Asymptotic total geodesy of local holomorphic curves on a bounded symmetric domain in its Harish-Chandra realization exiting the boundary

Shan Tai Chan and Ngaiming Mok

1 Introduction

Denote by Δ the open unit disk in $\mathbb{C}$. Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain of rank $r \geq 2$ in its Harish-Chandra realization. Denote by ds^2_Ω the Bergman metric for any bounded symmetric domain $\Omega \Subset \mathbb{C}^N$ in its Harish-Chandra realization.

Let $\mu : U \to \mathbb{C}^N$ be a holomorphic embedding such that $\mu(U \cap \Delta) \subset \Omega$, $\mu(U \cap \partial\Delta) \subset \partial\Omega$, where $U = B^1(b_0, \varepsilon)$ is an open neighborhood of a point $b_0 \in \partial\Delta$. Denote by $\sigma(x)$ the second fundamental form of $\mu(U \cap \Delta) = S$ in (Ω, ds^2_Ω) at $x \in S$. The main objective of this article is to prove that $\|\sigma(\mu(w))\| \to 0$ as $w \to b$ for any general point $b \in U \cap \partial\Delta$. One of the motivations of this study is to provide complete proof of Theorem 3.5.1. in [Mk11] as corollary of our Main Theorem and its applications as stated in [Mk11, p.254-255], which is also related to the study of compact complex-analytic subvarieties in the quotient Ω/Γ of bounded symmetric domain Ω by torsion-free discrete subgroup $\Gamma \subset \mathrm{Aut}_0(\Omega)$.

Remark. Note that such a holomorphic embedding μ is said to be asymptotically totally geodesic at general point $b \in U \cap \partial\Delta$ if $\|\sigma(\mu(w))\| \to 0$ as $w \to b$ for general point $b \in U \cap \partial\Delta$ (cf. [Mk09]).

Mok [Mk14] has proven that such local holomorphic curve μ is asymptotically totally geodesic at general point $b \in U \cap \partial\Delta$ under the assumption that μ exits at regular points of the boundary of Ω, and has provided the precise estimate of the norm of the second fundamental form as follows:

Proposition 1.1 (Main Theorem, [Mk14])**.** *Let $\mu : U \to \mathbb{C}^N$ be a holomorphic embedding such that $\mu(U \cap \Delta) \subset \Omega$, $\mu(U \cap \partial\Delta) \subset E_1 = \mathrm{Reg}(\partial\Omega)$, where U is an open neighborhood of a point $b_0 \in \partial\Delta$, $\Omega \Subset \mathbb{C}^N$ is a bounded symmetric domain of rank $r \geq 2$ in its Harish-Chandra realization. Then, μ is asymptotically totally geodesic at general point $b \in U \cap \partial\Delta$. More precisely, for any open neighborhood U_0 of b in $\mathbb{C}$ such that $U_0 \Subset U$, there is a positive constant C depending on U_0 such that $\|\sigma(\mu(w))\| \leq C\delta(w)$ for any $w \in U_0 \cap \Delta$, where $\delta(w) := 1 - |w|$ for $w \in \Delta$.*

Our main result is the following theorem:

Theorem 1.2. *Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization equipping with the Bergman metric ds^2_Ω. Let $\mu : U = B^1(b_0, \varepsilon) \to \mathbb{C}^N$ be a holomorphic embedding such that $\mu(U \cap \Delta) \subset \Omega$ and $\mu(U \cap \partial\Delta) \subset \partial\Omega$. Denote by $\sigma(z)$ the second fundamental form of $\mu(U\cap\Delta)$ in (Ω, ds^2_Ω) at $z = \mu(w)$, then $\lim_{w\in U\cap\Delta,\, w\to b}\|\sigma(\mu(w))\| = 0$ for general point $b \in U\cap\partial\Delta$.*

Before proving Theorem 1.2 in the general situation, we will first prove Theorem 1.2 under the assumption that Ω is irreducible and of tube type. The reason of considering irreducible bounded symmetric domain of tube type and of rank ≥ 2 is due to the idea coming from the proof of Theorem 1 in [Mk02]. After that, the complete proof of Theorem 1.2 will follow from routine construction and the procedure of reducing the problem to the case where Ω is of tube type.

