,\ldots,\xi_n\in\mathbb{R}^d$,

$$\begin{aligned}\phi_{\xi_1,\ldots,\xi_n}(t_1,\ldots,t_n)&:=\mathcal{F}f_n(t_1,\cdot,\ldots,t_n,\cdot,t,x)(\xi_1,\ldots,\xi_n)\\&=e^{-i(\xi_1+\ldots+\xi_n)\cdot x}\overline{\mathcal{F}G(t_2-t_1,\cdot)(\xi_1)}\,\overline{\mathcal{F}G(t_3-t_2,\cdot)(\xi_1+\xi_2)}\ldots\overline{\mathcal{F}G(t-t_n,\cdot)(\xi_1+\ldots+\xi_n)}\\&=e^{-i(\xi_1+\ldots+\xi_n)\cdot x}\frac{\sin((t_2-t_1)|\xi_1|)}{|\xi_1|}\cdot\frac{\sin((t_3-t_2)|\xi_1+\xi_2|)}{|\xi_1+\xi_2|}\cdots\frac{\sin((t-t_n)|\xi_1+\ldots+\xi_n|)}{|\xi_1+\ldots+\xi_n|}\end{aligned}$$

if $0<t_1<\ldots<t_n<t$ and $\phi_{\xi_1,\ldots,\xi_n}(t_1,\ldots,t_n)=0$ otherwise. Hence,

$$|\phi_{\xi_1,\ldots,\xi_n}(t_1,\ldots,t_n)|\leq(t_2-t_1)\ldots(t-t_n)1_{\{0<t_1<\ldots<t_n<t\}}\leq t^n 1_{[0,t]^n}.$$

It follows that $f_n(\cdot,t,x)$ satisfies conditions (i) and (ii) of Theorem 2.10.

Similarly to the calculations done in the proof of Theorem 3.4 below, one can prove that $I_t(n)<\infty$, under condition (2). By Theorem 2.10, we conclude that $f_n(\cdot,t,x)\in\mathcal{H}^{\otimes n}$ and $\|f_n(\cdot,t,x)\|_{\mathcal{H}^{\otimes n}}^2=I_t(n)$. $\square$

Remark 2.12. Theorem 2.10.c) can also be applied to the function $S = f_n(\cdot, t, x)$ where $f_n(\cdot, t, x)$ is defined by (21) and G is the fundamental solution of the heat equation, given by (18). Using the same argument as in the proof of Theorem 2.11, we infer that, if μ satisfies (2), then $f_n(\cdot, t, x) \in \mathcal{H}^{\otimes n}$ for all $t > 0$ and $x \in \mathbb{R}^d$.

3 Existence of mild solution

In this section, we establish the existence of a solution to equation (1) under condition (2) (by applying Theorem 2.9), and we show that this solution is $L^2(\Omega)$-continuous and has uniformly bounded moments of order $p \geq 2$.

We need to recall an important analytical result. (See also relation (3.4) of [11] for a related result.)

Lemma 3.1. *Let μ be a tempered measure on $\mathbb{R}^d$ whose Fourier transform in $\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^d)$ is a locally-integrable function $f : \mathbb{R}^d \to [0, \infty]$ such that $f(x) < \infty$ if and only $x \neq 0$. Then for any $\beta > 0$,*

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi + \eta|^2} \right)^{\beta} \mu(d\xi) = \int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi|^2} \right)^{\beta} \mu(d\xi). \tag{26}$$

Proof: We prove the result in a similar way as in Remark 5.8 in [24]. We assume that the right hand side of (26) is finite, otherwise it is trivial. Note that for $c > 0$ and $\beta > 0$,

$$c^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^{\infty} t^{\beta-1} e^{-ct} dt. \tag{27}$$

Fix $\eta \in \mathbb{R}^d$. We apply (27) to $c = 1 + |\xi + \eta|^2$ and then integrate $\mu(d\xi)$. Using Fubini's theorem, we obtain:

$$\int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi + \eta|^2} \right)^{\beta} \mu(d\xi) = \frac{1}{\Gamma(\beta)} \int_0^{\infty} t^{\beta-1} e^{-t} \left(\int_{\mathbb{R}^d} e^{-t|\xi+\eta|^2} \mu(d\xi) \right) dt.$$

Let $p_t(x) = (2\pi t)^{-d/2} e^{-|x|^2/(2t)}$. Note that for any $\xi, \eta \in \mathbb{R}^d$,

$$\mathcal{F}(e^{-i\eta \cdot} p_{2t})(\xi) = \int_{\mathbb{R}^d} e^{-i(\xi+\eta)\cdot x} p_{2t}(x) dx = \mathcal{F} p_{2t}(\xi + \eta) = e^{-t|\xi+\eta|^2}.$$

By applying Parseval's identity (4) to the function $\varphi = e^{-i\eta \cdot} p_{2t} \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^d)$, we see that

$$\int_{\mathbb{R}^d} e^{-i\eta \cdot x} p_{2t}(x) f(x) dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-t|\xi+\eta|^2} \mu(d\xi)$$

Hence, by applying Fubini's theorem,

$$\begin{aligned} \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\frac{1}{1 + |\xi + \eta|^2} \right)^{\beta} \mu(d\xi) &= \frac{1}{\Gamma(\beta)} \int_0^{\infty} t^{\beta-1} e^{-t} \left(\int_{\mathbb{R}^d} e^{-i\eta \cdot x} p_{2t}(x) f(x) dx \right) dt \\ &= \frac{1}{\Gamma(\beta)} \int_{\mathbb{R}^d} e^{-i\eta \cdot x} G_{d,\beta}(x) f(x) dx, \end{aligned} \tag{28}$$

where $G_{d,\beta}$ is the Bessel kernel:

$$G_{d,\beta}(x) = \frac{1}{\Gamma(\beta)} \int_0^{\infty} t^{\beta-1} e^{-t} p_{2t}(x) dt > 0.$$

We take the modulus on both sides of (28) and we use the fact that the left-hand side of this relation is non-negative. We use the inequality $|\int \dots| \leq \int |\dots|$ on the right-hand side. Since $|e^{-i\eta\cdot x}| = 1$ and f is non-negative, we obtain that

$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi-\eta|^2} \right)^{\beta} \mu(d\xi) \leq \int_{\mathbb{R}^d} G_{d,k}(x) f(x) dx = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{\beta} \mu(d\xi).$$

□

Based on the previous lemma, we obtain the following result.

