Temporal asymptotics for fractional parabolic Anderson model

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Abstract

In this paper, we consider fractional parabolic equation of the form $\frac{\partial u}{\partial t} = -(-\Delta)^{\frac{\alpha}{2}} u + u\dot{W}(t,x)$, where $-(-\Delta)^{\frac{\alpha}{2}}$ with $\alpha \in (0,2]$ is a fractional Laplacian and $\dot{W}$ is a Gaussian noise colored in space and time. The precise moment Lyapunov exponents for the Stratonovich solution and the Skorohod solution are obtained by using a variational inequality and a Feynman-Kac type large deviation result for space-time Hamiltonians driven by α-stable process. As a byproduct, we obtain the critical values for θ and η such that $\mathbb{E}\exp\left(\theta\left(\int_0^1\int_0^1 |r-s|^{-\beta_0}\gamma(X_r - X_s)drds\right)^\eta\right)$ is finite, where X is d-dimensional symmetric α-stable process and $\gamma(x)$ is $|x|^{-\beta}$ or $\prod_{j=1}^d |x_j|^{-\beta_j}$.

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1 Introduction

Let $\left\{\dot{W}(t,x)\,, t\geq 0\,, x\in\mathbb{R}^d\right\}$ be a general mean zero Gaussian noise on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ whose covariance function is given by

$$\mathbb{E}[\dot{W}(r,x)\dot{W}(s,y)] = |r-s|^{-\beta_0}\gamma(x-y),$$

where $\beta_0 \in [0,1)$ and

$$\gamma(x) = \begin{cases} |x|^{-\beta} & \text{where } \beta\in[0,d) \ \text{ or} \\ \prod_{j=1}^d |x_j|^{-\beta_j} & \text{where } \beta_j \in [0,1), j=1,\dots,d. \end{cases}$$

If we abuse the notation $\beta = \sum_{i=1}^d \beta_i$, the spatial covariance function has the following scaling property

$$\gamma(cx) = |c|^{-\beta}\gamma(x) \tag{1.1}$$

for both cases. In this paper, we shall study the following fractional parabolic Anderson model,

$$\begin{cases} \dfrac{\partial u}{\partial t} = -(-\Delta)^{\frac{\alpha}{2}} u + u\dot{W}(t,x),\ t>0, x\in\mathbb{R}^d \\ u(0,x) = u_0(x),\ x\in\mathbb{R}^d\,, \end{cases} \tag{1.2}$$

where $-(-\Delta)^{\frac{\alpha}{2}}$ with $0<\alpha\le 2$ is the fractional Laplacian and where the initial condition satisfies $0<\delta\le|u_0(x)|\le M<\infty$. Without loss of of generality, we assume $u_0(x)\equiv 1$ when we study the long-term asymptotics of $u(t,x)$. The product $u\dot{W}(t,x)$ appearing in the above equation will be understood in the sense of Skorohod and in the sense of Stratonovich.

Let us recall some results from [28] for the SPDE (1.2).

(i) Theorem 5.3 in [28] implies that, under the following condition:

$$\beta<\alpha\,, \tag{1.3}$$

Eq. (1.2) in the Skorohod sense has a unique mild solution $\tilde{u}(t,x)$, and its n-th moment can be represented as (see [28, Theorem 5.6])

$$\mathbb{E}[\tilde{u}(t,x)^n] = \mathbb{E}_X\left[\prod_{j=1}^n u_0(X_t^j+x)\exp\left(\sum_{1\le j<k\le n}\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^j-X_s^k)drds\right)\right]\,, \tag{1.4}$$

where $X_1,\ldots,X_n$ are n independent copies of d-dimensional *symmetric α-stable process* and are independent of W, and $\mathbb{E}_X$ denotes the expectation with respect to $(X_t^x, t\ge 0)$.

(ii) Under a more restricted condition:

$$\alpha\beta_0+\beta<\alpha \tag{1.5}$$

the following Feynman-Kac formula

$$u(t,x) = \mathbb{E}_X\left[u_0(X_t^x)\exp\left(\int_0^t\int_{\mathbb{R}^d}\delta_0(X_{t-r}^x-y)W(dr,dy)\right)\right]\,, \tag{1.6}$$

is a mild Stratonovich solution to (1.2) (see [28, Theorem 4.6]), where $\delta_0(x)$ is the Dirac delta function. Consequently, Theorem 4.8 in [28] provides a Feynman-Kac type representation for n-th moment of $u(t,x)$

$$\mathbb{E}[u(t,x)^n] = \mathbb{E}\left[\prod_{j=1}^n u_0(X_t^j+x)\exp\left(\frac{1}{2}\sum_{j,k=1}^n\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^j-X_s^k)drds\right)\right]\,. \tag{1.7}$$

The more restricted condition (1.5) is to ensure that the "diagonal" terms, i.e., the sum $\sum_{k=1}^n\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r^k-X_s^k)drds$ appearing in (1.7) are exponentially integrable (see Lemma 2.1 and Theorem 2.3, or [28, Theorem 3.3] in a more general setting). To deal with the moments given by (1.4) and that given by (1.7) simultaneously, we introduce, under the condition (1.5), for a positive $\rho \in [0,1]$,

$$u^\rho(t,x) := \mathbb{E}_X\Bigg[u_0(X_t^x)\exp\bigg(\int_0^t\int_{\mathbb{R}^d}\delta_0(X_{t-r}^x - y)W(dr,dy) \\ -\frac{\rho}{2}\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r - X_s)drds\bigg)\Bigg]. \tag{1.8}$$

When $\rho = 0$, $u^\rho(t,x)$ is the Stratonovich solution $u(t,x)$ to (1.2), and when $\rho = 1$, $u^\rho(t,x)$ is the Skorohod solution $\tilde{u}(t,x)$ to (1.2). The n-th moment of $u^\rho(t,x)$ for a positive integer n is given by

$$\mathbb{E}[|u^\rho(t,x)|^n] = \mathbb{E}\Bigg[\prod_{j=1}^n u_0(X_t^j + x)\exp\bigg(\frac{1}{2}\sum_{j,k=1}^n\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^j - X_s^k)drds \\ -\frac{\rho}{2}\sum_{j=1}^n\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^j - X_s^j)drds\bigg)\Bigg]. \tag{1.9}$$

Let us point out that when $\rho = 1$, $\mathbb{E}[|u^\rho(t,x)|^n]$ is finite under the weaker condition (1.3).

The goal of this article is to obtain the precise asymptotics, as $t \to \infty$, of the p-th moment $\mathbb{E}\left[|u^\rho(t,x)|^p\right]$ for any (fixed) positive real number p. To describe our main result, we recall the definition of Fourier transform and introduce some notations. Denote by $\mathcal{S}(\mathbb{R}^d)$ the Schwartz space of smooth functions that are rapidly decreasing on $\mathbb{R}^d$, and let $\mathcal{S}'(\mathbb{R}^d)$ be its dual space, i.e., the space of tempered distributions. Let $\widehat{f}(\xi)$ or $(\mathcal{F}f)(\xi)$ denote the Fourier transform of f, for f in the space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions. In particular, we set

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)dx, \text{ for } f \in L^1(\mathbb{R}^d).$$

We will also need the following notations.

$$\mathcal{E}_\alpha(f,f) := \int_{\mathbb{R}^d} |\xi|^\alpha |\widehat{f}(\xi)|^2 d\xi\,;$$

$$\mathcal{F}_{\alpha,d} := \left\{f \in L^2(\mathbb{R}^d) : \|f\|_2 = 1 \text{ and } \mathcal{E}_\alpha(f,f) < \infty\right\}\,; \tag{1.10}$$

$$\mathcal{A}_{\alpha,d} := \left\{g(s,x) : \int_{\mathbb{R}^d} g^2(s,x)dx = 1, \forall s \in [0,1] \text{ and } \int_0^1\int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(s,\xi)|^2 d\xi ds < \infty\right\}\,; \tag{1.11}$$

$$\mathbf{M}(\alpha,\beta_0,d,\gamma) := \sup_{g\in\mathcal{A}_{\alpha,d}}\Bigg\{\frac{1}{2}\int_0^1\int_0^1\int_{\mathbb{R}^{2d}}\frac{\gamma(x-y)}{|r-s|^{\beta_0}}g^2(s,x)g^2(r,y)dxdydrds \\ -\int_0^1\int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(s,\xi)|^2 d\xi ds\Bigg\}. \tag{1.12}$$

The finiteness of the variational representation $\mathbf{M}(\alpha,\beta_0,d,\gamma)$, when $\beta < \alpha$, is established in the Appendix. Note that $\mathbf{M}(\alpha,\beta_0,d,\gamma)$ has the scaling property, for any $\theta > 0$,

$$\mathbf{M}(\alpha,\beta_0,d,\theta\gamma) = \theta^{\frac{\alpha}{\alpha-\beta}}\mathbf{M}(\alpha,\beta_0,d,\gamma), \tag{1.13}$$

which can be derived in the same way as Lemma 4.1 in [12]. The following is the main result in this paper.

Theorem 1.1 *Let $\rho \in [0,1]$ and assume the condition* (1.5) *(and when $\rho = 1$, the condition* (1.5) *is replaced by the condition* (1.3)*). Let $p \geq 2$ be any real number or $p = 1$. Then*

$$\lim_{t\to\infty} t^{-\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}} \log \|u^\rho(t,x)\|_p = (p-\rho)^{\frac{\alpha}{\alpha-\beta}} \mathbf{M}(\alpha, \beta_0, d, \gamma),$$

where $\|u^\rho(t,x)\|_p = \Big(\mathbb{E}[|u^\rho(t,x)|^p]\Big)^{1/p}$.

We conclude this introduction with some remarks on the motivation of our work and a brief literature review for the related results. The following three points motivate us to obtain the above asymptotics.

(i) The limit related to the long-term asymptotics is known as the *moment Lyapunov exponent* in literature and the problem is closely related to the issue of *intermittency* (see, e.g., [23]). To illustrate our idea, write the limit in Theorem 1.1 in the following form:

$$\lim_{t\to\infty} t^{-\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}} \log \mathbb{E}\exp\Big(p\log|u^\rho(t,x)|\Big) = \Lambda(p).$$

The system satisfies the usual definition of intermittency, i.e., the function $\Lambda(p)/p$ is strictly increasing on $[2,\infty)$. By the large deviation theory (see, e.g., Theorem 1.1.4 in [11] and its proof for the lower bound), for any sufficiently large $l > 0$

$$\lim_{t\to\infty} t^{-\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}} \log \mathbb{P}(A_{t,l}) = -\sup_{p>0}\Big\{lp - \Lambda(p)\Big\} < 0$$

and there is $p_l > 0$ such that

$$\lim_{t\to\infty} \frac{\mathbb{E}\big[|u^\rho(t,x)|^{p_l} 1_{A_{t,l}}\big]}{\mathbb{E}[|u^\rho(t,x)|^{p_l}]} = 1$$

where

$$A_{t,l} = \Big\{\log|u^\rho(t,x)| \geq l t^{\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}}\Big\}.$$

This observation shows that as in other cases of intermittency, it is rare for the solution $u(t,x)$ to take large values but that the impact of taking large values should not be ignored.

(ii) When the noise $\dot{W}$ is the space-time white noise with dimension one in space, the parabolic Anderson model (1.2) is the model for the *continuum directed polymer in random environment* (see [1] for the case $\alpha = 2$ and [6] for the case $\alpha < 2$), where (1.2) is understood in the Skorohod sense, the solution $\tilde{u}(t,x)$ is the *partition function* for the polymer measure, and $\log\tilde{u}(t,x)$ is the *free energy* for the polymer (see, e.g., [15]).

Similarly, if we consider an α-stable motion X in the random environment modelled by $\dot{W}$, one may consider the Hamiltonian

$$H^\rho(t,x) := \int_0^t \int_{\mathbb{R}^d} \delta_0(X_{t-r}^x - y) W(dr,dy) - \frac{\rho}{2} \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) drds \,. \quad (1.14)$$

Then, $u^\rho(t,x) = \mathbb{E}_X[e^{H^\rho(t,x)}]$ is the partition function for the polymer measure, and $\log u^\rho(t,x)$ is the free energy for the polymer.

(iii) The equation (1.2), as one of the basic SPDEs, describes a variety of models, such as the parabolic Anderson model (see, e.g. [7]) and the model for continuum directed polymer in random environment (see, e.g., [1]), in which the long-term asymptotic property of the solution is desirable. In the recent publication [8], the space-time fractional diffusion equation of the form

$$\left(\partial^\beta + \frac{\nu}{2}(-\Delta)^{\alpha/2}\right) u(t,x) = \lambda u(t,x) \dot{W}(t,x) \,,$$

has been studied, where ∂^β is the Caputo derivative in time t. It is highly non-trivial to obtain precise asymptotics in general case. Our model (1.2) corresponds to the case $\beta = 1$, and our result may provide some perspective for the general situation.

The moment Lyapunov exponent has been studied extensively with vast literature. To our best knowledge, however, the investigation in the setting of white/fractional space-time Gaussian noise started only recently, especially at the level of precision given in this paper. When the driving processes are Brownian motion instead of stable process, i.e., the operator in (1.2) is $\frac{1}{2}\Delta$ instead of the fractional Laplacian, the long-term asymptotic lower and upper bounds for the moments of the solution were studied in [4] for the Skorohod solution and in [29] for the Stratonovich solution; the *precise* moment Lyapunov exponents were obtained in recent publications [9, 10] for the Skorohod solutions, and [12] for the Stratonovich solution. In [3], the authors obtained the intermittency property for the fractional heat equation in the Skorohod sense, by studying the lower and upper asymptotic bounds of the solution.

In the present paper, we aim to obtain the *precise* p-th moment Lyapunov exponents for both Stratonovich solution and Skorohod solution to the fractional heat equation in a unified way, for any real positive number $p \geq 2$. The mathematical challenges and/or the originality of this work come from the following aspects. First, compared with case of the heat equation, the fact that the fractional Laplacian is not a local operator makes the computations and analysis more sophisticated. New ideas and methodologies are required. In particular, Fourier analysis is involved in a more substantial way. Second, the Feynman-Kac large deviation result for stable process (Theorem 3.1) is a key to our approach. However, the method used to derive a similar result for Brownian motion in [12] can no longer be applied, as the behavior of stable process is totally different from the behavior of Brownian motion. Third, we obtain the precise long-term asymptotics for $u^\rho(t,x)$ with $\rho \in [0,1]$, which enables us to get the precise moment Lyapunov exponents for the Stratonovich solution and the Skorohod solution simultaneously. Finally, the existing results on precise Lyapunov exponents were mainly for n-th moment with n a positive integer, due to the fact that the

Feynman-Kac type representation is valid only for the moment of integer orders. We are able to extend the result from positive integers to real numbers $p \geq 2$. The idea is to use the variational inequality and the hypercontractivity of the Ornstein-Uhlenbeck semigroup operators.

The paper is organized as follows. In Section 2, we establish some rough bounds for the long-term asymptotics of the Stratonovich solution by comparison method. The rough bounds will be used in the derivation of the precise upper bound in Section 6. The critical exponential integrability of $\int_0^1 \int_0^1 |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds$ is also studied. In Section 3, we obtain an Feynman-Kac type large deviation result for α-stable processes, which plays a critical role in obtaining the variational representation for the precise moment Lyapunov exponent. In Section 4, we establish a lower bound for the p-th moment of $u^\rho(t,x)$ which is also valid if the α-stable process is replaced by some general symmetric Lévy process. In Sections 5 and 6, we validate the lower bound and the upper bound in Theorem 1.1, respectively. Finally, in Appendix, the well-posedness of the variation given in (1.12) which appears in Theorem 1.1 is justified, and the proof of a technical lemma that is used in Section 6 is provided.

2 Asymptotic bounds by comparison method

In this section we derive long-term asymptotic bounds by comparison method for $\log \mathbb{E}[u(t,x)] = \log \mathbb{E} \exp \left(\int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds \right)$. Note that by the self-similarity property of the stable process X, the integral inside the above exponential has the following scaling property,

$$\begin{aligned} &\int_0^{at} \int_0^{at} |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds \\ \overset{d}{=} &a^{2-\frac{\beta}{\alpha}-\beta_0} \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds, \text{ for all } a > 0. \end{aligned} \tag{2.1}$$

First, we present the following integrability result.

Lemma 2.1 $\mathbb{E} \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds < \infty$ *if and only if* $\alpha\beta_0 + \beta < \alpha$.

Proof Using the self-similarity of X, and the scaling property of $\gamma(x)$, we have $\mathbb{E}[\gamma(X_r - X_s)] = |r-s|^{-\frac{\beta}{\alpha}} \mathbb{E}[\gamma(X_1)]$, noting that $0 < \mathbb{E}[\gamma(X_1)] < \infty$, under the condition of this lemma. Hence, we have

$$\mathbb{E} \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds = \mathbb{E}[\gamma(X_1)] \int_0^t \int_0^t |r-s|^{-\beta_0} |r-s|^{-\frac{\beta}{\alpha}} dr ds,$$

which concludes the proof. □

Lemma 2.2 Under the condition (1.5), the process

$$Y_t = \int_0^t \int_0^t |r-u|^{-\beta_0} \gamma(X_r - X_u) dr du, \quad t \geq 0$$

has a continuous version.

Proof We shall use the notation $\|F\|_p = (\mathbb{E}[|F|^p])^{1/p}$. For any $0 \leq s < t \leq \infty$, we have for any $p \geq 1$,

$$\begin{aligned} \|Y_t - Y_s\|_p &\leq \int_s^t \int_0^s |r-u|^{-\beta_0} \|\gamma(X_r - X_u)\|_p dr du + \int_0^s \int_s^t |r-u|^{-\beta_0} \|\gamma(X_r - X_u)\|_p dr du \\ &\quad + \int_s^t \int_s^t |r-u|^{-\beta_0} \|\gamma(X_r - X_u)\|_p dr du \\ &=: I_1 + I_2 + I_3 \,. \end{aligned}$$

By scaling property, when $1 < p < \frac{\alpha}{\beta}$,

$$\|\gamma(X_r - X_u)\|_p = \left(\mathbb{E} \left[|X_r - X_u|^{-\frac{\beta}{\alpha} p}\right]\right)^{1/p} = c_p |r-u|^{-\frac{\beta}{\alpha}} \,.$$

Thus,

$$\begin{aligned} I_1 &\leq \int_s^t \int_0^s |r-u|^{-\beta_0 - \frac{\beta}{\alpha}} du dr \\ &\leq \frac{1}{1 - \beta_0 - \frac{\beta}{\alpha}} \int_s^t r^{1-\beta_0-\frac{\beta}{\alpha}} dr \\ &\leq C \int_s^t t^{1-\beta_0-\frac{\beta}{\alpha}} dr = C t^{1-\beta_0-\frac{\beta}{\alpha}} |t-s| \,. \end{aligned}$$

This means

$$I_1^p \leq C t^{p(1-\beta_0-\frac{\beta}{\alpha})} |t-s|^p \,.$$

Similar estimates for I_2 and I_3 can also be obtained. Thus for $0 \leq s, t \leq T$, there is a constant C_T depending only on $(\alpha, \beta, \beta_0, T)$ such that

$$\mathbb{E} \; |Y_t - Y_s|^p = \|Y_t - Y_s\|_p^p \leq C_T |t-s|^p \,.$$

It follows from Kolmogorov continuity criterion that $\{Y_t \,, t \geq 0\}$ has a continuous version.