The first application of Theorem 1.2 is to prove the following theorem, which is precisely Theorem 3.5.1. in [Mk11, p. 254].

Theorem 1.3 (Theorem 3.5.1. [Mk11])**.** *Let $f : (\Delta, \lambda ds^2_\Delta) \to (\Omega, ds^2_\Omega)$ be a holomorphic isometric embedding, where λ is a positive real constant and $\Omega \Subset \mathbb{C}^N$ is a bounded symmetric domain in its Harish-Chandra realization. Then f is asymptotically totally geodesic at general point $b \in \partial\Delta$.*

Proof. It follows from [Mk12] that f may be extended holomorphically around b for general point $b \in \partial\Delta$, namely there is an open neighborhood U_b of b and a holomorphic embedding $f^\sharp : U_b \to \mathbb{C}^N$ such that $f^\sharp|_{U_b\cap\Delta} = f|_{U_b\cap\Delta}$ and $f^\sharp(U_b \cap \partial\Delta) \subset \partial\Omega$ because f is proper holomorphic. Note that there are only finitely many points $\hat{b}$ on $\partial\Delta$ such that f could not extend holomorphically around $\hat{b} \in \partial\Delta$. Denote by $\sigma(z)$ the second fundamental form of $f(\Delta)$ in (Ω, ds^2_Ω). Then Theorem 1.2 asserts that $\lim_{w\in U_b\cap\Delta,\, w\to b'}\|\sigma(f(w))\| = 0$ for general point $b' \in U_b \cap \partial\Delta$. We may suppose that $b \in \partial\Delta$ is a general point chosen so that $\lim_{w\in U_b\cap\Delta,\, w\to b}\|\sigma(f(w))\| = 0$ as there are only finitely many potentially bad boundary points on $\partial\Delta$ (cf [Mk09]). The result follows. $\square$

2 Preliminaries

Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank r. We may identify $\Omega \cong G_0/K$ as Hermitian symmetric space of the non-compact type, where $G_0 = \mathrm{Aut}_0(\Omega)$ and $K \subset G_0$ is the isotropy subgroup at $\mathbf{0} \in \Omega$ (cf [Wo72], [Mk14]). We follow some basic terminologies introduced in [Wo72] (cf [Mk89], [Mk14]). Let $G^{\mathbb{C}}$ be the complexification of G_0 and $\mathfrak{g}^{\mathbb{C}}$ be the complex Lie algebra of $G^{\mathbb{C}}$. Let $\mathfrak{g}_0 \subset \mathfrak{g}^{\mathbb{C}}$ be the real Lie algebra of G_0, which is a non-compact real form of $\mathfrak{g}^{\mathbb{C}}$, and $\mathfrak{k} \subset \mathfrak{g}_0$ be the Lie algebra of K. Fixing a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{k}$, the complexification $\mathfrak{h}^{\mathbb{C}}$ of $\mathfrak{h}$ lies in the complexification $\mathfrak{k}^{\mathbb{C}}$ of $\mathfrak{k}$. Then $\mathfrak{h}^{\mathbb{C}} \subset \mathfrak{g}^{\mathbb{C}}$ is also a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$, and the set of all roots of $\mathfrak{g}^{\mathbb{C}}$ lies in $\sqrt{-1}\mathfrak{h}^*$. Let Δ^+_M be the set of non-compact positive roots as a subset of the set of all roots of $\mathfrak{g}^{\mathbb{C}}$, then $\mathfrak{m}^+ = \bigoplus_{\varphi\in\Delta^+_M} \mathbb{C}e_\varphi$ and $\mathfrak{g}_\varphi = \mathbb{C}e_\varphi$ with e_φ being of unit length with respect to the canonical Kähler-Einstein metric h. We let $\Psi = \{\psi_1, \ldots, \psi_r\}$ be a maximal strongly orthogonal set of non-compact positive roots. From the Polydisk Theorem (cf [Wo72], [Mk14]), there is a maximal polydisk $\Delta^r \cong \Pi \subset \Omega$ given by $\Pi = \left(\bigoplus_{j=1}^r \mathfrak{g}_{\psi_j}\right) \cap \Omega$ such that $(\Pi, h|_\Pi) \subset (\Omega, h)$ is totally geodesic, $\Omega = \bigcup_{\gamma \in K} \gamma \cdot \Pi$.