Lemma 3.2. *For any $t > 0$,*

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi+\eta)|^2 \mu(d\xi) \leq 4t^2 \int_{\mathbb{R}^d} \frac{1}{1+t^2|\xi|^2} \mu(d\xi) \tag{29}$$

and

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi+\eta)|^2 \mu(d\xi) \leq 2(t^2 \vee 1) \int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \mu(d\xi). \tag{30}$$

Proof: We first prove (29). Note that $\frac{\sin x}{x} \leq \frac{2}{1+x}$ for any $x > 0$. (This can be seen as follows: if $x \leq 1$, then $\frac{\sin x}{x} \leq 1 \leq \frac{2}{1+x}$, and if $x > 1$, then $\frac{\sin x}{x} \leq \frac{1}{x} \leq \frac{2}{1+x}$.) Hence

$$|\mathcal{F}G(t,\cdot)(\xi)|^2 = \frac{\sin^2(t|\xi|)}{|\xi|^2} \leq \frac{4t^2}{(1+t|\xi|)^2} \leq \frac{4t^2}{1+t^2|\xi|^2}.$$

It follows that

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi+\eta)|^2 \mu(d\xi) \leq \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{4t^2}{1+t^2|\xi+\eta|^2} \mu(d\xi) =$$
$$4t^2 \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1+|t\xi+\eta|^2} \mu(d\xi) = 4t^2 \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{1}{1+|\xi+\eta|^2} \mu_t(d\xi),$$

where $\mu_t = \mu \circ h_t^{-1}$ and $h_t(\xi) = t\xi$. We now apply Lemma 3.1 (with $\beta = 1$) to the measure μ_t. To justify the application of this result, we note that the Fourier transform in $\mathcal{S}'(\mathbb{R}^d)$ of the measure μ_t is the non-negative definite function f_t defined by $f_t(x) = f(tx), x \in \mathbb{R}^d$, since for any $\varphi \in \mathcal{S}(\mathbb{R}^d)$ we have:

$$\begin{aligned}\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi) \mu_t(d\xi) &= \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}\varphi(t\xi) \mu(d\xi) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \mathcal{F}\varphi^{(t)}(\xi) \mu(d\xi) \\ &= \int_{\mathbb{R}^d} \varphi^{(t)}(x) f(x) dx = \int_{\mathbb{R}^d} \varphi(x) f_t(x) dx,\end{aligned}$$

where $\varphi^{(t)}(x) = t^{-d}\varphi(x/t)$. It follows that

$$\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{1}{1+|\xi+\eta|^2}\mu_t(d\xi) = \int_{\mathbb{R}^d}\frac{1}{1+|\xi|^2}\mu_t(d\xi) = \int_{\mathbb{R}^d}\frac{1}{1+t^2|\xi|^2}\mu(d\xi).$$

Inequality (30) follows similarly, by observing that

$$\frac{\sin^2(t|\xi|)}{|\xi|^2} \le t^2 \le t^2\frac{2}{1+|\xi|^2} \quad \text{if } |\xi|\le 1$$

and

$$\frac{\sin^2(t|\xi|)}{|\xi|^2} \le \frac{1}{|\xi|^2} \le \frac{2}{1+|\xi|^2} \quad \text{if } |\xi| > 1.$$

□

We will need the following elementary result.

Lemma 3.3. *For any $n \ge 1$ and for any function $h : [0,t]^n \to \mathbb{R}$ which is either non-negative or integrable,*

$$\int_{[0,t]^n}\int_{[0,t]^n}\prod_{j=1}^n \gamma(t_j - s_j)h(t_1,\ldots,t_n)dt_1\ldots dt_n ds_1\ldots ds_n \le \Gamma_t^n \int_{[0,t]^n}|h(t_1,\ldots,t_n)|dt_1\ldots dt_n, \tag{31}$$

where $\Gamma_t = \int_{-t}^t \gamma(s)ds = 2\int_0^t \gamma(s)ds$.

Proof: We consider only the case when h is a non-negative function. The proof for an integrable function h is similar. We use an induction argument on $n \ge 1$. For $n = 1$, we note that $\int_0^t \gamma(r-s)dr = \int_{-s}^{t-s}\gamma(r)dr \le \Gamma_t$ and hence

$$\int_0^t h(s)\left(\int_0^t \gamma(r-s)dr\right)ds \le \Gamma_t\int_0^t h(s)ds.$$

For the induction step, we assume that the inequality holds for $n-1$. Then

$$\int_0^t\int_0^t \gamma(t_n - s_n)\left(\int_{[0,t]^{2(n-1)}} h(t_1,\ldots,t_n)\prod_{j=1}^{n-1}\gamma(t_j-s_j)dt_1ds_1\ldots dt_{n-1}ds_{n-1}\right)dt_nds_n \le$$
$$\int_0^t\int_0^t \gamma(t_n-s_n)\left(\Gamma_t^{n-1}\int_{[0,t]^{n-1}} h(t_1,\ldots,t_n)dt_1\ldots dt_{n-1}\right)dt_nds_n =$$
$$\Gamma_t^{n-1}\int_{[0,t]^{n-1}}\left(\int_0^t\int_0^t \gamma(t_n-s_n)h(t_1,\ldots,t_n)dt_nds_n\right)dt_1\ldots dt_{n-1} \le$$
$$\Gamma_t^{n-1}\int_{[0,t]^{n-1}}\left(\Gamma_t\int_0^t h(t_1,\ldots,t_n)dt_n\right)dt_1\ldots dt_{n-1}$$

where we used the induction hypothesis for the first inequality, and inequality (31) for the case $n = 1$ for the last inequality. For the equality above, we used Fubini's theorem whose application is justified since h is non-negative. □

The next result is the analogue of Theorem 3.2 of [17] (or Theorem 5.2 of [24]) for the wave equation.

Theorem 3.4. *Suppose that the measure μ satisfies condition (2). If $d \geq 4$, suppose in addition that μ satisfies Hypothesis A. Then equation (1) has a mild solution u which is $L^2(\Omega)$-continuous and satisfies: for any $p \geq 2$ and $T > 0$*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} E|u(t,x)|^p < \infty.$$

Proof: *Step 1.* We first show the existence of a mild solution u.