□

Theorem 2.3 *Under the condition (1.5), there exists a constant $\delta > 0$ such that when $\theta \in (0, \delta)$,*

$$\mathbb{E} \exp \left(\theta \left(\int_0^1 \int_0^1 |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds \right)^{\frac{\alpha}{\alpha\beta_0 + \beta}} \right) < \infty, \tag{2.2}$$

and consequently, for all $\lambda > 0$,

$$\mathbb{E} \exp \left(\lambda \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r - X_s) dr ds \right) < \infty. \tag{2.3}$$

Remark 2.4 The inequality (2.3) is a consequence of Theorem 3.3 in [28] which was proved by using moment method. Below we will provide another approach to prove (2.3) by using the techniques from the theory of large deviations, and it turns out that this approach enables us to get a stronger result (see Remark 2.6).

Proof Denote

$$Z_t = \left(\int_0^t \int_0^t |s-r|^{-\beta_0} \gamma(X_s - X_r) ds dr \right)^{\frac{1}{2}}, \text{ for } t \geq 0 . \tag{2.4}$$

First we shall show that Z_t is sub-additive and hence exponentially integrable by [11, theorem 1.3.5].

The following identity holds

$$|s-r|^{-\beta_0} = C_0 \int_{\mathbb{R}} |s-u|^{-\frac{\beta_0+1}{2}} |r-u|^{-\frac{\beta_0+1}{2}} du, \tag{2.5}$$

where $C_0 > 0$ depends on β_0 only. Similarly, for the function $\gamma(x)$ we also have

$$\gamma(x) = C(\gamma) \int_{\mathbb{R}^d} K(y-x) K(y) dy, \quad x \in \mathbb{R}^d , \tag{2.6}$$

where $C(\gamma) > 0$ is a constant and

$$K(x) = \begin{cases} \prod_{j=1}^d |x_j|^{-\frac{1+\beta_j}{2}} & \text{if } \gamma(x) = \prod_{j=1}^d |x_j|^{-\beta_j}, \\ |x|^{-\frac{d+\beta}{2}} & \text{if } \gamma(x) = |x|^{-\beta}. \end{cases} \tag{2.7}$$

With these identities, we can rewrite Z_t as

$$Z_t = \left(\int_{\mathbb{R}\times\mathbb{R}^d} \xi_t^2(u,x) du dx \right)^{1/2},$$

where

$$\xi_t(u,x) = C_0 C(\gamma) \int_0^t |s-u|^{-\frac{\beta_0+1}{2}} K(X_s - x) ds .$$

For $t_1, t_2 > 0$, by the triangular inequality

$$Z_{t_1+t_2} \leq Z_{t_1} + \left(\int_{\mathbb{R}\times\mathbb{R}^d} \left[\xi_{t_1+t_2}(u,x) - \xi_{t_1}(u,x) \right]^2 du dx \right)^{1/2} .$$

Let $\widetilde{X}_s = X_{t_1+s} - X_{t_1}$, which is independent of $\{X_r, 0 \leq r \leq t_1\}$, and we have

$$\begin{aligned} &\xi_{t_1+t_2}(u,x) - \xi_{t_1}(u,x) \\ =& C_0 C(\gamma) \int_{t_1}^{t_1+t_2} |s-u|^{-\frac{\beta_0+1}{2}} K(X_s - x) ds \\ =& C_0 C(\gamma) \int_0^{t_2} |s+t_1-u|^{-\frac{\beta_0+1}{2}} K(\widetilde{X}_s + X_{t_1} - x) ds . \end{aligned}$$

The translation invariance of the integral on $\mathbb{R}^{d+1}$ implies that

$$\int_{\mathbb{R}\times\mathbb{R}^d}\Big[\xi_{t_1+t_2}(u,x)-\xi_{t_1}(u,x)\Big]^2 dudx=\int_{\mathbb{R}\times\mathbb{R}^d}\Big[\widetilde{\xi}_{t_2}(u,x)\Big]^2 dudx\,,$$

where

$$\widetilde{\xi}_{t_2}(u,x)=C_0C(\gamma)\int_0^{t_2}|s-u|^{-\frac{\beta_0+1}{2}}K(\widetilde{X}_s-x)ds.$$

Therefore, the process Z_t is sub-additive, which means that for any $t_1,t_2>0$, $Z_{t_1+t_2}\le Z_{t_1}+\widetilde{Z}_{t_2}$, where $\widetilde{Z}_{t_2}$ is independent of $\{Z_s, 0\le s\le t_1\}$ and has the same distribution as Z_{t_2}.

Notice that Z_t is non-negative, non-decreasing, and pathwise continuous by Lemma 2.2. Thus it follows from [11, Theorem 1.3.5] that, for any $t>0$ and $\theta>0$

$$\mathbb{E}\exp\big[\big(\theta Z_t\big)\big]<\infty,$$

and

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\big(\theta Z_t\big)\right]=\Psi(\theta), \tag{2.8}$$

for some $\Psi(\theta)\in[0,\infty)$. Moreover, by the scaling property (2.1) we have $Z_{at}\overset{d}{=}a^{\kappa}Z_t$ with $\kappa=1-\frac{\beta}{2\alpha}-\frac{\beta_0}{2}\in(1/2,1)$, and hence for all $\theta>0$,

$$\Psi(\theta)=\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\big[\big(\theta Z_t\big)\big]=\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\big(Z_{t\theta^{\frac{1}{\kappa}}}\big)\right]=\theta^{\frac{1}{\kappa}}\Psi(1). \tag{2.9}$$

Chebyshev inequality implies that

$$\exp(\theta t)\mathbb{P}(Z_t\ge t)\le\mathbb{E}\exp(\theta Z_t)\quad\text{and then}\quad\theta t+\log\mathbb{P}(Z_t\ge t)\le\log\mathbb{E}\exp(\theta Z_t)\,.$$

Taking the limit yields, for any $\theta>0$,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(Z_t\ge t)\le\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp(\theta Z_t)\right]-\theta=\theta^{\frac{1}{\kappa}}\Psi(1)-\theta.$$

Therefore

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{P}(Z_t\ge t)\le\inf_{\theta\in(0,1)}\{\theta^{\frac{1}{\kappa}}\Psi(1)-\theta\}, \tag{2.10}$$

where the term on the right-hand side is strictly negative noting that $1/\kappa\in(1,2)$ and $\Psi(1)\ge0$, and is denoted by $-a$ for some $a>0$. Hence there exists a constant $T>0$ such that when $t\ge T$,

$$\mathbb{P}(Z_1\ge t^{1-\kappa})=\mathbb{P}(Z_t\ge t)\le\exp\left(-at/2\right). \tag{2.11}$$

Consequently,

$$\begin{aligned}\mathbb{E}\left[\exp(\theta Z_1^{\frac{1}{1-\kappa}})\right] &= \int_0^\infty\mathbb{P}(\theta Z_1^{\frac{1}{1-\kappa}}\ge y)e^y dy+1\\ &\le \int_0^T e^y dy+\int_T^\infty e^{-a\theta^{-1}y/2}e^y dy+1,\end{aligned}$$

where the last term is finite if $\theta\in(0,a/2)$. This implies (2.2).

Finally (2.3) is obtained by (2.2), the scaling property (2.1) and the fact that the condition (1.5) implies $\frac{\alpha}{\alpha\beta_0+\beta}>1$. □

Remark 2.5 Note that by(2.9), $\Psi(\theta) = \theta^{\frac{1}{\kappa}}\Psi(1)$ with $\Psi(1) \in [0, \infty)$. Actually, $\Psi(1) > 0$ when $\beta_0 = 0$, by (2.21) in the proof of Lemma 2.8. However, when $\beta_0 \in (0, 1)$, $\Psi(1)$ must be 0, which means that the asymptotics given by (2.8) is not optimal. Indeed, if $\Psi(1) \neq 0$, Gärtner-Ellis theorem for non-negative random variable ([11, Corollary 1.2.5]) and equation (2.9) imply that for $\lambda > 0$,

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Z_t^2 \geq \lambda t^{2-2\kappa}) = -\sup_{\theta>0} \left\{\theta\lambda^{\frac{1}{2}} - \theta^{\frac{1}{\kappa}}\Psi(1)\right\} = C_1\lambda^{\frac{1}{2-2\kappa}},$$

where $C_1 = \Psi(1)^{\frac{\kappa}{\kappa-1}}(\kappa^{\frac{\kappa}{1-\kappa}} - \kappa^{\frac{1}{1-\kappa}})$. Note that the assumption $\Psi(1) > 0$ guarantees that $\theta^{\frac{1}{\kappa}}\Psi(1)$ is an essentially smooth function ([11, Definition 1.2.3]), and hence the Gärtner-Ellis theorem can be applied. Then, by the Varadhan's integral lemma ([11, Theorem 1.1.6] or [17, Section 4.3]),

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E} \exp(\theta t^{2\kappa-1} Z_t^2) = \sup_{\lambda>0} \left\{\lambda\theta - C_1\lambda^{\frac{1}{2-2\kappa}}\right\} = C_2\theta^{\frac{1}{2\kappa-1}},$$

where C_2 is a positive constant depending on C_1 and κ. By the scaling property (2.1), this limit is equal to

$$\lim_{t\to\infty} t^{-1} \log \mathbb{E}\left[\exp(\theta Z_{t^\eta}^2)\right] = \lim_{t\to\infty} t^{-\frac{1}{\eta}} \log \mathbb{E}\left[\exp(\theta Z_t^2)\right] = C_2\theta^{\frac{1}{2\kappa-1}},$$

where $\eta = \frac{2\kappa-1}{2\kappa}$ and $\frac{1}{\eta} = \frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta-\alpha\beta_0}$. This contradicts with Proposition 2.9 when $\beta_0 \in (0, 1)$.

Remark 2.6 We observe that the restriction $\theta \in (0, \delta)$ for (2.2) in Theorem 2.3 can be removed when $\beta_0 \in (0, 1)$. Indeed, the inequality (2.10) in the proof can be replaced by

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Z_t \geq t) \leq \inf_{\theta>0}\{\theta^{\frac{1}{\kappa}}\Psi(1) - \theta\}.$$

Noting that by Remark 2.5, $\Psi(1) = 0$ when $\beta_0 \in (0, 1)$, we have

$$\limsup_{t\to\infty} \frac{1}{t} \log \mathbb{P}(Z_t \geq t) = -\infty.$$

This enables us to choose any positive number for a in (2.11), and hence (2.2) holds for any $\theta > 0$. Moreover, using Theorem 1.1 (note that Theorem 1.1 is proved without quoting Theorem 2.7), the critical exponential integrability and the corresponding critical exponent for $\int_0^1\int_0^1 |r - s|^{-\beta_0}\gamma(X_r - X_s)drds$ can be obtained.

Theorem 2.7 *Let $C_0 := C\big(\alpha, \beta, \beta_0, d, \gamma(\cdot)\big)$ be given in* (2.16)*. Then under the condition* (1.5)*, we have*

$$\mathbb{E} \exp\left(\theta\left(\int_0^1\int_0^1 |r - s|^{-\beta_0}\gamma(X_r - X_s)drds\right)^{\frac{\alpha}{\beta}}\right) < \infty, \ \textit{for any } \theta < C_0, \tag{2.12}$$

and

$$\mathbb{E}\exp\left(\theta\left(\int_0^1\int_0^1 |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)^{\frac{\alpha}{\beta}}\right)=\infty,\ \textit{for any}\ \theta>C_0. \tag{2.13}$$

Furthermore,

$$\mathbb{E}\exp\left(\theta\left(\int_0^1\int_0^1 |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)^{\eta}\right)<\infty,\ \textit{for any}\ \theta>0\ \textit{and}\ \eta<\frac{\alpha}{\beta}. \tag{2.14}$$

and

$$\mathbb{E}\exp\left(\theta\left(\int_0^1\int_0^1 |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)^{\eta}\right)=\infty,\ \textit{for any}\ \theta>0\ \textit{and}\ \eta>\frac{\alpha}{\beta}. \tag{2.15}$$

Proof Recall that Z_t is defined in (2.4). Theorem 1.1 implies that, when $p=1$ and $\rho=0$,

$$\lim_{t\to\infty} t^{-\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}}\log\mathbb{E}\exp\left(\frac{1}{2}Z_t^2\right)=\mathbb{M}(\alpha,\beta_0,d,\gamma).$$

By the scaling property (2.1) of Z_t^2 and the change of variable $s=t^{\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}}$, the above equation is equivalent to

$$\lim_{s\to\infty}\frac{1}{s}\log\mathbb{E}\exp\left(\theta s^{1-\beta/\alpha}Z_1^2\right)=(2\theta)^{\frac{\alpha}{\alpha-\beta}}\mathbb{M}(\alpha,\beta_0,d,\gamma).$$

Then the Gärtner-Ellis theorem implies

$$\begin{aligned}\lim_{s\to\infty}\frac{1}{s}\log\mathbb{P}(s^{-\beta/\alpha}Z_1^2\geq\lambda)&=-\sup_{\theta>0}\left\{\theta\lambda-(2\theta)^{\frac{\alpha}{\alpha-\beta}}\mathbb{M}(\alpha,\beta_0,d,\gamma)\right\}\\&=-\lambda^{\frac{\alpha}{\beta}}\frac{\beta}{\alpha-\beta}\left(\frac{\alpha-\beta}{2\alpha}\right)^{\frac{\alpha}{\beta}}\Big(\mathbb{M}(\alpha,\beta_0,d,\gamma)\Big)^{\frac{\beta-\alpha}{\beta}}.\end{aligned}$$

Denote

$$C_0:=\frac{\beta}{\alpha-\beta}\left(\frac{\alpha-\beta}{2\alpha}\right)^{\frac{\alpha}{\beta}}\Big(\mathbb{M}(\alpha,\beta_0,d,\gamma)\Big)^{\frac{\beta-\alpha}{\beta}}, \tag{2.16}$$

and hence C_0 is a finite positive constant. Then we have

$$\lim_{s\to\infty}\frac{1}{s}\log\mathbb{P}(Z_1^{\frac{2\alpha}{\beta}}\geq s)=-C_0. \tag{2.17}$$

For any fixed $\sigma\in(0,1)$, there exists $T_\sigma>0$ such that when $t>T_\sigma$,

$$\mathbb{P}(Z_1^{\frac{2\alpha}{\beta}}\geq t)\leq\exp(-t\sigma C_0),$$

and therefore,

$$\begin{aligned}\mathbb{E}\left[\exp(\theta Z_1^{\frac{2\alpha}{\beta}})\right]&=1+\int_0^\infty\mathbb{P}(\theta Z_1^{\frac{2\alpha}{\beta}}\geq t)e^t dt\\&\leq 1+\int_0^{T_\sigma}e^t dt+\int_{T_\sigma}^\infty e^{-t\sigma\theta^{-1}C_0}e^t dy,\end{aligned}$$

where the right-hand side is finite when $\theta < \sigma C_0$. Since $\sigma \in (0,1)$ can be arbitrarily chosen, the first result (2.12) is obtained. Finally the inequalitys (2.13), (2.14), (2.15) can be proved in a similar way by using (2.17), and the proof is concluded. $\square$

To obtain the optimal asymptotics for $\mathbb{E}\exp\Big(\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\Big)$, we first investigate the asymptotics of $\mathbb{E}\exp\Big(\int_0^t\int_0^t \gamma(X_s-X_r)dsdr\Big)$.

Lemma 2.8 *Under the condition* (1.3)*, there exists a constant* $C\in(0,\infty)$*, such that*

$$\lim_{t\to\infty} t^{-\frac{2\alpha-\beta}{\alpha-\beta}}\log\mathbb{E}\exp\left(\theta\int_0^t\int_0^t \gamma(X_r-X_s)drds\right) = C\theta^{\frac{\alpha}{\alpha-\beta}}, \quad \forall\,\theta>0. \tag{2.18}$$

Let $\tilde X$ *be an independent copy of* X*. Then under the condition* (1.3)*, there exist* $0<D_1\le D_2<\infty$ *such that for all* $\theta>0$*,*

$$\begin{aligned} D_1\theta^{\frac{\alpha}{\alpha-\beta}} &\le \liminf_{t\to\infty} t^{-\frac{2\alpha-\beta}{\alpha-\beta}}\log\mathbb{E}\exp\left(\theta\int_0^t\int_0^t \gamma(X_r-\tilde X_s)drds\right)\\ &\le \limsup_{t\to\infty} t^{-\frac{2\alpha-\beta}{\alpha-\beta}}\log\mathbb{E}\exp\left(\theta\int_0^t\int_0^t \gamma(X_r-\tilde X_s)drds\right)\le D_2\theta^{\frac{\alpha}{\alpha-\beta}}. \end{aligned}\tag{2.19}$$

Proof When $\gamma(x)=|x|^{-\beta}$, (2.18) is a direct consequence of [14, Equation (1.18)] using the scaling property of the $\int_0^t\int_0^t\gamma(X_r-X_s)drds$. When $\gamma(x)=\prod_{j=1}^d |x_j|^{-\beta_j}$, it suffices to show that there there exists a constant $C_1<\infty$ such that

$$\lim_{t\to\infty} t^{-\frac{2\alpha-\beta}{\alpha-\beta}}\log\mathbb{E}\exp\left(\theta\int_0^t\int_0^t \gamma(X_r-X_s)drds\right) = C_1\theta^{\frac{\alpha}{\alpha-\beta}}\,. \tag{2.20}$$

This is because that $\prod_{j=1}^d |x_j|^{-\beta_j}\ge |x|^{-\beta}$ and hence C_1 is greater than or equal to the constant $C>0$ in (2.18) when $\gamma(x)=|x|^{-\beta}$. This means that if $C_1<\infty$ satisfies (2.20), then it will be automatically positive.

From now on the generic constant C may be different in different places.