2.1 Canonical Kähler-Einstein metric on irreducible bounded symmetric domains

Given an irreducible bounded symmetric domain $\Omega \Subset \mathbb{C}^N$ in its Harish-Chandra realization, denote by g_Ω the canonical Kähler-Einstein metric on Ω normalized so that minimal disks are of constant Gaussian curvature -2. Note that the Bergman kernel of Ω may be written as

$$K_\Omega(z,z) = \frac{1}{\mathrm{Vol}(\Omega)} h_\Omega(z,z)^{-(p(\Omega)+2)},$$

where $h_\Omega(z,z)$ is some polynomial in $(z_1,\ldots,z_N,\overline{z_1},\ldots,\overline{z_N})$ with $h_\Omega(0,z) \equiv 1$, $\mathrm{Vol}(\Omega)$ is the Euclidean volume of Ω in $\mathbb{C}^N$ with respect to the standard Euclidean metric on $\mathbb{C}^N$ and $p(\Omega) := p(X_c) = \dim_{\mathbb{C}} \mathscr{C}_o(X_c)$ is the complex dimension of the VMRTs $\mathscr{C}_o(X_c)$ of $X_c \cong G_c/K$ at $o = eK$ (cf [Mk89]). Then the Kähler form ω_{g_Ω} respect to g_Ω on Ω is given by

$$\omega_{g_\Omega} = \sqrt{-1}\partial\overline{\partial}(-\log(-\rho)),$$

where $\rho(z) := -h_\Omega(z,z)$.

Lemma 2.4 (cf [Mk14, Mk15])**.** *Let $\mu : U \to \mathbb{C}^N$ be a holomorphic embedding such that $\mu(U\cap\Delta) \subset \Omega$, $\mu(U \cap \partial\Delta) \subset \partial\Omega$, where $U \subset \mathbb{C}$ is an open neighborhood of some point $\hat{b} \in \partial\Delta$ and Ω is an irreducible bounded symmetric domain of rank $r \geq 2$. For general point $b \in U \cap \partial\Delta$, there is an integer m depending on b such that $(U \cap \Delta, \mu^* g_\Omega|_{U\cap\Delta})$ is asymptotically of Gaussian curvature $-\frac{2}{m}$ along $U_b \cap \partial\Delta$ for some open neighborhood U_b of b in U. More precisely, denote by $\kappa(w)$ the Gaussian curvature of $(U \cap \Delta, \mu^* g_\Omega|_{U\cap\Delta})$ at $w \in U \cap \Delta$, then there is an integer m depending on b such that*

$$\kappa(w) = -\frac{2}{m} + O(\delta(w)^2)$$

as $w \to b$ for general point $b \in U \cap \partial\Delta$, where $\delta(w) = 1 - |w|$ for $w \in \Delta$.

Proof. From [Mk14] and [Mk15], for general point $b \in U \cap \partial\Delta$, the real-analytic function $-\rho(\mu(w))$ vanishes to the order m on an open neighborhood of b in $U \cap \partial\Delta$ for some integer $m \geq 1$ depending on b. Then, we have $-\rho(\mu(w)) = (1-|w|^2)^m \chi(w)$ on U_b for some smooth positive function χ defined on some neighborhood of $\overline{U_b}$ and some positive integer m, where U_b is some open neighborhood of b in U such that $U_b \Subset U$, say $U_b = B^1(b, \varepsilon_b)$ for some small $\varepsilon_b > 0$. Then, on $U_b \cap \Delta$, we have

$$\mu^*\omega_{g_\Omega} = -\sqrt{-1}\partial\overline{\partial}\log h_\Omega(\mu(w),\mu(w)) = m \cdot \omega_{g_\Delta} - \sqrt{-1}\partial\overline{\partial}\log\chi(w)$$

