Nonlinear Feynman-Kac formulae for SPDEs with space-time noise

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Abstract

We study a class of backward doubly stochastic differential equations (BDSDEs) involving martingales with spatial parameters, and show that they provide probabilistic interpretations (Feynman-Kac formulae) for certain semilinear stochastic partial differential equations (SPDEs) with space-time noise. As an application of the Feynman-Kac formulae, random periodic solutions and stationary solutions to certain SPDEs are obtained.

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1 Introduction

The existence and uniqueness of solutions to general backward stochastic differential equations (BSDEs) was obtained by Pardoux and Peng in their pioneering work [18], and they found in [19] that solutions to BSDEs provide probabilistic interpretations for solutions to semilinear parabolic PDEs, which is an extension of the classical Feynman-Kac formula. Furthermore, Pardoux and Peng [20] introduced and studied the so-called backward doubly stochastic differential equations (BDSDEs), the solutions to which serve as nonlinear Feynman-Kac formulae for associated semilinear SPDEs driven by white noise in time. Along this line, this article concerns probabilistic interpretations (nonlinear Feynman-Kac formulae) for solutions to a class of semilinear SPDEs driven by space-time noise.

Let $(\Omega, \mathcal{F}, P)$ be a probability space satisfying the usual conditions. Let $W = (W_t, t \geq 0)$ be standard d-dimensional Brownian motion and $(B(t,x), t \geq 0)$ be a one-dimensional local martingale with spatial parameter $x \in \mathbb{R}^d$ which is independent of W. Consider the following BDSDE,

$$Y_s^{t,x} \;=\; \phi(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr$$

$$+\int_s^T g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \overleftarrow{B}(dr, X_r^{t,x}) - \int_s^T Z_r^{t,x} dW_r, \quad s \in [t,T], \quad (1.1)$$

where $X_s^{t,x}$ is the unique strong solution to

$$dX_s^{t,x} = b(X_s^{t,x})ds + \sigma(X_s^{t,x})dW_s, \quad s \in [t,T], \quad X_t^{t,x} = x \in \mathbb{R}^d. \quad (1.2)$$

Here $\phi : \mathbb{R}^d \to \mathbb{R}$; $f, g : [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$; $b : \mathbb{R}^d \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ are measurable functions. Denote by $\mathscr{L}$ the infinitesimal generator of X, i.e.

$$(\mathscr{L}u)(x) = \frac{1}{2}\sum_{i,j=1}^d a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j}u(x) + \sum_{i=1}^d b_i(x)\frac{\partial}{\partial x_i}u(x)\,,$$

with $a_{ij}(x) = \sum_{k=1}^d \sigma_{ik}(x)\sigma_{jk}(x)$.

There are three major goals in this article. Firstly, under suitable conditions, we obtain an existence and uniqueness theorem for BDSDE (1.1). Secondly, we establish the connection between BDSDE (1.1) and the following semilinear SPDE,

$$\begin{cases} -\partial_t u(t,x) = \Big[\mathscr{L}u(t,x) + f(t,x,u(t,x),\nabla u(t,x)\sigma(x))\Big]dt \\ \qquad\qquad +g(t,x,u(t,x),\nabla u(t,x)\sigma(x))\overleftarrow{B}(dt,x), \qquad (t,x) \in [0,T]\times\mathbb{R}^d, \\ u(T,x) = \phi(x)\,, \end{cases} \quad (1.3)$$

where $\nabla u = (\partial_{x_1}u, \dots, \partial_{x_d}u)$ and $\int \overleftarrow{B}(dt,x)$ is a backward Itô integral. Thirdly, as an application of this connection, we construct periodic and stationary solutions for SPDEs via infinite horizon BDSDEs.

We would like to give a brief review for the background and some remarks on the connection between our work and some related literature. After the introduction of BDSDEs driven by two independent Brownian motions in [20], BDSDEs and probabilistic interpretations (nonlinear Feynman-Kac formulae) for SPDEs have been extensively investigated in several directions, and we list a few of them which is far from complete.

For SPDEs driven by temporal white noise, Bally and Matoussi gave probabilistic interpretations for solutions in Sobolev spaces (weak solutions) in [3]; Buckdahn and Ma introduced viscosity solutions and established Feynman-Kac formulae in [4, 5]. SPDEs driven by temporal colored noise and the associated BDSDEs driven by Brownian motion and fractional Brownian motion were studied by Jing and León in [11, 12].

Feynman-Kac formulae for linear SPDEs with space-time noise were obtained in [8, 9, 10, 22]. On the other hand, the related results for nonlinear SPDEs with space-time noise seem to be very limited, and we only find one paper [15] which is due to Matoussi and Scheutzow. In [15], the authors dealt with a general type of SPDEs with nonlinear space-time noise which includes equation (1.3). However, the conditions imposed in [15] for the associated BDSDEs are rather restrictive in our situation. In the present article, we obtain the existence and uniqueness of the solution to BSDE (1.1) under relatively general conditions (see Theorem 3.2 and Remark 3.3 for its relationship with the result in [15]). In comparison with BDSDEs driven by (fractional) Brownian motions in [20, 3, 5, 12], technically it is more difficult to establish an existence and uniqueness theorem under general conditions for BDSDE (1.1) due to the spatial dependence of $B(t,x)$. The key step is to combine the Itô's formula with contraction mapping theorem in a proper way to obtain the existence and uniqueness of the solution in a suitable Banach space (i.e. $\mathcal{M}^{2,\beta}$ given in (3.7), see the proof of Theorem 3.2).

A remarkable application of the nonlinear Feynman-Kac formula is the construction of random periodic and stationary solutions to SPDEs via the associated BDSDEs. Unlike the deterministic situation, in which elliptic PDEs give the steady status of parabolic PDEs when time tends to infinity, "elliptic SPDEs" do not exist, and we need to use other equations to take the role of "elliptic SPDEs". It turns out that the solutions to associated infinite horizon BDSDEs can be used to represent the periodic/stationary solutions to SPDEs.[1] In Section 6, we aim to find the random periodic solution to the following infinite horizon SPDE without terminal value which has a form on the interval $[0,T]$ for arbitrary $T>0$:

$$\begin{aligned} u(t,x) &= u(T,x)+\int_t^T [\mathscr{L}u(s,x)+f\big(s,x,u(s,x),(\sigma^T\nabla u)(s,x)\big)]ds \\ &\quad -\int_t^T g\big(s,x,u(s,x),(\sigma^T\nabla v)(s,x)\big)\overleftarrow{B}(ds,x). \end{aligned} \tag{1.4}$$

For a given $\tau>0$, if a solution u to SPDE (1.4) satisfies

$$\theta_\tau\circ u(t,\cdot)=u(t+\tau,\cdot)\quad \text{for all } t\geq 0,\quad \text{a.s.}, \tag{1.5}$$

where θ is the shift operator defined in Section 6, we call u a random periodic solution.

Periodicity is a common phenomenon in our world which is exhibited in, for instance, change of seasons, long-duration oscillation of ocean temperature, and migration pattern of birds. Many efforts have been made by mathematicians, physicists, oceanographers, biologists, etc., to depict and study periodicity in systems perturbed by noises. Considering the significance of periodic solution in deterministic dynamical system, the importance of (random) periodic solution in random dynamical system is obvious. However, unlike in deterministic dynamical systems, the perturbations caused by the noises in random dynamical systems break the strict periodicity, which had brought difficulty to give a rigorous mathematical definition of periodicity for a long time. Observing that the random periodic solution is a stationary solution (stochastic fixed solution) of fixed discrete times with an equal interval as the period, Zhao and Zheng [25] put forward the concept of random periodic solution for C^1-cocycles, and later Feng, Zhao and Zhou proposed random periodic solution for semi-flows in [6].

Nevertheless, random periodic solutions can be obtained in few cases for SPDEs due to the partial differential operator and the noises. To our best knowledge, the only known result was obtained by Feng, Wu and Zhao in [7] based on the definition of random periodic solution for semi-flows. In [7] , the authors identified random periodic solutions to SPDEs

[1] Note that Peng [21] first discovered that the solutions to semilinear elliptic PDEs can be represented by the solutions to infinite horizon BSDEs, and Zhang and Zhao [24] obtained random stationary solutions to SPDEs driven by cylindrical Brownian motion via BDSDEs.

driven by temporal white noise with solutions to infinite horizon random integral equations. In this article, random periodic solutions to SPDEs driven by local martingales with spatial parameters are constructed by the associated BDSDEs. We would like to point out that our result is not an immediate extension of [7] or [24], since the noise in SPDE (1.3) also depends on the space variable x, which makes the analysis more challenging.

This article is organized as follows. In Section 2, we recall some preliminaries on Itô-Kunita's stochastic integral and provide some lemmas which shall be used later. The existence and uniqueness of the solution to BDSDE (1.1) is studied in Section 3, and the p-moments of the solution is estimated in Section 4. In Section 5, with the help of the finite p-moments of the solution, we obtain the regularity of the solution to the BDSDE and then establish the connection between BDSDE (1.1) and SPDE (1.3). Finally, in Section 6 we construct the periodic and stationary solution to the SPDE via the infinite horizon BDSDE.

Throughout the paper, C is a generic constant which may vary in different places.

2 Some preliminaries

In this section we provide preliminaries on the integrals against local martingales with spatial parameters and some useful lemmas. For more details on the Itô-Kunita's integral, we refer to [14].

Denote

$$\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B\,, \qquad \mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_T^B\,, \tag{2.1}$$

where $\mathcal{F}_{s,t}^B = \sigma\{B(r,x) - B(s,x), s \le r \le t, x \in \mathbb{R}^d\}$, $\mathcal{F}_t^B = \mathcal{F}_{0,t}^B$, and $\mathcal{F}_{s,t}^W$ and $\mathcal{F}_t^W$ are defined in a similar way. Note that $\mathcal{G}_t$ is a filtration, while $\mathcal{F}_t$ is not. The joint quadratic variation of $(B(s,x), t \ge 0, x \in \mathbb{R}^d)$ is

$$\langle B(\cdot,x)\,, B(\cdot,y)\rangle_t = \int_0^t q(s,x,y)ds\,. \tag{2.2}$$

Throughout the paper, we assume the following condition on $q(s,x,y)$.

(H) The function $q(s,x,y)$ given in (2.2) satisfies

$$\sup_{0\le s\le T} |q(s,x,y)| \le K(1 + |x|^\kappa + |y|^\kappa),$$

for some $0 < \kappa < 2$ and $0 < K < \infty$.

Let $(f_t, 0 \le t \le T)$ be a predictable process with respect to the backward filtration $\mathcal{F}_{t,T}^B$ satisfying

$$\int_0^T q(s, f_s, f_s)ds < \infty \qquad a.s. \tag{2.3}$$

Then the stochastic integral $\int_0^T \overleftarrow{B}(ds, f_s)$ is well-defined (see e.g. [14, Chapter 3]). In particular, if the paths of f_t are a.s. continuous, then assuming **(H)**, condition (2.3) is satisfied, and the integral can be approximated by Riemann sums ([14, 16])

$$\int_t^T \overleftarrow{B}(ds, f_s) = \lim_{|\Delta|\to 0} \sum_{k=0}^{n-1} \left[B(t_{k+1}, f_{t_{k+1}}) - B(t_k, f_{t_{k+1}})\right],$$

where $\Delta = \{t = t_0 < \cdots < t_n = T\}$ and $|\Delta| = \sup_{0\le k\le n-1} |t_{k+1} - t_k|$.

Let $(X_s^{t,x}, t \le s \le T)$ be the solution to equation (1.2). Note that $X^{t,x}$ is independent of B and it is a.s. continuous. Thus $\int_s^T \overleftarrow{B}(dr, X_r^{t,x})$ for $t \le s \le T$ is well defined under **(H)**, and its quadratic variation is given by ([14, Theorem 3.2.4])

$$\left\langle \int_\cdot^T \overleftarrow{B}(dr, X_r^{t,x}) \right\rangle_{s,T} = \int_s^T q(r, X_r^{t,x}, X_r^{t,x}) dr\,. \tag{2.4}$$

In the sequel, we shall use the following generalized Itô's formula, which is an extension of [20, Lemma 1.3].

Lemma 2.1 *Suppose that f, g and h are $\mathcal{F}_t$-measurable processes such that*

$$\int_0^T |f_s|ds + \int_0^T g_s^2 q(s, X_s, X_s)ds + \int_0^T h_s^2 ds < \infty, \text{ a.s.},$$

where $X_s = X_s^{0,x}$. Let S_t be $\mathcal{F}_t$-measurable and of the form

$$S_t = S_0 + \int_0^t f_s ds + \int_0^t g_s d\overleftarrow{B}(ds, X_s) + \int_0^t h_s dW_s,\ 0 \le t \le T,$$

and then we have

$$\begin{aligned} S_t^2 &= S_0^2 + 2\int_0^t S_s dS_s - \int_0^t |g_s|^2 q(s, X_s, X_s)ds + \int_0^t |h_s|^2 ds \\ &= S_0^2 + 2\int_0^t S_s f_s ds + 2\int_0^t S_s g_s d\overleftarrow{B}(ds, X_s) + 2\int_0^t S_s h_s dW_s \\ &\quad - \int_0^t |g_s|^2 q(s, X_s, X_s)ds + \int_0^t |h_s|^2 ds\,. \end{aligned} \tag{2.5}$$

More generally, for any function $\varphi \in C^2(\mathbb{R})$, we have the following Itô's formula,

$$\begin{aligned} \varphi(S_t) &= \varphi(S_0) + \int_0^t \varphi'(S_s)dS_s - \frac{1}{2}\int_0^t \varphi''(S_s)|g_s|^2 q(s, X_s, X_s)ds + \frac{1}{2}\int_0^t \varphi''(S_s)|h_s|^2 ds \\ &= \varphi(S_0) + \int_0^t \varphi'(S_s) f_s ds + \int_0^t \varphi'(S_s) g_s d\overleftarrow{B}(ds, X_s) + \int_0^t \varphi'(S_s) h_s dW_s \\ &\quad -\frac{1}{2}\int_0^t \varphi''(S_s)|g_s|^2 q(s, X_s, X_s)ds + \frac{1}{2}\int_0^t \varphi''(S_s)|h_s|^2 ds\,. \end{aligned} \tag{2.6}$$

Proof. The Itô's formula (2.6) can be proven by the standard methods of approximation and localization (see e.g. [13, Theorem 3.3]). Here we provide a sketch of proof for (2.5), and (2.6) can be proven in a similar spirit.

Fixing $t > 0$ and a partition $\Delta = \{0 = t_0, t_1, \ldots, t_n = t\}$ of $[0, t]$, we have

$$\begin{aligned}
S_t^2 - S_0^2 =& \sum_{i=0}^{n-1} \left[S_{t_{i+1}}^2 - S_{t_i}^2 \right] \\
=& 2 \sum_{i=0}^{n-1} S_{t_i} \left[S_{t_{i+1}} - S_{t_i} \right] + \sum_{i=0}^{n-1} \left[S_{t_{i+1}} - S_{t_i} \right]^2 \\
=& 2 \sum_{i=0}^{n-1} S_{t_i} \left(\int_{t_i}^{t_{i+1}} f_s ds + \int_{t_i}^{t_{i+1}} h_s dW_s \right) + 2 \sum_{i=0}^{n-1} S_{t_{i+1}} \int_{t_i}^{t_{i+1}} g_s d\overleftarrow{B}(ds, X_s) \\
& - 2 \sum_{i=0}^{n-1} (S_{t_{i+1}} - S_{t_i}) \int_{t_i}^{t_{i+1}} g_s d\overleftarrow{B}(ds, X_s) + \sum_{i=0}^{n-1} \left[S_{t_{i+1}} - S_{t_i} \right]^2 .
\end{aligned} \tag{2.7}$$

When the mesh size $|\Delta|$ goes to zero,

$$\sum_{i=0}^{n-1} S_{t_i} \left(\int_{t_i}^{t_{i+1}} f_s ds + \int_{t_i}^{t_{i+1}} h_s dW_s \right) + \sum_{i=0}^{n-1} S_{t_{i+1}} \int_{t_i}^{t_{i+1}} g_s d\overleftarrow{B}(ds, X_s)$$

converges to $\int_0^t S_s dS_s$, and the rest terms on the right-hand side of (2.7)

$$\begin{aligned}
& - 2 \sum_{i=0}^{n-1} (S_{t_{i+1}} - S_{t_i}) \int_{t_i}^{t_{i+1}} g_s d\overleftarrow{B}(ds, X_s) + \sum_{i=0}^{n-1} \left[S_{t_{i+1}} - S_{t_i} \right]^2 \\
&= \sum_{i=0}^{n-1} \left(S_{t_{i+1}} - S_{t_i} \right) \left(\int_{t_i}^{t_{i+1}} h_s dW_s - \int_{t_i}^{t_{i+1}} g_s d\overleftarrow{B}(ds, X_s) \right) + \sum_{i=0}^{n-1} \rho_i \\
&= \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} h_s dW_s + \int_{t_i}^{t_{i+1}} g_s d\overleftarrow{B}(ds, X_s) \right) \left(\int_{t_i}^{t_{i+1}} h_s dW_s - \int_{t_i}^{t_{i+1}} g_s d\overleftarrow{B}(ds, X_s) \right) + \sum_{i=0}^{n-1} \rho_i' \\
&= \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} h_s dW_s \right)^2 - \sum_{i=0}^{n-1} \left(\int_{t_i}^{t_{i+1}} g_s d\overleftarrow{B}(ds, X_s) \right)^2 + \sum_{i=0}^{n-1} \rho_i'
\end{aligned}$$

converges to

$$\int_0^t h_s^2 ds - \int_0^t g_s^2 q(s, X_s, X_s) ds,$$

since $\sum_{i=0}^{n-1} \rho_i'$ converges to zero based on the fact that the covariation between a martingale and an absolutely continuous process is zero. ■

Similarly, we also have the following product rule.