We claim that (2.20) is equivalent to

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left(\theta\Big(\int_0^t\int_0^t \gamma(X_r-X_s)drds\Big)^{1/2}\right) = C\theta^{\frac{2\alpha}{2\alpha-\beta}}\,, \qquad \forall\,\theta>0 \tag{2.21}$$

for some constant $C\in(0,\infty)$, which can be proved in the same way as we did to get (2.9) in the proof of the Theorem 2.3. Indeed, by the scaling property (2.1) with $\beta_0=0$, and by a Gärtner-Ellis type result for non-negative random variables ([11, Corollary 1.2.5]), both (2.20) and (2.21) are equivalent to the tail asymptotics

$$\lim_{t\to\infty} t^{-1}\log\mathbb{P}\Big(\int_0^1\int_0^1 \gamma(X_r-X_s)drds\ge\lambda t^{\frac{\beta}{\alpha}}\Big) = -\sup_{\theta>0}\Big\{\sqrt{\lambda}\theta - C\theta^{\frac{2\alpha}{2\alpha-\beta}}\Big\} = -C\lambda^{\frac{\alpha}{\beta}}\,,\ \forall\,\lambda>0.$$

Now we prove (2.19). The upper bound can be obtained by (2.18) and the observation that

$$\mathbb{E}\exp\left(\theta\int_0^t\int_0^t\gamma(X_r-\tilde{X}_s)drds\right)=\mathbb{E}\exp\left(\theta C(\gamma)\int_{\mathbb{R}^d}\int_0^t K(X_r-x)dr\int_0^t K(\tilde{X}_s-x)dsdx\right)$$
$$\le\mathbb{E}\exp\left(\frac{\theta}{2}C(\gamma)\int_{\mathbb{R}^d}\left(\int_0^t K(X_r-x)dr\right)^2+\left(\int_0^t K(\tilde{X}_r-x)dr\right)^2dx\right)$$
$$=\left[\mathbb{E}\exp\left(\frac{\theta}{2}C(\gamma)\int_{\mathbb{R}^d}\left(\int_0^t K(X_r-x)dr\right)^2dx\right)\right]^2\le\mathbb{E}\exp\left(\theta\int_0^t\int_0^t\gamma(X_r-X_s)drds\right).$$

For the lower bound, if suffices to consider the case $\gamma(x)=|x|^{-\beta}$. By [5, Theorem 1.2],

$$\lim_{t\to\infty}t^{-\frac{\alpha}{\beta}}\log\mathbb{P}\left(\int_0^1\int_0^1\gamma(X_r-\tilde{X}_s)drds\ge t\right)=-a,$$

where a is a positive constant. By the scaling property (2.1), the above equality is equivalent to

$$\lim_{t\to\infty}t^{-\frac{2\alpha-\beta}{\alpha-\beta}}\log\mathbb{P}\left(t^{-\frac{2\alpha-\beta}{\alpha-\beta}}\int_0^t\int_0^t\gamma(X_r-\tilde{X}_s)drds\ge\lambda\right)=-a\lambda^{\frac{\alpha}{\beta}},\ \text{for all }\lambda>0.$$

Then by Varadhan's integral lemma, we have

$$\lim_{t\to\infty}t^{-\frac{2\alpha-\beta}{\alpha-\beta}}\log\exp\left(\theta\int_0^t\int_0^t\gamma(X_r-\tilde{X}_s)drds\right)=\sup_{\lambda>0}\{\theta\lambda-a\lambda^{\frac{\alpha}{\beta}}\}=b\theta^{\frac{\alpha}{\alpha-\beta}},$$

for some $b>0$. □

Based on the above result, we shall derive the following asymptotics for $\mathbb{E}\exp\left(\theta\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)$ by comparison method.

Proposition 2.9 *Under the condition 1.5, there is* $0<C_1<C_2<\infty$ *such that for any* $\theta>0$,

$$\begin{aligned}C_1\theta^{\frac{\alpha}{\alpha-\beta}}&\le\liminf_{t\to\infty}t^{-\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}}\log\mathbb{E}\exp\left(\theta\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)\\&\le\limsup_{t\to\infty}t^{-\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}}\log\mathbb{E}\exp\left(\theta\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)\le C_2\theta^{\frac{\alpha}{\alpha-\beta}}.\end{aligned}\tag{2.22}$$

Similarly, under the condition (1.3), *there is* $0<D_1<D_2<\infty$ *such that for any* $\theta>0$,

$$\begin{aligned}D_1\theta^{\frac{\alpha}{\alpha-\beta}}&\le\liminf_{t\to\infty}t^{-\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}}\log\mathbb{E}\exp\left(\theta\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r-\tilde{X}_s)drds\right)\\&\le\limsup_{t\to\infty}t^{-\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}}\log\mathbb{E}\exp\left(\theta\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r-\tilde{X}_s)drds\right)\le D_2\theta^{\frac{\alpha}{\alpha-\beta}}.\end{aligned}\tag{2.23}$$

Remark 2.10 *By the scaling property* (2.1), *the above asymptotics* (2.22) *is equivalent to*

$$C_1\theta^{\frac{\alpha}{\alpha-\beta}} \le \liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left(\frac{\theta}{t}\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)$$
$$\le \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\exp\left(\frac{\theta}{t}\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right) \le C_2\theta^{\frac{\alpha}{\alpha-\beta}}, \qquad (2.24)$$

respectively. We also have a similar result for (2.23).

Proof The proof is similar to [12, Proposition 2.1], but we include details for the reader's convenience. First we prove the lower bound in (2.22). Note that

$$\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r-X_s)drds \ge t^{-\beta_0}\int_0^t\int_0^t \gamma(X_r-X_s)drds,$$

where the term on the right-hand side has the same distribution as

$$\int_0^{t^{\frac{2\alpha-\beta-\alpha\beta_0}{2\alpha-\beta}}}\int_0^{t^{\frac{2\alpha-\beta-\alpha\beta_0}{2\alpha-\beta}}} \gamma(X_r-X_s)drds$$

by the scaling property (2.1). Then the lower bound is an immediate consequence of Lemma 2.8.

Now we show the upper bound of (2.22). By the symmetry of the integrand function, we have

$$\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r-X_s)drds = 2\iint_{[0\le s\le r\le t]} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\,.$$

Thus, the inequality (2.22) is equivalent to

$$\limsup_{t\to\infty} t^{-\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}}\log\mathbb{E}\exp\left(\theta\iint_{[0\le s\le r\le t]} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right) \le C\theta^{\frac{\alpha}{\alpha-\beta}}\,. \qquad (2.25)$$

Compared with lower bound, the estimation (2.25) is more difficult to obtain because $|r-s|^{-\beta_0}$ is unbounded when r and s are close. We shall decompose the integral $\int_{[0\le s\le r\le t]}|r-s|^{-\beta_0}\gamma(X_r-X_s)drds$ and then apply Hölder inequality to obtain the desired result. More precisely, let $[0\le s\le r\le t] = I_1\cup I_2\cup I_3$, where $I_1=[0\le s\le t\le t/2], I_2=[t/2\le s\le r\le t]$ and $I_3=[0,t/2]\times[t/2,t]$. Noting that $\iint_{I_1}|r-s|^{-\beta_0}\gamma(X_r-X_s)drds$ and $\iint_{I_2}|r-s|^{-\beta_0}\gamma(X_r-X_s)drds$ are mutually independent and are equal in law, by the Hölder inequality,

$$\mathbb{E}\exp\left(\theta\iint_{[0\le s\le r\le t]} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)$$
$$\le \left(\mathbb{E}\exp\left(\theta p\iint_{I_1} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)\right)^{2/p}$$
$$\times\left(\mathbb{E}\exp\left(\theta q\iint_{I_3} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)\right)^{1/q},$$

where $p^{-1}+q^{-1}=1$. Furthermore, by the scaling property (2.1),

$$\iint_{I_1} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds \overset{d}{=} \left(\frac{1}{2}\right)^{\frac{2\alpha-\beta-\alpha\beta_0}{\alpha}} \iint_{[0\le s\le r\le t]} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\,.$$

Taking $p=2^{\frac{2\alpha-\beta-\alpha\beta_0}{\alpha}}$, we have

$$\mathbb{E}\exp\left(\theta\iint_{[0\le r\le s\le t]} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right) \le \left(\mathbb{E}\exp\left(\theta q\int_0^{t/2}\int_{t/2}^{t} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)\right)^{\frac{1}{q}\frac{p}{p-2}}.$$

Now to obtain (2.22), it suffices to show

$$\limsup_{t\to\infty} t^{-\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}} \log\mathbb{E}\exp\left(\theta\int_0^{t/2}\int_{t/2}^{t} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right) \le C\theta^{\frac{\alpha}{\alpha-\beta}}\,. \tag{2.26}$$

Actually, decomposing $[0,t/2]\times[t/2,t]$ as $A\cup B$, where $A=[t/4,t/2]\times[t/2,3t/4]$ and $B=[0,t/2]\times[t/2,t]\setminus A$, we have

$$\begin{aligned}\mathbb{E}\exp&\left(\theta\int_0^{t/2}\int_{t/2}^{t} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)\\ &\le \left(\mathbb{E}\exp\left(\theta p\iint_A |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)\right)^{1/p}\\ &\quad\times\left(\mathbb{E}\exp\left(\theta q\iint_B |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)\right)^{1/q},\end{aligned} \tag{2.27}$$

where $1/p+1/q=1$ and $p,q>0$ are to be determined later. Since X has stationary increments and by (2.1), we have

$$\begin{aligned}\iint_A |r-s|^{-\beta_0}\gamma(X_r-X_s)drds &\overset{d}{=} \int_0^{t/4}\int_{t/4}^{t/2} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\\ &\overset{d}{=} \left(\frac{1}{2}\right)^{\frac{2\alpha-\beta-\alpha\beta_0}{\alpha}} \int_0^{t/2}\int_{t/2}^{t} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\,.\end{aligned}$$

Now let us choose $p=2^{\frac{2\alpha-\beta-\alpha\beta_0}{\alpha}}$, and the above identity combined with (2.27) yields

$$\begin{aligned}\mathbb{E}\exp\left(\theta\int_0^{t/2}\int_{t/2}^{t} |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right) &\le \mathbb{E}\exp\left(\theta q\iint_B |r-s|^{-\beta_0}\gamma(X_r-X_s)drds\right)\\ \le \mathbb{E}\exp\left(\theta q\left(\frac{t}{4}\right)^{-\beta_0}\int_0^t\int_0^t \gamma(X_r-X_s)drds\right) &\le \mathbb{E}\exp\left(\theta q4^{\beta_0}\int_0^{t^\eta}\int_0^{t^\eta}\gamma(X_r-X_s)drds\right),\end{aligned}$$

where $\eta = \frac{2\alpha-\beta-\alpha\beta_0}{2\alpha-\beta}$. Thus (2.26) follows from Lemma 2.8 with t being replaced by t^η and (2.22) is obtained.

The lower bound in (2.23) can be obtained in a similar way as for the lower bound in (2.22), by using the second half of Lemma 2.8. Now we show the upper bound. Noting that the stable process has stationary increments which are independent over disjoint time intervals, we have

$$\int_0^{t/2}\int_{t/2}^t |r-s|^{-\beta_0}\gamma(X_r-X_s)drds \overset{d}{=} \int_0^{t/2}\int_0^{t/2} |r-s|^{-\beta_0}\gamma(X_r-\tilde{X}_s)drds.$$

By Remark 5.7 in [28], under the condition (1.3),

$$\mathbb{E}\exp\left(\lambda\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r-\tilde{X}_s)drds\right) < \infty \text{ for all } \lambda > 0.$$

Hence (2.26) still holds under the condition (1.3), and therefore the upper bound in (2.23) is obtained. The proof is concluded. □

3 Feynman-Kac large deviation for stable process

In this section, we will obtain a Feynman-Kac large deviation result (Proposition 3.1 below) for symmetric α-stable process, which is a space-time extension of Lemma 6 in [13] and will play a critical role in the derivation of our main result. In [12] a similar result for Brownian motion was obtained (Proposition 3.1 in that paper) in order to get the precise moment Lyapunov exponent for the Stratonovich solution of heat equation. The approach in [12] heavily depends on the local property of the Laplacian operator and the property of Brownian motion such as the continuity of paths and the Gaussian tail probability, and hence cannot be adapted to our situation, as the fractional Laplacian is a non-local operator, the stable process is a pure jump process, and the stable distribution is fat-tailed. Inspired by the idea in [13], instead of considering the stable process itself, we shall consider the stable process restricted in bounded domains by taking its image of quotient map, which will be elaborated below.

Fix a positive number M. Let $\mathbb{T}^d_M = \mathbb{R}^d/M\mathbb{Z}^d$ be the d-dimensional torus and X^M_t be the image of X_t under the quotient map from $\mathbb{R}^d$ to $\mathbb{T}^d_M$. Then, X^M is a Markov process with independent increments on $\mathbb{T}^d_M$, and its associated Dirichlet form is given by

$$\mathcal{E}_{\alpha,M}(f,f) := \frac{1}{M^{d+\alpha}}\sum_{k\in\mathbb{Z}^d} |k|^\alpha|\widehat{f}(k)|^2, \tag{3.1}$$

where $\widehat{f}$ denotes the usual Fourier transform for functions on $\mathbb{T}^d_M$, i.e., for $k\in\mathbb{Z}^d$,

$$\widehat{f}(k) := \int_{\mathbb{T}^d_M} f(x)e^{-2\pi i k\cdot x/M}dx = \int_{[0,M]^d} f(x)e^{-2\pi i k\cdot x/M}dx.$$

Here the function f on $\mathbb{T}_M^d$ is considered as an M-periodic function (with the same symbol f) on $\mathbb{R}^d$, which means that $f(x+kM)=f(x)$ for any $k\in\mathbb{Z}^d$. Let

$$\mathcal{F}_{\alpha,M} := \{f\in L^2(\mathbb{T}_M^d) : \|f\|_{2,\mathbb{T}_M^d}=1 \text{ and } \mathcal{E}_{\alpha,M}(f,f)<\infty\}, \tag{3.2}$$

where

$$\|f\|_{2,\mathbb{T}_M^d} = \Big(\langle f,f\rangle_{2,\mathbb{T}_M^d}\Big)^{1/2} := \left(\int_{\mathbb{T}_M^d}|f(x)|^2dx\right)^{1/2} = \left(\int_{[0,M]^d}|f(x)|^2dx\right)^{1/2}$$

is the L^2-norm on $\mathbb{T}_M^d$ endowed with the Lebesgue measure.

Proposition 3.1 *Let $f(t,x):[0,1]\times\mathbb{T}_M^d\to\mathbb{R}$ be a continuous function. Then, we have*

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left(\int_0^t f(\frac{s}{t},X_s^M)ds\right)\right] = \int_0^1\lambda_M(f(s,\cdot))ds, \tag{3.3}$$

where $\lambda_M(f) := \sup_{g\in\mathcal{F}_{\alpha,M}}\Big\{\langle g,fg\rangle_{2,\mathbb{T}_M^d}-\mathcal{E}_{\alpha,M}(g,g)\Big\}$.

Proof Let $\{0=s_0<s_1<\cdots<s_{n-1}<s_n=1\}$ be a uniform partition of the interval $[0,1]$. First, we consider the functions of the form

$$f(s,x)=\sum_{i=0}^{n-1}f_i(x)I_{[s_i,s_{i+1})}(s)+f_{n-1}(x)I_{\{1\}}(s)\,.$$

By the Markov property, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\int_0^t f(\frac{s}{t},X_s^M)ds\right)\right] = \mathbb{E}\left[\exp\left(\int_0^{\frac{t}{n}} f(\frac{s}{t},X_s^M)ds\right)\exp\left(\int_{\frac{t}{n}}^t f(\frac{s}{t},X_s^M)ds\right)\right]\\
&= \mathbb{E}\left[\exp\left(\int_0^{\frac{t}{n}} f_0(X_s^M)ds\right)\mathbb{E}_{X^M_{\frac{t}{n}}}\left[\exp\left(\int_0^{(1-\frac{1}{n})t} f(\frac{s}{t}+\frac{1}{n},X_s^M)ds\right)\right]\right]\\
&\geq \mathbb{E}\left[\exp\left(\int_0^{\frac{t}{n}} f_0(X_s^M)ds\right);|X^M_{\frac{t}{n}}|<\delta\right]\inf_{|x|<\delta}\mathbb{E}_x\left[\exp\left(\int_0^{(1-\frac{1}{n})t} f(\frac{s}{t}+\frac{1}{n},X_s^M)ds\right)\right],
\end{aligned}$$

where $\mathbb{E}_x$ denotes the expectation with respect to the stable process staring from x.

Repeating the above procedure, we can get

$$\prod_{i=0}^{n-1}\inf_{|x|<\delta}\mathbb{E}_x\left[\exp\left(\int_0^{\frac{t}{n}} f_i(X_s^M)ds\right);|X^M_{\frac{t}{n}}|<\delta\right] \leq \mathbb{E}\left[\exp\left(\int_0^t f(\frac{s}{t},X_s^M)ds\right)\right]. \tag{3.4}$$

Similarly, we have

$$\mathbb{E}\left[\exp\left(\int_0^t f(\frac{s}{t},X_s^M)ds\right)\right] \leq \prod_{i=0}^{n-1}\sup_{x\in\mathbb{T}_M^d}\mathbb{E}_x\left[\exp\left(\int_0^{\frac{t}{n}} f_i(X_s^M)ds\right)\right]. \tag{3.5}$$

First, we show that

$$\lim_{t\to\infty}\frac{1}{t}\log\inf_{|x|<\delta}\mathbb{E}_x\left[\exp\left(\int_0^t f_i(X_s^M)ds\right); |X_t^M|<\delta\right]\geq \lambda_M(f_i). \tag{3.6}$$

By boundedness of f_i and the Markov property, we have

$$\begin{aligned}&\mathbb{E}_x\left[\exp\left(\int_0^t f_i(X_s^M)ds\right); |X_t^M|<\delta\right]\geq C\mathbb{E}_x\left[\exp\left(\int_1^{t-1} f_i(X_s^M)ds\right); |X_t^M|<\delta\right]\\&= C\int_{\mathbb{T}_M^d}\bar{p}(y-x)\mathbb{E}_y\left[\exp\left(\int_0^{t-2} f_i(X_s^M)ds\right)\mathbb{E}_{X_{t-2}^M}\left[I_{[|X_1^M|<\delta]}\right]\right]dy, \end{aligned}\tag{3.7}$$

where $\bar{p}(y)$ is the density function of X_1^M. Note that $\bar{p}(y)$ is strictly positive and continuous on $\mathbb{T}_M^d$, and Then, there exists $\varepsilon>0$ such that $\inf_{y\in\mathbb{R}^M}\bar{p}(y)\geq\varepsilon$ and consequently $\inf_{x\in\mathbb{R}^M}\mathbb{E}_x\left[I_{[|X_1^M|<\delta]}\right]\geq\varepsilon\delta^d$. Therefore,

$$\mathbb{E}_x\left[\exp\left(\int_0^t f_i(X_s^M)ds\right); |X_t^M|<\delta\right]\geq C\varepsilon^2\delta^d\int_{\mathbb{T}_M^d}\mathbb{E}_y\left[\exp\left(\int_0^{t-2} f_i(X_s^M)ds\right)\right]dy. \tag{3.8}$$

On the other hand, for any $g\in\mathcal{F}_{\alpha,M}$,

$$\begin{aligned}&\int_{\mathbb{T}_M^d}\mathbb{E}_y\left[\exp\left(\int_0^{t-2} f_i(X_s^M)ds\right)\right]dy\\&\geq \|g\|_\infty^{-2}\int_{\mathbb{T}_M^d}g(y)\mathbb{E}_y\left[\exp\left(\int_0^{t-2} f_i(X_s^M)ds\right)g(X_{t-2}^M)\right]dy\\&=\|g\|_\infty^{-2}\langle g, e^{-(t-2)(T_{\alpha,M}-V_{f_i})}g\rangle_{2,\mathbb{T}_M^d},\end{aligned}\tag{3.9}$$

where in the last step $T_{\alpha,M}$ is the self-adjoint operator associated with the Dirichlet form $\mathcal{E}_{\alpha,M}$, V_f is the operator of the multiplication of the function f, and the equality follows from [13, Lemma 5]. By spectral representation theory, there exists a probability measure $\mu_g(d\lambda)$ such that

$$\langle g, f_ig\rangle_{\alpha,M}-\mathcal{E}_{\alpha,M}(g,g)=\langle g,-(T_{\alpha,M}-V_{f_i})g\rangle_{\alpha,M}=\int_{-\infty}^{\infty}\lambda\mu_g(d\lambda), \tag{3.10}$$

and

$$\langle g, e^{-(t-2)(T_{\alpha,M}-V_{f_i})}g\rangle_{\alpha,M}=\int_{-\infty}^{\infty}e^{-(t-2)\lambda}\mu_g(d\lambda)\geq\exp\left((t-2)\int_{-\infty}^{\infty}\lambda\mu_g(d\lambda)\right). \tag{3.11}$$

Combining (3.10) and (3.11), we have

$$\liminf_{t\to\infty}\frac{1}{t}\log\langle g, e^{(t-2)(T_{\alpha,M}-V_{f_i})}g\rangle_{\alpha,M}\geq\langle g, f_ig\rangle_{\alpha,M}-\mathcal{E}_{\alpha,M}(g,g), \tag{3.12}$$

and then, by choosing g arbitrarily, (3.6) follows from (3.8), (3.9) and (3.12).