(cf [Mk14]), where $\omega_{g_\Delta} = -\sqrt{-1}\partial\overline{\partial}\log(1-|w|^2)$. Then,

$$\mu^*\omega_{g_\Omega} = \left(\frac{m}{(1-|w|^2)^2} + q(w)\right) \cdot \sqrt{-1}dw \wedge d\overline{w},$$

where $q(w) = -\frac{\partial^2 \log \chi}{\partial w \partial \overline{w}}$ is a smooth function defined on a neighborhood of $\overline{U_b}$.

From [Mk14], p. 13, it suffices to show that $q(w) \cdot (1-|w|^2)^2 = O(\delta(w)^2)$ on $U_b \cap \Delta$, where $\delta(w) = 1 - |w|$ is the distance between $w \in \Delta$ and $\partial\Delta$. Since $q(w)$ is a smooth function defined on a neighborhood of $\overline{U_b}$ and $\overline{U_b}$ is compact, so $|q(w)|^2$ is bounded on $\overline{U_b}$, i.e. $0 \le |q(w)|^2 \le C_1$ on $\overline{U_b}$ for some real constants C_1 independent of w. It is clear that $(1+|w|)^2$ is bounded above by some positive real number for any $w \in U_b$ because U_b is bounded. Now, on $U_b \cap \Delta$, we have

$$\mu^* \omega_{g_\Omega} = \frac{u}{(1-|w|^2)^2} \cdot \sqrt{-1} dw \wedge d\overline{w} = u \cdot \omega_g,$$

where $u = m + q(w)(1-|w|^2)^2$. After shrinking U_b if necessary, we can suppose that $u \neq 0$ on an neighborhood of $\overline{U_b}$ because $|q(w)|^2$ is bounded and $(1-|w|^2)^2$ vanishes on $U_b \cap \partial\Delta$. Denote by $\kappa(w)$ the Gaussian curvature of $(U \cap \Delta, \mu^* g_\Omega|_{U\cap\Delta})$ at $w \in U \cap \Delta$. For $w \in U_b \cap \Delta$, we have

$$\kappa(w) \cdot \frac{u}{(1-|w|^2)^2} = -\frac{\partial^2}{\partial w \partial \overline{w}} \log \frac{u}{(1-|w|^2)^2} = -\frac{\partial^2}{\partial w \partial \overline{w}} \log u - \frac{2}{(1-|w|^2)^2}.$$

In particular, for $w \in U_b \cap \Delta$, we have

$$\begin{aligned} \kappa(w) &= -\frac{1}{u}\frac{\partial^2 \log u}{\partial w \partial \overline{w}}(1-|w|^2)^2 - \frac{2}{u} \\ &= -\frac{2}{m} + \left(\frac{2q(w)}{m \cdot u} - \frac{1}{u}\frac{\partial^2 \log u}{\partial w \partial \overline{w}}\right)(1-|w|^2)^2 \\ &= -\frac{2}{m} + \left(\frac{2q(w)}{m \cdot u} - \frac{1}{u}\frac{\partial^2 \log u}{\partial w \partial \overline{w}}\right)(1+|w|)^2 \cdot \delta(w)^2 \end{aligned}$$

For general point $b \in U \cap \partial\Delta$, there is an open neighborhood U_b of b in U such that $U_b \Subset U$ and $u > 0$ on U_b. Then, $\frac{2q(w)}{m \cdot u} - \frac{1}{u}\frac{\partial^2 \log u}{\partial w \partial \overline{w}}$ is smooth and real-valued on U_b. Thus, we have

$$\kappa(w) = -\frac{2}{m} + O(\delta(w)^2)$$

as $w \to b$ for general point $b \in U \cap \partial\Delta$. □

2.2 Convention

Let M be a smooth manifold and E is a differentiable vector bundle over M, then we denote by $\Gamma(M,E)$ (resp. $\Gamma_{\mathrm{loc},x}(M,E)$) the space of smooth sections (resp. local smooth sections around $x \in M$) of E. We also denote by $\Gamma_{\mathrm{loc}}(M,E)$ the space of local smooth sections around some point in M. If M is a complex manifold and E is a holomorphic vector bundle over M, then we also denote by $\Gamma_{\mathrm{loc}}(M,E)$ as the space of local holomorphic sections of E around some point in M. For a complex manifold X and $x \in X$, we always identify $T_x(X)$ with $T_x^{1,0}(X)$, namely $\xi \in T_x(X)$ can be written as $\xi = v + \overline{v}$ for some $v \in T_x^{1,0}(X)$.