Note that $f_n(\cdot,t,x) \in \mathcal{H}_n$ for all $t > 0$, $x \in \mathbb{R}^d$ and $n \geq 1$ (by Theorem 2.11). Therefore, by Theorem 2.9, it suffices to show that (24) holds, i.e.

$$\sum_{n\geq 0} \frac{1}{n!}\alpha_n(t) < \infty, \tag{32}$$

where

$$\alpha_n(t) = E|J_n(t,x)|^2 = E|I_n(f_n(\cdot,t,x))|^2 = (n!)^2 \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}. \tag{33}$$

To prove (32), we proceed as in the proof of Theorem 3.2 of [17]. In the integrals below, we use the notation $\mathbf{t} = (t_1,\ldots,t_n)$, $\mathbf{s} = (s_1,\ldots,s_n)$, $\mathbf{x} = (x_1,\ldots,x_n)$ and $\mathbf{y} = (y_1,\ldots,y_n)$. Then

$$\alpha_n(t) = \int_{[0,t]^{2n}} \prod_{j=1}^n \gamma(t_j - s_j)\psi_n(\mathbf{t},\mathbf{s})d\mathbf{t}d\mathbf{s}, \tag{34}$$

where

$$\psi_n(\mathbf{t},\mathbf{s}) = \frac{1}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}} \mathcal{F}g_{\mathbf{t}}^{(n)}(\cdot,t,x)(\xi_1,\ldots,\xi_n)\overline{\mathcal{F}g_{\mathbf{s}}^{(n)}(\cdot,t,x)(\xi_1,\ldots,\xi_n)}\mu(d\xi_1)\ldots\mu(d\xi_n)$$

and

$$g_{\mathbf{t}}^{(n)}(\cdot,t,x) = n!\widetilde{f}_n(t_1,\cdot,\ldots,t_n,\cdot,t,x). \tag{35}$$

If the permutation ρ of $\{1,\ldots,n\}$ is chosen such that $t_{\rho(1)} < \ldots < t_{\rho(n)}$, then

$$\begin{aligned}\mathcal{F}g_{\mathbf{t}}^{(n)}(\xi_1,\ldots,\xi_n) &= e^{-i\sum_{j=1}^n \xi_j\cdot x}\overline{\mathcal{F}G(t_{\rho(2)}-t_{\rho(1)},\cdot)(\xi_{\rho(1)})}\ \overline{\mathcal{F}G(t_{\rho(3)}-t_{\rho(2)},\cdot)(\xi_{\rho(1)}+\xi_{\rho(2)})} \\ &\quad \ldots\overline{\mathcal{F}G(t-t_{\rho(n)},\cdot)(\xi_{\rho(1)}+\ldots+\xi_{\rho(n)})} \end{aligned}\tag{36}$$

By the Cauchy-Schwarz inequality and the inequality $ab \leq (a^2+b^2)/2$, we obtain:

$$\psi_n(\mathbf{t},\mathbf{s}) \leq \psi_n(\mathbf{t},\mathbf{t})^{1/2}\psi_n(\mathbf{s},\mathbf{s})^{1/2} \leq \frac{1}{2}\Big(\psi_n(\mathbf{t},\mathbf{t}) + \psi_n(\mathbf{s},\mathbf{s})\Big).$$

Using (34) and the symmetry of the function γ, it follows that

$$\begin{aligned}\alpha_n(t) &\leq \frac{1}{2}\int_{[0,t]^{2n}} \prod_{j=1}^n \gamma(t_j-s_j)\Big(\psi_n(\mathbf{t},\mathbf{t}) + \psi_n(\mathbf{s},\mathbf{s})\Big)d\mathbf{t}d\mathbf{s} \\ &= \int_{[0,t]^{2n}} \prod_{j=1}^n \gamma(t_j-s_j)\psi_n(\mathbf{t},\mathbf{t})d\mathbf{t}d\mathbf{s}.\end{aligned}$$

Using Lemma 3.3 for the function $h(\mathbf{t}) = \psi_n(\mathbf{t},\mathbf{t})$, we obtain:

$$\alpha_n(t) \leq \Gamma_t^n \int_{[0,t]^n} \psi_n(\mathbf{t},\mathbf{t}) d\mathbf{t}. \tag{37}$$

We now estimate $\psi_n(\mathbf{t},\mathbf{t})$. We denote $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ for $j = 1, \ldots, n$. We have:

$$\begin{aligned}
\psi_n(\mathbf{t},\mathbf{t}) &= \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} |\mathcal{F}G(u_1,\cdot)(\xi_{\rho(1)})|^2 \, |\mathcal{F}G(u_2,\cdot)(\xi_{\rho(1)}+\xi_{\rho(2)})|^2 \ldots \\
&\qquad |\mathcal{F}G(u_n,\cdot)(\xi_{\rho(1)}+\ldots+\xi_{\rho(n)})|^2 \mu(d\xi_1)\ldots\mu(d\xi_n) \\
&= \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^d} \mu(d\xi_1') |\mathcal{F}G(u_1,\cdot)(\xi_1')|^2 \left(\int_{\mathbb{R}^d} \mu(d\xi_2') |\mathcal{F}G(u_2,\cdot)(\xi_1'+\xi_2')|^2 \ldots \right. \\
&\qquad \left. \left(\int_{\mathbb{R}^d} |\mathcal{F}G(u_n,\cdot)(\xi_1'+\ldots+\xi_n')|^2 \mu(d\xi_n') \right) \ldots \right),
\end{aligned}$$

where for the last equality we used the change of variable $\xi_j' = \xi_{\rho(j)}$ for $j = 1, \ldots, n$. Using Lemma 3.2 it follows that

$$\begin{aligned}
\psi_n(\mathbf{t},\mathbf{t}) &\leq \frac{1}{(2\pi)^{nd}} \prod_{j=1}^n \left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G(u_j,\cdot)(\xi_j+\eta)|^2 \mu(d\xi_j) \right) \\
&\leq \frac{1}{(2\pi)^{nd}} \prod_{j=1}^n \int_{\mathbb{R}^d} \frac{4u_j^2}{1+u_j^2|\xi_j|^2} \mu(d\xi_j). \qquad (38)
\end{aligned}$$