Now we show that

$$\limsup_{t\to\infty}\frac{1}{t}\log\sup_{x\in\mathbb{T}_M^d}\mathbb{E}_x\left[\exp\left(\int_0^t f_i(X_s^M)ds\right)\right]\le\lambda_M(f_i). \tag{3.13}$$

Actually, by the uniform boundedness of f_i on $\mathbb{T}_M^d$ and the Markov property of X^M,

$$\begin{aligned}\mathbb{E}_x\left[\exp\left(\int_0^t f_i(X_s^M)ds\right)\right]&\le C\mathbb{E}_x\left[\exp\left(\int_1^t f_i(X_s^M)ds\right)\right]\\&=C\int_{R_M}\bar{p}(y-x)\mathbb{E}_y\left[\exp\left(\int_0^{t-1} f_i(X_s^M)ds\right)\right]dy\\&=C\langle\bar{p},e^{-(t-1)\langle T_{\alpha,M}-V_{f_i}\rangle}1\rangle_{2,R_M}.\end{aligned}$$

By spectral representation, for any $g\in\mathcal{F}_{\alpha,M}$,

$$\langle g,e^{-(t-1)(T_{\alpha,M}-V_{f_i})}g\rangle_{\alpha,M}=\int_{-\sigma_0}^{\infty}e^{-(t-1)\lambda}\mu_g(d\lambda)\le e^{(t-1)\sigma_0},$$

where $-\sigma_0=-\lambda_M(f_i)$ is the infimum of the spectrum of the operator $T_{\alpha,M}-V_{f_i}$. Hence

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}_x\left[\exp\left(\int_0^t f_i(X_s^M)ds\right)\right]\le\lambda_M(f_i).$$

Combining (3.4), (3.5), (3.6) and (3.13), we have

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left(\int_0^t f(\frac{s}{t},X_s^M)ds\right)\right]=\sum_{i=0}^{n-1}\lambda_M(f_i). \tag{3.14}$$

Finally, for general continuous function $f(s,x)$ on $[0,1]\times\mathbb{T}_M^d$, let

$$f_n(s,x)=\sum_{i=0}^{n-1}f(s_i,x)I_{[s_i,s_{i+1})}(s)+f(s_{n-1},x)I_{\{1\}}(s).$$

Then, by the uniform continuity of f on $[0,1]\times\mathbb{T}_M^d$, f_n converges to f uniformly. By letting n go to infinity in (3.14), we can obtain (3.3). □

In the meantime, the lower bound in (3.3) also holds for the original stable process X.

Proposition 3.2 *For the stable process X on the whole $\mathbb{R}^d$, if we assume that $f(s,x)$ is continuous in (s,x) on $[0,1]\times\mathbb{R}^d$ and that the family $\{f(\cdot,x),x\in\mathbb{R}^d\}$ of functions is equicontinuous, Then, we can obtain the lower bound*

$$\liminf_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left(\int_0^t f(\frac{s}{t},X_s)ds\right)\right]\ge\int_0^1\lambda(f(s,\cdot))ds, \tag{3.15}$$

where $\lambda(f)=\sup_{g\in\mathcal{F}_\alpha}\left\{\langle g,fg\rangle_{2,\mathbb{R}^d}-\mathcal{E}_\alpha(g,g)\right\}$.

Proof The proof is similar to the lower bound part of the proof for Proposition (3.3). We shall only sketch the idea.

We still start with the functions of the form $f(s,x) = \sum_{i=0}^{n-1} f_i(x) I_{[s_i, s_{i+1})}(s) + f_{n-1}(x) I_{\{1\}}(s)$. Fix a compact set $D \subset \mathbb{R}^d$, Then, there exists a positive ε such that the density function $p(y)$ of X_1 is bigger than ε for all $y \in D$. For any $g \in \mathcal{F}_\alpha$ with support inside D, using a similar argument as (3.8) – (3.12), we can get

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x \left[\exp\left(\int_0^t f_i(X_s) ds \right) ; |X_t| < \delta \right] \geq \langle g, f_i g \rangle_{\alpha, \mathbb{R}^d} - \mathcal{E}_\alpha(g,g).$$

Therefore, for any $g \in \mathcal{F}_\alpha$ with compact support, we have

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E} \left[\exp\left(\int_0^t f_i(X_s) ds \right) \right] \geq \langle g, f_i g \rangle_{\alpha, \mathbb{R}^d} - \mathcal{E}_\alpha(g,g),$$

and hence

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E} \left[\exp\left(\int_0^t f_i(X_s) ds \right) \right] \geq \lambda(f_i).$$

Finally, (3.15) follows from a limiting argument. □

4 A variational inequality

In this section, we will establish a lower bound for $\|u^\rho(t,x)\|_p$ for $p \geq 1, \rho \in [0,1]$, where u^ρ is given by (1.8) when $\rho \in [0,1)$ under the condition (1.5) and $u^1(t,x)$ is the Skorohod solution $\tilde{u}(t,x)$ under the condition (1.3). This will be used to obtain the lower bound in Theorem 1.1.

First let us introduce some notations by recalling the Dalang's approach (see [16]) of defining stochastic integral with respect to the Gaussian noise $\dot{W}$. Let $\mathcal{D}(\mathbb{R}^{d+1})$ be the set of smooth functions on $\mathbb{R}^{d+1}$ with compact support, and $\mathcal{H}$ be the Hilbert space spanned by $\mathcal{D}(\mathbb{R}^{d+1})$ under the inner product

$$\langle \varphi, \psi \rangle_{\mathcal{H}} := \int_{\mathbb{R}^2} \int_{\mathbb{R}^{2d}} |r-s|^{-\beta_0} \gamma(x-y) \varphi(r,x) \psi(s,y) dr ds dx dy, \; \forall\, \varphi, \psi \in \mathcal{D}(\mathbb{R}^{d+1}). \tag{4.1}$$

In the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $W = \{W(h), h \in \mathcal{H}\}$ be an isonormal Gaussian process with covariance function give by $\mathbb{E}[W(h)W(g)] = \langle h, g \rangle_{\mathcal{H}}$. We also write, for $h \in \mathcal{H}$,

$$W(h) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} h(s,x) W(ds, dx).$$

Denote the Fourier transforms of $|s|^{-\beta_0}$ and $\gamma(x)$ by $\mu_0(d\tau)$ and $\mu(d\xi)$, respectively, then

$$\mu_0(d\tau) = C_{\beta_0} |\tau|^{\beta_0 - 1} d\tau; \tag{4.2}$$

$$\mu(d\xi) = \begin{cases} C_{\beta,d} |\xi|^{\beta-d} d\xi, & \text{for } \gamma(x) = |x|^{-\beta}, \\ \prod_{j=1}^d C_{\beta_j} |\xi|^{\beta_j - 1} d\xi, & \text{for } \gamma(x) = \prod_{j=1}^d |x_j|^{-\beta_j}. \end{cases} \tag{4.3}$$

The Parseval's identity provides an alternative representation for the inner product,

$$\mathbb{E}[W(\varphi)W(\psi)] = \langle \varphi, \psi \rangle_{\mathcal{H}} = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \varphi(\tau, \xi) \overline{\widehat{\psi}(\tau, \xi)} \mu_0(d\tau)\mu(d\xi), \text{ for } \varphi, \psi \in \mathcal{S}(\mathbb{R}^{d+1}).$$

With the above notations (1.3) is equivalent to the following general form of the Dalang's condition

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^\alpha} \mu(d\xi) < \infty, \tag{4.4}$$

and (1.5) is equivalent to

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^{\alpha(1-\beta_0)}} \mu(d\xi) < \infty\,. \tag{4.5}$$

Now we recall the approximation procedure used in [20, 21, 28], which we shall use in the proof of the main result in this section. Denote $g_\delta(t) := \frac{1}{\delta} I_{[0,\delta]}(t)$ for $t \geq 0$ and $p_\varepsilon(x) = \frac{1}{\varepsilon^d} p(\frac{x}{\varepsilon})$ for $x \in \mathbb{R}^d$, where $p(x) \in \mathcal{D}(\mathbb{R}^d)$ is a symmetric probability density function and its Fourier transform $\widehat{p}(\xi) \geq 0$ for all $\xi \in \mathbb{R}^d$. For positive numbers ε and δ, define

$$\dot{W}^{\epsilon,\delta}(t,x) := \int_0^t \int_{\mathbb{R}^d} g_\delta(t-s) p_\epsilon(x-y) W(ds, dy) = W(\phi_{t,x}^{\varepsilon,\delta}), \tag{4.6}$$

where

$$\phi_{t,x}^{\varepsilon,\delta}(s,y) := g_\delta(t-s) p_\varepsilon(x-y) \cdot I_{[0,t]}(s).$$

Consider the following approximation of (1.2)

$$\begin{cases} u^{\varepsilon,\delta}(t,x) = -(-\Delta)^{\frac{\alpha}{2}} u^{\varepsilon,\delta}(t,x) + u^{\varepsilon,\delta}(t,x) \dot{W}^{\varepsilon,\delta}(t,x), \\ u^{\varepsilon,\delta}(0,x) = u_0(x). \end{cases} \tag{4.7}$$

Then, Feynman-Kac formula for the Stratonovich solution $u^{\varepsilon,\delta}$ is

$$u^{\varepsilon,\delta}(t,x) = \mathbb{E}_X \left[u_0(X_t^x) \exp\left(\int_0^t \dot{W}^{\varepsilon,\delta}(r, X_{t-r}^x) dr \right) \right],$$

and the Feynman-Kac formula for the Skorohod solution $\tilde{u}^{\varepsilon,\delta}(t,x)$ is

$$\tilde{u}^{\varepsilon,\delta}(t,x) = \mathbb{E}_X \left[u_0(X_t^x) \exp\left(\int_0^t \dot{W}^{\varepsilon,\delta}(r, X_{t-r}^x) dr - \frac{1}{2} \int_{\mathbb{R}^{d+1}} |\mathcal{F}\Phi_{t,x}^{\varepsilon,\delta}(\tau,\xi)|^2 \mu_0(d\tau)\mu(d\xi) \right) \right],$$

where

$$\Phi_{t,x}^{\varepsilon,\delta}(u,y) := \int_0^t g_\delta(t-u-s) p_\varepsilon(X_s^x - y) ds \cdot I_{[0,t]}(u). \tag{4.8}$$

Notet that

$$\int_0^t \dot{W}^{\varepsilon,\delta}(r, X_{t-r}^x) dr = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \Phi_{t,x}^{\varepsilon,\delta}(u,y) W(du, dy),$$

by stochastic Fubini's theorem.

For $\rho \in [0,1]$, define the following random Hamiltonian,

$$H^\rho_{\varepsilon,\delta}(t,x) := \int_0^t \dot{W}^{\varepsilon,\delta}(r, X^x_{t-r})dr - \frac{\rho}{2}\int_{\mathbb{R}^{d+1}} |\mathcal{F}\Phi^{\varepsilon,\delta}_t(\tau,\xi)|^2 \mu_0(d\tau)\mu(d\xi),$$

and denote

$$u^\rho_{\varepsilon,\delta}(t,x) := \mathbb{E}_X\left[\exp\left(H^\rho_{\varepsilon,\delta}(t,x)\right)\right]. \tag{4.9}$$

Then, for all fixed $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, under the condition (1.5), for all $\rho \in [0,1]$, $H^\rho_{\varepsilon,\delta}(t,x)$ converges to $H^\rho(t,x)$ given in (1.14) (see Theorem 4.1 in [28]) and $u^\rho_{\varepsilon,\delta}(t,x)$ converges to $u^\rho(t,x) := \mathbb{E}_X\left[\exp\left(H^\rho(t,x)\right)\right]$ in L^p for all $p \geq 1$ (see Theorem 4.6 in [28]). Under the less restricted condition (1.3), when $\rho = 1$, $u^1_{\varepsilon,\delta}(t,x)$ converges to the Skorohod solution $\tilde{u}(t,x)$ of (1.2) in L^p for all $p \geq 1$ (see Theorem 5.6 in [28]).

The following is the main result in this section.

Proposition 4.1 *We assume one of the following conditions*

(i) The condition (1.5) *is satisfied and* $\rho \in [0,1]$.

(ii) Dalang's condition (1.3) *is satisfied and* $\rho = 1$.

Let $p \geq 1$, *and when* $p = 1$ *we assume* $\rho \in [0,1)$. *Then, for any* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^d$,

$$\begin{aligned}&\left(\mathbb{E}|u^\rho(t,x)|^p\right)^{1/p} \\ &\geq \sup_{g\in\mathcal{S}_H(\mathbb{R}^{d+1})} \mathbb{E}_X\left[\exp\left(\int_0^t (\widetilde{\mathcal{F}}g)(s,X_s)ds - \frac{1}{2(p-\rho)}\int_{\mathbb{R}^{d+1}} |g(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)\right)\right],\end{aligned}$$

where

$$\mathcal{S}_H(\mathbb{R}^{d+1}) = \left\{g \in \mathcal{S}(\mathbb{R}^{d+1});\ g(-\tau,-\xi) = \overline{g(\tau,\xi)}\right\}, \tag{4.10}$$

and

$$(\widetilde{\mathcal{F}}g)(s,x) = \int_{\mathbb{R}^{d+1}} e^{-2\pi i(\tau s + \xi\cdot x)} g(\tau,\xi)\mu_0(d\tau)\mu(d\xi). \tag{4.11}$$

Proof First, we consider the case $p > 1$ and $\rho \in [0,1]$. Let $q := p(p-1)^{-1}$ be the conjugate of p. Let $\varphi(t,x) \in \mathcal{S}(\mathbb{R}^{d+1})$ be a real function, and denote

$$X_\varphi = \exp\left(\int_{\mathbb{R}}\int_{\mathbb{R}^d} \varphi(s,y)W(ds,dy) - \frac{q}{2}\int_{\mathbb{R}^{d+1}} |\widehat{\varphi}(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)\right).$$

Note that $X_\varphi \in L^q(\Omega)$ and $\|X_\varphi\|_q = 1$. Hence, by Hölder's inequality, we see

$$
\begin{aligned}
&\|u^\rho_{\varepsilon,\delta}(t,x)\|_p \geq \mathbb{E}\left[u^\rho_{\varepsilon,\delta}(t,x)X_\varphi\right] \\
&= \mathbb{E}_W\mathbb{E}_X\Bigg[\exp\Bigg(\int_{\mathbb{R}}\int_{\mathbb{R}^d}\left[\Phi^{\varepsilon,\delta}_{t,x}(s,y)+\varphi(s,y)\right]W(ds,dy) \\
&\quad -\frac{\rho}{2}\int_{\mathbb{R}^{d+1}}|\mathcal{F}\Phi^{\varepsilon,\delta}_{t,x}(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)-\frac{q}{2}\int_{\mathbb{R}^{d+1}}|\widehat{\varphi}(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)\Bigg)\Bigg] \\
&= \mathbb{E}_X\Bigg[\exp\Bigg(\frac{1-\rho}{2}\int_{\mathbb{R}^{d+1}}|\mathcal{F}\Phi^{\varepsilon,\delta}_{t,x}(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi) \\
&\quad +\int_{\mathbb{R}^{d+1}}\overline{\mathcal{F}\Phi^{\varepsilon,\delta}_{t,x}(\tau,\xi)}\widehat{\varphi}(\tau,\xi)\mu_0(d\tau)\mu(d\xi)-\frac{q-1}{2}\int_{\mathbb{R}^{d+1}}|\widehat{\varphi}(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)\Bigg)\Bigg].
\end{aligned}
$$

Note that for any $x \geq 1$,

$$
\begin{aligned}
(1-\rho)a^2+2ab-(q-1)b^2 &= (1-\rho)a^2+2(1-x)ab+2xab-(q-1)b^2 \\
&\geq -\frac{(x-1)^2}{1-\rho}b^2+2xab-(q-1)b^2 = 2xab-\left((q-1)+\frac{(x-1)^2}{1-\rho}\right)b^2\,.
\end{aligned}
$$

If we choose the optimal value $c_0 = 1+(1-\rho)(q-1)$ for x, Then, we have

$$
(1-\rho)a^2+2ab-(q-1)b^2 \geq 2a(c_0b)-\frac{1}{p-\rho}(c_0b)^2\,.
$$

This argument also works with the product ab replaced by inner products $\langle\cdot,\cdot\rangle_{\mathcal{H}}$, noting that $\int_{\mathbb{R}^{d+1}}\overline{\mathcal{F}\Phi^{\varepsilon,\delta}_{t,x}(\tau,\xi)}\widehat{\varphi}(\tau,\xi)\mu_0(d\tau)\mu(d\xi)$ is a real (random) number. Therefore,

$$
\begin{aligned}
\|u^\rho_{\varepsilon,\delta}(t,0)\|_p \geq &\mathbb{E}_X\Bigg[\exp\Bigg(\int_{\mathbb{R}^{d+1}}\overline{\mathcal{F}\Phi^{\varepsilon,\delta}_{t,x}(\tau,\xi)}\Big(c_0\widehat{\varphi}(\tau,\xi)\Big)\mu_0(d\tau)\mu(d\xi) \\
&-\frac{1}{2}\frac{1}{p-\rho}\int_{\mathbb{R}^{d+1}}|c_0\widehat{\varphi}(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)\Bigg)\Bigg]. \qquad (4.12)
\end{aligned}
$$

Note that

$$
\mathcal{F}\Phi^{\varepsilon,\delta}_{t,x}(\tau,\xi) = \int_0^t \exp(-2\pi i(\tau(t-s)+\xi\cdot X_s))\mathcal{F}\left(\frac{1}{\delta}I_{[0,(t-s)\wedge\delta]}(\cdot)\right)(\tau)\widehat{p_\varepsilon}(\xi)ds
$$

which converges to $\int_0^t \exp(-2\pi i(\tau(t-s)+\xi\cdot X_s))ds$ as ε and δ go to 0. Letting ε and δ go to 0 in (4.12) yields

$$
\begin{aligned}
\|u^\rho(t,0)\|_p \geq &\mathbb{E}_X\Bigg[\exp\Bigg(\int_0^t\int_{\mathbb{R}^{d+1}}\exp(-2\pi i(\tau(t-s)+\xi\cdot X_s))\Big(c_0\widehat{\varphi}(\tau,\xi)\Big)\mu_0(d\tau)\mu(d\xi)ds \\
&-\frac{1}{2}\frac{1}{p-\rho}\int_{\mathbb{R}^{d+1}}|c_0\widehat{\varphi}(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)\Bigg)\Bigg].
\end{aligned}
$$

The proof is concluded for the case $p > 1$, noting that $\mathcal{F}(\mathcal{S}(\mathbb{R}^{d+1})) = \mathcal{S}(\mathbb{R}^{d+1})$, and $\widehat{\varphi}(-\tau,-\xi) = \overline{\widehat{\varphi}(\tau,\xi)}$ since φ is a real function.