3 Construction of holomorphic isometric embedding

Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank r in its Harish-Chandra realization. Let $\mu : U = B^1(b_0, \varepsilon) \to \mathbb{C}^N$ be a holomorphic embedding such that $\mu(U \cap \Delta) \subset \Omega$ and

$\mu(U\cap\partial\Delta)\subset\partial\Omega$, where $b_0\in\partial\Delta$. For general point $b\in U\cap\partial\Delta$, $\|\sigma(\mu(w))\|^2$ is real-analytic around b (cf [Mk09]). Let $\{w_k\}_{k=1}^{+\infty}$ be a sequence of points in $U\cap\Delta$ such that $w_k\to b$ as $k\to+\infty$. Let $\varphi_k\in\mathrm{Aut}(\Delta)$ be the map $\varphi_k(\zeta)=\frac{\zeta+w_k}{1+\overline{w_k}\zeta}$ and $\Phi_k\in\mathrm{Aut}(\Omega)$ such that $\Phi_k(\mu(w_k))=\mathbf{0}$, where $k=1,2,3,\dots$. Then we have $\Phi_k(\mu(\varphi_k(0)))=0$. Consider the sequence $\{\Phi_k\circ(\mu\circ\varphi_k)\}_{k=1}^{+\infty}$ of germs of holomorphic maps $(\Delta;0)\to(\Omega;\mathbf{0})$. That means all $\Phi_k\circ(\mu\circ\varphi_k)$ are defined on some small open neighborhood $U'=B^1(0,\varepsilon')$ of 0 in Δ, which is valid by choosing some suitable sequence $\{w_k\}_{k=1}^{+\infty}$ in $U\cap\Delta$ converging to $b\in U\cap\partial\Delta$ and for sufficiently small $\varepsilon'>0$.

Lemma 3.5. *By choosing some suitable sequence $\{w_k\}_{k=1}^{+\infty}$ of points in $U\cap\Delta$ converging to $b\in U\cap\partial\Delta$, then there is a subsequence of $\{\widetilde{\mu}_j=\Phi_j\circ(\mu\circ\varphi_j)\}_{j=1}^{+\infty}$ converges to some holomorphic map $\widetilde{\mu}$ on U' after shrinking U' if necessary such that $\widetilde{\mu}:(\Delta,m_0g_\Delta;0)\to(\Omega,g_\Omega;\mathbf{0})$ is a germ of holomorphic isometry for some integer $m_0\geq 1$.*

Proof. It is clear that the sequence $\{\widetilde{\mu}_j=\Phi_j\circ(\mu\circ\varphi_j)\}_{j=1}^{+\infty}$ is bounded on compact subsets of $B^1(0,\varepsilon')$, so it should contain a subsequence $\{\widetilde{\mu}_{j_k}\}_{k=1}^{+\infty}$ converging uniformly on compact subsets of $B^1(0,\varepsilon')=U'$ to some holomorphic map $\widetilde{\mu}$ by Montel's Theorem and Weierstrass' Theorem [Na71, p. 7-8]. After shrinking U' if necessary, we may suppose that such a sequence $\{\widetilde{\mu}_{j_k}\}_{k=1}^{+\infty}$ converges uniformly to $\widetilde{\mu}$ on $\overline{U'}$ because we only need to consider the germ of holomorphic map $\widetilde{\mu}:(\Delta;0)\to(\Omega;\mathbf{0})$.