Lemma 2.2 *Let Q_t be a continuous $\mathcal{F}_t$-measurable process with bounded variation and S_t be given in Lemma 2.1. Then the following product rule holds*

$$d(S_t Q_t) = S_t dQ_t + Q_t dS_t.$$

The following result of exponential integrability will be used in the proof the existence and uniqueness of the solutions to (1.1) in Section 3.

Lemma 2.3 *If b and σ are bounded measurable functions, we have*

$$\mathbb{E}\int_0^T \exp\left(pq(t,X_t,X_t)\right)dt < \infty, \text{ for all } p>0.$$

In particular,

$$\mathbb{E}\exp\left(p\int_0^T q(t,X_t,X_t)dt\right) < \infty \text{ and } \mathbb{E}\int_0^T q(t,X_t,X_t)^p dt < \infty, \text{ for all } p>0.$$

Proof. Note that $X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dW_s$. Since b is bounded and $|q(t,x,y)| \le K(1+|x|^\kappa + |y|^\kappa)$ with $\kappa \in (0,2)$ for all $t\in[0,T]$ by condition **(H)**, it suffices to show that

$$\mathbb{E}\int_0^T \exp\left(p\left|\int_0^t \sigma(X_s)dW_s\right|^\kappa\right)dt < \infty, \text{ for all } p>0,$$

which can be reduced to show that

$$\mathbb{E}\left[\sup_{0\le t\le T}\exp\left(p\left|\int_0^t \sigma(X_s)dW_s\right|^\kappa\right)dt\right] < \infty. \tag{2.8}$$

Denoting $N_t = \int_0^t \sigma(X_s)dW_s$, by the exponential inequality for martingales (see, e.g. [17, Formula (A.5)]), we have for any $x>0$,

$$P\left(\sup_{0\le t\le T}|N_t| \ge x\right) \le 2\exp\left(-\frac{x^2}{2D_0}\right),$$

where $D_0 = T\|\sigma\|_\infty^2 < \infty$. Let $\tilde{N} = \sup_{0\le t\le T]}|N_t|$. The left-hand side of (2.8) is estimated as follows,

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\le t\le T}\exp\left(p\left|\int_0^t \sigma(X_s)dW_s\right|^\kappa\right)dt\right] = \mathbb{E}\left[e^{p|\tilde{N}|^\kappa}\right]\\ &= \int_{\mathbb{R}} P\left(|\tilde{N}|^\kappa \ge \frac{y}{p}\right)e^y dy \le 1 + \int_0^\infty 2\exp\left(-\frac{y^{\frac{2}{\kappa}}p^{-\frac{2}{\kappa}}}{2D_0}+y\right)dy,\end{aligned}$$

where the integral on the right-hand side is finite for all $p>0$ since $\kappa<2$. The proof is concluded. ∎

3 Existence and uniqueness of solutions to BDSDEs

This section concerns the existence and uniqueness theorem for the following BDSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)\overleftarrow{B}(ds, X_s) - \int_t^T Z_s dW_s,\ t \in [0, T], \qquad (3.1)$$

where $(X_s = X_s^{0,x}, 0 \le s \le T)$ is the unique solution to (1.2). Denote, for $p \ge 1$,

$$S^p([0,T];\mathbb{R}) = \left\{h : \Omega \times [0,T] \to \mathbb{R};\ \text{continuous}, h(t) \text{ is } \mathcal{F}_t\text{-measurable, and } \mathbb{E}\big[\sup_{0\le t\le T} |h(t)|^p\big] < \infty\right\},$$

$$M^p([0,T];\mathbb{R}^l) = \left\{\varphi : \Omega \times [0,T] \to \mathbb{R}^l;\ \varphi(t) \text{ is } \mathcal{F}_t\text{-measurable and } \mathbb{E}\int_0^T |\varphi(t)|^p dt < \infty\right\},$$

$$Q^{2p}([0,T];\mathbb{R}) = \left\{g : \Omega \times [0,T] \to \mathbb{R};\ g(t) \text{ is } \mathcal{F}_t\text{-measurable and } \mathbb{E}\int_0^T |g(t)|^{2p}|q(t, X_t, X_t)|^p dt < \infty\right\},$$

where $\mathcal{F}_t$ is given in (2.1).

We will follow the standard procedure in [20]. First, as a preparation, we prove the following existence and uniqueness result when f and g are independent of Y and Z.

Proposition 3.1 *Let $f \in M^2([0,T];\mathbb{R}), g \in Q^2([0,T];\mathbb{R})$ and $\xi \in L^2(\mathcal{F}_T)$. Then the equation*

$$Y_t = \xi + \int_t^T f(s)ds + \int_t^T g(s)d\overleftarrow{B}(ds, X_s) - \int_t^T Z_s dW_s,\ \ t \in [0,T], \qquad (3.2)$$

has a unique solution $(Y, Z) \in S^2([0,T];\mathbb{R}) \times M^2([0,T];\mathbb{R}^d)$.

Proof. First we discuss the uniqueness. Suppose $(Y^i, Z^i), i = 1, 2$, are two solutions in $M^2([0,T];\mathbb{R}) \times M^2([0,T];\mathbb{R}^d)$. Denote $\bar{Y} = Y^1 - Y^2$ and $\bar{Z} = Z^1 - Z^2$. Then

$$\bar{Y}_t + \int_t^T \bar{Z}_s dW_s = 0,$$

and hence

$$\mathbb{E}(\bar{Y}_t^2) + \mathbb{E}\int_t^T |\bar{Z}_s|^2 ds = 0$$

because $\bar{Y}_t \in \mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$ and $\mathbb{E}^W(\bar{Y}_t \int_t^T \bar{Z}_s dW_s) = 0$, where $\mathbb{E}^W$ means the expectation taken in the probability space generated by W. This immediately implies the uniqueness.

Now, we consider the existence. Denote $d\overleftarrow{B}(ds, X_s)$ by dM_s and let

$$N_t = \mathbb{E}\left[\xi + \int_0^T f(s)ds + \int_0^T g(s)dM_s \middle| \mathcal{G}_t\right],$$

which is a martingale with respect to the filtration $\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_T^B$. Then by the martingale representation theorem, there exists a square integrable process $Z_t \in \mathcal{G}_t$ such that

$$N_t = N_0 + \int_0^t Z_s dW_s. \tag{3.3}$$

Letting $t = T$ we have $N_T = N_0 + \int_0^T Z_s dW_s$. On the other hand, from the definition of N_t, we have

$$N_T = \xi + \int_0^T f(s)ds + \int_0^T g(s)dM_s\,.$$

Thus, we have

$$\begin{aligned} N_t &= N_0 + \int_0^t Z_s dW_s = N_T - \int_t^T Z_s dW_s \\ &= \xi + \int_0^T f(s)ds + \int_0^T g(s)dM_s - \int_t^T Z_s dW_s\,. \end{aligned}$$

Namely, we have

$$\mathbb{E}\left[\xi + \int_0^T f(s)ds + \int_0^T g(s)dM_s \middle| \mathcal{G}_t\right] = \xi + \int_0^T f(s)ds + \int_0^T g(s)dM_s - \int_t^T Z_s dW_s. \tag{3.4}$$

Let

$$Y_t = \mathbb{E}\left[\xi + \int_t^T f(s)ds + \int_t^T g(s)dM_s \middle| \mathcal{G}_t\right]. \tag{3.5}$$

Then by (3.4),

$$Y_t = \xi + \int_t^T f(s)ds + \int_t^T g(s)dM_s - \int_t^T Z_s dW_s.$$

Thus (Y_t, Z_t) given by (3.3) and (3.5) satisfies (3.2).

Now we show $(Y_t, Z_t) \in \mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$. Note that $Y_t \in \mathcal{F}_t$ because $\xi + \int_t^T f(s)ds + \int_t^T g(s)dM_s \in \mathcal{F}_T^W \vee \mathcal{F}_{t,T}^B$ and its expectation conditional on $\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_T^B$ is measurable with respect to $\mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$. Similarly, we have $\int_t^T Z_s dW_s = \xi + \int_t^T f(s)ds + \int_t^T g(s)dM_s - Y_t$ is $\mathcal{F}_T^W \vee \mathcal{F}_{t,T}^B$ measurable since the right-hand side is. By the martingale representation theory, Z_s is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$ measurable for $s \in [t, T]$, and hence Z_t is $\mathcal{F}_t$-measurable.

Finally, we show the square integrability of Y and Z. By equations (2.4) and (3.5), Hölder inequality and Burkholder-Davis-Gundy inequality, there exists a constant C depending only on T such that

$$\begin{aligned} \mathbb{E}\left(\sup_{0\le t\le T} |Y_t|^2\right) \le& C\left(\mathbb{E}(|\xi|^2) + \mathbb{E}\int_0^T |f(s)|^2 ds + \mathbb{E}\int_0^T g^2(s) q(s, X_s, X_s) ds\right) \\ <&\infty. \end{aligned}$$

Hence $Y \in S^2([0,T];\mathbb{R})$. On the other hand, noting

$$\int_0^T Z_s dW_s = -Y_0 + \xi + \int_0^T f(s)ds + \int_0^T g(s)dM_s,$$

by Burkholder-Davis-Gundy inequality, we have $Z \in \mathbb{M}^2([0,T];\mathbb{R}^d)$. ∎

Now we are ready to prove the main result Theorem 3.2 in this section. Assume the following conditions.

(A1) Let the functions b and σ be bounded and satisfy the global Lipschitz condition:

$$|b(x) - b(y)| + |\sigma(x) - \sigma(y)| \le K|x - y|\,,$$

where we use $|\cdot|$ to denote both the Euclidean norm for a vector in $\mathbb{R}^d$ and the Hilbert-Schmidt norm for a matrix in $\mathbb{R}^{d\times d}$.

(B1) Let f and g be two given functions such that $f(\cdot,0,0) \in M^2([0,T];\mathbb{R})$, $g(\cdot,0,0) \in Q^2([0,T];\mathbb{R})$, and

$$\begin{aligned} |f(t,y_1,z_1) - f(t,y_2,z_2)|^2 &\le K(|y_1-y_2|^2 + |z_1-z_2|^2)\,; \\ |g(t,y_1,z_1) - g(t,y_2,z_2)|^2 &\le K|y_1-y_2|^2 + \alpha_t(\omega)|z_1-z_2|^2\,, \end{aligned}$$

where $\alpha_t(\omega)q(t,X_t,X_t) \le \alpha$ a.s. for some constant $\alpha \in (0,1)$.

Theorem 3.2 *Let the conditions* **(H)**, **(A1)** *and* **(B1)** *be satisfied. Assume $\xi \in L^p(\mathcal{F}_T)$ for some $p > 2$, the BDSDE (3.1) has a unique solution $(Y,Z) \in \mathcal{M}^{2,\beta}$ for some $\beta > 0$, where the space $\mathcal{M}^{2,\beta}$ is defined in (3.7). Furthermore, $Y \in S^2([0,T];\mathbb{R})$.*

Remark 3.3 *In [15], the authors considered the following BDSDE*

$$Y_s = \xi + \int_s^T f(r,Y_r,Z_r)dr + \int_s^T M(dr, \tilde{g}(r,Y_r,Z_r)) - \int_s^T Z_r dW_r. \qquad (3.6)$$

Consider their case when $k = 1$ and $l = d+1$. If we let $M(t,x,y) = yB(t,x)$ for $x \in \mathbb{R}^d, y \in \mathbb{R}$ and $\tilde{g}(r,Y_r,Z_r) = (X_r, g(r,Y_r,Z_r))$, then equation (3.6) is reduced to BDSDE (3.1). Now the joint quadratic variation of M is given by

$$\langle M(\cdot,x,y), M(\cdot,x',y')\rangle_t = \langle yB(\cdot,x), y'B(\cdot,x')\rangle_t = yy'\int_0^t q(s,x,x')ds,$$

and thus the characteristic of the family of the local martingales $\{M(\cdot,x,y),(x,y) \in \mathbb{R}^{d+1}\}$ is $a(s,(x,y),(x',y')) = yy'q(s,x,x')$. In [15], to obtain the existence and uniqueness of the solution to BDSDE (3.6)*, the authors imposed the following condition (inequality (3) on page 5) on the characteristic a when $k = 1$*

$$|a(s,z,z) - a(s,z,z') - a(s,z',z) + a(s,z',z')| \le |z - z'|^2,$$

where $z = (x,y)$ and $z' = (x',y')$. This condition implies that $\sup_{0\le s\le T,\, x\in\mathbb{R}^d} q(s,x,x) \le 1$ if we let $x = x'$, and that $\sup_{0\le s\le T,\, y\in\mathbb{R}} y^2|q(s,x,x) - 2q(s,x,x') + q(s,x',x')| \le |x-x'|^2$ if we let $y = y'$, which further implies that $q(s,x,y) \equiv C(s)$ for some deterministic function $C(s) \in [0,1]$.

Proof. Define

$$\mathcal{M}^{2,\beta} = \left\{ (y,z) : (\mathcal{F}_t)\text{-adapted and } \mathbb{E}\int_0^T \tilde{q}_s \exp\left(\beta \int_s^T \tilde{q}_r dr\right) |y_s|^2 ds \right.$$
$$\left. + \mathbb{E}\int_0^T \exp\left(\beta \int_s^T \tilde{q}_r dr\right) |z_s|^2 ds < \infty \right\} \tag{3.7}$$

with $\beta > 0$ to be determined later and $\tilde{q}_r = q(r, X_r, X_r) \vee 1$.

For any $(y,z) \in \mathcal{M}^{2,\beta} \subset M^2([0,T];\mathbb{R}) \times M^2([0,T];\mathbb{R}^d)$, by condition **(B1)** and Proposition 3.1, there exists a unique pair $(Y,Z) \in S^2([0,T];\mathbb{R}) \times M^2([0,T];\mathbb{R}^d)$ so that

$$Y_t = \xi + \int_t^T f(s, y_s, z_s) ds + \int_t^T g(s, y_s, z_s) dM_s - \int_t^T Z_s dW_s, \tag{3.8}$$

where we take the notation $dM_s = d\overleftarrow{B}(s, X_s)$.

Step 1. In this step, we shall show the mapping defined by (3.8) maps $\mathcal{M}^{2,\beta}$ to itself, i.e. $(Y,Z) \in \mathcal{M}^{2,\beta}$, for all $\beta > 0$.

Denote $f_s := f(s, y_s, z_s)$ and $g_s := g(s, y_s, z_s)$. Applying Lemma 2.1 and Lemma 2.2 to $Y_t^2 \exp\left(\beta \int_0^t \tilde{q}_s^{(n)} ds\right)$ where $\tilde{q}_s^{(n)} = \tilde{q}_s \wedge n$, we have

$$\begin{aligned}
&Y_t^2 \exp\left(\beta \int_0^t \tilde{q}_s^{(n)} ds\right) \\
=&\xi^2 \exp\left(\beta \int_0^T \tilde{q}_s^{(n)} ds\right) - \int_t^T \beta \tilde{q}_s^{(n)} \exp\left(\beta \int_0^s \tilde{q}_r^{(n)} dr\right) Y_s^2 ds - \int_t^T \exp\left(\beta \int_0^s \tilde{q}_r^{(n)} dr\right) |Z_s|^2 ds \\
&+ 2\int_t^T \exp\left(\beta \int_0^s \tilde{q}_r^{(n)} dr\right) Y_s f_s ds + \int_t^T \exp\left(\beta \int_0^s \tilde{q}_r^{(n)} dr\right) g_s^2 q(s, X_s, X_s) ds \\
&- 2\int_t^T \exp\left(\beta \int_0^s \tilde{q}_r^{(n)} dr\right) Y_s Z_s dW_s + 2\int_t^T \exp\left(\beta \int_0^s \tilde{q}_r^{(n)} dr\right) Y_s g_s dM_s.
\end{aligned} \tag{3.9}$$

The expectations of these two stochastic integrals on the right-hand side of the above equation are equal to zero. Here we only show that

$$\mathbb{E}\int_t^T \exp\left(\beta \int_0^s \tilde{q}_r^{(n)} dr\right) Y_s g_s dM_s = 0, \tag{3.10}$$

and the other one can be proven in a similar way. In fact, by Burkholder-Davis-Gundy inequality,

$$\begin{aligned}
&\mathbb{E} \sup_{0 \le t \le T} \int_t^T \exp\left(\beta \int_0^s \tilde{q}_r^{(n)} dr\right) Y_s g_s dM_s \\
\le &C\mathbb{E}\left(\int_0^T Y_t^2 g_t^2 q(t, X_t, X_t) dt\right)^{\frac{1}{2}}
\end{aligned}$$

$$\le C\mathbb{E}\left(\sup_{0\le t\le T}|Y_t|^2\int_0^T(g^2(t,0,0)+y_t^2+\alpha_t(\omega)|z_t|^2)q(t,X_t,X_t)dt\right)^{\frac{1}{2}}$$
$$\le C\left(\mathbb{E}(\sup_{0\le t\le T}|Y_t|^2)+\mathbb{E}\int_0^T(g^2(t,0,0)+y_t^2+\alpha_t(\omega)|z_t|^2)q(t,X_t,X_t)dt\right)$$
$$<\infty,$$

where the last inequality follows from the facts $Y\in S^2([0,T];\mathbb{R}),\ (y,z)\in\mathcal{M}^{2,\beta}$, condition **(B1)**, Lemma 2.3 and Hölder inequality. This implies (3.10).