When $p = 1$ and $\rho \in [0,1)$, we have

$$\begin{aligned}\mathbb{E}[u^\rho_{\varepsilon,\delta}(t,x)] =& \mathbb{E}_X\left[\exp\left(\frac{1-\rho}{2}\int_{\mathbb{R}^{d+1}} |\mathcal{F}\Phi^{\varepsilon,\delta}_{t,x}(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)\right)\right]\\ \geq& \mathbb{E}_X\left[\exp\left(\int_{\mathbb{R}^{d+1}} \overline{\mathcal{F}\Phi^{\varepsilon,\delta}_{t,x}(\tau,\xi)}\Big(c_0\widehat{\varphi}(\tau,\xi)\Big)\mu_0(d\tau)\mu(d\xi)\right.\right.\\ &\left.\left. -\frac{1}{2}\frac{1}{1-\rho}\int_{\mathbb{R}^{d+1}} |c_0\widehat{\varphi}(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)\right)\right].\end{aligned}$$

where the last step follows from $(1-\rho)a^2 \geq 2ab - \frac{1}{1-\rho}b^2$. The result can be deduced in a similar way. □

Remark 4.2 The result still holds if the α-stable process X in $u^\rho(t,x)$ is replaced by a general symmetric Lévy process with characteristic function $\mathbb{E}[e^{i\xi\cdot X_t}] = e^{-t\Psi(\xi)}$. In this case, the conditions (1.5) and (1.3) are $\int_{\mathbb{R}^d}\frac{1}{1+[\Psi(\xi)]^{1-\beta_0}}\mu(d\xi) < \infty$ and $\int_{\mathbb{R}^d}\frac{1}{1+\Psi(\xi)}\mu(d\xi) < \infty$, respectively.

5 On the lower bound

In this section, we establish the lower bound in Theorem 1.1 for all $p \geq 1$.

Note that $\mu_0(d(c\tau)) = c^{\beta_0}\mu_0(d\tau)$ and $\mu(d(c\xi)) = c^\beta\mu(d\xi)$ for any $c > 0$, by (4.2) and (4.3). Consequently, for $h \in \mathcal{S}_H(\mathbb{R}^{d+1})$, where $\mathcal{S}_H(\mathbb{R}^{d+1})$ is given in (4.10), we have

$$(\widetilde{\mathcal{F}}h(a\cdot, b*))(s,x) = a^{-\beta_0}b^{-\beta}(\widetilde{\mathcal{F}}h(\cdot,*))(a^{-1}s, b^{-1}x),\ a>0, b>0, \tag{5.1}$$

where $\widetilde{\mathcal{F}}g$ is defined by (4.11).

Now let

$$t_p = t^\chi(p-\rho)^{\frac{\alpha}{\alpha-\beta}} \text{ for } p\geq 1, \text{ with } \chi = \frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}, \tag{5.2}$$

and for any $h \in \mathcal{S}_H(\mathbb{R}^{d+1})$ denote

$$h_t(\tau,\xi) = t(p-\rho)h\Big(t\tau, (p-\rho)^{-\frac{1}{\alpha-\beta}}t^{-\frac{\chi-1}{\alpha}}\xi\Big).$$

Then, by (5.1), change of variables and the self-similarity of the α-stable process, we have

$$\int_0^{t_p}(\widetilde{\mathcal{F}}h)(\frac{s}{t_p}, X_s)ds \overset{d}{=} \int_0^t(\widetilde{\mathcal{F}}h_t)(s,X_s)ds,$$

and

$$\int_{\mathbb{R}^{d+1}}|h_t(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi) = (p-\rho)t_p\int_{\mathbb{R}^{d+1}}|h(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi).$$

Clearly, $h_t \in \mathcal{S}_H(\mathbb{R}^{d+1})$. Proposition 4.1 and the above two identities imply

$$\begin{aligned}\|u^\rho(t,x)\|_p &\ge \mathbb{E}_X\left[\exp\left(\int_0^t (\widetilde{\mathcal{F}}h_t)(s,X_s)ds - \frac{1}{2(p-\rho)}\int_{\mathbb{R}^{d+1}} |h_t(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)\right)\right]\\ &= \mathbb{E}_X\left[\exp\left(\int_0^{t_p} (\widetilde{\mathcal{F}}h)(\frac{s}{t_p},X_s)ds - \frac{t_p}{2}\int_{\mathbb{R}^{d+1}} |h(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)\right)\right].\end{aligned}$$

By Proposition 3.2,

$$\begin{aligned}&\liminf_{t\to\infty}\frac{1}{t_p}\log\mathbb{E}_X\left[\exp\left(\int_0^{t_p}(\widetilde{\mathcal{F}}h)(\frac{s}{t_p},X_s)ds\right)\right] \ge \int_0^1 \lambda((\widetilde{\mathcal{F}}h)(s,\cdot))ds\\ &= \int_0^1 \sup_{g\in\mathcal{F}_\alpha}\left\{\int_{\mathbb{R}^d}(\widetilde{\mathcal{F}}h)(s,x)g^2(x)dx - \int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(\xi)|^2d\xi\right\}ds\\ &= \sup_{g\in\mathcal{A}_{\alpha,d}}\left\{\int_0^1\int_{\mathbb{R}^d}(\widetilde{\mathcal{F}}h)(s,x)g^2(s,x)dxds - \int_0^1\int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(s,\xi)|^2d\xi ds\right\},\end{aligned}$$

where $\mathcal{A}_{\alpha,d}$ is given by (1.11). Therefore,

$$\begin{aligned}&\liminf_{t\to\infty} t^{-\chi}\log\|u^\rho(t,x)\|_p\\ &\ge (p-\rho)^{\frac{\alpha}{\alpha-\beta}}\sup_{g\in\mathcal{A}_{\alpha,d}}\left\{\Gamma(h,g) - \int_0^1\int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(s,\xi)|^2d\xi ds\right\}\\ &\ge (p-\rho)^{\frac{\alpha}{\alpha-\beta}}\sup_{g\in\mathcal{A}_{\alpha,d}}\left\{\sup_{h\in\mathcal{S}_H(\mathbb{R}^{d+1})}\Gamma(h,g) - \int_0^1\int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(s,\xi)|^2d\xi ds\right\},\end{aligned}\tag{5.3}$$

where

$$\begin{aligned}\Gamma(h,g) &= \int_0^1\int_{\mathbb{R}^d}(\widetilde{\mathcal{F}}h)(s,x)g^2(s,x)dxds - \frac{1}{2}\int_{\mathbb{R}^{d+1}}|h(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)\\ &= \int_{\mathbb{R}^{d+1}} h(\tau,\xi)(\mathcal{F}g^2)(\tau,\xi)\mu_0(d\tau)\mu(d\xi) - \frac{1}{2}\int_{\mathbb{R}^{d+1}}|h(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi).\end{aligned}$$

Since $\mathcal{S}_H(\mathbb{R}^{d+1})$ is dense in $L^2(\mathbb{R}^{d+1},\mu_0\otimes\mu)$ (see, e.g., [24]), and $\Gamma(\cdot,g)$ is continuous with respect to the $L^2(\mathbb{R}^{d+1},\mu_0\otimes\mu)$-norm, we have

$$\begin{aligned}&\sup_{h\in\mathcal{S}_H(\mathbb{R}^{d+1})}\Gamma(h,g) \ge \Gamma\Big(\mathcal{F}(g^2)(-\tau,-\xi),g\Big) = \frac{1}{2}\int_{\mathbb{R}^{d+1}}|(\mathcal{F}g^2)(\tau,\xi)|^2\mu_0(d\tau)\mu(d\xi)\\ &= \frac{1}{2}\int_0^1\int_0^1\int_{\mathbb{R}^{2d}}\frac{\gamma(x-y)}{|s-r|^{\beta_0}}g^2(s,x)g^2(r,y)dxdydrds.\end{aligned}$$

Summarizing the computations starting from (5.3), we have

$$\begin{aligned}&\liminf_{t\to\infty} t^{-\chi}\log\|u^\rho(t,x)\|_p\\ &\ge (p-\rho)^{\frac{\alpha}{\alpha-\beta}}\sup_{g\in\mathcal{A}_{\alpha,d}}\left\{\frac{1}{2}\int_0^1\int_0^1\int_{\mathbb{R}^{2d}}\frac{\gamma(x-y)}{|s-r|^{\beta_0}}g^2(s,x)g^2(r,y)dxdydrds\right.\\ &\qquad\left. - \int_0^1\int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(s,\xi)|^2d\xi ds\right\}\\ &= (p-\rho)^{\frac{\alpha}{\alpha-\beta}}\mathbf{M}(\alpha,\beta_0,d,\gamma),\end{aligned}$$

and the lower bound is established.

6 On the upper bound

In this section, we provide a proof for the upper bound in Theorem 1.1. In Subsections 6.1 and 6.2, we shall obtain the upper bound for any positive integer $n \geq 1$, i.e.,

$$\begin{aligned}\limsup_{t\to\infty} t^{-\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}} \log \mathbb{E} \exp \Bigg(\frac{1}{2} \sum_{j,k=1}^{n} \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r^j - X_s^k) drds \\ - \frac{\rho}{2} \sum_{j=1}^{n} \int_0^t \int_0^t |r-s|^{-\beta_0} \gamma(X_r^j - X_s^j) drds \Bigg) \\ \leq n(n-\rho)^{\frac{\alpha}{\alpha-\beta}} \mathbf{M}(\alpha, \beta_0, d, \gamma). \qquad (6.1)\end{aligned}$$

The proof for real number $p \geq 2$ is inspired by the idea in [26]. We shall compare $\|u^\rho(t,x)\|_p$ with $\|u^\rho(t,x)\|_2$ by using the Mehler's formula and hypercontractivity of the Ornstein-Uhlenbeck semigroup operators. First, we address the case when $\rho \in [0,1]$, under the condition (1.5).

Let $W' = \{W'(h), h \in \mathcal{H}\}$ be an independent copy of $W = \{W(h), h \in \mathcal{H}\}$, and let $W : \Omega \to \mathbb{R}^{\mathcal{H}}$ and $W' : \Omega \to \mathbb{R}^{\mathcal{H}}$ be the canonical mappings associated with W and W', respectively. For any $F \in L^2(\Omega)$, there is a measurable mapping ψ_F from $\mathbb{R}^{\mathcal{H}}$ to $\mathbb{R}$ such that $F = \psi_F \circ W$. Denote by $\{T_\tau, \tau \geq 0\}$ the Ornstein-Uhlenbeck semigroup associated with W. By Mehler's formula (see, e.g., [27]),

$$T_\tau(F) = \mathbb{E}' \left[\psi_F(e^{-\tau} W + \sqrt{1-e^{-2\tau}} W') \right],$$

where $\mathbb{E}'$ denotes the expectation with respect to W'. For $p \in (1,\infty)$ and $\tau \geq 0$, define $q = 1 + e^{2\tau}(p-1)$, Then, the Ornstein-Uhlenbeck semigroup operators possess the following hypercontractivity property (see, e.g., [27]),

$$\|T_\tau F\|_q \leq \|F\|_p. \qquad (6.2)$$

Now fix $q \geq 2$. Let $e^{2\tau} = q-1$, Then, $\|T_\tau F\|_q \leq \|F\|_2$. Let $\tilde{\rho} = \frac{\rho+q-2}{q-1} \in [0,1)$. By (1.8) and

Mehler's formula,

$$T_\tau u^{\tilde\rho}(t,x) = \mathbb{E}'\mathbb{E}_X\Bigg[\exp\Bigg(e^{-\tau}\int_0^t\int_{\mathbb{R}^d}\delta_0(X^x_{t-r}-y)W(dr,dy)$$

$$+\sqrt{1-e^{-2\tau}}\int_0^t\int_{\mathbb{R}^d}\delta_0(X^x_{t-r}-y)W'(dr,dy)-\frac{\tilde\rho}{2}\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r-X_s)drds\Bigg)\Bigg]$$

$$=\mathbb{E}_X\Bigg[\exp\Bigg(e^{-\tau}\int_0^t\int_{\mathbb{R}^d}\delta_0(X^x_{t-r}-y)W(dr,dy)$$

$$+\frac{1}{2}(1-\tilde\rho-e^{-2\tau})\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X_r-X_s)drds\Bigg)\Bigg]$$

$$=\mathbb{E}_X\Bigg[\exp\Bigg(\int_0^t\int_{\mathbb{R}^d}\delta_0(X^x_{t-r}-y)W_\tau(dr,dy)$$

$$-\frac{\rho}{2}\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma_\tau(X_r-X_s)drds\Bigg)\Bigg],$$

where in the last step $W_\tau = e^{-\tau}W$ and $\gamma_\tau(x)=e^{-2\tau}\gamma(x)$. By (6.2) with $p=2$, (6.1) with $n=2$, and the scaling property for $\mathbf{M}(\alpha,\beta_0,d,\gamma)$ defined by (1.13), we have

$$\|T_\tau u^{\tilde\rho}(t,x)\|_q \le (2-\tilde\rho)^{\frac{\alpha}{\alpha-\beta}}\mathbf{M}(\alpha,\beta_0,d,\gamma)$$
$$=(2-\tilde\rho)^{\frac{\alpha}{\alpha-\beta}}e^{\frac{2\tau\alpha}{\alpha-\beta}}\mathbf{M}(\alpha,\beta_0,d,\gamma_\tau)=(q-\rho)^{\frac{\alpha}{\alpha-\beta}}\mathbf{M}(\alpha,\beta_0,d,\gamma_\tau).$$

Observing that

$$T_\tau u^{\tilde\rho}(t,x)=\mathbb{E}_X\Bigg[\exp\Bigg(\int_0^t\int_{\mathbb{R}^d}\delta_0(X^x_{t-r}-y)W_\tau(dr,dy)-\frac{\rho}{2}\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma_\tau(X_r-X_s)drds\Bigg)\Bigg],$$

the upper bound in Theorem 1.1 for any real number $q\ge 2$ follows from the scaling property (1.13).

Finally, for the case $\rho=1$ under the condition (1.3), in which $u^\rho(t,x)$ is the Skorohod solution to (1.2), we can apply the approach in [26] and obtain the upper bound for all real numbers $p\ge 2$.

6.1 Upper bound under the condition (1.5).

In this subsection, we deal with the case $\rho\in[0,1]$ under the condition (1.5). The proof will be split into four steps.

Step 1. In this step, we will reduce the study of n-th moment to the study of first moment. Recall that (2.5) and (2.6) imply

$$\int_0^t\int_0^t|r-s|^{-\beta_0}\gamma(X^j_r-X^k_s)drds$$
$$=C_0C(\gamma)\int_{\mathbb{R}^{d+1}}\left(\int_0^t|s-u|^{-\frac{\beta_0+1}{2}}K(x-X^k_s)ds\int_0^t|r-u|^{-\frac{\beta_0+1}{2}}K(x-X^j_r)dr\right)dudx\,. \quad (6.3)$$

Therefore, by the inequality $(\sum_{j=1}^n a_j)^2 \le n\sum_{j=1}^n a_j^2$, we have

$$\begin{aligned}&\sum_{j,k=1}^n \int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^j - X_s^k)drds - \rho\sum_{j=1}^n\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^j - X_s^j)drds\\ &\le (n-\rho)\sum_{j=1}^n\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^j - X_s^j)drds.\end{aligned}$$

Consequently, to obtain the upper bound in Theorem 1.1, it suffices to show

$$\begin{aligned}&\limsup_{t\to\infty} t^{-\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}}\log\mathbb{E}\left[\exp\left(\frac{n-\rho}{2}\sum_{j=1}^n\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^j - X_s^j)drds\right)\right]\\ &\le n(n-\rho)^{\frac{\alpha}{\alpha-\beta}}\mathbf{M}(\alpha,\beta_0,d,\gamma). \end{aligned} \tag{6.4}$$

By the scaling property (2.1), we see

$$\int_0^t\int_0^t |r-s|^{-\beta_0}\gamma(X_r^j - X_s^j)drds \overset{d}{=} \frac{1}{t_n}\frac{1}{n-\rho}\int_0^{t_n}\int_0^{t_n}\frac{\gamma(X_r^j - X_s^j)}{|t_n^{-1}(r-s)|^{\beta_0}}drds,$$

where $t_n = t^{\frac{2\alpha-\beta-\alpha\beta_0}{\alpha-\beta}}(n-\rho)^{\frac{\alpha}{\alpha-\beta}}$ is given in (5.2). Therefore, (6.4) is equivalent to

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left(\frac{1}{2t}\int_0^t\int_0^t\frac{\gamma(X_r - X_s)}{|t^{-1}(r-s)|^{\beta_0}}drds\right)\right] \le \mathbf{M}(\alpha,\beta_0,d,\gamma). \tag{6.5}$$

Now, to obtain the upper bound, it suffices to prove (6.5). To this goal, we shall use the representations (2.5) and (2.6) for the covariance functions. But in these two representations, the integrals are over infinite domains. We shall approximate them by bounded, continuous, and locally supported functions, and this will enable us to apply Hahn-Banach theorem in Step 4.

Step 2. In this step, we will replace the temporal covariance function by a smooth function with compact support. Let the function $\varrho : \mathbb{R}^+ \to [0,1]$ be a smooth function such that $\varrho(u) = 1, u\in[0,1]$, $\varrho(u) = 0$ for $u \ge 2$, and $-1 \le \varrho'(u) \le 0$. Define the following truncated functions

$$k_A(u) = |u|^{-\frac{1+\beta_0}{2}}\varrho(A^{-1}|u|),\ k_{A,a}(u) = |u|^{-\frac{1+\beta_0}{2}}\varrho(A^{-1}|u|)(1-\varrho(a^{-1}|u|)), \tag{6.6}$$

with $A > 0$ being a large number and $a > 0$ being a number close to zero.