Recall that $\mu^*\omega_{g_\Omega}=m_0\omega_{g_\Delta}+q(w)\sqrt{-1}dw\wedge d\overline{w}$ on $U_b\cap\Delta$ for some $U_b=B^1(b,\varepsilon_b)$ due to $\mu(U\cap\partial\Delta)\subset\partial\Omega$ and $\mu(U\cap\Delta)\subset\Omega$, where m_0 is some positive integer, $q(w)$ is a smooth (real-valued) function on U_b such that $|q(w)|$ is bounded from above on U_b for some open neighborhood U_b of b in $\mathbb{C}$.

For k sufficiently large and $w\in U'$ after shrinking U' if necessary, we have $\varphi_k(U')\subset U_b\cap\Delta$ by choosing some suitable sequence $\{w_k\}_{k=1}^{+\infty}$ in $U\cap\Delta$ converging to $b\in\partial\Delta$ and

$$\begin{aligned}\partial\overline{\partial}\log(-\rho(\widetilde{\mu}_k(w)))&=\partial\overline{\partial}\log(-\rho(\mu(\varphi_k(w))))\\&=m_0\partial\overline{\partial}\log(1-|\varphi_k(w)|^2)+q(\varphi_k(w))|\varphi_k'(w)|^2dw\wedge d\overline{w}\\&=m_0\partial\overline{\partial}\log(1-|w|^2)+q(\varphi_k(w))|\varphi_k'(w)|^2dw\wedge d\overline{w}\end{aligned}$$

so that $\frac{\partial^2}{\partial w\partial\overline{w}}\log(-\rho(\widetilde{\mu}_k(w)))=m_0\frac{\partial^2}{\partial w\partial\overline{w}}\log(1-|w|^2)+q(\varphi_k(w))|\varphi_k'(w)|^2$. Taking limit as $k\to+\infty$ (passing to some subsequence of $\{\widetilde{\mu}_k\}_{k=1}^{+\infty}$ if necessary) and since $\widetilde{\mu}:(\Delta;0)\to(\Omega;\mathbf{0})$ is a germ of holomorphic map, we have $\frac{\partial^2}{\partial w\partial\overline{w}}\log(-\rho(\widetilde{\mu}(w)))=m_0\frac{\partial^2}{\partial w\partial\overline{w}}\log(1-|w|^2)$ so that $\widetilde{\mu}^*g_\Omega=m_0g_\Delta$ on U''. That means $\widetilde{\mu}:(\Delta,m_0g_\Delta;0)\to(\Omega,g_\Omega;\mathbf{0})$ is a germ of holomorphic isometry, and thus it extends to a holomorphic isometry $(\Delta,m_0g_\Delta)\to(\Omega,g_\Omega)$ by the extension theorem of Mok [Mk12]. The extension is still denoted by $\widetilde{\mu}$. □

We have the following basic lemma from analysis:

Lemma 3.6. *Let $\phi(\tau)=\frac{p(\tau)}{q(\tau)}$ be a quotient of some real-valued, real-analytic functions p,q on $\hat{U}$, where $\hat{U}$ is some open neighborhood of 0 in $\mathbb{C}$. Denote by $\mathcal{H}=\{\tau\in\mathbb{C}:\mathrm{Im}\tau>0\}$ the upper-half* *plane in $\mathbb{C}$. Suppose that $\phi(\tau)$ is bounded from above and below on $\hat{U} \cap \mathcal{H}$, then $\phi(\tau)$ extends real-analytically around a general point $b \in \hat{U} \cap \partial\mathcal{H}$.*