Now taking expectation in the equation (3.9), we have

$$\mathbb{E}\left(Y_t^2\exp\left(\beta\int_0^t\tilde{q}_s^{(n)}ds\right)\right)+\mathbb{E}\int_t^T\beta\tilde{q}_s^{(n)}\exp\left(\beta\int_0^s\tilde{q}_r^{(n)}dr\right)Y_s^2ds+\mathbb{E}\int_t^T\exp\left(\beta\int_0^s\tilde{q}_r^{(n)}dr\right)|Z_s|^2ds$$
$$=\mathbb{E}\left(\xi^2\exp\left(\beta\int_0^T\tilde{q}_s^{(n)}ds\right)\right)+2\mathbb{E}\int_t^T\exp\left(\beta\int_0^s\tilde{q}_r^{(n)}dr\right)Y_sf_sds$$
$$+\mathbb{E}\int_t^T\exp\left(\beta\int_0^s\tilde{q}_r^{(n)}dr\right)g_s^2q(s,X_s,X_s)ds$$
$$\le\mathbb{E}\left(\xi^2\exp\left(\beta\int_0^T\tilde{q}_s^{(n)}ds\right)\right)+2\sqrt{K}\mathbb{E}\int_t^T\exp\left(\beta\int_0^s\tilde{q}_r^{(n)}dr\right)|Y_s|\left(|f(s,0,0)|+|y_s|+|z_s|\right)ds$$
$$+2\mathbb{E}\int_t^T\exp\left(\beta\int_0^s\tilde{q}_r^{(n)}dr\right)\left(g^2(s,0,0)+K(y_s^2+\alpha_s(\omega)|z_s|^2)\right)q(s,X_s,X_s)ds$$
$$\le\mathbb{E}\left(\xi^2\exp\left(\beta\int_0^T\tilde{q}_s^{(n)}ds\right)\right)+\sqrt{K}\mathbb{E}\int_t^T\tilde{q}_s^{(n)}\exp\left(\beta\int_0^s\tilde{q}_r^{(n)}dr\right)\left(\delta Y_s^2+\frac{3}{\delta}(|f(s,0,0)|^2+|y_s|^2+|z_s|^2)\right)ds$$
$$+K\mathbb{E}\int_t^T\exp\left(\beta\int_0^s\tilde{q}_r^{(n)}dr\right)(g^2(s,0,0)+y_s^2+\alpha_s(\omega)|z_s|^2)q(s,X_s,X_s)ds,$$

where in the last step we used the fact $2ab\le\delta a^2+\frac{1}{\delta}b^2$ for any $\delta>0$. Choose $\delta>0$ sufficiently small such that $\beta>\sqrt{K}\delta$, and then we have

$$(\beta-\sqrt{K}\delta)\mathbb{E}\int_0^T\tilde{q}_s^{(n)}\exp\left(\beta\int_0^s\tilde{q}_r^{(n)}dr\right)Y_s^2ds+\mathbb{E}\int_0^T\exp\left(\beta\int_0^s\tilde{q}_r^{(n)}dr\right)|Z_s|^2ds$$
$$\le\mathbb{E}\left(\xi^2\exp\left(\beta\int_0^T\tilde{q}_s^{(n)}ds\right)\right)+\frac{3\sqrt{K}}{\delta}\int_0^T\exp\left(\beta\int_s^T\tilde{q}_r^{(n)}dr\right)(|f(s,0,0)|^2+|y_s|^2+|z_s|^2)ds$$
$$+2\mathbb{E}\int_0^T\exp\left(\beta\int_0^s\tilde{q}_r^{(n)}dr\right)\left(g^2(s,0,0)+K(y_s^2+\alpha_s(\omega)|z_s|^2)\right)q(s,X_s,X_s)ds$$
$$\le\mathbb{E}\left(\xi^2\exp\left(\beta\int_0^T\tilde{q}_sds\right)\right)+\frac{3\sqrt{K}}{\delta}\mathbb{E}\int_0^T\exp\left(\beta\int_0^s\tilde{q}_rdr\right)(|f(s,0,0)|^2+|y_s|^2+|z_s|^2)ds$$
$$+2\mathbb{E}\int_0^T\exp\left(\beta\int_0^s\tilde{q}_rdr\right)\left(g^2(s,0,0)+K(y_s^2+\alpha_s(\omega)|z_s|^2)\right)q(s,X_s,X_s)ds. \tag{3.11}$$

Here $\mathbb{E}\left(\xi^2\exp\left(\beta\int_0^T\tilde{q}_sds\right)\right)<\infty$ because of Lemma 2.3 and the condition $\xi\in L^p$ for some $p>2$. The last two integrals in (3.11) are both finite because of Lemma 2.3, conditions **(B1)** and the fact $(y,z)\in\mathcal{M}^{2,\beta}$.

Now let n go to infinity. Then denoting by $C(\delta)$ the sum of the terms on the right-hand side of (3.11), by the Monotone Convergence Theorem we have

$$(\beta-\sqrt{K}\delta)\mathbb{E}\int_0^T \tilde{q}_s \exp\left(\beta\int_t^T \tilde{q}_r dr\right) Y_s^2 ds+\mathbb{E}\int_0^T \exp\left(\beta\int_t^T \tilde{q}_r dr\right)|Z_s|^2 ds\le C(\delta)<\infty.$$

Therefore $(Y,Z)\in\mathcal{M}^{2,\beta}$, and the mechanism (3.8) defines a mapping Ψ from $\mathcal{M}^{2,\beta}$ to itself: $(Y,Z)=\Psi(y,z)$.

Step 2. In this step, we shall prove that Ψ is a contraction mapping on $\mathcal{M}^{2,\beta}$ for sufficiently large β.

Let $(Y^i,Z^i)=\Psi(y^i,z^i)$ for $i=1,2$, $\bar{f}_s=f(s,y_s^1,z_s^1)-f(s,y_s^2,z_s^2)$, and $\bar{g}_s=g(s,y_s^1,z_s^1)-g(s,y_s^2,z_s^2)$. Then

$$Y_t^1-Y_t^2=\int_t^T \bar{f}_s ds+\int_t^T \bar{g}_s dM_s-\int_t^T \bar{Z}_s dW_s.$$

We shall use a generic notation $\bar{h}=h^1-h^2$, where h can be Y,Z,y and z. Applying Lemma 2.1 and Lemma 2.2 to $\bar{Y}_t^2\exp\left(\beta\int_0^t \tilde{q}_s^{(n)}ds\right)$, we have

$$\begin{aligned}
&\bar{Y}_t^2\exp\left(\beta\int_0^t \tilde{q}_s^{(n)}ds\right)\\
=&-\int_t^T \beta\tilde{q}_s^{(n)}\exp\left(\beta\int_0^s \tilde{q}_r^{(n)}dr\right)\bar{Y}_s^2 ds-\int_t^T \exp\left(\beta\int_0^s \tilde{q}_r^{(n)}dr\right)|\bar{Z}_s|^2 ds\\
&+2\int_t^T \exp\left(\beta\int_0^s \tilde{q}_r^{(n)}dr\right)\bar{Y}_s\bar{f}_s ds+\int_t^T \exp\left(\beta\int_0^s \tilde{q}_r^{(n)}dr\right)\bar{g}_s^2 q(s,X_s,X_s)ds\\
&-2\int_t^T \exp\left(\beta\int_0^s \tilde{q}_r^{(n)}dr\right)\bar{Y}_s\bar{Z}_s dW_s+2\int_t^T \exp\left(\beta\int_0^s \tilde{q}_r^{(n)}dr\right)\bar{Y}_s\bar{g}_s dM_s.
\end{aligned}$$

As in **Step 1**, we can show that these two stochastic integrals on the right-hand side of the above equation are integrable and hence the expectations of them are zero. Taking expectation and letting n go to infinity, we have

$$\begin{aligned}
&\mathbb{E}\left(\bar{Y}_t^2\exp\left(\beta\int_0^t \tilde{q}_s ds\right)\right)+\mathbb{E}\int_t^T \beta\tilde{q}_s\exp\left(\beta\int_0^s \tilde{q}_r dr\right)\bar{Y}_s^2 ds+\mathbb{E}\int_t^T \exp\left(\beta\int_0^s \tilde{q}_r dr\right)|\bar{Z}_s|^2 ds\\
=&2\mathbb{E}\left(\int_t^T \exp\left(\beta\int_0^s \tilde{q}_r dr\right)\bar{Y}_s\bar{f}_s ds+\int_t^T \exp\left(\beta\int_0^s \tilde{q}_r dr\right)\bar{g}_s^2 q(s,X_s,X_s)ds\right)\\
\le&\mathbb{E}\left[\int_t^T \exp\left(\beta\int_0^s \tilde{q}_r dr\right)\left(2\sqrt{K}|\bar{Y}_s|(|\bar{y}_s|+|\bar{z}_s|)+K|\bar{y}_s|^2+\alpha_s(\omega)|\bar{z}_s|^2 q(s,X_s,X_s)\right)ds\right]\\
\le&\mathbb{E}\left[\int_t^T \exp\left(\beta\int_0^s \tilde{q}_r dr\right)\left(\frac{2}{a}|\bar{Y}_s|^2+aK|\bar{y}_s|^2+aK|\bar{z}_s|^2+K|\bar{y}_s|^2 q(s,X_s,X_s)+\alpha|\bar{z}_s|^2\right)ds\right]\\
\le&\frac{2}{a}\,\mathbb{E}\int_t^T \tilde{q}_s\exp\left(\beta\int_0^s \tilde{q}_r dr\right)|\bar{Y}_s|^2 ds+(Ka+K)\mathbb{E}\int_t^T \tilde{q}_s\exp\left(\beta\int_0^s \tilde{q}_r dr\right)|\bar{y}_s|^2 ds\\
&+(Ka+\alpha)\mathbb{E}\int_t^T \exp\left(\beta\int_0^s \tilde{q}_r dr\right)|\bar{z}_s|^2 ds,
\end{aligned}$$

where in the second inequality we used the fact $2xy \le \frac{1}{a}x^2 + ay^2$ for any $a > 0$. This implies

$$\left(\beta - \frac{2}{a}\right)\mathbb{E}\int_0^t \tilde{q}_s \exp\left(\beta\int_s^T \tilde{q}_r dr\right)\bar{Y}_s^2 ds + \mathbb{E}\int_0^t \exp\left(\beta\int_s^T \tilde{q}_r dr\right)|\bar{Z}_s|^2 ds$$
$$\le K(a+1)\mathbb{E}\int_0^t \tilde{q}_s \exp\left(\beta\int_s^T \tilde{q}_r dr\right)|\bar{y}_s|^2 ds + (Ka+\alpha)\mathbb{E}\int_0^t \exp\left(\beta\int_s^T \tilde{q}_r dr\right)|\bar{z}_s|^2 ds.$$

Since $0 < \alpha < 1$, we may choose $a > 0$ so that $Ka + \alpha < 1$. Choose β such that $\beta - \frac{2}{a} = \frac{K(a+1)}{Ka+\alpha}$. Let $\rho = \frac{K(a+1)}{\beta - \frac{2}{a}} = Ka + \alpha < 1$, and then we have

$$\|(\bar{Y}, \bar{Z})\|_{2,\beta}^2 \le \rho\|(\bar{y}, \bar{z})\|_{2,\beta}^2,$$

where

$$\|(y,z)\|_{2,\beta}^2 = \mathbb{E}\int_0^T \tilde{q}_s \exp\left(\beta\int_s^T \tilde{q}_r dr\right)|y_s|^2 ds + \frac{Ka+\alpha}{K(a+1)}\mathbb{E}\int_0^T \exp\left(\beta\int_s^T \tilde{q}_r dr\right)|z_s|^2 ds.$$

Therefore Ψ is a contraction mapping in the space $\mathcal{M}^{2,\beta}$ endorsed with the norm $\|\cdot\|_{2,\beta}$.

Step 3. Finally, the fact that $Y \in S^2([0,T];\mathbb{R})$ can be proven in the same way as in Proposition 3.1. ∎

4 Moments of the solution

In this section, we show that under suitable conditions, the solution to (3.1) has finite p-moments for $p > 2$, which will be used in Section 5 to obtain the regularity of the solution.

Basic calculations yield the following lemma on $\phi_n(x)$, where $\phi_n(x)$ is an approximation of $|x|^{2p}$ at quadratic growth as $|x|$ tends to infinity. The lemma will be used in the proof of Theorem 4.2.

Lemma 4.1 *Fix $p > 1$. For $x \in \mathbb{R}$, define*

$$\varphi_n(x) = (|x| \wedge n)^p + pn^{p-1}(|x| - n)^+.$$

Then $\varphi_n(x)$ is a convex function with

$$\varphi_n'(x) = p|x|^{p-1}I_{[|x|\le n]} + pn^{p-1}I_{[|x|>n]}, \quad \varphi_n''(x) = p(p-1)|x|^{p-2}I_{[|x|\le n]},$$

where $\varphi_n''(x)$ is defined as the Randon-Nikodym derivative $d\varphi_n'(x)/dx$.

Let $\phi_n(x) = \varphi_n(x^2)$. Then we have

$$|\phi_n'(x)\cdot x| \le 2p\phi_n(x), \quad |\phi_n''(x)\cdot x^2| \le 2p(2p-1)\phi_n(x).$$

Furthermore, we also have the estimations

$$
\begin{aligned}
&|\phi_n'(x)| \le (2p)^{\frac{1}{2}} \left(\phi_n(x)\right)^{\frac{1}{2}} \left(\phi_n''(x)\right)^{\frac{1}{2}} ,\\
&|\phi_n'(x)|^{\frac{2p}{2p-1}} \le (2p)^{\frac{2p}{2p-1}} \phi_n(x),\\
&|\phi_n''(x)|^{\frac{p}{p-1}} \le (2p)^{\frac{p}{p-1}} \phi_n(x),
\end{aligned}
$$

and for any $\gamma \in (0,1)$*,*

$$
\phi_n''(x)|x|^{2\gamma} \le C_{p,\gamma}(\phi_n(x))^{1-\frac{1}{p}(1-\gamma)},
$$

where $C_{p,\gamma}$ *is a constant depending only on* (p,γ)*.*

Theorem 4.2 *In addition to the conditions* **(H)**, **(A1)** *and* **(B1)** *we assume*

(i) $|g(t,y,z)|^2 \le C(g^2(t,0,0) + |y|^{2\gamma}) + \alpha_t(\omega)|z|^2$, $\gamma \in (0,1)$, $\alpha_t(\omega) q(t,X_t,X_t) \le \alpha$ *a.s. for some constant* $\alpha < 1$*;*

(ii) for some $p > 1, \xi \in L^{2p}(\Omega, \mathcal{F}_T, \mathbb{P}), f(\cdot,0,0) \in M^{2p}([0,T];\mathbb{R})$ *and* $g(\cdot,0,0) \in Q^{2p}([0,T];\mathbb{R})$*.*

Then

$$
\mathbb{E}\left(\sup_{0\le t\le T} |Y_t|^{2p} + \left(\int_0^T |Z_t|^2 dt \right)^p \right) < \infty \, . \tag{4.1}
$$

Proof. Let $\phi_n(x)$ be the one defined in Lemma 4.1. Note that $\phi_n(x)$ is convex and $d\phi_n'(x) = \phi_n''(x)dx$. Applying Lemma 2.1 to $\phi_n(Y_t)$, we have

$$
\phi_n(Y_t) + \frac{1}{2}\int_t^T \phi_n''(Y_s)|Z_s|^2 ds = I_1 + \int_t^T \phi_n'(Y_s) \left[g(s,Y_s,Z_s)\overleftarrow{B}(ds,X_s) - Z_s dW_s \right] , \tag{4.2}
$$

where

$$
\begin{aligned}
I_1 :=& \phi_n(\xi) + \int_t^T \phi_n'(Y_s) f(s,Y_s,Z_s)ds + \frac{1}{2}\int_t^T \phi_n''(Y_s) g^2(s,Y_s,Z_s) q(s,X_s,X_s)ds\\
\le& |\xi|^{2p} + C\int_t^T |\phi_n'(Y_s)|(|f(s,0,0)| + |Y_s| + |Z_s|)ds\\
&+ C\int_t^T \phi_n''(Y_s)(g^2(s,0,0) + |Y_s|^{2\gamma}) q(s,X_s,X_s)ds + \frac{1}{2}\alpha \int_t^T \phi_n''(Y_s)|Z_s|^2 ds.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
&\phi_n(Y_t) + \frac{1}{2}\int_t^T \phi_n''(Y_s)|Z_s|^2 ds\\
\le& |\xi|^{2p} + C\int_t^T |\phi_n'(Y_s)|(|f(s,0,0)| + |Y_s| + |Z_s|)ds\\
&+ C\int_t^T \phi_n''(Y_s)(g^2(s,0,0) + |Y_s|^{2\gamma}) q(s,X_s,X_s)ds + \frac{1}{2}\alpha \int_t^T \phi_n''(Y_s)|Z_s|^2 ds
\end{aligned}
$$

$$+\int_t^T \phi_n'(Y_s)\left[g(s,Y_s,Z_s)\overleftarrow{B}(ds,X_s)-Z_s dW_s\right]. \tag{4.3}$$