We now go back to the estimate (37) for $\alpha_n(t)$. We decompose the set $[0,t]^n$ into $n!$ disjoint regions of the form $t_{\rho(1)} < \ldots < t_{\rho(n)}$ with $\rho \in S_n$. Using (38), it follows that

$$\begin{aligned}
\alpha_n(t) &\leq \Gamma_t^n \frac{1}{(2\pi)^{nd}} \sum_{\rho\in S_n} \int_{t_{\rho(1)}<\ldots<t_{\rho(n)}} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \frac{4(t_{\rho(j+1)}-t_{\rho(j)})^2}{1+(t_{\rho(j+1)}-t_{\rho(j)})^2|\xi_j|^2} \mu(d\xi_1)\ldots\mu(d\xi_n) d\mathbf{t} \\
&= \Gamma_t^n n! \frac{1}{(2\pi)^{nd}} \int_{t_1<\ldots<t_n} \int_{\mathbb{R}^{nd}} \prod_{j=1}^n \frac{4(t_{j+1}-t_j)^2}{1+(t_{j+1}-t_j)^2|\xi_j|^2} \mu(d\xi_1)\ldots\mu(d\xi_n) d\mathbf{t} \\
&= \Gamma_t^n n! \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} \int_{S_{t,n}} \prod_{j=1}^n \frac{4w_j^2}{1+w_j^2|\xi_j|^2} d\mathbf{w} \mu(d\xi_1)\ldots\mu(d\xi_n),
\end{aligned}$$

where $S_{t,n} = \{(w_1,\ldots,w_n) \in [0,t]^n; w_1+\ldots+w_n \leq t\}$ and $\mathbf{w} = (w_1,\ldots,w_n)$. As in the proof of Lemma 3.3 of [17], since $S_{t,n} \subset S_t^I \times S_t^{I^c}$, the last integral is smaller than

$$\begin{aligned}
J &:= \sum_{I\subset\{1,\ldots,n\}} \int_{\mathbb{R}^{d|I|}} \prod_{j\in I} 1_{\{|\xi_j|\leq N\}} \left(\int_{S_t^I} \prod_{j\in I} \frac{4w_j^2}{1+w_j^2|\xi_j|^2} d\mathbf{w}_I \right) \prod_{j\in I} \mu(d\xi_j) \\
&\qquad \int_{\mathbb{R}^{d|I^c|}} \prod_{j\in I^c} 1_{\{|\xi_j|>N\}} \left(\int_{S_t^{I^c}} \prod_{j\in I^c} \frac{4w_j^2}{1+w_j^2|\xi_j|^2} d\mathbf{w}_{I^c} \right) \prod_{j\in I^c} \mu(d\xi_j),
\end{aligned}$$

where $S_t^I = \{\mathbf{w}_I = (w_j)_{j\in I}; w_j \geq 0, \sum_{j\in I} w_j \leq t\}$ and $S_t^{I^c} = \{\mathbf{w}_{I^c} = (w_j)_{j\in I^c}; w_j \geq 0, \sum_{j\in I^c} w_j \leq t\}$. Here $|I|$ is the cardinality of I and $N > 0$ is arbitrary.

For the integral over the set S_t^I we use the bound

$$\frac{4w_j^2}{1+w_j^2|\xi_j|^2} \le 4w_j^2 \le 4t^2,$$

and so, this integral is bounded by $(4t^2)^{|I|}\int_{S_t^I} d\mathbf{w}_I = 4^{|I|}t^{3|I|}/|I|!$. For the integral over $S_t^{I^c}$, we have:

$$\begin{aligned}
&\int_{S_t^{I^c}} \prod_{j\in I^c} \frac{4w_j^2}{1+w_j^2|\xi_j|^2} d\mathbf{w}_{I^c} \le \prod_{j\in I^c} \int_0^t \frac{4w_j^2}{1+w_j^2|\xi_j|^2} dw_j \\
&= \prod_{j\in I^c} \int_0^t \frac{4}{|\xi_j|^2} dw_j = 4^{|I^c|} t^{|I^c|} \prod_{j\in I^c} \frac{1}{|\xi_j|^2}.
\end{aligned}$$

We denote

$$C_N = \int_{\{|\xi|>N\}} \frac{1}{|\xi|^2}\mu(d\xi) \quad \text{and} \quad D_N = \int_{\{|\xi|\le N\}} \mu(d\xi).$$

It follows that

$$\begin{aligned}
J &\le 4^n \sum_{I\subset\{1,\dots,n\}} \frac{t^{3|I|}}{|I|!} D_N^{|I|} \cdot t^{|I^c|} C_N^{|I^c|} = 4^n \sum_{k=0}^n \binom{n}{k} \frac{t^{3k}}{k!} D_N^k t^{n-k} C_N^{n-k} \\
&\le 4^n \sum_{k=0}^n 2^n \frac{t^{n+2k}}{k!} D_N^k C_N^{n-k}.
\end{aligned}$$

Hence

$$\alpha_n(t) \le \Gamma_t^n n! \frac{1}{(2\pi)^{nd}} 8^n \sum_{k=0}^n \frac{t^{n+2k}}{k!} D_N^k C_N^{n-k} \tag{39}$$

and

$$\begin{aligned}
\sum_{n\ge0} \frac{1}{n!}\alpha_n(t) &\le \sum_{n\ge0} \Gamma_t^n \frac{1}{(2\pi)^{nd}} 8^n \sum_{k=0}^n \frac{t^{n+2k}}{k!} D_N^k C_N^{n-k} \\
&= \sum_{k\ge0} \frac{t^{2k}}{k!} D_N^k C_N^{-k} \sum_{n\ge k} (8(2\pi)^{-d} C_N \Gamma_t t)^n \\
&= \sum_{k\ge0} \frac{t^{2k}}{k!} D_N^k C_N^{-k} (8(2\pi)^{-d} C_N \Gamma_t t)^k . \sum_{n\ge0} (8(2\pi)^{-d} C_N \Gamma_t t)^n.
\end{aligned}$$

Due to condition (2), $C_N \to 0$ as $N \to \infty$. Hence, $8(2\pi)^{-d} C_N \Gamma_t t < 1$ for $N \ge N_t$. It follows that:

$$\begin{aligned}
\sum_{n\ge0} \frac{1}{n!}\alpha_n(t) &\le \frac{1}{1-8(2\pi)^{-d} C_N \Gamma_t t} \sum_{k\ge0} \frac{1}{k!} (8(2\pi)^{-d} D_N \Gamma_t t^3)^k \\
&= \frac{1}{1-8(2\pi)^{-d} C_N \Gamma_t t} \exp\left(8(2\pi)^{-d} D_N \Gamma_t t^3\right) < \infty.
\end{aligned}$$

This concludes the proof of (32).

Step 2. We show that the p-th moments of u are uniformly bounded.