Then, by Hölder's inequality, we have for any $\varepsilon > 0$

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{1}{2t}\int_0^t\int_0^t \frac{\gamma(X_r - X_s)}{|t^{-1}(r-s)|^{\beta_0}}drds\right)\right]\\
=&\mathbb{E}\left[\exp\left(C_0C(\gamma)\frac{1}{2t}\int_{\mathbb{R}^{d+1}}\left(\int_0^t |t^{-1}(s-u)|^{-\frac{\beta_0+1}{2}}K(x-X_s)ds\right)^2 dudx\right)\right]\\
\le&\left(\mathbb{E}\left[\exp\left((1+\varepsilon)C_0C(\gamma)\frac{p}{2t}\int_{\mathbb{R}^{d+1}}\left(\int_0^t k_{A,a}(t^{-1}(s-u))K(x-X_s)ds\right)^2 dudx\right)\right]\right)^{1/p}\\
&\times\left(\mathbb{E}\left[\exp\left((1+\frac{1}{\varepsilon})C_0C(\gamma)\frac{q}{2t}\int_{\mathbb{R}^{d+1}}\left(\int_0^t \tilde{k}_{A,a}(t^{-1}(s-u))K(x-X_s)ds\right)^2 dudx\right)\right]\right)^{1/q},
\end{aligned} \tag{6.7}$$

where

$$\tilde{k}_{A,a}(u) = |u|^{-\frac{1+\beta_0}{2}} - k_{A,a}(u).$$

Note that

$$\begin{aligned}
\tilde{k}_{A,a}(u) &= (|u|^{-\frac{1+\beta_0}{2}} - k_A(u)) + (k_A(u) - k_{A,a}(u))\\
&\le |u|^{-\frac{1+\beta_0}{2}} I_{[|u|\ge A]} + |u|^{-\frac{1+\beta_0}{2}} I_{[|u|\le 2a]}\\
&\le A^{-\frac{\beta_0-\beta_0'}{2}}|u|^{-\frac{\beta_0'+1}{2}} + (2a)^{\frac{\tilde{\beta}_0-\beta_0}{2}}|u|^{-\frac{\tilde{\beta}_0+1}{2}},
\end{aligned} \tag{6.8}$$

for $0 < \beta_0' < \beta_0 < \tilde{\beta}_0 < 1$. We may choose β_0' and $\tilde{\beta}_0$ such that $(\alpha, \beta_0', \beta)$ and $(\alpha, \tilde{\beta}_0, \beta)$ satisfy the condition (1.5) if $\rho \in [0, 1)$ or the condition (1.3) if $\rho = 1$.

Combining (2.5) and (6.8), for the second term in (6.7), we have

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left((1+\frac{1}{\varepsilon})C_0C(\gamma)\frac{q}{2t}\int_{\mathbb{R}^{d+1}}\left(\int_0^t \tilde{k}_{A,a}(t^{-1}(s-u))K(x-X_s)ds\right)^2 dudx\right)\right]\\
\le&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\Bigg[\exp\Bigg(C(\varepsilon,q)\Bigg[A^{-(\beta_0-\beta_0')}\frac{1}{2t}\int_0^t\int_0^t |r-s|^{-\beta_0'}\gamma(X_r-X_s)drds\\
&\qquad+(2a)^{\tilde{\beta}_0-\beta_0}\frac{1}{2t}\int_0^t\int_0^t |r-s|^{-\tilde{\beta}_0}\gamma(X_r-X_s)drds\Bigg]\Bigg)\Bigg]\\
\le& C\Big(\alpha,\beta,\varepsilon,q,\gamma(\cdot)\Big)\left(A^{-\frac{\alpha(\beta_0-\beta_0')}{\alpha-\beta}} + (2a)^{\frac{\alpha(\tilde{\beta}_0-\beta_0)}{\alpha-\beta}}\right)
\end{aligned} \tag{6.9}$$

where the last step follows from Hölder's inequality and (2.24). Therefore, for fixed (ε, q), this term can be as small as we wish if we choose A sufficiently large and a sufficiently small. On the other hand, we can choose ε arbitrarily close to 0 and p arbitrarily close to 1. Consequently, to prove (6.5), it suffices to prove

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\exp\left(C_0C(\gamma)\frac{1}{2t}\int_{\mathbb{R}^{d+1}}\left(\int_0^t k_{A,a}(t^{-1}(s-u))K(x-X_s)ds\right)^2 dudx\right)\right]\\
&\le \mathbf{M}(\alpha,\beta_0,d,\gamma).
\end{aligned} \tag{6.10}$$

Step 3. In this step, we will replace the spatial covariance function by a smooth function with compact support. Similarly to the truncation for the temporal covariance function, for $0 < b < B < \infty$, we let

$$K_{B,b}(x) = K(x)\varrho(B^{-1}|x|)(1 - \varrho(b^{-1}|x|)),$$

where $K(x)$ is given in (2.7). Then, $0 \le K_{B,b}(x) \le K(x)$ and $K_{B,b}(x) \to K(x)$ when $B \to \infty$ and $b \to 0$. Now the left-hand side of (6.10) can be estimated in the similar way as in (6.7), i.e.,

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(C_0 C(\gamma)\frac{1}{2t}\int_{\mathbb{R}^{d+1}}\left(\int_0^t k_{A,a}(t^{-1}(s-u))K(x-X_s)ds\right)^2 dudx\right)\right] \\
\le&\left(\mathbb{E}\left[\exp\left((1+\varepsilon)C_0 C(\gamma)\frac{p}{2t}\int_{\mathbb{R}^{d+1}}\left(\int_0^t k_{A,a}(t^{-1}(s-u))K_{B,b}(x-X_s)ds\right)^2 dudx\right)\right]\right)^{1/p} \\
&\times\left(\mathbb{E}\left[\exp\left((1+\frac{1}{\varepsilon})C_0 C(\gamma)\frac{q}{2t}\int_{\mathbb{R}^{d+1}}\left(\int_0^t k_{A,a}(t^{-1}(s-u))\tilde{K}_{B,b}(x-X_s)ds\right)^2 dudx\right)\right]\right)^{1/q},
\end{aligned}$$

where $\tilde{K}_{B,b}(x) = K(x) - K_{B,b}(x)$. Noting that $k_{A,a}(u)$ is supported on $[-2A, 2A]$ and is uniformly bounded (say, by L), we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left((1+\frac{1}{\varepsilon})C_0 C(\gamma)\frac{q}{2t}\int_{\mathbb{R}^{d+1}}\left(\int_0^t k_{A,a}(t^{-1}(s-u))\tilde{K}_{B,b}(x-X_s)ds\right)^2 dudx\right)\right] \\
\le&\mathbb{E}\left[\exp\left((1+\frac{1}{\varepsilon})C_0 C(\gamma)L^2(4A+2)\frac{q}{2t}\int_{\mathbb{R}^d}\left(\int_0^t \tilde{K}_{B,b}(x-X_s)ds\right)^2 dx\right)\right].
\end{aligned}$$

Using that $\frac{(a+b)^2}{t+s} \le \frac{a^2}{t} + \frac{b^2}{s}$, we have

$$\begin{aligned}
&\frac{1}{t+s}\int_{\mathbb{R}^d}\left(\int_0^{t+s} \tilde{K}_{B,b}(x-X_s)ds\right)^2 dx \\
\le&\frac{1}{t}\int_{\mathbb{R}^d}\left(\int_0^t \tilde{K}_{B,b}(x-X_s)ds\right)^2 dx + \frac{1}{s}\int_{\mathbb{R}^d}\left(\int_t^{t+s} \tilde{K}_{B,b}(x-X_s)ds\right)^2 dx \\
=&\frac{1}{t}\int_{\mathbb{R}^d}\left(\int_0^t \tilde{K}_{B,b}(x-X_s)ds\right)^2 dx + \frac{1}{s}\int_{\mathbb{R}^d}\left(\int_0^s \tilde{K}_{B,b}(x-(X_{t+s}-X_t))ds\right)^2 dx,
\end{aligned}$$

where the last equality follows from a change of variable for s and the fact that the Lebesgue measure on $\mathbb{R}^d$ is invariant under the translation $x \to x + X_t$. Hence, by the independent and stationary properties of the increments of Lévy processes, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{C}{t+s}\int_{\mathbb{R}^d}\left(\int_0^{t+s} \tilde{K}_{B,b}(x-X_s)ds\right)^2 dx\right)\right] \\
\le&\mathbb{E}\left[\exp\left(\frac{C}{t}\int_{\mathbb{R}^d}\left(\int_0^t \tilde{K}_{B,b}(x-X_s)ds\right)^2 dx\right)\right]\mathbb{E}\left[\exp\left(\frac{C}{s}\int_{\mathbb{R}^d}\left(\int_0^s \tilde{K}_{B,b}(x-X_s)ds\right)^2 dx\right)\right].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left(\frac{C}{t+s}\int_{\mathbb{R}^d}\left(\int_0^{t+s}\tilde{K}_{B,b}(x-X_s)ds\right)^2dx\right)\right]\\
\le&\limsup_{t\to\infty}\frac{1}{t}\log\left(\mathbb{E}\left[\exp\left(C\int_{\mathbb{R}^d}\left(\int_0^1\tilde{K}_{B,b}(x-X_s)ds\right)^2dx\right)\right]\right)^t\\
=&\log\mathbb{E}\left[\exp\left(C\int_{\mathbb{R}^d}\left(\int_0^1\tilde{K}_{B,b}(x-X_s)ds\right)^2dx\right)\right]. \qquad (6.11)
\end{aligned}$$

By Theorem 2.3 we have by Dalang's condition (1.3)

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\theta C(\gamma)\int_{\mathbb{R}^d}\left(\int_0^1 K(x-X_s)ds\right)^2dx\right)\right]\\
&=\mathbb{E}\left[\exp\left(\theta\int_0^1\int_0^1\gamma(X_r-X_s)drds\right)\right]<\infty
\end{aligned}$$

for any $\theta>0$. Now letting $B\to\infty$ and $b\to 0$, by the dominated convergence theorem we see that the term on the right-hand side of (6.11) goes to 0.

Now combining all the inequalities after (6.10), noting that we can choose ε arbitrarily close to 0, and p arbitrarily close to 1, we have that (6.10) can be reduced to

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\exp\left(C_0C(\gamma)\frac{1}{2t}\int_{\mathbb{R}^{d+1}}\left(\int_0^t k_{A,a}(t^{-1}(s-u))K_{B,b}(x-X_s)ds\right)^2dudx\right)\right]\\
&\le\mathbf{M}(\alpha,\beta_0,d,\gamma).
\end{aligned}$$

Step 4. Summarizing the arguments in Step 2 and Step 3, we see that to obtain the upper bound in Theorem 1.1, it suffices to show

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\exp\left(\frac{\theta}{2t}C_0C(\gamma)\int_{\mathbb{R}^{d+1}}\left[\int_0^t k_{A,a}(t^{-1}(s-u))K_{B,b}(x-X_s)ds\right]^2dudx\right)\right]\\
&\le\theta^{\frac{\alpha}{\alpha-\beta}}\mathbf{M}(\alpha,\beta_0,d,\gamma). \qquad (6.12)
\end{aligned}$$

In this final step, we will prove the above inequality. Fix positive constants A,a,B,b and choose arbitrarily $M>2\max\{A,B\}$.

$$\begin{aligned}
&\int_{\mathbb{R}^{d+1}}\left[\int_0^t k_{A,a}(u-t^{-1}s)K_{B,b}(x-X_s)ds\right]^2dudx\\
&=\sum_{k\in\mathbb{Z}}\sum_{z\in\mathbb{Z}^d}\int_{[0,M]^{d+1}}\left[\int_0^t k_{A,a}(Mk+u-t^{-1}s)K_{B,b}(Mz+x-X_s)ds\right]^2dudx\\
&\le\int_{[0,M]^{d+1}}\left[\sum_{j\in\mathbb{Z}}\sum_{z\in\mathbb{Z}^d}\int_0^t k_{A,a}(Mj+u-t^{-1}s)K_{B,b}(Mz+x-X_s)ds\right]^2dudx\\
&=\int_{[0,M]^{d+1}}\left[\int_0^t\widetilde{k}_M(u-t^{-1}s)\widetilde{K}_M(x-X_s)ds\right]^2dudx\,, \qquad (6.13)
\end{aligned}$$

where

$$\widetilde{k}_M(u) = \sum_{j\in\mathbb{Z}} k_{A,a}(Mj+u) \quad \text{and} \quad \widetilde{K}_M(x) = \sum_{z\in\mathbb{Z}^d} K_{B,b}(Mz+x) \tag{6.14}$$

are M-periodic functions. Note that the summations in (6.14) are well-defined, since the supports of $k_{A,a}(\cdot)$ and $K_{B,b}(\cdot)$ are bounded domains. The process

$$\phi_t(u,x) := \frac{1}{t}\int_0^t \widetilde{k}_M(u-t^{-1}s)\widetilde{K}_M(x-X_s)ds\,, \qquad (u,x)\in[0,M]^{d+1}, \tag{6.15}$$

can be considered as a process taking values in the Hilbert space $L^2([0,M]^{d+1})$ with the norm denoted by $\|\cdot\|$. Since $\widetilde{k}_M$ and $\widetilde{K}_M$ are bounded, smooth functions with bounded derivatives, there is a constant $C>0$, such that

$$\|\phi_t(\cdot,\cdot)\| \le C \quad \text{and} \quad \|\phi_t(\cdot+u_1,\ \cdot+x_1) - \phi_t(\cdot+u_2,\ \cdot+x_2)\| \le C|(u_1,x_1)-(u_2,x_2)|$$

for all t and $(u_1,x_1),(u_2,x_2)\in[0,M]^{d+1}$. Let $\mathbb{K}$ be the closure of the following set in $L^2([0,M]^{d+1})$:

$$\Big\{f\in L^2([0,M]^{d+1}):\ \|f\|\le C \text{ and } \|f(\cdot+u_1,\ \cdot+x_1)-f(\cdot+u_2,\ \cdot+x_2)\| \\ \le C|(u_1,x_1)-(u_2,x_2)| \text{ for } (u_1,x_1),(u_2,x_2)\in[0,M]^{d+1}\Big\}.$$

Then, ϕ_t defined in (6.15) belongs to $\mathbb{K}$, and it follows from [19, Theorem IV8.21] that $\mathbb{K}$ is compact in $L^2([0,M]^{d+1})$.

Let $\delta>0$ be fixed. For any $g\in\mathbb{K}$, noting that the set of bounded and continuous functions are dense in $L^2([0,M]^{d+1})$, the Hahn-Banach theorem ([30]) implies that there is a bounded and continuous function $f\in L^2([0,M]^{d+1})$ such that $\|g\|^2 < -\|f\|^2+2\langle f,g\rangle+\delta$. By the finite cover theorem for compact sets, one can find finitely many bounded and continuous functions $f_1,\cdots,f_m$ such that $\|g\|^2<\delta+\max_{1\le i\le m}\{-\|f_j\|^2+2\langle f_i,g\rangle\}$ for all $g\in\mathbb{K}$. In particular, we have, noting that $\phi_t\in\mathbb{K}$,

$$\mathbb{E}\left[e^{\frac{1}{2}\theta t\|\phi_t\|^2}\right] \le e^{\frac{1}{2}\delta\theta t}\sum_{i=1}^m e^{-\frac{1}{2}\theta t\|f_i\|^2}\mathbb{E}\left[e^{\theta t\langle f_i,\phi_t\rangle}\right]\,.$$

Therefore,

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[e^{\frac{1}{2}\theta t\|\phi_t\|^2}\right] \le \frac{1}{2}\delta + \max_{1\le i\le m}\left\{-\frac{1}{2}\theta\|f_i\|^2 + \limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[e^{\theta t\langle f_i,\phi_t\rangle}\right]\right\}. \tag{6.16}$$

Notice that, for $i=1,\dots,m$,

$$t\langle f_i,\phi_t\rangle = \int_0^t\left[\int_{[0,M]^{d+1}} f_i(u,x)\widetilde{k}_M(u-t^{-1}s)\widetilde{K}_M(x-X_s)dudx\right]ds = \int_0^t \bar{f}_i\Big(\frac{s}{t},X_s\Big)ds\,,$$

where

$$\bar{f}_i(s,x) = \int_{[0,M]^{d+1}} f_i(u,y)\widetilde{k}_M(u-s)\widetilde{K}_M(y-x)dudy \quad (s,x)\in[0,1]\times\mathbb{R}^d.$$

Since $\widetilde{K}_M$ is a periodic function and $\widetilde{K}_M(x-X_s)=\widetilde{K}_M(x-X_s^M)$, we have that

$$t\langle f_i,\phi_t\rangle=\int_0^t \bar{f}_i\Big(\frac{s}{t},X_s^M\Big)ds\,.$$

It is easy to check that $\bar{f}_i$ satisfies the condition in Proposition 3.1. Hence,

$$\lim_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[e^{\theta t\langle f_i,\phi_t\rangle}\right]=\sup_{g\in\mathcal{A}^M_{\alpha,d}}\left\{\theta\int_0^1\int_{\mathbb{T}^d_M}\bar{f}_i(s,x)g^2(s,x)dxds-\int_0^1\mathcal{E}_{\alpha,M}(g(s,\cdot),g(s,\cdot))ds\right\},$$

where

$$\mathcal{A}^M_{\alpha,d}=\left\{g(s,\cdot)\in L^2(\mathbb{T}^d_M):\|g(s,\cdot)\|_{\mathbb{T}^d_M}=1,\forall s\in[0,1]\text{ and }\int_0^1\mathcal{E}_{\alpha,M}(g(s,\cdot),g(s,\cdot))ds<\infty\right\}.$$

Notice that

$$\begin{aligned}&\int_0^1\int_{\mathbb{R}^d}\bar{f}_i(s,x)g^2(s,x)dxds\\&=\int_{[0,M]^{d+1}}f_i(u,y)\Big[\int_0^1\int_{\mathbb{T}^d_M}\widetilde{k}_M(u-s)\widetilde{K}_M(y-x)g^2(s,x)dxds\Big]dudy\\&\le\frac{1}{2}\|f_i\|^2+\frac{1}{2}\int_{[0,M]^d}\int_{\mathbb{R}}\left[\int_0^1\int_{\mathbb{T}^d_M}|u-s|^{-\frac{1+\beta_0}{2}}\widetilde{K}_M(y-x)g^2(s,x)dxds\right]^2dudy.\end{aligned}\tag{6.17}$$

Since δ in (6.16) can be arbitrarily small and M in (6.17) can be arbitrarily large, the desired inequality (6.12) follows from inequalities (6.13) – (6.17) and Lemma 7.3.