Since $\phi_n'(x)$ is at linear growth when $|x|$ goes to infinity, by the conditions on g and the facts $(Y,Z)\in\mathcal{M}^{2,\beta}, Y\in S^2([0,T];\mathbb{R})$, we can prove

$$\mathbb{E}\int_t^T \phi_n'(Y_s)\left[g(s,Y_s,Z_s)\overleftarrow{B}(ds,X_s)-Z_s dW_s\right]=0$$

in the same spirit of proof for equation (3.10). Therefore, by taking expectation of (4.3) we have

$$\begin{aligned}
&\mathbb{E}(\phi_n(Y_t))+\frac{1}{2}\mathbb{E}\int_t^T \phi_n''(Y_s)|Z_s|^2 ds\\
\le&\mathbb{E}|\xi|^{2p}+C\mathbb{E}\int_t^T |\phi_n'(Y_s)|(|f(s,0,0)|+|Y_s|+|Z_s|)ds\\
&+C\mathbb{E}\int_t^T \phi_n''(Y_s)(g^2(s,0,0)+|Y_s|^{2\gamma})q(s,X_s,X_s)ds+\frac{1}{2}\alpha\int_t^T \phi_n''(Y_s)|Z_s|^2 ds\\
\le&\mathbb{E}|\xi|^{2p}+C\mathbb{E}\int_t^T \left(|\phi_n(Y_s)|^{\frac{2p-1}{2p}}|f(s,0,0)|+\phi_n(Y_s)+(\phi_n(Y_s))^{\frac{1}{2}}(\phi_n''(Y_s))^{\frac{1}{2}}|Z_s|\right)ds\\
&+C\mathbb{E}\int_t^T \left((\phi_n(Y_s))^{\frac{p-1}{p}}g^2(s,0,0)+(\phi_n(Y_s))^{1-\frac{1}{p}(1-\gamma)}\right)q(s,X_s,X_s)ds+\frac{1}{2}\alpha\int_t^T \phi_n''(Y_s)|Z_s|^2 ds,
\end{aligned} \tag{4.4}$$

where the last inequality follows from Lemma 4.1. By Young's inequality, we have

$$|\phi_n(Y_s)|^{\frac{2p-1}{2p}}|f(s,0,0)|\le\frac{2p-1}{2p}|\phi_n(Y_s)|+\frac{1}{2p}|f(s,0,0)|^{2p}, \tag{4.5}$$

$$(\phi_n(Y_s))^{\frac{1}{2}}(\phi_n''(Y_s))^{\frac{1}{2}}|Z_s|\le\frac{1}{a}\phi_n(Y_s)+a\phi_n''(Y_s)|Z_s|^2,\quad\text{for any } a>0, \tag{4.6}$$

and

$$\begin{aligned}
&(\phi_n(Y_s))^{\frac{p-1}{p}}g^2(s,0,0)q(s,X_s,X_s)+(\phi_n(Y_s))^{1-\frac{1}{p}(1-\gamma)}q(s,X_s,X_s)\\
\le\ &\frac{p-1}{p}\phi_n(Y_s)+\frac{1}{p}|g(s,0,0)|^{2p}q(s,X_s,X_s)^p+\frac{p-1+\gamma}{p}\phi_n(Y_s)+\frac{1-\gamma}{p}q(s,X_s,X_s)^{\frac{p}{1-\gamma}}.
\end{aligned} \tag{4.7}$$

Choosing sufficiently small a, we may find constants $\theta<1$ and $C<\infty$ independent of n, such that after substituting (4.5)-(4.7) into (4.4), by Lemma 2.3 and conditions for $f(\cdot,0,0)$ and $g(\cdot,0,0)$, we have

$$\begin{aligned}
&\mathbb{E}(\phi_n(Y_t))+\frac{1}{2}\mathbb{E}\int_t^T \phi_n''(Y_s)|Z_s|^2 ds\\
\le&C+C\int_t^T \mathbb{E}(\phi_n(Y_s))ds+\frac{1}{2}\theta\mathbb{E}\int_t^T \phi_n''(Y_s)|Z_s|^2 ds.
\end{aligned}$$

All the terms in the above equation are finite, since $\phi_n(x)$ is at quadratic growth when $|x| \to \infty$ and $\phi_n''(x)$ is bounded. Then it follows from the Gronwall's Lemma that, for all $n \in \mathbb{N}^+$,

$$\sup_{0\le t\le T} \mathbb{E}\left(\phi_n(Y_t)\right) < Ce^{CT},$$

and hence

$$\sup_{0\le t\le T} \mathbb{E}\left(\phi_n(Y_t)\right) + \mathbb{E}\int_0^T \phi_n''(Y_s)|Z_s|^2 ds < C,$$

for some constant C independent of n. Letting n go to infinity and noting that the derivatives $\phi_n^{(i)}(x) \nearrow (|x|^{2p})^{(i)}$ for $i = 0, 1, 2$, we have

$$\sup_{0\le t\le T} \mathbb{E}(|Y_t|^{2p}) + \mathbb{E}\int_0^T |Y_t|^{2p-2}|Z_t|^2 dt < \infty. \tag{4.8}$$

Therefore, by (4.2) and the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
\mathbb{E}\left(\sup_{0\le t\le T} \phi_n(Y_t)\right) \le & C + C\mathbb{E}\left(\int_0^T (\phi_n'(Y_s))^2 \left(g^2(s, Y_s, Z_s)q(s, X_s, X_s) + |Z_s|^2\right) ds\right)^{\frac{1}{2}} \\
\le & C + C\mathbb{E}\left[\sup_{0\le t\le T} (\phi_n(Y_t))^{\frac{1}{2}} \left(\int_0^T |Y_s|^{2p-2} \left(g^2(s, Y_s, Z_s)q(s, X_s, X_s) + |Z_s|^2\right) ds\right)^{\frac{1}{2}}\right] \\
\le & C + \frac{1}{2}\mathbb{E}\left(\sup_{0\le t\le T} \phi_n(Y_t)\right) + C\mathbb{E}\int_0^T |Y_s|^{2p-2} \left(g^2(s, Y_s, Z_s)q(s, X_s, X_s) + |Z_s|^2\right) ds \\
\le & A + \frac{1}{2}\mathbb{E}\left(\sup_{0\le t\le T} \phi_n(Y_t)\right),
\end{aligned}$$

where $A < \infty$ is a constant independent of n, noting that

$$\begin{aligned}
& \mathbb{E}\int_0^T |Y_s|^{2p-2} \left(g^2(s, Y_s, Z_s)q(s, X_s, X_s) + |Z_s|^2\right) ds \\
\le & \mathbb{E}\int_0^T |Y_s|^{2p-2} \left(\left[C(g(s,0,0)^2 + |Y_s|^{2\gamma}) + \alpha_t|Z_s|^2\right] q(s, X_s, X_s) + |Z_s|^2\right) ds \\
\le & C\mathbb{E}\int_0^T |Y_s|^{2p-2} g^2(s,0,0)q(s, X_s, X_s) ds + C\mathbb{E}\int_0^T |Y_s|^{2p-2+2\gamma} ds + 2\mathbb{E}\int_0^T |Y_s|^{2p-2}|Z_s|^2 ds
\end{aligned}$$

is finite by Hölder's inequality, (4.8) and the condition that $g(\cdot, 0, 0) \in Q^{2p}([0, T]; \mathbb{R})$.

Thus we have, letting $n \to \infty$,

$$\mathbb{E}\left(\sup_{0\le t\le T} |Y_t|^{2p}\right) < \infty. \tag{4.9}$$

Let $(Y^{(n)}, Z^{(n)})_{n\in\mathbb{N}}$, where $(Y^{(0)}, Z^{(0)}) = (0, 0)$, be a sequence of processes generated by the mapping (3.8). Then $(Y^{(n)}, Z^{(n)})$ converges to the solution of (3.1) in the space $\mathcal{M}^{2,\beta}$

by Theorem 3.2. Denote $(Y^{(n-1)}, Z^{(n-1)})$ by (y, z) and $(Y^{(n)}, Z^{(n)})$ by (Y, Z). Applying Itô's formula to Y_T^2, we have

$$\int_0^T |Z_s|^2 ds \leq |\xi|^2 + 2\int_0^T Y_s f(s, y_s, z_s) ds + \int_0^T g^2(s, y_s, z_s) q(s, X_s, X_s) ds \\ + 2\int_0^T Y_s \left[g(s, y_s, z_s) \overleftarrow{B}(ds, X_s) - Z_s dW_s \right].$$

For any $\delta > 0$, using condition (i) and the fact that we can find $C(\delta)$ for any $\delta > 0$ such that $(b + a_1 + \cdots + a_n)^p \leq (1+\delta) b^p + C(\delta)(a_1^p + \cdots + a_n^p)$, we have

$$\left(\int_0^T |Z_s|^2 ds\right)^p \leq (1+\delta)\left(\alpha \int_0^T |z_s|^2 ds\right)^p + C(\delta)\Bigg[1 + |\xi|^{2p} + \left|\int_0^T Y_s g(s, y_s, z_s) \overleftarrow{B}(ds, X_s)\right|^p \\ + \left(\int_0^T (g^2(s,0,0) + |y_s|^{2\gamma}) q(s, X_s, X_s) ds\right)^p + \left(\int_0^T |Y_s||f(s, y_s, z_s)| ds\right)^p + \left|\int_0^T Y_s Z_s dW_s\right|^p\Bigg].$$

Taking expectation on both sides and noting that $\sup_n \sup_{0\leq t\leq T} \mathbb{E}|Y_t^{(n)}|^{2p} < \infty$, by (4.9), Lemma 2.3 and conditions (i), (ii), we can find some constants $C'(\delta)$ and $C''(\delta)$ such that

$$\left(\int_0^T |Z_s|^2 ds\right)^p \\ \leq (1+\delta)\left(\alpha \int_0^T |z_s|^2 ds\right)^p + C'(\delta)\Bigg[1 + \mathbb{E}\left(\int_0^T |Y_s|(f(s,0,0) + |y_s| + |z_s|) ds\right)^p \\ + \mathbb{E}\left(\int_0^T |Y_s|^2 (1 + |y_s|^{2\gamma} + \alpha_s |z_s|^2) q(s, X_s, X_s) ds\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T |Y_s|^2 |Z_s|^2 ds\right)^{\frac{p}{2}}\Bigg] \\ \leq (1+\delta)\alpha^p \mathbb{E}\left(\int_0^T |z_s|^2 ds\right)^p + C''(\delta)\Bigg[1 + \mathbb{E}\left(\int_0^T |Y_s||z_s| ds\right)^p \\ + \mathbb{E}\left(\int_0^T |Y_s|^2 |z_s|^2 ds\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T |Y_s|^2 |Z_s|^2 ds\right)^{\frac{p}{2}}\Bigg].$$

By Young's inequality, for any $a > 0$,

$$\mathbb{E}\left(\int_0^T |Y_s||z_s| ds\right)^p + \mathbb{E}\left(\int_0^T |Y_s|^2 |z_s|^2 ds\right)^{\frac{p}{2}} + \mathbb{E}\left(\int_0^T |Y_s|^2 |Z_s|^2 ds\right)^{\frac{p}{2}} \\ \leq \frac{1}{a}\left[\mathbb{E}\left(\int_0^T |Y_s|^2 ds\right)^p + 2T\mathbb{E}\sup_{0\leq s\leq T} |Y_s|^{2p}\right] + 2a\mathbb{E}\left(\int_0^T |z_s|^2 ds\right)^p + a\mathbb{E}\left(\int_0^T |Z_s|^2 ds\right)^p \\ \leq C(a) + 2a\mathbb{E}\left(\int_0^T |z_s|^2 ds\right)^p + a\mathbb{E}\left(\int_0^T |Z_s|^2 ds\right)^p.$$

Therefore,

$$(1-aC''(\delta))\mathbb{E}\left(\int_0^T |Z_s|^2 ds\right)^p \le [(1+\delta)\alpha^p + 2aC''(\delta)]\mathbb{E}\left(\int_0^T |z_s|^2 ds\right)^p + A(\delta,a).$$

One can choose δ and a small enough such that $(1+\delta)\alpha^p + 3aC''(\delta) < 1$. Denote $\rho = \frac{(1+\delta)\alpha^p+2aC''(\delta)}{1-aC''(\delta)}$ and $A'(\delta,a) = \frac{A(\delta,a)}{1-aC''(\delta)}$. Then $0<\rho<1$ and $A'(\delta,a)<\infty$, and we have

$$\mathbb{E}\left(\int_0^T |Z_s^{(n)}|^2 ds\right)^p \le \rho\mathbb{E}\left(\int_0^T |Z_s^{(n-1)}|^2 ds\right)^p + A'(\delta,a).$$

This yields

$$\mathbb{E}\left(\lim_{n\to\infty}\int_0^T |Z_s^{(n)}|^2 ds\right)^p \le \liminf_{n\to\infty}\mathbb{E}\left(\int_0^T |Z_s^{(n)}|^2 ds\right)^p < \infty. \tag{4.10}$$

The inequality (4.1) now follows from (4.9) and (4.10). ∎

5 BDSDEs and semilinear SPDEs

In this section, under proper conditions, we will obtain the regularity of the solution to the BDSDE, and then establish the relationship between the SPDE

$$\begin{aligned} u(t,x) = \phi(x) + &\int_t^T [\mathscr{L}u(s,x) + f(s,x,u(s,x),\nabla u(t,x)\sigma(x))]\, ds \\ &+ \int_t^T g(s,x,u(s,x),\nabla u(t,x)\sigma(x)) \overleftarrow{B}(ds,x), \quad t\in[0,T], \end{aligned} \tag{5.1}$$

and the BDSDE

$$\begin{aligned} Y_s^{t,x} =& \phi(X_T^{t,x}) + \int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr \\ &+ \int_s^T g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\overleftarrow{B}(dr,X_r^{t,x}) - \int_s^T Z_r^{t,x}dW_r, \quad s\in[t,T]. \end{aligned} \tag{5.2}$$

Assume the following condition for f and g.

(B2) Let $f,g:[0,\infty)\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}$ be two given functions satisfying for $t\in[0,T]$

$$|f(t,x,y_1,z_1)-f(t,x,y_2,z_2)|^2 \le K(|y_1-y_2|^2+|z_1-z_2|^2);$$

$$|g(t,x,y_1,z_1)-g(t,x,y_2,z_2)|^2 \le K|y_1-y_2|^2+\alpha_t(x)|z_1-z_2|^2,$$

where $\alpha_t(x)q(t,x,x)\le\alpha$ for some constant $\alpha\in(0,1)$.

Theorem 5.1 *Assume* **(H)**, **(A1)**, **(B2)** *and that ϕ is of class C^2. Let $\{u(t,x); 0\le t\le T, x\in\mathbb{R}^d\}$ be a random field such that $u(t,x)$ is $\mathcal{F}_{t,T}^B$-measurable for each (t,x), $u\in C^{0,2}([0,T]\times\mathbb{R}^d;\mathbb{R})$ a.s., and $u(t,x)$ satisfies (5.1). Then $u(t,x)=Y_t^{t,x}$, where $(Y_s^{t,x},Z_s^{t,x})_{t\le s\le T}$ is the unique solution to (5.2).*

Proof. We shall borrow the idea from the proof for [20, Theorem 3.1]. To prove the result, it suffices to show that $(u(s, X_s^{t,x}), \sigma(x)^T \nabla u(s, X_s^{t,x}); t \le s \le T)$ solves BDSDE (5.2). Letting $t = t_0 < t_1 < t_2 < \cdots < t_n = T$, by Itô's formula and equation (5.1), we have

$$\begin{aligned}
&\sum_{i=0}^{n-1}[u(t_i, X_{t_i}^{t,x}) - u(t_{i+1}, X_{t_{i+1}}^{t,x})] \\
=&\sum_{i=0}^{n-1}[u(t_i, X_{t_i}^{t,x}) - u(t_i, X_{t_{i+1}}^{t,x})] + \sum_{i=0}^{n-1}[u(t_i, X_{t_{i+1}}^{t,x}) - u(t_{i+1}, X_{t_{i+1}}^{t,x})] \\
=&\sum_{i=0}^{n-1}\Bigg[- \int_{t_i}^{t_{i+1}} \mathscr{L} u(t_i, X_s^{t,x}) ds - \int_{t_i}^{t_{i+1}} \sigma(x)^T \nabla u(t_i, X_s^{t,x}) dW_s \\
&\quad + \int_{t_i}^{t_{i+1}} \left[\mathscr{L} u(s, X_{t_{i+1}}^{t,x}) + f(s, X_{t_{i+1}}^{t,x}, u(s, X_{t_{i+1}}^{t,x}), \sigma(x)^T \nabla u(s, X_{t_{i+1}}^{t,x}))\right] ds \\
&\quad + \int_{t_i}^{t_{i+1}} g(s, X_{t_{i+1}}^{t,x}, u(s, X_{t_{i+1}}^{t,x}), \sigma(X_{t_{i+1}}^{t,x})^T \nabla u(s, X_{t_{i+1}}^{t,x})) \overleftarrow{B}(ds, X_{t_{i+1}}^{t,x}) \Bigg].
\end{aligned}$$

Let the mesh size go to zero and the result is concluded. ■

To get the converse of the above theorem, we need more path regularity of (Y, Z), for which we impose the following conditions.