We proceed as in the proof of Theorem 4.2 of [1]. We denote by $\|\cdot\|_p$ the $L^p(\Omega)$-norm. We use the fact that for any $F \in \mathcal{H}_n$ and $p \geq 2$,

$$\|F\|_p \leq (p-1)^{n/2}\|F\|_2 \tag{40}$$

(see last line of page 62 of [22]). Using Minkowski's inequality, applying (40) for $F = J_n(t,x)$, and invoking (33) and (39), we see that:

$$\begin{aligned} \|u(t,x)\|_p &\leq \sum_{n\geq 0}\|J_n(t,x)\|_p \leq \sum_{n\geq 0}(p-1)^{n/2}\|J_n(t,x)\|_2 = \sum_{n\geq 0}(p-1)^{n/2}\left(\frac{1}{n!}\alpha_n(t)\right)^{1/2} \\ &\leq \sum_{n\geq 0}(p-1)^{n/2}\Gamma_t^{n/2}\frac{1}{(2\pi)^{nd/2}}8^{n/2}\sum_{k=0}^{n}\frac{t^{n/2+k}}{(k!)^{1/2}}D_N^{k/2}C_N^{(n-k)/2}, \end{aligned}$$

which is uniformly bounded for $(t,x) \in [0,T]\times\mathbb{R}^d$ (using the same argument as above).

Step 3. We show that u is $L^2(\Omega)$-continuous.

The argument is Step 2 above shows that for any $T>0$ and $p \geq 2$,

$$\sum_{n\geq 0}\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\|J_n(t,x)\|_p \leq C_{T,p} < \infty.$$

Hence $\{u_n(t,x) = \sum_{k=0}^{n} J_k(t,x)\}_{n\geq 1}$ converges to $u(t,x)$ in $L^p(\Omega)$, uniformly in $(t,x) \in [0,T]\times\mathbb{R}^d$. By Lemma 3.6 below, J_n is $L^2(\Omega)$-continuous, and hence u_n is $L^2(\Omega)$-continuous. Therefore, u is $L^2(\Omega)$-continuous. □

Remark 3.5. In the proof of Theorem 3.4, we expressed $\alpha_n(t)$ as an integral which depends on the measure μ (instead of the kernel f); see (34). However, the fact that the Fourier transform of μ is the *locally integrable non-negative function* f was used in Lemma 3.1.

The following result is an extension of Lemma 4.2 of [1] to the case of an arbitrary covariance function $\gamma(t)$.

Lemma 3.6. *Under the conditions of the Theorem 3.4, we have:*
a) for any $n \geq 1$ and $t > 0$,

$$E|J_n(t+h,x) - J_n(t,x)|^2 \to 0 \quad \textit{as } h \to 0, \ \textit{uniformly in } x \in \mathbb{R}^d;$$

b) for any $n \geq 1$, $t > 0$ and $x \in \mathbb{R}^d$

$$E|J_n(t,x+z) - J_n(t,x)|^2 \to 0 \quad \textit{as } |z| \to 0, z \in \mathbb{R}^d.$$

Proof: a) We assume that $|h| \leq 1$ and $h > 0$. (The case $h < 0$ is similar.) We have:

$$\begin{aligned} E|J_n(t+h,x) - J_n(t,x)|^2 &= n!\,\|\widetilde{f}_n(\cdot,t+h,x) - \widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} \\ &\leq \frac{2}{n!}\left(A_n(t,h) + B_n(t,h)\right), \end{aligned} \tag{41}$$

where

$$
\begin{aligned}
A_n(t,h) &= (n!)^2 \|\widetilde{f}_n(\cdot,t+h,x)1_{[0,t]^n} - \widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}} && (42)\\
B_n(t,h) &= (n!)^2 \|\widetilde{f}_n(\cdot,t+h,x)1_{[0,t+h]^n\setminus[0,t]^n}\|^2_{\mathcal{H}^{\otimes n}}. && (43)
\end{aligned}
$$

We evaluate $A_n(t,h)$ first. We have:

$$
A_n(t,h) = \int_{[0,t]^{2n}} \prod_{j=1}^n \gamma(t_j - s_j)\psi^{(n)}_{t,h}(\mathbf{t},\mathbf{s})d\mathbf{t}d\mathbf{s},
$$

where

$$
\begin{aligned}
\psi^{(n)}_{t,h}(\mathbf{t},\mathbf{s}) &= \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} \mathcal{F}\left[g^{(n)}_{\mathbf{t}}(\cdot,t,x+h) - g^{(n)}_{\mathbf{t}}(\cdot,t,x)\right](\xi_1,\ldots,\xi_n)\\
&\quad \overline{\mathcal{F}\left[g^{(n)}_{\mathbf{s}}(\cdot,t,x+h) - g^{(n)}_{\mathbf{s}}(\cdot,t,x)\right](\xi_1,\ldots,\xi_n)}\mu(d\xi_1)\ldots\mu(d\xi_n)
\end{aligned}
$$

and $g^{(n)}_{\mathbf{t}}(\cdot,t,x)$ is given by (35). By the Cauchy-Schwarz inequality and the inequality $ab \leq (a^2+b^2)/2$,

$$
\psi^{(n)}_{t,h}(\mathbf{t},\mathbf{s}) \leq \left(\psi^{(n)}_{t,h}(\mathbf{t},\mathbf{t})\right)^{1/2}\left(\psi^{(n)}_{t,h}(\mathbf{s},\mathbf{s})\right)^{1/2} \leq \frac{1}{2}\left(\psi^{(n)}_{t,h}(\mathbf{t},\mathbf{t}) + \psi^{(n)}_{h}(\mathbf{s},\mathbf{s})\right).
$$

Using the symmetry of the function γ and Lemma 3.3, it follows that

$$
A_n(t,h) \leq \int_{[0,t]^{2n}} \prod_{j=1}^n \gamma(t_j - s_j)\psi^{(n)}_{t,h}(\mathbf{t},\mathbf{t})d\mathbf{t}d\mathbf{s} \leq \Gamma_t^n \int_{[0,t]^n} \psi^{(n)}_{t,h}(\mathbf{t},\mathbf{t})d\mathbf{t}. \tag{44}
$$

Using definition (36)) of the Fourier transform of $g^{(n)}_{\mathbf{t}}(\cdot,t,x)$, we see that

$$
\begin{aligned}
\psi^{(n)}_{t,h}(\mathbf{t},\mathbf{t}) &= \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} |\mathcal{F}G(u_1,\cdot\,)(\xi_{\rho(1)})|^2 \ldots |\mathcal{F}G(u_{n-1},\cdot\,)(\xi_{\rho(1)}+\ldots+\xi_{\rho(n-1)})|^2\\
&\quad |\mathcal{F}[G(u_n+h,\cdot\,) - G(u_n,\cdot\,)](\xi_{\rho(1)}+\ldots+\xi_{\rho(n)})|^2 \mu(d\xi_1)\ldots\mu(d\xi_n),
\end{aligned}
$$

where $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ and $0 < t_{\rho(1)} < \ldots < t_{\rho(n)} < t = t_{\rho(n+1)}$. It follows that

$$
\begin{aligned}
\psi^{(n)}_{t,h}(\mathbf{t},\mathbf{t}) &\leq \frac{1}{(2\pi)^{nd}} \prod_{j=1}^{n-1}\left(\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G(u_j,\cdot)(\xi_j+\eta)|^2 \mu(d\xi_j)\right) && (45)\\
&\quad \cdot \sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G(u_n+h,\cdot)(\xi_n+\eta) - \mathcal{F}G(u_n,\cdot)(\xi_n+\eta)|^2 \mu(d\xi_n).
\end{aligned}
$$

By applying the dominated convergence theorem twice, we infer first that $\psi^{(n)}_{t,h}(\mathbf{t},\mathbf{t}) \to 0$ as $h \to 0$, and then that $A_n(t,h) \to 0$ as $h \to 0$.