6.2 When $\rho=1$ under the condition (1.3)

In this subsection, we consider the Skorohod case, i.e., $\rho=1$, under the condition (1.3), by applying the methodology used in Section 6.1. However, under condition (1.3), there will be a technical issue in step 1, since the left-hand side of (6.5) is infinity if condition (1.5) is violated. To deal with this issue, we will first, do step 2 for n-th moments which reduces $|s|^{-\beta_0}$ to a smooth function with compact support, and then, we do step 1 to reduce the n-th moment to first moment.

More precisely, as in Step 1 in Section 6.1, when $\rho=1$, (6.1) is equivalent to

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left(\frac{1}{(n-1)t}\sum_{1\le j<k\le n}\int_0^t\int_0^t\frac{\gamma(X_r^j-X_s^k)}{|t^{-1}(r-s)|^{\beta_0}}drds\right)\right]\\&\le\mathbf{M}(\alpha,\beta_0,d,\gamma)\end{aligned}\tag{6.18}$$

Recall that $k_{A,a}(u)$ is defined in (6.6). Let

$$\psi_{A,a}(u)=C_0\int_{\mathbb{R}}k_{A,a}(u-v)k_{A,a}(v)dv$$

and

$$\tilde{\psi}_{A,a}(u) = |u|^{-\beta_0} - \psi_{A,a}(u)\,.$$

Then, by Hölder's inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\exp\left(\frac{1}{(n-1)t}\sum_{1\le j<k\le n}\int_0^t\int_0^t \frac{\gamma(X_r^j - X_s^k)}{|t^{-1}(r-s)|^{\beta_0}}drds\right)\right] \\
&\le \left(\mathbb{E}\left[\exp\left(p\frac{C_0C(\gamma)}{(n-1)t}\sum_{1\le j<k\le n}\int_0^t\int_0^t \psi_{A,a}(t^{-1}(r-s))\gamma(X_r^j - X_s^k)drds\right)\right]\right)^{1/p} \\
&\quad\times\left(\mathbb{E}\left[\exp\left(q\frac{C_0C(\gamma)}{(n-1)t}\sum_{1\le j<k\le n}\int_0^t\int_0^t \tilde{\psi}_{A,a}(t^{-1}(r-s))\gamma(X_r^j - X_s^k)drds\right)\right]\right)^{1/q}. \qquad (6.19)
\end{aligned}$$

Therefore, using a similar argument which reduces (6.5) to (6.10), one can show that to prove (6.18), it is suffices to prove

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left(\frac{1}{(n-1)t}\sum_{1\le j<k\le n}\int_0^t\int_0^t \psi_{A,a}(t^{-1}(r-s))\gamma(X_r^j - X_s^k)drds\right)\right] \\
&\le \mathbf{M}(\alpha,\beta_0,d,\gamma)\,, \qquad (6.20)
\end{aligned}$$

provided that, for any $\lambda > 0$

$$\lim_{\substack{A\to\infty\\ a\to 0}}\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left(\lambda\int_0^t\int_0^t \tilde{\psi}_{A,a}(t^{-1}(r-s))\gamma(X_r^j - X_s^k)drds\right)\right] = 0. \qquad (6.21)$$

Recalling that $\tilde{k}_{A,a}(u) = |u|^{-\frac{1+\beta_0}{2}} - k_{A,a}(u)$,

$$\begin{aligned}
&|u|^{-\beta_0} - \psi_{A,a}(u) = C_0\int_{\mathbb{R}}|u-v|^{-\frac{1+\beta_0}{2}}|v|^{-\frac{1+\beta_0}{2}}dv - C_0\int_{\mathbb{R}}k_{A,a}(u-v)k_{A,a}(v)dv \\
&\le C\left(\int_{\mathbb{R}}\tilde{k}_{A,a}(u-v)|v|^{-\frac{1+\beta_0}{2}}dv + \int_{\mathbb{R}}k_{A,a}(u-v)\tilde{k}_{A,a}(v)dv\right) \\
&\le 2C\int_{\mathbb{R}}\tilde{k}_{A,a}(u-v)|v|^{-\frac{1+\beta_0}{2}}dv \\
&\le 2C\left(A^{-\frac{\beta_0-\beta_0'}{2}}\int|u-v|^{-\frac{\beta_0'+1}{2}}|v|^{-\frac{1+\beta_0}{2}}dv + (2a)^{\frac{\tilde{\beta}_0-\beta_0}{2}}\int|u-v|^{-\frac{\tilde{\beta}_0+1}{2}}|v|^{-\frac{1+\beta_0}{2}}dv\right)
\end{aligned}$$

where $0 < \beta_0' < \beta_0 < \tilde{\beta}_0 < 1$ and the last inequality follows from (6.8). Hence we have

$$\tilde{\psi}_{A,a}(u) = |u|^{-\beta_0} - \psi_{A,a}(u) \le C(\beta_0,\beta',\tilde{\beta})\left(A^{-\frac{\beta_0-\beta_0'}{2}}u^{\frac{\beta_0+\beta_0'}{2}} + (2a)^{\frac{\tilde{\beta}_0-\beta_0}{2}}u^{\frac{\beta_0+\tilde{\beta}_0}{2}}\right). \qquad (6.22)$$

Therefore, (6.21) holds because of (6.22) and the second half of Proposition 2.9, and hence (6.18) now is reduced to (6.20).

By a similar argument used in Step 1, in order to show (6.18) that has been reduced to (6.20), it suffices to prove

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}\left[\exp\left(\frac{1}{2t}\int_0^t\int_0^t\psi_{A,a}(t^{-1}(r-s))\gamma(X_r-X_s)drds\right)\right]$$
$$\le \mathbf{M}(\alpha,\beta_0,d,\gamma). \tag{6.23}$$

The left-hand side now is finite under condition (1.3) since $\psi_{A,a}$ is a bounded function. Noting that (6.23) is identical to (6.10), we may prove it in the exact same way as in Step 3 and Step 4 in Subsection 6.1.

7 Appendix

First, we will prove the finiteness of $\mathbf{M}(\alpha,\beta_0,d,\gamma)$ defined in (1.12). Consider a general non-negative definite (generalized) function $\gamma(x)\in\mathcal{S}'(\mathbb{R}^d)$. By the Bochner-Schwartz Theorem, there exists a tempered measure μ on $\mathbb{R}^d$ such that γ is the Fourier transform of μ in $\mathcal{S}'(\mathbb{R}^d)$, i.e.

$$\int_{\mathbb{R}^d}\varphi(x)\gamma(x)dt=\int_{\mathbb{R}^d}\mathcal{F}\varphi(x)\mu(dx)\quad\text{for all}\quad\varphi\in\mathcal{S}(\mathbb{R}^d).$$

It follows that for $f,g\in\mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\gamma(x-y)f(x)g(y)dxdy=\int_{\mathbb{R}^d}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\mu(d\xi). \tag{7.1}$$

Lemma 7.1 *Under the Dalang's condition* (4.4),

$$\sup_{g\in\mathcal{F}_{\alpha,d}}\left\{\theta\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy-\int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(\xi)|^2d\xi\right\}<\infty,$$

for any $\theta>0$*, where* $\mathcal{F}_{\alpha,d}$ *is given in* (1.10)

Proof It suffices to consider $g\in\mathcal{F}_{\alpha,d}\cap\mathcal{S}(\mathbb{R}^d)$, since $\mathcal{S}(\mathbb{R}^d)$ is dense in $\mathcal{F}_{\alpha,d}$ endowed with the norm

$$\|g\|^2=\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy\right)^{1/2}+\int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(\xi)|^2d\xi\,.$$

By (7.1) and noting that $\|\mathcal{F}(g^2)(\cdot)\|_\infty\le 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\gamma(x-y)g^2(x)g^2(y)dxdy&=\int_{\mathbb{R}^d}|\mathcal{F}(g^2)(\xi)|^2\mu(d\xi)\\ &\le\mu([|\xi|\le N])+\int_{[|\xi|>N]}|(\widehat{g}*\widehat{g})(\xi)|^2|\xi|^\alpha\frac{\mu(d\xi)}{|\xi|^\alpha}\\ &\le\mu([|\xi|\le N])+\big\||(\widehat{g}*\widehat{g})(\cdot)|^2|\cdot|^\alpha\big\|_\infty\int_{[|\xi|>N]}\frac{\mu(d\xi)}{|\xi|^\alpha}.\end{aligned}$$

Since $\alpha \in (0,2]$ we see $|\xi|^{\alpha/2} \le |\xi-\eta|^{\alpha/2} + |\eta|^{\alpha/2}$ for all $\eta \in \mathbb{R}^d$. Thus, we have

$$\left|(\widehat{g} * \widehat{g})(\xi)|\xi|^{\alpha/2}\right| \le \int_{\mathbb{R}^d} |\widehat{g}|(\xi-\eta)|\widehat{g}|(\eta)\left(|\eta|^{\alpha/2} + |\xi-\eta|^{\alpha/2}\right)d\eta$$
$$\le 2\left(|\widehat{g}|(\cdot) * \left(|\widehat{g}|(\cdot)|\cdot|^{\alpha/2}\right)\right)(\xi).$$

By Young's inequality and Parseval's identity,

$$\left\| |\widehat{g}|(\cdot) * \left(|\widehat{g}|(\cdot)|\cdot|^{\alpha/2}\right)\right\|_\infty^2 \le \|\widehat{g}\|_2^2 \int_{\mathbb{R}^d} |\xi|^\alpha |\widehat{g}(\xi)|^2 d\xi = \int_{\mathbb{R}^d} |\xi|^\alpha |\widehat{g}(\xi)|^2 d\xi.$$

Therefore, for any $\theta > 0$,

$$\theta \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \gamma(x-y) g^2(x) g^2(y) dxdy - \int_{\mathbb{R}^d} |\xi|^\alpha |\widehat{g}(\xi)|^2 d\xi$$
$$\le \theta\mu([|\xi| \le N]) + \left(\theta \int_{[|\xi|>N]} \frac{\mu(d\xi)}{|\xi|^\alpha} - 1\right)\int_{\mathbb{R}^d} |\xi|^\alpha |\widehat{g}(\xi)|^2 d\xi.$$

Since $\mu(d\xi)$ is tempered and hence locally integrable, $\mu([|\xi| \le N])$ is finite for any $0 < N < \infty$. On the other hand, the Dalang's condition (4.4) implies that $\lim_{N\to\infty} \int_{[|\xi|>N]} \frac{\mu(d\xi)}{|\xi|^\alpha} = 0$. Therefore, for any $\theta > 0$, one can always find N sufficiently large such that

$$\theta \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \gamma(x-y) g^2(x) g^2(y) - \int_{\mathbb{R}^d} |\xi|^\alpha |\widehat{g}(\xi)|^2 d\xi \le \theta\mu([|\xi| \le N]) < \infty.$$

This concludes the proof. □

Lemma 7.2 *Let $\gamma_0(u), u \in \mathbb{R}$ be a locally integrable function. Then, under the Dalang's condition* (4.4),

$$\sup_{g \in \mathcal{A}_{\alpha,d}} \left\{ \theta \int_0^1 \int_0^1 \int_{\mathbb{R}^{2d}} \gamma_0(r-s)\gamma(x-y) g^2(s,x) g^2(r,y) dxdydrds \right.$$
$$\left. - \int_0^1 \int_{\mathbb{R}^d} |\xi|^\alpha |\widehat{g}(s,\xi)|^2 d\xi ds \right\} < \infty,$$

for any $\theta > 0$.

Proof The result will be proven by using a similar argument as that in the proof [10, Lemma 5.2]. Similar as in Lemma 7.1. Consider $g \in \mathcal{A}_{\alpha,d} \cap \mathcal{S}(\mathbb{R}^{d+1})$, and extend $g(s,x)$ periodically in s from $[0,1] \times \mathbb{R}^d$ to $[0,\infty) \times \mathbb{R}^d$, still denoted by the same notation $g(s,x)$. Then, we have

$$\int_0^1 \int_0^1 \int_{\mathbb{R}^{2d}} \gamma_0(r-s)\gamma_0(x-y) g^2(r,x) g^2(s,y) dxdydrds$$
$$= 2\int_0^1 \int_0^r \int_{\mathbb{R}^{2d}} \gamma_0(r-s)\gamma(x-y) g^2(r,x) g^2(s,y) dxdydrds$$
$$= 2\int_0^1 \gamma_0(r) \int_0^{1-r} \int_{\mathbb{R}^{2d}} \gamma(x-y) g^2(r+s,x) g^2(s,y) dxdydsdr$$
$$\le 2\int_0^1 |\gamma_0(r)| \int_0^1 \int_{\mathbb{R}^{2d}} \gamma(x-y) g^2(r+s,x) g^2(s,y) dxdydsdr.$$

By (7.1), we can write

$$\int_{\mathbb{R}^{2d}}\gamma(x-y)g^2(r+s,x)g^2(s,y)dxdy=\int_{\mathbb{R}^d}\left(\mathcal{F}g^2(r+s,\cdot)\right)(\xi)\overline{\left(\mathcal{F}g^2(s,\cdot)\right)(\xi)}\mu(d\xi)$$
$$\le\left(\int_{\mathbb{R}^d}\left|\left(\mathcal{F}g^2(r+s,\cdot)\right)(\xi)\right|^2\mu(d\xi)\right)^{1/2}\left(\int_{\mathbb{R}^d}\left|\left(\mathcal{F}g^2(s,\cdot)\right)(\xi)\right|^2\mu(d\xi)\right)^{1/2}$$
$$=\left(\int_{\mathbb{R}^{2d}}\gamma(x-y)g^2(r+s,x)g^2(r+s,y)dxdy\right)^{1/2}\left(\int_{\mathbb{R}^{2d}}\gamma(x-y)g^2(s,x)g^2(s,y)dxdy\right)^{1/2}.$$

Noting that g is periodic in time, we see by Hölder inequality,

$$\int_0^1\int_{\mathbb{R}^{2d}}\gamma(x-y)g^2(r+s,x)g^2(s,y)dxdyds\le\int_0^1\int_{\mathbb{R}^{2d}}\gamma(x-y)g^2(s,x)g^2(s,y)dxdyds.$$

Summarizing the above computations, we obtain

$$\int_0^1\int_0^1\int_{\mathbb{R}^{2d}}\gamma_0(r-s)\gamma(x-y)g^2(r,x)g^2(s,y)dxdydrds$$
$$\le 2\int_0^1|\gamma_0(u)|du\int_0^1\int_{\mathbb{R}^{2d}}\gamma(x-y)g^2(s,x)g^2(s,y)dxdyds.$$

Hence,

$$\sup_{g\in\mathcal{A}_{\alpha,d}}\Bigg\{\theta\int_0^1\int_0^1\int_{\mathbb{R}^{2d}}\gamma_0(r-s)\gamma(x-y)g^2(s,x)g^2(r,y)dxdydrds$$
$$-\int_0^1\int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(s,\xi)|^2d\xi ds\Bigg\}$$
$$\le\sup_{g\in\mathcal{A}_{\alpha,d}}\Bigg\{2\theta\int_0^1|\gamma_0(u)|du\int_0^1\int_{\mathbb{R}^{2d}}\gamma(x-y)g^2(s,x)g^2(s,y)dxdyds$$
$$-\int_0^1\int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(s,\xi)|^2d\xi ds\Bigg\}$$
$$=\int_0^1\sup_{g\in\mathcal{A}_{\alpha,d}}\Bigg\{2\theta\int_0^1|\gamma_0(u)|du\int_0^1\int_{\mathbb{R}^{2d}}\gamma(x-y)g^2(s,x)g^2(s,y)dxdy$$
$$-\int_0^1\int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(s,\xi)|^2d\xi\Bigg\}ds$$
$$=\sup_{g\in\mathcal{F}_{\alpha,d}}\Bigg\{2\theta\int_0^1|\gamma_0(u)|du\int_0^1\int_{\mathbb{R}^{2d}}\gamma(x-y)g^2(x)g^2(y)dxdy$$
$$-\int_0^1\int_{\mathbb{R}^d}|\xi|^\alpha|\widehat{g}(\xi)|^2d\xi\Bigg\}ds,$$

where the variation on the right-hand side is finite by Lemma 7.1. □

The following lemma was used in the proof of upper bound.

Lemma 7.3 *Let $\widetilde{K}_M$ be defined by* (6.14). *Then*

$$\limsup_{M\to\infty} \sup_{g\in\mathcal{A}^M_{\alpha,d}} \left\{ \frac{1}{2}C_0C(\gamma)\int_{[0,M]^d}\int_{\mathbb{R}}\left[\int_0^1\int_{\mathbb{T}^d_M}|u-s|^{-\frac{1+\beta_0}{2}}\widetilde{K}_M(y-x)g^2(s,x)dxds\right]^2 dudy \right.$$
$$\left. -\int_0^1 \mathcal{E}_{\alpha,M}(g(s,\cdot),g(s,\cdot))ds\right\} \le \mathbf{M}(\alpha,\beta_0,d,\gamma)\,. \qquad (7.2)$$

Proof By [22, Lemma A.1], there exists a positive constant $C_{\alpha,d}$, depending on (α,d) only, such that

$$|\xi|^\alpha = C_{\alpha,d}\int_{\mathbb{R}^d}\frac{1-\cos(2\pi\xi\cdot y)}{|y|^{d+\alpha}}dy,$$

where $C_{\alpha,d}=\int_{\mathbb{R}^d}\frac{1-\cos(\eta\cdot y)}{|y|^{d+\alpha}}dy$ for any $\eta\in\mathbb{R}^d$ with $|\eta|=2\pi$. By Lemma 7.4, we have

$$\mathcal{E}_\alpha(f,f)=\frac{C_{\alpha,d}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{|f(y)-f(x)|^2}{|y-x|^{d+\alpha}}dydx, \qquad (7.3)$$

and for any M-periodic function h,

$$\mathcal{E}_{\alpha,M}(h,h)=\frac{C_{\alpha,d}}{2}\int_{[0,M]^d}\int_{\mathbb{R}^d}\frac{|h(y)-h(x)|^2}{|y-x|^{d+\alpha}}dydx. \qquad (7.4)$$

To prove (7.2), for any fixed M-periodic (in space) function $g\in\mathcal{A}^M_{\alpha,d}$, we shall construct a function $f\in\mathcal{A}_{\alpha,d}$ such that $f\equiv g$ on $[0,1]\times[M^{1/2},M-M^{1/2}]$ and the difference between g and f on $[0,1]\times(\mathbb{R}^d\setminus[M^{1/2},M-M^{1/2}])$ is negligible in some suitable sense as M goes to infinity.