(A2) $b \in C_b^3(\mathbb{R}^d; \mathbb{R}^d)$, $\sigma \in C_b^3(\mathbb{R}^d; \mathbb{R}^{d\times d})$, i.e. b and σ are bounded functions of class C^3 whose partial derivatives are also bounded.

(B3) Besides condition **(B2)**, we also assume

(i) For any $s \in [0, T]$, $f(s, \cdot, \cdot, \cdot)$ and $g(s, \cdot, \cdot, \cdot)$ are of class C^3, and all their partial derivatives are bounded on $[0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$.

(ii) g is uniformly bounded, $|g_z(t, x, y, z)|^2 q(t, x, x) \le \alpha < 1$, and $|g_y(t, x, y, z)|^2 q(t, x, x) < C < \infty$, for $(t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$.

(iii) $\phi \in C_p^3(\mathbb{R}^d; \mathbb{R})$, i.e. ϕ is of class C^3 whose partial derivatives are of polynomial growth.

Under the condition **(A2)**, it is known (see e.g. [23]) that the random field $\{X_s^{0,x}; 0 \le s \le T, x \in \mathbb{R}^d\}$ has a version of class C^2 in x, and of class C^1 in (s, x). Moreover, for fixed (t, x),

$$\sup_{t\le s\le T}(|X_s^{t,x}| + |\nabla X_s^{t,x}| + |\nabla^2 X_s^{t,x}|) \in \bigcap_{p\ge 1} L^p(\Omega).$$

First we establish the relationship between Y and Z. Denote by $D = (D^1, D^2, \cdots, D^d)$ the Malliavin derivative operator with respect to the Brownian motion $W = (W^1, W^2, \cdots, W^d)$.

Proposition 5.2 *Assume* **(H), (A2)** *and* **(B3)**. *Then* $Z_s^{t,x} = D_s Y_s^{t,x}$ *a.s., and furthermore the random process* $\{Z_s^{t,x}, t \le s \le T\}$ *has a continuous version given by*

$$Z_s^{t,x} = \nabla Y_s^{t,x} (\nabla X_s^{t,x})^{-1} \sigma(X_s^{t,x}),$$

and in particular,

$$Z_t^{t,x} = \nabla Y_t^{t,x} \sigma(x),$$

where $\nabla X_s^{t,x} := \left(\frac{\partial (X_s^{t,x})_i}{\partial x_j}\right)_{1 \le i,j \le d}$ *is the matrix of first order derivatives of* $X_s^{t,x}$ *with respect to the initial value* x *of* $X_s^{t,x}$, *and* $(\nabla Y_s^{t,x}, \nabla Z_s^{t,x})$ *is the unique solution to BDSDE (5.5). Moreover, for* $p \ge 1$,

$$\mathbb{E}\left(\sup_{0 \le s \le t} |Z_s^{t,x}|^p\right) < \infty. \tag{5.3}$$

Remark 5.3 *In this proposition,* $(\nabla Y_s^{t,x}, \nabla Z_s^{t,x})$ *is just a notation for the solution to BDSDE (5.5). After the regularity of* $Y_s^{t,x}$ *is obtained in Theorem 5.4, one shall see that* $(\nabla Y_s^{t,x}, \nabla Z_s^{t,x})$ *also means the matrix of the first order derivatives of* $(Y_s^{t,x}, Z_s^{t,x})$.

Proof. The proof is similar to the combination of the proofs of Proposition 2.2, Lemma 2.5 and Lemma 2.6 in [19]. Here we provide a sketch of the proof for the reader's convenience.

For a general function $\phi(x) = (\phi_1(x), \cdots, \phi_d(x))^T : \mathbb{R}^d \to \mathbb{R}^d$, we denote

$$\phi'(x) := \left(\frac{\partial \phi_i(x)}{\partial x_j}\right)_{1 \le i,j \le d}.$$

By the chain rule for vector-valued functions, $\nabla X_s^{t,x}$ is the unique solution to the following linear SDE,

$$\nabla X_s^{t,x} = I + \int_t^s b'(X_r^{t,x}) \nabla X_r^{t,x} dr + \sum_{k=1}^d \int_t^s \sigma_k'(X_r^{t,x}) \nabla X_r^{t,x} dW_r^k, \; s \in [t,T],$$

where σ_k is the k-th column of the matrix σ. On the other hand, the Malliavin derivative $D_\theta X_s^{t,x}$ satisfies the following linear SDE, for $s \in [t,T]$,

$$D_\theta X_s^{t,x} = \sigma(X_\theta^{t,x}) + \int_\theta^s b'(X_r^{t,x}) D_\theta X_r^{t,x} dr + \sum_{k=1}^d \int_\theta^s \sigma_k'(X_r^{t,x}) D_\theta X_r^{t,x} dW_r^k, \quad \theta \in [t,s].$$

By the uniqueness of the solutions to linear SDEs, we have

$$D_\theta X_s^{t,x} = \nabla X_s^{t,x} (\nabla X_\theta^{t,x})^{-1} \sigma(X_\theta^{t,x}), \quad t \le \theta < s \le T. \tag{5.4}$$

Let $(\nabla Y_s^{t,x}, \nabla Z_s^{t,x})$ be the unique solution to the linear BDSDE, for $s \in [t,T]$,

$$\nabla Y_s^{t,x} = \phi'(X_T^{t,x}) \nabla X_T^{t,x} + \int_s^T \Big[f_x(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \nabla X_r^{t,x}$$

$$+ f_y(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\nabla Y_r^{t,x} + f_z(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\nabla Z_r^{t,x}\Big] dr$$

$$+ \int_s^T \Big[g_x(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\nabla X_r^{t,x} + g_y(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\nabla Y_r^{t,x}$$

$$+ g_z(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\nabla Z_r^{t,x}\Big] B(dr, X_r^{t,x}) - \int_s^T \nabla Z_r^{t,x} dW_r. \tag{5.5}$$

On the other hand, as in the proof of Proposition 2.2 in [19], one can show that under the conditions **(H)**, **(A2)** and **(B3)**, $X_s^{t,x}, Y_s^{t,x}$ and $Z_s^{t,x}$ are in $\mathbb{D}^{1,2}$, and $(D_\theta Y_s^{t,x}, D_\theta Z_s^{t,x})$ solves uniquely the linear BDSDE (5.5) as well, i.e. for $t \le \theta < s \le T$,

$$D_\theta Y_s^{t,x} = \phi'(X_T^{t,x}) D_\theta X_T^{t,x} + \int_s^T \Big[f_x(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) D_\theta X_r^{t,x}$$

$$+ f_y(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) D_\theta Y_r^{t,x} + f_z(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) D_\theta Z_r^{t,x}\Big] dr$$

$$+ \int_s^T \Big[g_x(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) D_\theta X_r^{t,x} + g_y(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) D_\theta Y_r^{t,x}$$

$$+ g_z(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) D_\theta Z_r^{t,x}\Big] B(dr, X_r^{t,x}) - \int_s^T D_\theta Z_r^{t,x} dW_r.$$

By (5.4) and the uniqueness of the solution to linear BDSDEs, we have

$$D_\theta Y_s^{t,x} = \nabla Y_s^{t,x} (\nabla X_\theta^{t,x})^{-1} \sigma(X_\theta^{t,x}),\ t \le \theta < s \le T.$$

Letting s decrease to θ, we have

$$Z_\theta^{t,x} = \lim_{s \to \theta^+} D_\theta Y_s^{t,x} = \nabla Y_\theta^{t,x} (\nabla X_\theta^{t,x})^{-1} \sigma(X_\theta^{t,x}).$$

The continuity of $Z_s^{t,x}$ follows from the continuities of $\nabla Y_s^{t,x}, \nabla X_s^{t,x}$ and $X_s^{t,x}$. Finally, we may obtain L^p estimates for $\sup_{0 \le s \le t} |\nabla Y_s^{t,x}|$ as we have done for $\sup_{0 \le s \le t} |Y_s^{t,x}|$ in Theorem 4.2, and thus (5.3) is deduced. ■

Theorem 5.4 *Assume* **(H)**, **(A2)** *and* **(B3)**, *and additionally assume that for some* $\gamma > 0$ *and* $K > 0$,

$$|q(t, x, x) - q(t, x, y)| \le K|x - y|^\gamma, \ \text{ for all } t \in [0, T].$$

Then the random field $\{Y_s^{t,x}; 0 \le s \le t \le T, x \in \mathbb{R}^d\}$ *has a version whose trajectories belong to* $C^{0,0,2}([0,T]^2 \times \mathbb{R}^d)$.

Proof. The proof follows from the approach used in the proof of Theorem 2.1 in [20], from which we also borrow some notations.

First we show that for fixed $(t, x) \in [0, T] \times \mathbb{R}^d$, $\{Y_s^{t,x}; s \in [t, T]\}$ has a continuous version. For $t \le s_1 \le s_2 \le T$ and $p > 1$,

$$\mathbb{E}(|Y_{s_2}^{t,x} - Y_{s_1}^{t,x}|^{2p}) \le C \Bigg[\mathbb{E} \left(\int_{s_1}^{s_2} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) dr \right)^{2p}$$

$$
\begin{aligned}
&+\mathbb{E}\left(\int_{s_1}^{s_2} g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\overleftarrow{B}(dr,X_r^{t,x})\right)^{2p}+\mathbb{E}\left(\int_{s_1}^{s_2} Z_r^{t,x}dW_r\right)^{2p}\Bigg]\\
\le& C\Bigg[\left(\int_{s_1}^{s_2}\left(\mathbb{E}|f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})|^{2p}\right)^{1/2p}dr\right)^{2p}\\
&+\left(\mathbb{E}\int_{s_1}^{s_2} g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})^2 q(r,X_r^{t,x},X_r^{t,x})dr\right)^{p}+\left(\mathbb{E}\int_{s_1}^{s_2}|Z_r^{t,x}|^2dr\right)^{p}\Bigg]\\
\le& C|s_1-s_2|^p,
\end{aligned}
$$

where the last inequality follows from the boundedness of f and g, the integrability of $q(r,X_r^{t,x},X_r^{t,x})$ by Lemma 2.3, and the integrability of Z by Proposition 5.2. Then the continuity follows from the Kolmogorov's continuity theorem.

For $t_1,t_2\in[s,T]$, denoting $X_r^i=X_r^{t_i,x_i}, Y_r^i=Y_r^{t_i,x_i}, Z_r^i=Z_r^{t_i,x_i}, i=1,2$, we have

$$
\begin{aligned}
Y_s^2-Y_s^1=&\int_0^1\phi'(X_T^1+\lambda(X_0^2-X_0^1))d\lambda\ (X_T^2-X_T^1)\\
&+\int_s^T\int_0^1 f_x(r,X_r^1+\lambda(X_r^2-X_r^1),Y_r^1,Z_r^1)(X_r^2-X_r^1)d\lambda dr\\
&+\int_s^T\int_0^1 f_y(r,X_r^2,Y_r^1+\lambda(Y_r^2-Y_r^1),Z_r^1)(Y_r^2-Y_r^1)d\lambda dr\\
&+\int_s^T\int_0^1 f_z(r,X_r^2,Y_r^2,Z_r^1+\lambda(Z_r^2-Z_r^1))(Z_r^2-Z_r^1)d\lambda dr\\
&+\int_s^T\int_0^1 g_x(r,X_r^1+\lambda(X_r^2-X_r^1),Y_r^1,Z_r^1)(X_r^2-X_r^1)d\lambda\ \overleftarrow{B}(dr,X_r^2)\\
&+\int_s^T\int_0^1 g_y(r,X_r^2,Y_r^1+\lambda(Y_r^2-Y_r^1),Z_r^1)(Y_r^2-Y_r^1)d\lambda\ \overleftarrow{B}(dr,X_r^2)\\
&+\int_s^T\int_0^1 g_z(r,X_r^2,Y_r^2,Z_r^1+\lambda(Z_r^2-Z_r^1))(Z_r^2-Z_r^1)d\lambda\ \overleftarrow{B}(dr,X_r^2)\\
&+\int_s^T g(r,X_r^1,Y_r^1,Z_r^1)[\overleftarrow{B}(dr,X_r^2)-\overleftarrow{B}(dr,X_r^1)]\\
&+\int_s^T Z_r^2-Z_r^1dW_r.
\end{aligned}
\tag{5.6}
$$

We adopt the following notations:

$q_{i,j}(r):=q(r,X_r^i,X_r^j),\ i,j=1,2,$

$A_r:=$ summation of the inner integrals of the 2nd, 3rd and 4th terms on the right-hand side of (5.6)

and

$B_r:=$ summation of the inner integrals of the 5th, 6th and 7th terms on the right-hand side of (5.6).

For $p>1$, applying Itô's formula to $|Y_s^2-Y_s^1|^{2p}$ and taking expectation, we have

$$
\begin{aligned}
&\mathbb{E}(|Y_s^2-Y_s^1|^{2p})+p(2p-1)\mathbb{E}\int_s^T |Y_r^2-Y_r^1|^{2p-2}|Z_r^2-Z_r^1|^2dr\\
=&\mathbb{E}\Bigg\{\left|\int_0^1 \phi'(X_T^1+\lambda(X_T^2-X_T^1))d\lambda(X_T^2-X_T^1)\right|^{2p}+2p\int_s^T |Y_r^2-Y_r^1|^{2p-1}A_rdr\\
&+p(2p-1)\int_s^T |Y_r^2-Y_r^1|^{2p-2}\Big(B_r^2q_{2,2}(r)+g^2(r,X_r^1,Y_r^1,Z_r^1)[q_{1,1}(r)+q_{2,2}(r)-2q_{1,2}(r)]\\
&\qquad+2B_rg(r,X_r^1,Y_r^1,Z_r^1)[q_{2,2}(r)-q_{1,2}(r)]\Big)dr\Bigg\}. \qquad (5.7)
\end{aligned}
$$

By the boundedness of the derivatives of f and the fact that $|Y_r^2-Y_r^1|^{2p-1}|Z_r^2-Z_r^1|\leq \frac{1}{a}|Y_r^2-Y_r^1|^{2p}+a|Y_r^2-Y_r^1|^{2p-2}|Z_r^2-Z_r^1|^2$ for $a>0$, the second term on the right-hand-side can be bounded by

$$
\begin{aligned}
&C\int_s^T \left(|Y_r^2-Y_r^1|^{2p}+|Y_r^2-Y_r^1|^{2p-1}|X_r^2-X_r^1|+|Y_r^2-Y_r^1|^{2p-1}|Z_r^2-Z_r^1|\right)dr\\
\leq&C(a)\int_s^T \left(|Y_r^2-Y_r^1|^{2p}+|X_r^2-X_r^1|^{2p}\right)dr+a\int_s^T |Y_r^2-Y_r^1|^{2p-2}|Z_r^2-Z_r^1|^2dr \qquad (5.8)
\end{aligned}
$$

for some $a\in(0,1)$.