As for the term $B_n(t,h)$, note that

$$
B_n(t,h) = \int_{[0,t+h]^{2n}} \prod_{j=1}^n \gamma(t_j - s_j)\gamma^{(n)}_{t,h}(\mathbf{t},\mathbf{s})1_{D_{t,h}}(\mathbf{t})1_{D_{t,h}}(\mathbf{s})d\mathbf{t}d\mathbf{s},
$$

where $D_{t,h}=[0,t+h]^n\backslash[0,t]^n$ and

$$\gamma^{(n)}_{t,h}(\mathbf{t},\mathbf{s})=\frac{1}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}}\mathcal{F}g^{(n)}_{\mathbf{t}}(\cdot\ ,t+h,x)(\xi_1,\ldots,\xi_n)\overline{\mathcal{F}g^{(n)}_{\mathbf{s}}(\cdot\ ,t+h,x)(\xi_1,\ldots,\xi_n)}\mu(d\xi_1)\ldots\mu(d\xi_n).$$

By the Cauchy-Schwarz inequality and the inequality $ab\leq(a^2+b^2)/2$,

$$\gamma^{(n)}_{t,h}(\mathbf{t},\mathbf{s})\leq\left(\gamma^{(n)}_{t,h}(\mathbf{t},\mathbf{t})\right)^{1/2}\left(\gamma^{(n)}_{t,h}(\mathbf{s},\mathbf{s})\right)^{1/2}\leq\frac{1}{2}\left(\gamma^{(n)}_{t,h}(\mathbf{t},\mathbf{t})+\gamma^{(n)}_{h}(\mathbf{s},\mathbf{s})\right).$$

Using the symmetry of the function γ and Lemma 3.3, it follows that:

$$\begin{aligned}B_n(t,h) &\leq \int_{[0,t+h]^{2n}}\prod_{j=1}^{n}\gamma(t_j-s_j)\gamma^{(n)}_{t,h}(\mathbf{t},\mathbf{t})1_{D_{t,h}}(\mathbf{t})1_{D_{t,h}}(\mathbf{s})d\mathbf{t}d\mathbf{s}\\ &\leq \int_{[0,t+h]^{2n}}\prod_{j=1}^{n}\gamma(t_j-s_j)\gamma^{(n)}_{t,h}(\mathbf{t},\mathbf{t})1_{D_{t,h}}(\mathbf{t})d\mathbf{t}d\mathbf{s}\\ &\leq \Gamma^n_{t+h}\int_{[0,t+h]^{n}}\gamma^{(n)}_{t,h}(\mathbf{t},\mathbf{t})1_{D_{t,h}}(\mathbf{t})d\mathbf{t}. \qquad (46)\end{aligned}$$

We observe that for any $\mathbf{t}=(t_1,\ldots,t_n)\in[0,t+h]^n$, if we denote $u_j=t_{\rho(j+1)}-t_{\rho(j)}$ where $\rho\in S_n$ is such that $0<t_{\rho(1)}<\ldots<t_{\rho(n)}<t+h=t_{\rho(n+1)}$, then

$$\begin{aligned}\gamma^{(n)}_{t,h}(\mathbf{t},\mathbf{t}) &= \frac{1}{(2\pi)^{nd}}\int_{\mathbb{R}^{nd}}|\mathcal{F}G(u_1,\cdot)(\xi_{\rho(1)})|^2\ldots|\mathcal{F}G(u_{n-1},\cdot)(\xi_{\rho(1)}+\ldots+\xi_{\rho(n-1)})|^2\\ &\quad |\mathcal{F}G(u_n+h,\cdot)(\xi_{\rho(1)}+\ldots+\xi_{\rho(n)})|^2\mu(d\xi_1)\ldots\mu(d\xi_n)\\ &\leq \prod_{j=1}^{n-1}\left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\mathcal{F}G(u_j,\cdot)(\xi_j+\eta)|^2\mu(d\xi_j)\right)\\ &\quad \left(\sup_{\eta\in\mathbb{R}^d}\int_{\mathbb{R}^d}|\mathcal{F}G(u_n+h,\cdot)(\xi_n+\eta)|^2\mu(d\xi_n)\right) \qquad (47)\end{aligned}$$

which is bounded by a constant of the form C^n_t for any $h\in[0,1]$, due to (30). The fact that $B_n(t,h)\to0$ as $h\to0$ follows from (46) by the dominated convergence theorem, since $D_{t,h}\to\emptyset$ as $h\to0$.

b) Note that

$$E|J_n(t,x+z)-J_n(t,x)|^2=\frac{1}{n!}C_n(t,z), \qquad (48)$$

where

$$\begin{aligned}C_n(t,z) &= (n!)^2\|\widetilde{f}_n(\cdot,t,x+z)-\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}\\ &= \int_{[0,t]^{2n}}\prod_{j=1}^{n}\gamma(t_j-s_j)\psi^{(n)}_{t,z}(\mathbf{t},\mathbf{s})d\mathbf{t}d\mathbf{s} \qquad (49)\end{aligned}$$

and

$$\begin{aligned}\psi^{(n)}_{t,z}(\mathbf{t},\mathbf{s}) &= \frac{1}{(2\pi)^{nd}}\int_{\mathbb{R}^d}\mathcal{F}\left[g^{(n)}_{\mathbf{t}}(\cdot,t,x+z)-g^{(n)}_{\mathbf{t}}(\cdot,t,x)\right]\\ &\quad \overline{\mathcal{F}\left[g^{(n)}_{\mathbf{s}}(\cdot,t,x+z)-g^{(n)}_{\mathbf{s}}(\cdot,t,x)\right]}\mu(d\xi_1)\ldots\mu(d\xi_n).\end{aligned}$$