Denote

$$E_M := [0,M]^d\setminus[M^{1/2},M-M^{1/2}]\,. \qquad (7.5)$$

By Lemma 3.4 in [18], for fixed $s\in[0,1]$, there is an $a(s)\in\mathbb{R}^d$ such that

$$\int_{E_M} g^2(s,x+a(s))dx \le 2dM^{-1/2}.$$

We assume $a\equiv 0$, for otherwise we may replace $g(s,\cdot)$ with $g(s,a(s)+\cdot)$ without changing the value inside $\{\}$ in (7.2). Therefore, without loss of generality, we assume for all $s\in[0,1]$,

$$\int_{E_M} g^2(s,x)dx \le 2dM^{-1/2}. \qquad (7.6)$$

Define $\varphi(x)=\phi(x_1)\cdots\phi(x_d),\quad x=(x_1,\cdots,x_d)\in\mathbb{R}^d$, where

$$\phi(x)=\begin{cases} xM^{-1/2}, & 0\le x\le M^{1/2},\\ 1, & M^{1/2}\le x\le M-M^{1/2},\\ M^{1/2}-xM^{-\frac{1}{2}}, & M-M^{1/2}\le x\le M,\\ 0, & \text{otherwise,}\end{cases}$$

and let

$$f(s,x) = g(s,x)\varphi(x)/\sqrt{G(s)},$$

with

$$G(s) := \int_{\mathbb{R}^d} g^2(s,y)\varphi^2(y)dy.$$

Then,

$$|\phi| \le 1, |\phi'| \le M^{-1/2} \text{ and hence } |\varphi| \le 1, |\nabla\varphi| \le d^{1/2}M^{-1/2}.$$

Noting that

$$1 \ge G(s) = \int_{[0,M]^d} g^2(s,y)\varphi^2(y)dy \ge 1 - \int_{E_M} g^2(s,y)dy \ge 1 - 2dM^{-1/2},$$

we have

$$0 < 1 - 2dM^{-1/2} \le b_M := \inf_{s\in[0,1]} G(s) \le 1. \tag{7.7}$$

Firstly, we compare the second terms in the variations on both sides of (7.2), i.e., compare $J_1 := \int_0^1 \mathcal{E}_\alpha(f(s,\cdot), f(s,\cdot))ds$ with $J := \int_0^1 \mathcal{E}_{\alpha,M}(g(s,\cdot), g(s,\cdot))ds$. Note that

$$\begin{aligned}
&|g(s,y)\varphi(y) - g(s,x)\varphi(x)| = |(g(s,y) - g(s,x))\varphi(y) + g(s,x)(\varphi(y) - \varphi(x))|^2 \\
&\le (1+\varepsilon)|g(s,y) - g(s,x)|^2\varphi^2(y) + (1 + 1/\varepsilon)g^2(s,x)|\varphi(y) - \varphi(x)|^2,
\end{aligned}$$

for any $\varepsilon > 0$. Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|g(s,y)\varphi(y) - g(s,x)\varphi(x)|^2}{|y-x|^{d+\alpha}}dydx \\
&\le (1+\varepsilon)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|g(s,y) - g(s,x)|^2\varphi^2(y)}{|y-x|^{d+\alpha}}dydx \\
&+ (1+1/\varepsilon)\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{g^2(s,x)|\varphi(y) - \varphi(x)|^2}{|y-x|^{d+\alpha}}dydx.
\end{aligned} \tag{7.8}$$

Now we bound the above two integrals separately. For the first integral, it is easy to verify by (7.3) that

$$\frac{C_{\alpha,d}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|g(s,y) - g(s,x)|^2\varphi^2(y)}{|y-x|^{d+\alpha}}dydx \le \mathcal{E}_{\alpha,M}(g(s,\cdot), g(s,\cdot)). \tag{7.9}$$

For the second integral, we have first, for any $\sigma \in (0,2)$,

$$\begin{aligned}
&g^2(s,x)|\varphi(y) - \varphi(x)|^2 \\
&\le g^2(s,x)|\varphi(y) - \varphi(x)|^2(I_{[0,M]^d}(x) + I_{[0,M]^d}(y)) \\
&= g^2(s,x)|\varphi(y) - \varphi(x)|^{2-\sigma}|\varphi(y) - \varphi(x)|^\sigma(I_{[0,M]^d}(x) + I_{[0,M]^d}(y)) \\
&\le 2^{2-\sigma}d^{\sigma/2}M^{-\sigma/2}g^2(s,x)(I_{[0,M]^d}(x) + I_{[0,M]^d}(y))(|y-x|^\sigma \wedge |y-x|^2),
\end{aligned}$$

Consequently, we have

$$
\begin{aligned}
&\frac{C_{\alpha,d}}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\frac{g^2(s,x)|\varphi(y)-\varphi(x)|^2}{|y-x|^{d+\alpha}}dydx\\
&\le C_{\alpha,d}2^{2-\sigma}d^{\sigma/2}M^{-\sigma/2}\int_{[0,M]^d}\int_{\mathbb{R}^d}\frac{g^2(s,x)(|y-x|^\sigma\wedge|y-x|^2)}{|y-x|^{d+\alpha}}dydx\\
&\quad+C_{\alpha,d}2^{2-\sigma}d^{\sigma/2}M^{-\sigma/2}\int_{\mathbb{R}^d}\int_{[0,M]^d}\frac{g^2(s,x)(|y-x|^\sigma\wedge|y-x|^2)}{|y-x|^{d+\alpha}}dydx\\
&=C_{\alpha,d}2^{2-\sigma}d^{\sigma/2}M^{-\sigma/2}\int_{[0,M]^d}g^2(s,x)dx\int_{\mathbb{R}^d}\frac{|y|^\sigma\wedge|y|^2}{|y|^{d+\alpha}}dy\\
&\quad+C_{\alpha,d}2^{2-\sigma}d^{\sigma/2}M^{-\sigma/2}\int_{[0,M]^d}\int_{\mathbb{R}^d}\frac{g^2(s,x+y)(|x|^\sigma\wedge|x|^2)}{|x|^{d+\alpha}}dxdy\\
&\le CM^{-\sigma/2},
\end{aligned}
\tag{7.10}
$$

for some constant C depending only on (α,d), where in the last second step, the two integrals are finite for $\alpha\in(\sigma,2)$.

Combining (7.3), (7.4), (7.8), (7.9) and (7.10), and recalling b_M given in (7.7), we have

$$
\begin{aligned}
b_MJ_1&=b_M\int_0^1\mathcal{E}_\alpha(f(s,\cdot),f(s,\cdot))ds\\
&\le\int_0^1G(s)\mathcal{E}_\alpha(f(s,\cdot),f(s,\cdot))ds\\
&\le(1+\varepsilon)\int_0^1\mathcal{E}_{\alpha,M}(g(s,\cdot),g(s,\cdot))ds+C(1+1/\varepsilon)M^{-\sigma/2}\\
&=(1+\varepsilon)J+C(1+1/\varepsilon)M^{-\sigma/2}.
\end{aligned}
\tag{7.11}
$$

Secondly, we estimate the first term inside {} in (7.2). Recall that $K_{B,b}(\cdot)$ is supported on $[-2B,2B]^d$, hence for any fixed $y\in[0,M]^d$, $K_{B,b}(y-\cdot)$ is supported on $[-2B,M+2B]^d$. Therefore, for $y\in[0,M]^d$,

$$
\begin{aligned}
&\int_{[0,M]^d}\widetilde{K}_M(y-x)g^2(s,x)dx=\int_{[0,M]^d}\sum_{z\in\mathbb{Z}^d}K_{B,b}(y-x+zM)g^2(s,x)dx\\
&=\int_{\mathbb{R}^d}K_{B,b}(y-x)g^2(s,x)dx=\int_{[-2B,M+2B]^d}K_{B,b}(y-x)g^2(s,x)dx
\end{aligned}
\tag{7.12}
$$

where the second equality follows from the M-periodicity of $g(s,\cdot)$.

Denote

$$\tilde{E}_M:=[-2B,M+2B]^d\setminus[M^{1/2},M-M^{1/2}].\tag{7.13}$$

Then there exists a constant C depending only on d such that

$$\int_{\tilde{E}_M}g^2(s,x)dx\le CM^{-1/2},\ \forall s\in[0,1].\tag{7.14}$$

This is because of (7.6), the periodicity of $g(s,\cdot)$ and the fact that there is a partition of $[-2B, M+2B]^d \setminus [0,M]^d$ such that the number of parts in the partition is finite depending only on d and each part from this partition can be shifted by zM for some $z \in \mathbb{Z}^d$ to become a subset of $[0,M]^d \setminus [2B, M-2B]^d \subset [0,M]^d \setminus [\sqrt{M}, M-\sqrt{M}]^d$ when $M > 4B^2$.

Notice that

$$g^2(s,x) = G(s) f^2(s,x), \quad \forall\, x \in [M^{1/2}, M - M^{1/2}] = [-2B, M+2B]^d \setminus \tilde{E}_M\,,$$

where $\tilde{E}_M$ is defined by (7.13). We can bound the integral in (7.2) as follows, noting (7.12),

$$\begin{aligned}
I :=& \int_{[0,M]^d}\int_{\mathbb{R}}\left[\int_0^1\int_{[0,M]^d} |u-s|^{-\frac{1+\beta_0}{2}}\widetilde{K}_M(y-x)g^2(s,x)dxds\right]^2 dudy \\
=& \int_{[0,M]^d}\int_{\mathbb{R}}\left[\int_0^1\int_{[-2B,M+2B]^d} |u-s|^{-\frac{1+\beta_0}{2}}K_{B,b}(y-x)g^2(s,x)dxds\right]^2 dudy \\
\le& (1+\varepsilon)\int_{[0,M]^d}\int_{\mathbb{R}}\left[\int_0^1\int_{[-2B,M+2B]^d\setminus\tilde{E}_M} |u-s|^{-\frac{1+\beta_0}{2}}K_{B,b}(y-x)g^2(s,x)dxds\right]^2 dudy \\
&+ (1+1/\varepsilon)\int_{[0,M]^d}\int_{\mathbb{R}}\left[\int_0^1\int_{\tilde{E}_M} |u-s|^{-\frac{1+\beta_0}{2}}K_{B,b}(y-x)g^2(s,x)dxds\right]^2 dudy \\
\le& (1+\varepsilon)\max_{s\in[0,1]} G(s)\int_{[0,M]^d}\int_{\mathbb{R}}\left[\int_0^1\int_{\mathbb{R}^d} |u-s|^{-\frac{1+\beta_0}{2}}K_{B,b}(y-x)f^2(s,x)dxds\right]^2 dudy \\
&+ (1+1/\varepsilon)\int_{[0,M]^d}\int_{\mathbb{R}}\left[\int_0^1\int_{\tilde{E}_M} |u-s|^{-\frac{1+\beta_0}{2}}K_{B,b}(y-x)g^2(s,x)dxds\right]^2 dudy \\
\le& (1+\varepsilon)\left(C_0 C(\gamma)\right)^{-1/2} I_1 + (1+1/\varepsilon) I_2\,,
\end{aligned} \tag{7.15}$$

where

$$\begin{aligned}
I_1 &:= \int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{\gamma(x-y)}{|r-s|^{\beta_0}} f^2(s,x)f^2(r,y)dxdydrds \\
I_2 &:= \int_{[0,M]^d}\int_{\mathbb{R}}\left[\int_0^1\int_{\tilde{E}_M} |u-s|^{-\frac{1+\beta_0}{2}}K_{B,b}(y-x)g^2(s,x)dxds\right]^2 dudy\,.
\end{aligned}$$

We consider I_2. Note that the function $K_{B,b}(\cdot)$ is uniformly bounded, say, by D. Then we have

$$
\begin{aligned}
I_2 =& C_0^{-1}\int_{[0,M]^d}\int_0^1\int_0^1 |r-s|^{-\beta_0} drdsdy \int_{\tilde{E}_M} K_{B,b}(y-x_1)g^2(s,x_1)dx_1 \\
&\int_{\tilde{E}_M} K_{B,b}(y-x_2)g^2(r,x_2)dx_2 \\
\le& C_0^{-1}\int_{[0,M]^d}\int_0^1\int_0^1 |r-s|^{-\beta_0} drdsdy \int_{\tilde{E}_M} K_{B,b}(y-x_1)g^2(s,x_1)dx_1 \int_{\tilde{E}_M} Dg^2(r,x_2)dx_2 \\
\le& CC_0^{-1}M^{-1/2}\int_{\mathbb{R}^d} K_{B,b}(y)dy \int_0^1\int_0^1 |r-s|^{-\beta_0} drds \int_{\tilde{E}_M} g^2(s,x_1)dx_1 \\
\le& CC_0^{-1}M^{-1/2}\int_{\mathbb{R}^d} K_{B,b}(y)dy(1-\beta_0)^{-1}\int_0^1 \left[s^{1-\beta_0}+(1-s)^{1-\beta_0}\right]\int_{\tilde{E}_M} g^2(s,x_1)dx_1ds \\
\le& CC_0^{-1}M^{-1/2}\int_{\mathbb{R}^d} K_{B,b}(y)dy(1-\beta_0)^{-1}(2-\beta_0)^{-1}\int_0^1\int_{\tilde{E}_M} g^2(s,x_1)dx_1ds \\
\le& 2CC_0^{-1}M^{-1/2}\int_{\mathbb{R}^d} K_{B,b}(y)dy(1-\beta_0)^{-1}(2-\beta_0)^{-1}(2dM^{-1/2}) \\
=& C\Big(K_{B,b}(\cdot),d,\beta_0\Big)M^{-1}, \qquad (7.16)
\end{aligned}
$$

where the third step and the last second step follow from (7.14).

Finally, combing (7.11), (7.15) and (7.16), we can bound the quantity inside { } in (7.2) as follows (recall that J and J_1 are defined by (7.11) and b_M is given in (7.7))

$$
\begin{aligned}
\frac{1}{2}C_0C(\gamma)I - J \le& \frac{1+\varepsilon}{2}I_1 + C(1+1/\varepsilon)M^{-1} - \frac{b_M}{1+\varepsilon}J_1 + C\frac{1+1/\varepsilon}{1+\varepsilon}M^{-\sigma/2} \\
\le& \frac{b_M}{1+\varepsilon}\Bigg\{\frac{(1+\varepsilon)^2}{2b_M}\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\beta_0}}f^2(s,x)f^2(r,y)dxdydrds \\
&-\int_0^1 \mathcal{E}_\alpha(f(s,\cdot),f(s,\cdot))ds\Bigg\} + C(1+1/\varepsilon)M^{-1} + C\frac{1+1/\varepsilon}{1+\varepsilon}M^{-\sigma/2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\sup_{g\in\mathcal{A}^M_{\alpha,d}}\left\{\frac{1}{2}C_0C(\gamma)I-J\right\}\\
\le&\frac{b_M}{1+\varepsilon}\sup_{f\in\mathcal{A}_{\alpha,d}}\left\{\frac{(1+\varepsilon)^2}{2b_M}\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\beta_0}}f^2(s,x)f^2(r,y)dxdydrds\right.\\
&\left.-\int_0^1\mathcal{E}_\alpha(f(s,\cdot),f(s,\cdot))ds\right\}+C(1+1/\varepsilon)M^{-1}+C\frac{1+1/\varepsilon}{1+\varepsilon}M^{-\sigma/2}\\
=&\frac{b_M}{1+\varepsilon}\left(\frac{(1+\varepsilon)^2}{b_M}\right)^{\frac{\alpha}{\alpha-\beta}}\sup_{f\in\mathcal{A}_{\alpha,d}}\left\{\frac{1}{2}\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\beta_0}}f^2(s,x)f^2(r,y)dxdydrds\right.\\
&\left.-\int_0^1\mathcal{E}_\alpha(f(s,\cdot),f(s,\cdot))ds\right\}+C(1+1/\varepsilon)M^{-1}+C\frac{1+1/\varepsilon}{1+\varepsilon}M^{-\sigma/2},
\end{aligned}
$$

where the last step follows from (1.13). Noting that $\lim_{M\to\infty}b_M=1$, we have, by choosing ε arbitrarily small,

$$
\begin{aligned}
\lim_{M\to\infty}\sup_{g\in\mathcal{A}^M_{\alpha,d}}\left\{\frac{1}{2}C_0C(\gamma)I-J\right\}\le\sup_{f\in\mathcal{A}_{\alpha,d}}&\left\{\frac{1}{2}\int_0^1\int_0^1\int_{\mathbb{R}^d\times\mathbb{R}^d}\frac{\gamma(x-y)}{|r-s|^{\beta_0}}\right.\\
&\left.f^2(s,x)f^2(r,y)dxdydrds-\int_0^1\mathcal{E}_\alpha(f(s,\cdot),f(s,\cdot))ds\right\}.
\end{aligned}
$$

Hence (7.2) is proved, provided $\alpha\in(\sigma,2)$. Note that $\sigma\in(0,2)$ is arbitrary, therefore (7.2) holds for $\alpha\in(0,2)$.

The proof is concluded, noting that for the case $\alpha=2$, (7.2) can be proved in a similar way as in [12, Lemma A.3]. □

Lemma 7.4 *Let $f\in L^2(\mathbb{R}^d)$ and $h\in L^2(\mathbb{T}^d_M)$. Then,*

$$
2\int_{\mathbb{R}^d}\Big(1-\cos(2\pi\xi\cdot y)\Big)|\widehat{f}(\xi)|^2d\xi=\int_{\mathbb{R}^d}|f(x+y)-f(x)|^2dx, \tag{7.17}
$$

and

$$
\frac{2}{M^d}\sum_{k\in\mathbb{Z}^d}\Big(1-\cos(2\pi k\cdot y)\Big)|\widehat{h}(k)|^2=\int_{[0,M]^d}|h(x+My)-h(x)|^2dx. \tag{7.18}
$$

Proof We will prove (7.18) only, and (7.17) can be proved in the same spirit. Noting that $1-\cos(2\pi k\cdot y)=2\sin^2(\pi k\cdot y)$, we have

$$
\begin{aligned}
&\frac{2}{M^d}\sum_{k\in\mathbb{Z}^d}\Big(1-\cos(2\pi k\cdot y)\Big)|\widehat{h}(k)|^2=\frac{1}{M^d}\sum_{k\in\mathbb{Z}^d}|2\sin(\pi k\cdot y)\widehat{h}(k)|^2\\
=&\frac{1}{M^d}\sum_{k\in\mathbb{Z}^d}\left|\left(e^{i\pi k\cdot y}-e^{-i\pi k\cdot y}\right)\widehat{h}(k)\right|^2=\int_{[0,M]^d}\left|h(x+\frac{My}{2})-h(x-\frac{My}{2})\right|^2dx.
\end{aligned}
$$

The last equality holds because of the Parseval's identity

$$\frac{1}{M^d}\sum_{k\in\mathbb{Z}^d}|\widehat{g}(k)|^2=\int_{[0,M]^d}|g(x)|^2\,dx,$$

and the fact that for any $a\in\mathbb{R}^d$ and any M-periodic function g,

$$\begin{aligned}\widehat{g(\cdot+a)}(k)&=\int_{[0,M]^d}e^{-2\pi ik\cdot y/M}g(y+a)dy=\int_{[0,M]^d}e^{-2\pi ik\cdot(y+a)/M}g(y+a)dy\,e^{2\pi ik\cdot a/M}\\&=\int_{[0,M]^d}e^{-2\pi ik\cdot y/M}g(y)dy\,e^{2\pi ik\cdot a/M}=\widehat{g}(k)e^{2\pi ik\cdot a/M},\end{aligned}$$

where the third equality holds because $e^{-2\pi ik\cdot y/M}g(y)$ is an M-periodic function in y. □

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