By (ii) in condition **(B3)** and the fact that g_x is bounded, we have

$$
B_r^2q_{2,2}(r)\leq C\left[q_{2,2}(r)|X_r^2-X_r^1|^2+|Y_r^2-Y_r^1|^2\right]+\alpha'|Z_r^2-Z_r^1|^2
$$

for some $\alpha'\in(\alpha,1)$. Furthermore, noting that $|q_{i_1,j_1}(r)-q_{i_2,j_2}(r)|\leq C|X_r^2-X_r^1|^{\gamma}$ for $(i_1,j_1)\neq(i_2,j_2)$ and the fact that g and its first derivatives are bounded, we may bound the third term on the right-hand side of (5.7) by, for some $\alpha''\in(\alpha',1)$,

$$
\begin{aligned}
&p(2p-1)\int_s^T |Y_r^2-Y_r^1|^{2p-2}\Big[C\big(1+q_{2,2}(r)\big)|X_r^2-X_r^1|^2+C|X_r^2-X_r^1|^{\gamma}\\
&\qquad+C|X_r^2-X_r^1|^{2\gamma}+C|Y_r^2-Y_r^1|^2+\alpha''|Z_r^2-Z_r^1|^2\Big]dr\\
\leq&C\int_s^T \Big[|Y_r^2-Y_r^1|^{2p}+(1+q_{2,2}(r))^p|X_r^2-X_r^1|^{2p}+C(|X_r^2-X_r^1|^{p\gamma}+|X_r^2-X_r^1|^{2p\gamma})\Big]dr\\
&\qquad+p(2p-1)\alpha''\int_s^T |Y_r^2-Y_r^1|^{2p-2}|Z_r^2-Z_r^1|^2dr. \qquad (5.9)
\end{aligned}
$$

Noting that $q_{2,2}(r)$ has finite p-moments for any positive p, combining (5.7), (5.8) and (5.9) and choosing a sufficiently small, we may find $\beta\in(0,1)$ and $C>0$ such that for some $p'>p$,

$$
\mathbb{E}(|Y_s^2-Y_s^1|^{2p})+p(2p-1)\beta\int_0^s |Y_r^2-Y_r^1|^{2p-2}|Z_r^2-Z_r^1|^2dr
$$

$$\leq C\Bigg[\left(\mathbb{E}\left[\sup_{0\leq t\leq T}|X_t^1|^q+\sup_{0\leq t\leq T}|X_t^2|^q\right]\right)^{1/2}\left(\mathbb{E}|X_T^2-X_T^1|^{4p}\right)^{1/2}$$

$$+\int_s^T\left(\mathbb{E}(|X_r^2-X_r^1|^{2p'})\right)^{p/p'}+\mathbb{E}(|X_r^2-X_r^1|^{p\gamma}+|X_r^2-X_r^1|^{2p\gamma})dr+\int_s^T\mathbb{E}(|Y_r^2-Y_r^1|^{2p})dr\Bigg],$$

where q is determined by the polynomial growth of $\phi'(x)$ and p. By Gronwall's inequality and the following estimate, for $m>0$,

$$\mathbb{E}(\sup_{0\leq s\leq T}|X_s^2-X_s^1|^m)\leq C_m(1+|x_1|^m+|x_2|^m)(|x_2-x_1|^m+|t_2-t_1|^{m/2}),$$

we deduce that for $|x_1|\vee|x_2|\leq R,\ t_1,t_2\in[s,T]$, there exists a constant $C_{p,R,T}$ such that

$$\mathbb{E}(|Y_s^2-Y_s^1|^{2p})\leq C_{p,R,T}(|x_2-x_1|^{p\gamma}+|t_2-t_1|^{p\gamma/2}),\quad \text{for any } p\geq 1. \tag{5.10}$$

Therefore by Kolmogorov's continuity theorem, for any fixed $s\in[0,T)$, the process $\{Y_s^{t,x},t\in[s,T],x\in\mathbb{R}^d\}$ has a continuous version. Actually, using a similar argument as in the proof of Theorem 4.2, we may get for any $p\geq 1$,

$$\mathbb{E}(\sup_{0\leq s\leq T}|Y_s^2-Y_s^1|^{2p})+\mathbb{E}\left[\left(\int_0^T|Z_s^2-Z_s^1|^2ds\right)^p\right]\leq C(|x_2-x_1|^{p\gamma}+|t_2-t_1|^{p\gamma/2}). \tag{5.11}$$

Now we show the existence of a continuous version of the first derivative of $Y_s^{t,x}$ in x. Let $\{e_1,\cdots,e_d\}$ be an orthonormal basis of $\mathbb{R}^d$ and $h\neq 0$ be a constant. Define

$$\Delta_h^iY_s^{t,x}:=\frac{1}{h}(Y_s^{t,x+he_i}-Y_s^{t,x}),$$

and similarly define $\Delta_h^iX_s^{t,x}$ and $\Delta_h^iZ_s^{t,x}$. Setting $x_1=x,x_2=x+he_i$ and $t_1=t_2$, by (5.11), we have

$$\mathbb{E}(\sup_{0\leq s\leq T}|\Delta_h^iY_s^{t,x}|^{2p})+\mathbb{E}\left[\left(\int_0^T|\Delta_h^iZ_s^{t,x}|^2ds\right)^p\right]<\infty. \tag{5.12}$$

Finally, we consider $\Delta_h^iY_s^{t,x}-\Delta_{h'}^iY_s^{t,x'}$, $|x|\vee|x'|\leq R$, which satisfies an equation analogous to (5.6). With the help of (5.11), (5.12), the condition **(B3)** and the following two estimations

$$\mathbb{E}\left[\int_s^Tf(r)B(dr,X_r^{t,x+he_i})-\int_s^Tf(r)B(dr,X_r^{t,x'+h'e_i})\right]^{2p}$$

$$\leq C\mathbb{E}\left[\int_s^Tf^2(r)\left(q(r,X_r^{t,x+he_i},X_r^{t,x+he_i})+q(r,X_r^{t,x'+h'e_i},X_r^{t,x'+h'e_i})-2q(r,X_r^{t,x+he_i},X_r^{t,x'+h'e_i})\right)dr\right]^p$$

$$\leq C\mathbb{E}\left[\int_s^Tf^2(r)\left|X_r^{t,x+he_i}-X_r^{t,x'+h'e_i}\right|^\gamma dr\right]^p$$

for an $(\mathcal{F}_t)$-adapted function f and

$$\mathbb{E}(\sup_{0\leq s\leq T}|\Delta_h^iX_s^{t,x}-\Delta_{h'}^iX_s^{t,x'}|^m)\leq C(1+|x|^m+|x'|^m)(|x-x'|^m+|h-h'|^m),$$

we may get the following estimate in much the same spirit as to get estimate (5.10):

$$\mathbb{E}(|\Delta_h^i Y_s^{t,x} - \Delta_{h'}^i Y_s^{t,x'}|^{2p}) \leq C(|h-h'|^{p\gamma} + |x-x'|^{p\gamma}),$$

where C is a constant depending on p, R, T, the bounds for g and the derivatives of f and g. This implies that the derivative of $Y_s^{t,x}$ as well as a continuous version of it exists. Moreover, similarly as in the proof of Theorem 4.2, we also have the following

$$\mathbb{E}(\sup_{0\leq s\leq T} |\Delta_h^i Y_s^{t,x} - \Delta_{h'}^i Y_s^{t,x'}|^{2p}) + \mathbb{E}\left[\left(\int_0^T |\Delta_h^i Z_s^{t,x} - \Delta_{h'}^i Z_s^{t,x'}|^2 ds\right)^p\right] \leq C(|h-h'|^{p\gamma} + |x-x'|^{p\gamma}),$$

which implies that the derivative of $Z_s^{t,x}$ with respect to x exists and it is continuous in the mean-square sense. Finally, the existence of a continuous second derivative of $Y_s^{t,x}$ can be proven in a similar way. ■

The following result provides a nonlinear Feynman-Kac formula for SPDE (5.1).

Theorem 5.5 *Assume the same conditions as in Theorem 5.4. Then* $\{u(t,x) := Y_t^{t,x}; 0 \leq t \leq T, x \in \mathbb{R}^d\}$ *is the unique classical solution to SPDE (5.1).*

Proof. Uniqueness follows from Theorem 5.1, and we show that $u(t,x) = Y_t^{t,x}$ is a solution to (5.1). Noting that $u(t+h, X_{t+h}^{t,x}) = Y_{t+h}^{t+h, X_{t+h}^{t,x}} = Y_{t+h}^{t,x}$, applying Itô's formula and using (5.2), we have that for $h > 0$,

$$\begin{aligned} u(t+h,x) - u(t,x) =& u(t+h, X_t^{t,x}) - u(t+h, X_{t+h}^{t,x}) + u(t+h, X_{t+h}^{t,x}) - u(t,x) \\ =& -\int_t^{t+h} \mathscr{L}u(t+h, X_s^{t,x}) ds - \int_t^{t+h} \nabla u(t+h, X_s^{t,x}) \sigma(X_s^{t,x}) dW_s \\ & - \int_t^{t+h} f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) ds - \int_t^{t+h} g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \overleftarrow{B}(ds, X_s^{t,x}) \\ & + \int_t^{t+h} Z_s^{t,x} dW_s. \end{aligned} \tag{5.13}$$

Let π_n be a partition $0 = t_0 < t_1 < \cdots < t_n = t$. By (5.13), we have

$$\begin{aligned} \phi(x) - u(t,x) =& -\sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [\mathscr{L}u(t_i, X_s^{t,x}) + f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x})] ds \\ & - \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}) \overleftarrow{B}(ds, X_s^{t,x}) \\ & + \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} [Z_s^{t,x} - \nabla u(t_i, X_s^{t,x}) \sigma(X_s^{t,x})] dW_s. \end{aligned}$$

If we let mesh sizes of the partitions π_n go to zero, by Theorem 5.4 and Proposition 5.2, we have

$$u(t,x) = \phi(x) + \int_t^T [\mathscr{L}u(s,x) + f(s,x,u(s,x), \nabla u(s,x)\sigma(x))] ds$$

$$+\int_t^T g(s,x,u(s,x),\nabla u(s,x)\sigma(x))\overleftarrow{B}(ds,x),$$

and the proof concludes. ■

Remark 5.6 *Consider the linear BDSDE*

$$Y_s^{t,x}=\phi(X_T^{t,x})+\int_s^T(h_r+\alpha_r Y_r^{t,x})dr+\int_s^T\beta_r Y_r^{t,x}\overleftarrow{B}(dr,X_r^{t,x})-\int_s^T Z_r^{t,x}dW_r,\quad t\le s\le T,$$

where $h_r=h(r,X_r^{t,x})$, $\alpha_r=\alpha(r,X_r^{t,x})$ *and* $\beta_r=\beta(r,X_r^{t,x})$. *The solution is given by*

$$Y_s^{t,x}=\phi(X_T^{t,x})\Gamma_s^T+\int_s^T\Gamma_s^r h_r dr-\int_s^T\Gamma_s^r Z_r^{t,x}dW_r,$$

where

$$\Gamma_s^r=\exp\left(\int_s^r\alpha_\tau d\tau+\int_s^r\beta_\tau\overleftarrow{B}(d\tau,X_\tau^{t,x})-\frac{1}{2}\int_s^r\beta_\tau^2 q(\tau,X_\tau^{t,x},X_\tau^{t,x})d\tau\right).$$

For the corresponding SPDE, noting that $Y_t^{t,x}$ *is* $\mathcal{F}_{t,T}^B$*-measurable, we have*

$$u(t,x)=Y_t^{t,x}=\mathbb{E}\left[\phi(X_T^{t,x})\Gamma_t^T+\int_t^T\Gamma_t^r h_r dr\middle|\mathcal{F}_{t,T}^B\right].$$

When $h_r\equiv\alpha_r\equiv 0$, $\beta_r\equiv 1$ *and* $X_r^{t,x}=x+W_r-W_t$, *the Feynman-Kac formula is given by*

$$\begin{aligned}u(t,x)=&\mathbb{E}_W\Bigg[\phi(x+W_T-W_t)\exp\Bigg(\int_t^T\overleftarrow{B}(dr,x+W_r-W_t)\\&-\frac{1}{2}\int_t^T q(r,x+W_r-W_t,x+W_r-W_t)dr\Bigg)\Bigg],\end{aligned}$$

and it coincides with the Feynman-Kac formula provided in [9, Theorem 3.1].

6 Random periodic solutions to semilinear SPDEs

In this section, we will construct random periodic solutions to semilinear SPDEs via the corresponding *infinite horizon* BDSDEs. For this purpose, we first consider the solvability of the BDSDE on $[0,T]$ with T increasing to infinity,

$$\begin{cases}Y_s^{t,x}=Y_T^{t,x}+\int_s^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})dr\\ \qquad -\int_s^T g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\overleftarrow{B}(dr,X_r^{t,x})-\int_s^T Z_r^{t,x}dW_r,\quad s\in[t,T],\\ \lim_{T\to\infty}e^{-K'T}Y_T^{t,x}=0,\end{cases}\tag{6.1}$$

for some positive constant $K' < \infty$. This equation is equivalent to the following *infinite horizon* BDSDE

$$\begin{aligned}
e^{-K's}Y_s^{t,x} &= \int_s^\infty e^{-K'r} f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})dr + \int_s^\infty K' e^{-K'r} Y_r^{t,x} dr \qquad (6.2)\\
&\quad - \int_s^\infty e^{-K'r} g(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}) \overleftarrow{B}(dr, X_r^{t,x}) - \int_s^\infty e^{-K'r} Z_r^{t,x} dW_r.
\end{aligned}$$

To study the equation on $[0,\infty)$, we introduce the weighted spaces, for $p \geq 2$ and $q > 0$,

$$\begin{aligned}
S^{p,-q}([0,\infty);\mathbb{R}) &= \Big\{\phi : \Omega \times [0,\infty) \to \mathbb{R},\ \text{continuous, } \phi(t) \text{ is } \mathcal{F}_t\text{-measurable, and} \\
&\qquad \mathbb{E}\big[\sup_{0\leq t<\infty} e^{-qt}|\phi(t)|^p\big] < \infty\Big\};\\
M^{p,-q}([0,\infty);\mathbb{R}^d) &= \Big\{\phi : \Omega \times [0,\infty) \to \mathbb{R}^d, \phi(t) \text{ is } \mathcal{F}_t\text{-measurable and } \mathbb{E}\int_0^\infty e^{-qt}|\phi(t)|^p dt < \infty\Big\}.
\end{aligned}$$

Assume the following conditions

(H)' The function $q(s,x,y)$ is uniformly bounded, i.e. there exists $M < \infty$ such that

$$\sup_{s\in\mathbb{R}_+,\,(x,y)\in\mathbb{R}^{2d}} |q(s,x,y)| \leq M, \quad a.s.$$

(B2)' Let $f, g : [0,\infty)\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d \to \mathbb{R}$ be two functions such that $f(\cdot,0,0,0), g(\cdot,0,0,0) \in M^{2,-K'}([0,\infty);\mathbb{R})$ and

$$|f(t,x_1,y_1,z_1) - f(t,x_2,y_2,z_2)|^2 \leq K(|x_1-x_2|^2 + |y_1-y_2|^2 + |z_1-z_2|^2);$$

$$|g(t,x_1,y_1,z_1) - g(t,x_2,y_2,z_2)|^2 \leq K(|x_1-x_2|^2 + |y_1-y_2|^2) + \alpha_t(x)|z_1-z_2|^2,$$

where $\alpha_t(x)q(t,x,x) \leq \alpha$ for some constant $\alpha \in (0,1)$.

(M) There exists a positive constant μ such that $2\mu - K' - \frac{K}{1-\alpha} - KM > 0$ and

$$(y_1 - y_2)\big(f(t,x,y_1,z) - f(t,x,y_2,z)\big) \leq -\mu|y_1-y_2|^2$$

with α and K taken from condition **(B2)'** and M from condition **(H)'**.

We will need the following estimation for X (see e.g. [13]) in the proof the Theorem 6.2.

Lemma 6.1 *Assume* **(A1)***, for $p \geq 1$ and $K' > 0$, we have*

$$E\left[\int_t^s e^{-K'r}|X_r^{t,x}|^{2p}dr\right] \leq e^{-K't}|x|^{2p} + CE\left[\int_t^s e^{-K'r}(|b(0)|^{2p} + |\sigma(0)|^{2p})dr\right] < \infty,$$

where C is a constant only depending on given parameters.

The following theorem guarantees the existence and uniqueness of the solution to the infinite horizon BDSDE under suitable conditions.

Theorem 6.2 *Assume* **(H)', (A1), (B2)'** *and* **(M)**. *BDSDE (6.1) has a unique solution* $(Y, Z) \in S^{2,-K'} \bigcap M^{2,-K'}([0, \infty); \mathbb{R}) \times M^{2,-K'}([0, \infty); \mathbb{R}^d)$.

Proof. First we show the uniqueness of the solution.

Let $(Y_s^{t,x}, Z_s^{t,x})$ and $(\hat{Y}_s^{t,x}, \hat{Z}_s^{t,x})$ be two solutions to BDSDE (6.1). Denote, for $s \geq t$,

$$\begin{aligned}
&\bar{Y}_s^{t,x} = \hat{Y}_s^{t,x} - Y_s^{t,x}; \quad \bar{Z}_s^{t,x} = \hat{Z}_s^{t,x} - Z_s^{t,x};\\
&\bar{f}(s, x) = f(s, X_s^{t,x}, \hat{Y}_s^{t,x}, \hat{Z}_s^{t,x}) - f(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x});\\
&\bar{g}(s, x) = g(s, X_s^{t,x}, \hat{Y}_s^{t,x}, \hat{Z}_s^{t,x}) - g(s, X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}).
\end{aligned}$$

Applying Itô's formula to $\mathrm{e}^{-K's}|\bar{Y}_s^{t,x}|^2$ on $[s, T]$, we obtain

$$\begin{aligned}
& e^{-K's}|\bar{Y}_s^{t,x}|^2 - K' \int_s^T e^{-K'r}|\bar{Y}_r^{t,x}|^2 dr + \int_s^T e^{-K'r}|\bar{Z}_r^{t,x}|^2 dr\\
= \;& e^{-K'T}|\bar{Y}_T^{t,x}|^2 + 2\int_s^T e^{-K'r}\bar{Y}_r^{t,x}\big(f(r, X_r^{t,x}, \hat{Y}_r^{t,x}, \hat{Z}_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, \hat{Z}_r^{t,x})\big)dr\\
& +2\int_s^T e^{-K'r}\bar{Y}_r^{t,x}\big(f(r, X_r^{t,x}, Y_r^{t,x}, \hat{Z}_r^{t,x}) - f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\big)dr\\
& +\int_s^T e^{-K'r}|\bar{g}(r, x)|^2 q(r, X_r^{t,x}, X_r^{t,x})dr - 2\int_s^T e^{-K'r}\bar{Y}_r^{t,x}\bar{g}(r, x)d\overleftarrow{B}(dr, X_r^{t,x})\\
& -2\int_s^T e^{-K'r}\bar{Y}_r^{t,x}\bar{Z}_r^{t,x}dW_r\\
\leq \;& e^{-K'T}|\bar{Y}_T^{t,x}|^2 - 2\mu\int_s^T e^{-K'r}|\bar{Y}_r^{t,x}|^2 dr + \frac{K}{1-\alpha-\varepsilon}\int_s^T e^{-K'r}|\bar{Y}_r^{t,x}|^2 dr\\
& +(1-\alpha-\varepsilon)\int_s^T e^{-K'r}|\bar{Z}_r^{t,x}|^2 dr + K\int_s^T e^{-K'r}|\bar{Y}_r^{t,x}|^2 q(r, X_r^{t,x}, X_r^{t,x})dr\\
& +\int_s^T e^{-K'r}\alpha_r(\omega)|\bar{Z}_r^{t,x}|^2 q(r, X_r^{t,x}, X_r^{t,x})dr - 2\int_s^T e^{-K'r}\bar{Y}_r^{t,x}\bar{g}(r, x)d\overleftarrow{B}(dr, X_r^{t,x})\\
& -2\int_s^T e^{-K'r}\bar{Y}_r^{t,x}\bar{Z}_r^{t,x}dW_r,
\end{aligned}$$

where $\varepsilon > 0$ is a sufficiently small number, and the last step follows from the conditions **(B2)'**, **(M)** and Young's inequality.