By the Cauchy-Schwarz inequality and the inequality $ab \leq (a^2+b^2)/2$,

$$\psi_{t,z}^{(n)}(\mathbf{t},\mathbf{s}) \leq \left(\psi_{t,z}^{(n)}(\mathbf{t},\mathbf{t})\right)^{1/2} \left(\psi_{t,z}^{(n)}(\mathbf{s},\mathbf{s})\right)^{1/2} \leq \frac{1}{2}\left(\psi_{t,z}^{(n)}(\mathbf{t},\mathbf{t}) + \psi_{t,z}^{(n)}(\mathbf{s},\mathbf{s})\right).$$

Using the symmetry of γ and Lemma 3.3, it follows that

$$C_n(t,z) \leq \int_{[0,t]^{2n}} \prod_{j=1}^{n} \gamma(t_j - s_j)\psi_{t,z}^{(n)}(\mathbf{t},\mathbf{t})d\mathbf{t}d\mathbf{s} \leq \Gamma_t^n \int_{[0,t]^n} \psi_{t,z}^{(n)}(\mathbf{t},\mathbf{t})d\mathbf{t}. \quad (50)$$

Using the definition (36) of the Fourier transform of $g_{\mathbf{t}}^{(n)}(\cdot,t,x)$, we see that

$$\begin{aligned}\psi_{t,z}^{(n)}(\mathbf{t},\mathbf{t}) &= \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} |\mathcal{F}G(u_1,\cdot)(\xi_{\rho(1)})|^2 \ldots |\mathcal{F}G(u_{n-1},\cdot)(\xi_{\rho(1)} + \ldots \xi_{\rho(n-1)})|^2 \\ &\quad |\mathcal{F}G(u_n,\cdot)(\xi_{\rho(1)} + \ldots \xi_{\rho(n)})|^2 |1 - e^{-i(\xi_1+\ldots+\xi_n)\cdot z}|^2 \mu(d\xi_1)\ldots\mu(d\xi_n), \quad (51)\end{aligned}$$

where $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ and $0 < t_{\rho(1)} < \ldots < t_{\rho(n)} < t = t_{\rho(n+1)}$. By applying twice the dominated convergence theorem, we conclude first that $\psi_{t,z}^{(n)}(\mathbf{t},\mathbf{t}) \to 0$ when $|z| \to 0$, and then that $C_n(t,z) \to 0$ when $|z| \to 0$. $\Box$

4 Hölder continuity

In this section, we assume that the spectral measure μ satisfies (3) and we show that the solution of equation (1) has a Hölder continuous modification. Note that (3) implies (2).

We recall the following results.

Proposition 4.1 (Proposition 7.4 of [9]). *Let G be the fundamental solution of the wave equation in dimension $d \geq 1$. If μ satisfies* (3)*, then:*
(i) for any $T > 0$ and $M > 0$, there exists a constant $C > 0$ depending on T, d, M, β such that for any $h \in \mathbb{R}$ with $|h| \leq M$

$$\sup_{t \in [0, T\wedge(T-h)]} \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G(t+h,\cdot)(\xi+\eta) - \mathcal{F}G(t,\cdot)(\xi+\eta)|^2 \mu(d\xi) \leq C|h|^{2-2\beta}; \quad (52)$$

(ii) for any $T > 0$, there exists a constant $C > 0$ depending on T, d, β such that for any $t \in [0,T]$

$$\sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi+\eta)|^2 \mu(d\xi) \leq Ct^{2-2\beta}; \quad (53)$$

(iii) for any $T > 0$ and for any compact set $K \subset \mathbb{R}^d$, there exists a constant $C > 0$ depending on T, K, d, β such that for any $z \in K$,

$$\sup_{t \in [0,T]} \sup_{\eta \in \mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi+\eta)|^2 |1 - e^{-i(\xi+\eta)\cdot z}|^2 \mu(d\xi) \leq C|z|^{2-2\beta}. \quad (54)$$

Lemma 4.2 (Lemma 3.5 of [5]). *For any $t > 0$ and $h > -1$*

$$\mathcal{I}_n(t,h) := \int_{0<t_1<\ldots<t_n<t} \prod_{j=1}^{n-1} (t_{j+1}-t_j)^h (t-t_n)^h d\mathbf{t} = \frac{\Gamma(1+h)^{n+1}}{\Gamma(n(1+h)+1)} t^{n(1+h)}.$$

We are now ready to state our result about the Hölder continuity of the solution.

Theorem 4.3. *Suppose that μ satisfies (3). If $d \geq 4$, suppose in addition that μ satisfies Hypothesis A. Let u be the solution of equation (1). Then:*
a) for any $p \geq 2$ and $T > 0$ there exists a constant $C > 0$ depending on p, T, d and β such that for any $t, t' \in [0,T]$ and for any $x \in \mathbb{R}^d$,

$$\|u(t,x) - u(t',x)\|_p \leq C|t-t'|^{1-\beta}; \tag{55}$$

b) for any $p \geq 2$, $T > 0$ and compact set $K \subset \mathbb{R}^d$, there exists a constant $C > 0$ depending on p, T, K, d and β such that for any $t \in [0,T]$ and for any $x, x' \in K$,

$$\|u(t,x) - u(t,x')\|_p \leq C|x-x'|^{1-\beta}. \tag{56}$$

Consequently, for any $T > 0$ and for any compact set $K \subset \mathbb{R}^d$, the solution $\{u(t,x); t \in [0,T], x \in K\}$ to equation (1) has a modification which is jointly θ-Hölder continuous in time and space, for any $\theta \in (0, 1-\beta)$.

Remark 4.4. If $f(x) = |x|^{-\alpha}$ is the Riesz kernel for some $0 < \alpha < d$, then the spectral measure μ is given by $\mu(d\xi) = C_{\alpha,d}|\xi|^{-(d-\alpha)}d\xi$, where $C_{\alpha,d} > 0$ is a constant which depends on α and d. In this case, condition (2) holds for any $0 < \alpha < 2$ and condition (3) holds for any β with $\alpha/2 < \beta < 1$. Therefore, for any $T > 0$ and for any compact set $K \subset \mathbb{R}^d$, the solution $u = \{u(t,x); t \in [0,T], x \in K\}$ has a modification which is jointly θ-Hölder continuous in time and space, for any $\theta \in (0, \frac{2-\alpha}{2})$. This result coincides with Theorem 5.1 of [1].