Proof. We may regard p, q as functions of (x, y), where $\tau = x + \sqrt{-1}y$. We write $p(\tau) = p(x, y), q(\tau) = q(x, y)$ as real-analytic functions of (x, y). Locally around 0, we have $p(x, y) = \sum_{i,j=0}^{+\infty} a_{ij} x^i y^j$ and $q(x, y) = \sum_{i,j=0}^{+\infty} b_{ij} x^i y^j$ for some $a_{ij}, b_{ij} \in \mathbb{R}$. Then we have the local holomorphic functions on $\mathbb{C}^2$ around $(0, 0) \in \mathbb{C}^2$ given by $\hat{p}(\tau, \zeta) := \sum_{i,j=0}^{+\infty} a_{ij} \tau^i \zeta^j$ and $\hat{q}(\tau, \zeta) := \sum_{i,j=0}^{+\infty} b_{ij} \tau^i \zeta^j$ with $\mathrm{Re}\tau = x$, $\mathrm{Re}\zeta = y$. Consider $\hat{\phi}(\tau, \zeta) = \frac{\hat{p}(\tau,\zeta)}{\hat{q}(\tau,\zeta)}$, which is a quotient of holomorphic functions around $(0, 0) \in \mathbb{C}^2$. Thus $\hat{\phi}$ is a meromorphic function on an open neighborhood U of $(0, 0)$ in $\mathbb{C}^2$. The set of indeterminacy $I(\hat{\phi})$ of $\hat{\phi}$ is of dimension at most 0 because it is the intersection of the set $Z(\hat{\phi})$ of zeros and set $P(\hat{\phi})$ of poles of $\hat{\phi}$ (cf. Gunning [Gun90, p. 180]). Moreover, the restriction of $\hat{\phi}$ to $U' := \{(\tau, \zeta) \in U : \mathrm{Im}\,\tau = 0,\ \mathrm{Im}\,\zeta = 0\}$ is bounded after shrinking U if necessary, so U' does not intersect $P(\hat{\phi}) \smallsetminus I(\hat{\phi})$. Note that the set of singular points of $\hat{\phi}$ on $\hat{U}$ is $P(\hat{\phi}) \cup I(\hat{\phi}) = P(\hat{\phi})$, so the above arguments show that the set of potentially bad points of ϕ lies inside $I(\hat{\phi}) \cap U'$, which is of dimension at most 0. Hence, for general point $b \in \hat{U} \cap \partial\mathcal{H}$, $\phi(\tau)$ extends real-analytically around b. □

Given a non-zero tangent vector $v \in T_x(\Omega)$, $x \in \Omega$, then under G_0-action, there is an unique normal form $\eta = (\eta_1, \ldots, \eta_r) \in T_{\mathbf{0}}\Pi$ of v satisfying $\eta_j \in \mathbb{R}$ $(1 \le j \le r)$ and $\eta_1 \ge \cdots \ge \eta_r \ge 0$, where $\Pi \cong \Delta^r$ is a maximal polydisk in Ω containing $\mathbf{0}$ and $r = \mathrm{rank}(\Omega)$. For the notion of normal form of tangent vectors in $T_x(\Omega)$, $x \in \Omega$, one may refer to [Mk02, Mk89] for details.

Lemma 3.7. *Let $v \in T_x(\Omega)$ be a tangent vector of unit length with respect to h at some $x \in \Omega$ and $\eta = \sum_{j=1}^{r} \eta_j e_{\psi_j} \in T_{\mathbf{0}}(\Pi)$ be the normal form of v. Then, the Hermitian bilinear form H_η defined by $H_\eta(\alpha, \beta) = R_{\eta\overline{\eta}\alpha\overline{\beta}}(\Omega, g_\Omega)$ has real eigenvalues lying inside the closed interval $[-2, 0]$ and the corresponding Hermitian matrix $\hat{H}_\eta$ of H_η can be represented as a diagonal matrix with respect to the standard orthonormal basis $\{e_\varphi : \varphi \in \Delta_M^+\}$ of $\mathfrak{m}^+$.*

Proof. We write $R_{\alpha\overline{\alpha'}\beta\overline{\beta'}} = R_{\alpha\overline{\alpha'}\beta\overline{\beta'}}(\Omega, g_\Omega)$ for simplicity. From the assumption, we have $\sum_{j=1}^{r} \eta_j^2 = 1$ and $\eta_1 \ge \cdots \ge \eta_r \ge 0$ are real numbers. Writing $\alpha = \sum_{\varphi \in \Delta_M^+} \alpha_\varphi e_\varphi$, $\beta = \sum_{\varphi \in \Delta_M^+} \beta_\varphi e_\varphi \in T_{\mathbf{0}}(\Omega) \cong \mathfrak{m}^+$, we can compute