Taking expectation (in this proof, we shall omit the standard localization procedure for the conciseness), we have

$$\begin{aligned}
& \mathbb{E}^{-K's}|\bar{Y}_s^{t,x}|^2 + (2\mu - K' - \frac{K}{1-\alpha-\varepsilon} - KM)\int_s^T e^{-K'r}|\bar{Y}_r^{t,x}|^2 dr + \varepsilon\int_s^T e^{-K'r}|\bar{Z}_r^{t,x}|^2 dr\\
\leq \;& \mathbb{E}^{-K'T}|\bar{Y}_T^{t,x}|^2. \qquad (6.3)
\end{aligned}$$

Taking $K'' > K'$ such that $2\mu - K'' - \frac{K}{1-\alpha-\varepsilon} - KM$ as well, we can see that (6.3) remains true with K' replaced by K''. In particular,

$$\mathbb{E}^{-K''s}|\bar{Y}_s^{t,x}|^2 \leq \mathbb{E}^{-K''T}|\bar{Y}_T^{t,x}|^2,$$

and hence

$$\mathbb{E}^{-K''s}|\bar{Y}_s^{t,x}|^2 \leq \mathrm{e}^{-(K''-K')T}\mathbb{E}^{-K'T}|\bar{Y}_T^{t,x}|^2. \tag{6.4}$$

Since $\hat{Y}_s^{t,x}, Y_s^{t,x} \in S^{2,-K'} \bigcap M^{2,-K'}([0,\infty);\mathbb{R})$,

$$\sup_{T\geq 0}\mathbb{E}\mathrm{e}^{-K'T}|\bar{Y}_T^{t,x}|^2 \leq \mathbb{E}\sup_{T\geq 0}\mathrm{e}^{-K'T}(2|\hat{Y}_T^{t,x}|^2 + 2|Y_T^{t,x}|^2) < \infty.$$

Therefore, letting T go to infinity in (6.4), we have

$$\mathbb{E}\mathrm{e}^{-K''s}|\bar{Y}_s^{t,x}|^2 = 0,$$

which yields the uniqueness.

Now we deduce the existence of the solution. For each $n \in \mathbb{N}$, we define a sequence of BDSDEs (5.2) with $\phi = 0$ and $T = n$ and denote it by BDSDE (5.2_n). It is easy to verify that for each n, the BDSDE satisfies the conditions in Theorem 3.2. Therefore, for each n, the unique solution $(Y_s^{t,x,n}, Z_s^{t,x,n})$ of BDSDE (5.2_n) belongs to $S^2([t,n];\mathbb{R})\times M^2([t,n];\mathbb{R}^d)$ which is identical with the space $S^{2,-K}([t,n];\mathbb{R}) \times M^{2,-K}([t,n];\mathbb{R}^d)$. Let $(Y_s^{t,x,n}, Z_s^{t,x,n}) = (0,0)$ for $s \in (n,\infty)$. Then $(Y_s^{t,x,n}, Z_s^{t,x,n}) \in S^{2,-K} \bigcap M^{2,-K}([t,\infty);\mathbb{R}) \times M^{2,-K}([t,\infty);\mathbb{R}^d)$. In the following, we will prove that $(Y_s^{t,x,n}, Z_s^{t,x,n})$ is a Cauchy sequence.

Let $(Y_s^{t,x,m}, Z_s^{t,x,m})$ and $(Y_s^{t,x,n}, Z_s^{t,x,n})$ be the solutions to BDSDE (5.2_m) and BDSDE (5.2_n), respectively, and assume $m > n$. Denote, for $s \in [t,\infty)$,

$$\bar{Y}_s^{t,x,m,n} = Y_s^{t,x,m} - Y_s^{t,x,n}, \quad \bar{Z}_s^{t,x,m,n} = Z_s^{t,x,m} - Z_s^{t,x,n}.$$

We will estimate $(Y_s^{t,x,m,n}, Z_s^{t,x,m,n})$ for $s \in [n,m]$ and $s \in [t,n]$, respectively.

(I). When $n \leq s \leq m$, $(\bar{Y}_s^{t,x,m,n}, \bar{Z}_s^{t,x,m,n}) = (Y_s^{t,x,m}, Z_s^{t,x,m})$. Note that $(Y_s^{t,x,m}, Z_s^{t,x,m})$ is the solution to BDSDE (5.2_m), i.e. for $s \in [t,m]$,

$$\begin{cases} dY_s^{t,x,m} = -f(s,X_s^{t,x},Y_s^{t,x,m},Z_s^{t,x,m})ds + g(s,X_s^{t,x},Y_s^{t,x,m},Z_s^{t,x,m})\overleftarrow{B}(ds,X_s^{t,x}) + Z_s^{t,x,m}dW_s \\ Y_m^{t,x,m} = 0. \end{cases}$$

An application of Itô's formula to $e^{-K'r}|Y_r^{t,x,m}|^2$ on $[s,m]$ leads to

$$\begin{aligned} & e^{-K's}|Y_s^{t,x,m}|^2 - K'\int_s^m e^{-K'r}|Y_r^{t,x,m}|^2dr + \int_s^m e^{-K'r}|Z_r^{t,x,m}|^2dr \\ = \ & 2\int_s^m e^{-K'r}Y_r^{t,x,m}f(r,X_r^{t,x},Y_r^{t,x,m},Z_r^{t,x,m})dr \qquad (6.5) \\ & + \int_s^m e^{-K'r}|g(r,X_r^{t,x},Y_r^{t,x,m},Z_r^{t,x,m})|^2 q(r,X_r^{t,x},X_r^{t,x})dr \end{aligned}$$

$$-2\int_s^m e^{-K'r}Y_r^{t,x,m}g(r,X_r^{t,x},Y_r^{t,x,m},Z_r^{t,x,m})d\overleftarrow{B}(dr,X_r^{t,x})-2\int_s^m e^{-K'r}Y_r^{t,x,m}Z_r^{t,x,m}dW_r.$$

Taking expectation and using the conditions **(B2)'** and **(M)**, we have

$$\begin{aligned}
&\mathbb{E}\left[e^{-K's}|Y_s^{t,x,m}|^2-K'\int_s^m e^{-K'r}|Y_r^{t,x,m}|^2dr+\int_s^m e^{-K'r}|Z_r^{t,x,m}|^2dr\right]\\
=\ &\mathbb{E}\bigg[2\int_s^m e^{-K'r}Y_r^{t,x,m}\big(f(r,X_r^{t,x},Y_r^{t,x,m},Z_r^{t,x,m})-f(r,X_r^{t,x},0,Z_r^{t,x,m})\big)dr\\
&+2\int_s^m e^{-K'r}Y_r^{t,x,m}\big(f(r,X_r^{t,x},0,Z_r^{t,x,m})-f(r,0,0,Z_r^{t,x,m})\big)dr\\
&+2\int_s^m e^{-K'r}Y_r^{t,x,m}\big(f(r,0,0,Z_r^{t,x,m})-f(r,0,0,0)\big)dr\\
&+2\int_s^m e^{-K'r}Y_r^{t,x,m}f(r,0,0,0)dr+\int_s^m e^{-K'r}|g(r,X_r^{t,x},Y_r^{t,x,m},Z_r^{t,x,m})|^2q(r,X_r^{t,x},X_r^{t,x})dr\bigg]\\
\leq\ &\mathbb{E}\bigg[-\left(2\mu-2\varepsilon-\frac{K}{1-\alpha-\varepsilon}-(1+\varepsilon)KM\right)\int_s^m e^{-K'r}|Y_r^{t,x,m}|^2dr+C\int_s^m e^{-K'r}|X_r^{t,x}|^2dr\\
&+\big(1-\alpha-\varepsilon+(1+\varepsilon)\alpha\big)\int_s^m e^{-K'r}|Z_r^{t,x,m}|^2dr+C\int_s^m e^{-K'r}|f(r,0,0,0)|^2dr\\
&+C\int_s^m e^{-K'r}|g(r,0,0,0)|^2dr\bigg].
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\bigg[e^{-K's}|Y_s^{t,x,m}|^2+\left(2\mu-2\varepsilon-\frac{K}{1-\alpha-\varepsilon}-(1+\varepsilon)KM-K'\right)\int_s^m e^{-K'r}|Y_r^{t,x,m}|^2dr\\
&+(1-\alpha)\varepsilon\int_s^m e^{-K'r}|Z_r^{t,x,m}|^2dr\bigg] \qquad (6.6)\\
\leq\ &C\int_s^m e^{-K'r}\mathbb{E}[|X_r^{t,x}|^2]dr+C\int_s^m e^{-K'r}|f(r,0,0,0)|^2dr+C\int_s^m e^{-K'r}|g(r,0,0,0)|^2dr.
\end{aligned}$$

Note that the constant $\varepsilon>0$ can be chosen to be sufficiently small such that all the terms on the left-hand side of (6.6) are positive. By Lemma 6.1, we have

$$\begin{aligned}
&\mathbb{E}\int_n^m e^{-K'r}|Y_r^{t,x,m}|^2dr]+E\left[\int_n^m e^{-K'r}|Z_r^{t,x,m}|^2dr\right]\\
\leq\ &C\mathrm{e}^{-K'n}|x|^2+C\mathbb{E}\int_n^m \mathrm{e}^{-K'r}(|b(0)|^2+|\sigma(0)|^2)dr\\
&+C\mathbb{E}\int_n^m \mathrm{e}^{-K'r}(|f(r,0,0,0)|^2+|g(r,0,0,0)|^2)dr, \qquad (6.7)
\end{aligned}$$

where the right-hand side converges to zero as $n, m \to \infty$. Applying Burkholder-Davis-Gundy inequality to (6.5) on the interval $[n, m]$ and using a similar argument which was used to obtain (6.7), we have

$$\begin{aligned} & \mathbb{E} \sup_{n \le s \le m} e^{-K's} |Y_s^{t,x,m}|^2 \\ \le\ & C\mathrm{e}^{-K'n}|x|^2 + C\mathbb{E}\int_n^m \mathrm{e}^{-K'r}(|b(0)|^2 + |\sigma(0)|^2)dr \qquad (6.8) \\ & + C\mathbb{E}\int_n^m e^{-K'r}(|f(r,0,0,0)|^2 + |g(r,0,0,0)|^2)dr + C\mathbb{E}\int_n^m \mathrm{e}^{-K'r}(|Y_r^{t,x,m}|^2 + |Z_r^{t,x,m}|^2)dr, \end{aligned}$$

where the right-hand side goes to zero as $n, m \to \infty$.

(II). When $t \le s \le n$, taking the notations

$$\begin{aligned} \bar{f}^{m,n}(s,x) &= f(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m}) - f(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}), \\ \bar{g}^{m,n}(s,x) &= g(s, X_s^{t,x}, Y_s^{t,x,m}, Z_s^{t,x,m}) - g(s, X_s^{t,x}, Y_s^{t,x,n}, Z_s^{t,x,n}), \end{aligned}$$

we have

$$\bar{Y}_s^{t,x,m,n} = Y_n^{t,x,m} + \int_s^n \bar{f}^{m,n}(r,x)dr - \int_s^n \bar{g}^{m,n}(r,x)\overleftarrow{B}(ds, X_s^{t,x}) - \int_s^n \bar{Z}_r^{t,x,m,n}dW_r.$$

Apply Itô's formula to $\mathrm{e}^{-K'r}|\bar{Y}_r^{t,x,m,n}|^2$ on $[s, n]$, and then take expectation,

$$\begin{aligned} & \mathbb{E}\left[e^{-K's}|\bar{Y}_s^{t,x,m,n}|^2 - K'\int_s^n e^{-K'r}|\bar{Y}_r^{t,x,m,n}|^2dr + \int_s^n e^{-K'r}|\bar{Z}_r^{t,x,m,n}|^2dr\right] \\ =\ & \mathbb{E}\Bigg[e^{-K'n}|Y_n^{t,x,m}|^2 \\ & +2\int_s^n e^{-K'r}\bar{Y}_r^{t,x,m,n}\big(f(r, X_r^{t,x}, Y_r^{t,x,m}, Z_r^{t,x,m}) - f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,m})\big)dr \\ & +2\int_s^n e^{-K'r}\bar{Y}_r^{t,x,m,n}\big(f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,m}) - f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n})\big)dr \\ & + \int_s^n e^{-K'r}|\bar{g}^{m,n}(r,x)|^2 q(r, X_r^{t,x}, X_r^{t,x})dr\Bigg] \\ \le\ & \mathbb{E}\Bigg[e^{-K'n}|Y_n^{t,x,m}|^2 - 2\mu\int_s^n e^{-K'r}|\bar{Y}_r^{t,x,m,n}|^2dr + \frac{K}{1-\alpha-\varepsilon}\int_s^n e^{-K'r}|\bar{Y}_r^{t,x,m,n}|^2dr \\ & +(1-\alpha-\varepsilon)\int_s^n e^{-K'r}|\bar{Z}_r^{t,x,m,n}|^2dr + K\int_s^n e^{-K'r}|\bar{Y}_r^{t,x,m,n}|^2 q(r, X_r^{t,x}, X_r^{t,x})dr \\ & + \int_s^n e^{-K'r}\alpha_r(\omega)|\bar{Z}_r^{t,x,m,n}|^2 q(r, X_r^{t,x}, X_r^{t,x})dr\Bigg]. \end{aligned}$$

Hence

$$\mathbb{E}\Bigg[e^{-K's}|\bar{Y}_s^{t,x,m,n}|^2 + \left(2\mu - K' - \frac{K}{1-\alpha-\varepsilon} - KM\right)\int_s^n e^{-K'r}|\bar{Y}_r^{t,x,m,n}|^2dr$$

$$
+\varepsilon\int_s^n e^{-K'r}|\bar{Z}_r^{t,x,m,n}|^2dr\Bigg]
$$

$$
\leq \mathbb{E}\left[e^{-K'n}|Y_n^{t,x,m}|^2\right].
$$

Taking ε small enough, we have

$$
\mathbb{E}\int_s^n e^{-K'r}|\bar{Y}_r^{t,x,m,n}|^2dr+\mathbb{E}\int_s^n e^{-K'r}|\bar{Z}_r^{t,x,m,n}|^2dr\leq C\mathbb{E}\left[e^{-K'n}|Y_n^{t,x,m}|^2\right], \tag{6.9}
$$

where the right-hand side goes to zero as n,m go to infinity by (6.8). Also by the Burkholder-Davis-Gundy inequality, we obtain

$$
\mathbb{E}\sup_{t\leq s\leq n}\mathrm{e}^{-K's}|\bar{Y}_s^{t,x,m,n}|^2\leq C\mathbb{E}\left[e^{-K'n}|Y_n^{t,x,m}|^2\right]. \tag{6.10}
$$

Now combining (6.7) – (6.10), we have

$$
\mathbb{E}\sup_{s\geq t}e^{-Ks}|\bar{Y}_s^{t,x,m,n}|^2+\mathbb{E}\int_t^\infty e^{-Kr}|\bar{Y}_r^{t,x,m,n}|^2dr+\mathbb{E}\int_t^\infty \mathrm{e}^{-Kr}|\bar{Z}_r^{t,x,m,n}|^2dr
$$

goes to zero as $n,m\to\infty$.