Proof of Theorem 4.3: a) Let $t, t' \in [0,T]$ and $x \in \mathbb{R}^d$ be arbitrary. Assume that $h := t' - t > 0$. (The case $h < 0$ is similar.) By Minkowski's inequality, (40) and (41),

$$\begin{aligned} \|u(t+h,x) - u(t,x)\|_p &\leq \sum_{n\geq 0} (p-1)^{n/2} \|J_n(t+h,x)) - J_n(t,x)\|_2 \\ &\leq \sum_{n\geq 0} (p-1)^{n/2} \left(\frac{2}{n!} \left[A_n(t,h) + B_n(t,h) \right] \right)^{1/2}, \end{aligned} \tag{57}$$

where $A_n(t,h)$ and $B_n(t,h)$ are given by (42), respectively (43).

To estimate $A_n(t,h)$, we use (44). Note that by (45), (52) and (53), we have

$$\psi_{t,h}^{(n)}(\mathbf{t},\mathbf{t}) \leq C^n (u_1 \ldots u_{n-1} h)^{2-2\beta},$$

where $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ and $0 < t_{\rho(1)} < \ldots < t_{\rho(n)} < t = t_{\rho(n+1)}$. By invoking Lemma 4.2, it follows that

$$\begin{aligned} A_n(t,h) &\leq h^{2-2\beta}\Gamma_t^n C^n n! \int_{0<t_1<\ldots<t_n<t} \prod_{j=1}^{n-1}(t_{j+1}-t_j)^{2-2\beta} dt_1 \ldots dt_n \\ &= h^{2-2\beta}\Gamma_t^n C^n n! \int_0^t \mathcal{I}_{n-1}(t_n, 2-2\beta) dt_n \\ &= h^{2-2\beta}\Gamma_t^n C^n n! \frac{\Gamma(3-2\beta)^n}{\Gamma((n-1)(3-2\beta)+1)} \int_0^t t_n^{(n-1)(3-2\beta)} dt_n. \end{aligned}$$

We now use the fact that for all $a > 1$ there exists a constant $C > 0$ such that

$$\Gamma(an+1) \geq C(n!)^a \quad \text{for all } n \geq 1 \tag{58}$$

(see e.g. (68) in [4]). It follows that

$$A_n(t,h) \leq h^{2-2\beta}\Gamma_t^n C^n \frac{1}{(n!)^{2-2\beta}} t^{(n-1)(3-2\beta)+1}. \tag{59}$$

To estimate $B_n(t,h)$, we use (46). First note that by (47) and (53),

$$\gamma_{t,h}^{(n)}(\mathbf{t},\mathbf{t}) \leq C^n [u_1 \ldots u_{n-1}(u_n + h)]^{2-2\beta},$$

where $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ and $0 < t_{\rho(1)} < \ldots < t_{\rho(n)} < t = t_{\rho(n+1)}$. We observe that if $(t_1, \ldots, t_n) \in D_{t,h} = [0, t+h]^n \backslash [0,t]^n$ then there exists at least one index i with $t_i > t$. So,

$$D_{t,h} = \bigcup_{\rho \in S_n} \{(t_1, \ldots, t_n); 0 \leq t_{\rho(1)} \leq \ldots \leq t_{\rho(n-1)} \leq t_{\rho(n)}, t < t_{\rho(n)} \leq t+h\}.$$

By applying Lemma 4.2, it follows that

$$\begin{aligned} B_n(t,h) &\leq \Gamma_{t+h}^n C^n \sum_{\rho \in S_n} \int_t^{t+h} \int_{0<t_{\rho(1)}<\ldots<t_{\rho(n-1)}<t_{\rho(n)}} \prod_{j=1}^{n-1} (t_{\rho(j+1)} - t_{\rho(j)})^{2-2\beta} (t+h-t_{\rho(n)})^{2-2\beta} d\mathbf{t} \\ &= \Gamma_{t+h}^n C^n n! \int_t^{t+h} \mathcal{I}_{n-1}(t_n, 2-2\beta)\,(t+h-t_n)^{2-2\beta} dt_n \\ &= \Gamma_{t+h}^n C^n n! \frac{\Gamma(3-2\beta)^n}{\Gamma((n-1)(3-2\beta)+1)} \int_t^{t+h} t_n^{(n-1)(3-2\beta)} (t+h-t_n)^{2-2\beta} dt_n \\ &= \Gamma_{t+h}^n C^n n! \frac{\Gamma(3-2\beta)^n}{\Gamma((n-1)(3-2\beta)+1)} \int_0^h (t+h-u)^{(n-1)(3-2\beta)}\, u^{2-2\beta} du \\ &\leq \Gamma_T^n C^n n! \frac{\Gamma(3-2\beta)^n}{\Gamma((n-1)(3-2\beta)+1)} T^{(n-1)(3-2\beta)} \frac{1}{3-2\beta} h^{3-2\beta}. \end{aligned}$$

Using (58), it follows that

$$B_n(t,h) \leq h^{2-2\beta}\Gamma_T^n C^n \frac{1}{(n!)^{2-2\beta}} T^{(n-1)(3-2\beta)}. \tag{60}$$

Relation (55) follows from (57), (59) and (60).

b) Let $t \in [0,T]$ and $x, x' \in K$ be arbitrary. We denote $z = x' - x$. By Minkowski's inequality, (40) and (48), we have:

$$\|u(t,x+z)-u(t,x)\|_p \leq \sum_{n\geq 0}(p-1)^{n/2}\|J_n(t,x+z)-J_n(t,x)\|_2 = \sum_{n\geq 0}(p-1)^{n/2}\left(\frac{1}{n!}C_n(t,z)\right),$$

where $C_n(t,z)$ is defined by (49). To estimate $C_n(t,z)$ we use (50). Note that by (51), (53) and (54),

$$\psi_{t,z}^{(n)}(\mathbf{t},\mathbf{t}) \leq C^n |z|^{2-2\beta}(u_1 \ldots u_{n-1})^{2-2\beta},$$

where $u_j = t_{\rho(j+1)} - t_{\rho(j)}$ and $0 < t_{\rho(1)} < \ldots < t_{\rho(n)} < t = t_{\rho(n+1)}$. Hence

$$C_n(t,z) \leq |z|^{2-2\beta} C^n \Gamma_t^n n! \int_{0<t_1<\ldots<t_n<t} \prod_{j=1}^{n-1}(t_{j+1}-t_j)^{2-2\beta} d\mathbf{t}.$$

Using the same estimate for the last integral as above, we infer that

$$C_n(t,z) \leq |z|^{2-2\beta} C^n \Gamma_t^n \frac{1}{(n!)^{2-2\beta}} t^{(n-1)(3-2\beta)+1}.$$

Relation (56) follows. The final statement is a consequence of Kolmogorov's continuity theorem. □

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