Denote by $(Y_s^{t,x},Z_s^{t,x})$ the limit of $(Y_s^{t,x,n},Z_s^{t,x,n})$ in the space $S^{2,-K}\bigcap M^{2,-K}([t,\infty);\mathbb{R})\times M^{2,-K}([t,\infty);\mathbb{R}^d)$. We now show that $(Y_s^{t,x},Z_s^{t,x})$ is a solution to BDSDE (6.2). Since $(Y_s^{t,x,n},Z_s^{t,x,n})$ satisfies BDSDE (5.2_n), it suffices to verify that BDSDE (5.2_n) converges to BDSDE (6.2) in $L^2(\Omega)$ as $n\to\infty$. We only show the convergence of stochastic integral with respect to $\overleftarrow{B}$, and the convergence of the rest terms can be proven in a similar way. To see it, notice that

$$
\begin{aligned}
&\mathbb{E}\left|\int_s^n e^{-K'r}g(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})\overleftarrow{B}(dr,X_r^{t,x})-\int_s^\infty e^{-K'r}g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\overleftarrow{B}(dr,X_r^{t,x})\right|^2\\
\leq\ & 2\mathbb{E}|\int_s^n e^{-K'r}\big(g(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})-g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\big)\overleftarrow{B}(dr,X_r^{t,x})|^2\\
&+2\mathbb{E}|\int_n^\infty e^{-K'r}g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\overleftarrow{B}(dr,X_r^{t,x})|^2.
\end{aligned}
$$

As $n\to\infty$, each term on the right-hand side of the above inequality converges to zero, since

$$
\begin{aligned}
&\mathbb{E}|\int_s^n e^{-K'r}\big(g(r,X_r^{t,x},Y_r^{t,x,n},Z_r^{t,x,n})-g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\big)B(dr,X_r^{t,x})|^2\\
\leq\ & C\mathbb{E}\int_t^\infty \mathrm{e}^{-K'r}(|Y_r^{t,x,n}-Y_r^{t,x}|^2+|Z_r^{t,x,n}-Z_r^{t,x}|^2)dr
\end{aligned}
$$

and

$$
\mathbb{E}|\int_n^\infty e^{-K'r}g(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})B(dr,X_r^{t,x})|^2
$$

$$\leq \; C\mathbb{E}\int_n^\infty e^{-K'r}(|X_r^{t,x}|^2+|Y_r^{t,x}|^2+|Z_r^{t,x}|^2)dr + C\int_n^\infty \mathrm{e}^{-K'r}|g(r,0,0,0)|^2dr.$$

The proof is concluded. ■

From now on, we assume that the noise $(B(t,x), t\geq 0, x\in\mathbb{R}^d)$ in SPDEs (1.3) and (1.4) is a centered Gaussian random field with covariance function

$$\mathbb{E}[B(t,x)B(s,y)] = (t\wedge s)q(x,y),$$

where $q(x,y)$ is a positive-definite function (see e.g. [9, Section 5]). The condition **(H)'** for $q(x,y)$ now becomes $\sup_{(x,y)\in\mathbb{R}^{2d}}|q(x,y)|\leq M$. Note that $(B(\cdot,x), x\in\mathbb{R}^d)$ is a family of Brownian motions (up to a multiplicative constant) with covariance $q(x,y)$, and the joint quadratic variation is given by

$$\langle B(\cdot,x), B(\cdot,y)\rangle_t = tq(x,y).$$

We now construct a measurable metric dynamical system $(\Omega, \mathscr{F}, P, (\theta_t)_{t\geq 0})$, where $\theta_t:\Omega\to\Omega$ is a measurable and measure-preserving mapping defined by $\theta_t\circ B(s,x) = B(s+t,x)-B(t,x)$ for $x\in\mathbb{R}^d$, and $\theta_t\circ W_s = W_{s+t}-W_t$. Then for any $s,t\geq 0$,

(i) $P(\theta^{-1}(A)) = P(A)$, for all $A\in\mathcal{F}$;

(ii) $\theta_0 = I$, where I is the identity transformation on Ω;

(iii) $\theta_s\circ\theta_t = \theta_{s+t}$.

Set, for any $\mathscr{F}$-measurable mapping ϕ defined on Ω,

$$\theta\circ\phi(\omega) = \phi\big(\theta(\omega)\big).$$

For any $r\geq 0$, $s\geq t$, $x\in\mathbb{R}^d$, apply the transformation θ_r to SDE (1.2), and then it follows that

$$\theta_r\circ X_s^{t,x} = x + \int_{t+r}^{s+r} b(\theta_r\circ X_{u-r}^{t,x})du + \int_{t+r}^{s+r}\sigma(\theta_r\circ X_{u-r}^{t,x})dW_u.$$

So by the uniqueness of the solution and a perfection procedure (see e.g. [1]), we have

$$\theta_r\circ X_s^{t,x} = X_{s+r}^{t+r,x} \quad \text{for all } r,s,t,x, \quad \text{a.s.} \tag{6.11}$$

For a given period $\tau>0$, we consider the random periodic solution to BDSDE (6.1). For this, we assume the following random periodic condition on the coefficients.

(P) For any $t\in[0,\infty)$, $(x,y,z)\in\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^{1\times d}$,

$$f(t,x,y,z) = f(t+\tau,x,y,z) \;\text{ and }\; g(t,x,y,z) = g(t+\tau,x,y,z).$$

Proposition 6.3 *Assume* **(H)', (A1), (B2)', (M)** *and* **(P)**. *The unique solution* $(Y_s^{t,x}, Z_s^{t,x})_{s\geq t}$ *to BDSDE (6.1) is a "crude" random periodic solution, i.e. for any* $0 \leq t \leq s$,

$$\theta_\tau \circ Y_s^{t,x} = Y_{s+\tau}^{t+\tau,x}, \quad \theta_\tau \circ Z_s^{t,x} = Z_{s+\tau}^{t+\tau,x} \quad \text{a.s.}$$

In particular, for any $t \geq 0$,

$$\theta_\tau \circ Y_t^{t,\cdot} = Y_{t+\tau}^{t+\tau,\cdot} \quad \text{a.s.} \tag{6.12}$$

Proof. Let $\hat{B}(s,x) = B(T'-s,x) - B(T',x)$ for arbitrary $T' > 0$ and $-\infty < s \leq T'$. Then $\hat{B}(s,x)$ is a local martingales with the same joint quadratic variation as $B(s,x)$ and $\hat{B}(0,x) = 0$. For an $\mathscr{F}_t$-adapted square integrable process $\{h(s)\}_{s\geq 0}$ and any $r \geq 0$,

$$\int_{t+r}^{T+r} h(s-r)d\overleftarrow{B}(s, X_s^{t+r,x}) = -\int_{T'-T-r}^{T'-t-r} h(T'-s-r)d\hat{B}(s, X_{T'-s}^{t+r,x}) \tag{6.13}$$

for $r \geq 0$. Applying θ_τ to $\hat{B}(s, X_u^{t,x})$, we have

$$\begin{aligned} \theta_\tau \circ \hat{B}(s,x) &= \theta_\tau \circ (B(T'-s,x) - B(T',x)) = B(T'-s+\tau,x) - B(T'+\tau,x) \\ &= \big(B(T'-s+\tau,x) - B(T',x)\big) - \big(B(T'+\tau,x) - B(T',x)\big) \\ &= \hat{B}(s-\tau,x) - \hat{B}(-\tau,x). \end{aligned} \tag{6.14}$$

So for $0 \leq t \leq T \leq T'$, by (6.11), (6.13) and (6.14)

$$\begin{aligned} \theta_\tau \circ \int_t^T h(s)d\overleftarrow{B}(s, X_s^{t,x}) &= -\theta_\tau \circ \int_{T'-T}^{T'-t} h(T'-s)d\hat{B}(s, X_{T'-s}^{t,x}) \\ &= -\int_{T'-T}^{T'-t} \theta_\tau \circ h(T'-s)d\hat{B}(s-\tau, \theta_\tau \circ X_{T'-s}^{t,x}) \\ &= -\int_{T'-T}^{T'-t} \theta_\tau \circ h(T'-s)d\hat{B}(s-\tau, X_{T'-s+\tau}^{t+\tau,x}) \\ &= -\int_{T'-T-\tau}^{T'-t-\tau} \theta_\tau \circ h(T'-s-\tau)d\hat{B}(s, X_{T'-s}^{t+\tau,x}) \\ &= \int_{t+\tau}^{T+\tau} \theta_\tau \circ h(s-\tau)d\overleftarrow{B}(s, X_s^{t+\tau,x}). \end{aligned}$$

Therefore, by condition **(P)**,

$$\begin{aligned} &\theta_\tau \circ \int_t^T g(u, X_u^{t,x}, Y_u^{t,x}, Z_u^{t,x})\overleftarrow{B}(du, X_u^{t,x}) \\ = &\int_{t+\tau}^{T+\tau} g(u, X_u^{t+\tau,x}, \theta_\tau \circ Y_{u-\tau}^{t,x}, \theta_\tau \circ Z_{u-\tau}^{t,x})\overleftarrow{B}(du, X_u^{t+\tau,x}). \end{aligned} \tag{6.15}$$

We apply θ_τ to BDSDE (6.1), and then get, by (6.15),

$$\begin{cases} \theta_\tau \circ Y_s^{t,x} = \theta_\tau \circ Y_T^{t,x} + \int_{s+\tau}^{T+\tau} f(u, X_u^{t+\tau,x}, \theta_\tau \circ Y_{u-\tau}^{t,x}, \theta_\tau \circ Z_{u-\tau}^{t,x})du \\ \qquad - \int_{s+\tau}^{T+\tau} g(u, X_u^{t+\tau,x}, \theta_\tau \circ Y_{u-\tau}^{t,x}, \theta_\tau \circ Z_{u-\tau}^{t,x}) \overleftarrow{B}(du, X_u^{t+\tau,x}) - \int_{s+\tau}^{T+\tau} \theta_\tau \circ Z_{u-\tau}^{t,x} dW_u \\ \lim_{T\to\infty} e^{-K'(T+\tau)} \theta_\tau \circ Y_T^{t,x} = 0. \end{cases} \tag{6.16}$$

On the other hand, BDSDE (6.1) implies

$$\begin{cases} Y_{s+\tau}^{t+\tau,x} = Y_{T+\tau}^{t,x} + \int_{s+\tau}^{T+\tau} f(u, X_u^{t+\tau,x}, Y_u^{t+\tau,x}, Z_u^{t+\tau,x})du \\ \qquad - \int_{s+\tau}^{T+\tau} g(u, X_u^{t+\tau,x}, Y_u^{t+\tau,x}, Z_u^{t+\tau,x}) \overleftarrow{B}(du, X_u^{t+\tau,x}) - \int_{s+\tau}^{T+\tau} Z_u^{t+\tau,x} dW_u, \quad s \in [t, T], \\ \lim_{T\to\infty} e^{-K'(T+\tau)} Y_{T+\tau}^{t+\tau,x} = 0. \end{cases} \tag{6.17}$$

By the uniqueness of the solution to BDSDE (6.1), it follows from comparing (6.16) with (6.17) that for any $t \geq 0$,

$$\theta_\tau \circ Y_s^{t,x} = Y_{s+\tau}^{t+\tau,x}, \quad \theta_\tau \circ Z_s^{t,x} = Z_{s+\tau}^{t+\tau,x}, \quad \text{for } s \geq t.$$

The proof is concluded. ■

For the infinite horizon BDSDE, we shall assume the following condition instead of **(B3)**.

(B3)' (i) For any $t \in [0, \infty)$, $f(t, \cdot, \cdot, \cdot)$ and $g(t, \cdot, \cdot, \cdot)$ are of class C^3, and all their derivatives are bounded on $[0, \infty) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$.

(ii) g is uniformly bounded, $|g_z(t, x, y, z)|^2 q(t, x, x) \leq \alpha < 1$, and $|g_y(t, x, y, z)|^2 q(t, x, x) < C < \infty$, for $(t, x, y, z) \in [0, \infty) \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$.

Similar to Theorem 5.4, the solution to the infinite horizon BDSDE also possesses path regularity.

Theorem 6.4 *Assume* **(H)', (A2)**, **(B3)'** *and* **(M)**, *and additionally assume that, for some constant $K > 0$ and $\gamma > 0$,*

$$|q(x, x) - q(x, y)| \leq K|x - y|^\gamma, \quad \textit{for all } t \in [0, T].$$

Then the random field $\{Y_s^{t,x}; s \geq t \geq 0, x \in \mathbb{R}^d\}$, which is the solution to BDSDE (6.1), has a version whose trajectories belong to $C^{0,0,2}([0, \infty)^2 \times \mathbb{R}^d)$.

For $t \geq 0$, define $u(t, x) = Y_t^{t,x}$, where $(Y_s^{t,x}, Z_x^{t,x})_{s \geq t}$ is the solution to BDSDE (6.1). Then it follows from Theorem 6.4 that $u \in C^{0,2}([0, \infty) \times \mathbb{R}^d)$. For arbitrary $T > 0$, we consider SPDE (1.4) on the interval $[0, T]$. Note that there is no given terminal condition for SPDE (1.4), so **(B3)(iii)** is not needed for the solvability and regularity of its solution. By Theorem 5.5, $u(t, x) = Y_t^{t,x}$ is a classical solution to SPDE (1.4), and we have the following theorem.

Theorem 6.5 *Assume the same conditions in Theorem 6.4. Let* $u(t,x) \triangleq Y_t^{t,x}$, *where* $(Y_s^{t,x}, Z_s^{t,x})_{s\geq t}$ *is the solution to BDSDE (6.1). Then for arbitrary* T *and* $t \in [0,T]$, $u(t,x)$ *is a solution to SPDE (1.4).*

The following theorem is the main result in this section.

Theorem 6.6 *Assume condition* **(P)** *and the same conditions in Theorem 6.4. For any* $T > 0$, *define* $u(t,x) \triangleq Y_t^{t,x}$, *where* $(Y_s^{t,x}, Z_s^{t,x})_{s\geq t}$ *is the solution to BDSDE (6.1). Then* $u(t,x)$ *has a version which is a "perfect" random periodic solution to SPDE (1.4).*

Proof. By Theorem 6.5, we know that $u(t,x) \triangleq Y_t^{t,x}$ is the solution to SPDE (1.4), so we get from (6.12) that for any $t \geq 0$,

$$\theta_\tau \circ u(t,\cdot) = u(t+\tau,\cdot) \quad \text{a.s.}$$

The above equality is the so-called "crude" random periodic property for $u(t,\cdot)$. By the continuity of $u(t,\cdot)$ in t, one can find an indistinguishable version of $u(t,\cdot)$, still denoted by $u(t,\cdot)$, such that it is a "perfect" random periodic solution in the sense of (1.5). ∎

Finally, we consider stationary solutions to SPDEs, in a special case that the coefficient functions $f, g : \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ in **(B3)'** are independent of time variable. Let $(\tilde{B}(t,x), t \in \mathbb{R}, x \in \mathbb{R}^d)$ be a centered Gaussian field independent of W with covariance $\mathbb{E}[\tilde{B}(t,x)\tilde{B}(s,y)] = (|t| \wedge |s|)q(x,y)$. Consider the following infinite horizon SPDE,

$$\begin{aligned} v(t,x) &= v(0,x) + \int_0^t [\mathscr{L}v(s,x) + f\big(x, v(s,x), (\sigma^T\nabla v)(s,x)\big)]ds \\ &\quad + \int_0^t g\big(x, v(s,x), (\sigma^T\nabla v)(s,x)\big)\tilde{B}(ds,x). \end{aligned} \tag{6.18}$$

For any $T > 0$, if we choose $B(s,x) = \tilde{B}(T-s,x) - \tilde{B}(T,x)$ as the driven noise in SPDE (1.4), then SPDE (1.4) is a time reversal of SPDE (6.18).

Theorem 6.7 *Assume the same conditions in Theorem 6.4. For any* $T > 0$, *let* $v(t,x) \triangleq Y_{T-t}^{T-t,x}$, *where* $(Y_s^{t,x}, Z_s^{t,x})_{s\geq t}$ *is the solution to BDSDE (6.1) with* $B(s,x) = \tilde{B}(T-s,x) - \tilde{B}(T,x)$ *for* $s \geq 0$. *Then* $v(t,x)$ *has a version which is a "perfect" stationary solution to SPDE (6.18).*

Proof. Notice that SPDE (1.4) is a time reversal transformation of SPDE (6.18). By Theorem 6.4 $v(t,x) = Y_{T-t}^{T-t,x}$ is a solution to (6.18). Furthermore, one can show that $v(t,x)$ does not depend on the choice of T as in [24, Theorem 2.12].

We define $\tilde{\theta}_t = (\theta_t)^{-1}$, $t \geq 0$. Then $(\Omega, \mathscr{F}, P)$ and $\tilde{\theta}$ constitute a new measurable metric dynamical system. Moreover, $\tilde{\theta}_t \circ \tilde{B}(s,x) = \tilde{B}(s+t,x) - \tilde{B}(t,x)$ since $\tilde{B}$ is the time-reversal form of B.

Since we are assuming that f, g in BDSDEs (6.1) and (1.4) are independent of time variable, **(P)** holds for any $\tau \geq 0$. Therefore, (1.5) gives a stationary solution to SPDE (1.4) by perfection procedure (see e.g. [1, 2]), i.e.

$$\theta_r \circ u(t,\cdot) = u(t+r,\cdot) \quad \text{for all } t, r \geq 0 \ \text{ a.s.}$$

Therefore,

$$\begin{aligned} \tilde{\theta}_r \circ v(t,x) &= \theta_{-r} \circ u(T-t,x) = \theta_{-r} \circ \theta_r \circ u(T-t-r,x) \\ &= u(T-t-r,x) = v(t+r,x) \quad \text{ for all } t, r \geq 0 \ \text{ a.s.}, \end{aligned}$$

where T is chosen sufficiently large such that $t + r \leq T$. This yields that $v(t,x) = Y_{T-t}^{T-t,x}$ is a "perfect" stationary solution to SPDE (6.18) with respect to the shift operator $\tilde{\theta}$. ∎

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