$$
\begin{aligned}
H_\eta(\alpha, \beta) &= \sum_{j=1}^{r} \eta_j^2 R_{e_{\psi_j}\overline{e_{\psi_j}}\alpha\overline{\beta}} = \sum_{j=1}^{r} \sum_{\varphi \in \Delta_M^+} \eta_j^2 \alpha_\varphi \overline{\beta_\varphi} R_{e_{\psi_j}\overline{e_{\psi_j}}e_\varphi\overline{e_\varphi}} \\
&= -2 \sum_{j=1}^{r} \eta_j^2 \alpha_{\psi_j} \overline{\beta_{\psi_j}} + \sum_{\varphi \in \Delta_M^+ \smallsetminus \Psi} \left(\sum_{j=1}^{r} \eta_j^2 R_{e_{\psi_j}\overline{e_{\psi_j}}e_\varphi\overline{e_\varphi}} \right) \alpha_\varphi \overline{\beta_\varphi}
\end{aligned}
$$

From [Mk89], $R_{e_{\psi_j}\overline{e_{\psi_j}}e_\varphi\overline{e_\varphi}} = 0$ (resp. -1) whenever $\psi_j - \varphi$ is not a root (resp. $\psi_j - \varphi$ is a root). Eigenvalues of H_η are $-2\eta_j^2$, $1 \le j \le r$, and those of the form $-(\eta_{i_1}^2 + \ldots + \eta_{i_m}^2)$ corresponding to e_φ for some $\varphi \in \Delta_M^+ \smallsetminus \Psi$ such that $\psi_{i_j} - \varphi$ is a root for $1 \le j \le m$ and $\psi_l - \varphi$ is not a root for $l \notin \{i_j : 1 \le j \le m\}$. Here we have $-2 \le -2\eta_j^2 \le 0$ $(1 \le j \le r)$ and $0 \ge -(\eta_{i_1}^2 + \ldots + \eta_{i_m}^2) \ge -1$

because $\sum_{j=1}^r \eta_j^2 = 1$ and $\eta_j \geq 0$, $1 \leq j \leq r$. In particular, the eigenvector corresponding to the eigenvalue $-2\eta_j^2$ is precisely e_{ψ_j}, $1 \leq j \leq r$. Note that the above computations imply that the corresponding Hermitian matrix $\hat{H}_\eta$ can be represented as a diagonal matrix with diagonal $-2\eta_1^2, \ldots, -2\eta_r^2$ and those eigenvalues $-(\eta_{i_1}^2 + \ldots + \eta_{i_m}^2)$ mentioned above with respect to the standard orthonormal basis $\{e_\varphi : \varphi \in \Delta_M^+\}$ of $\mathfrak{m}^+$. $\square$

From the construction the sequence $\{\widetilde{\mu}_k\}_{k=1}^{+\infty}$, we realize that the limit $\widetilde{\mu}$ of some subsequence of $\{\widetilde{\mu}_k\}_{k=1}^{+\infty}$ should have some special properties locally around 0. Moreover, we can produce another holomorphic map from $\widetilde{\mu}$ by the same kind of construction and such a map also has those special properties on the unit disk.

Proposition 3.8. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank r in its Harish-Chandra realization. Let $\mu : U = B^1(b_0, \varepsilon) \to \mathbb{C}^N$ be a holomorphic embedding such that $\mu(U \cap \partial\Delta) \subset \partial\Omega$ and $\mu(U \cap \Delta) \subset \Omega$, where $b_0 \in \partial\Delta$. Let $\{w_k\}_{k=1}^{+\infty}$ be some sequence of points in $U \cap \Delta$ converging to some general point $b \in U \cap \partial\Delta$ as $k \to +\infty$, and we let $\varphi_k \in \mathrm{Aut}(\Delta)$ and $\Phi_k \in \mathrm{Aut}(\Omega)$ such that $\varphi_k(0) = w_k$ and $\Phi_k(\mu(w_k)) = \mathbf{0}$, $k = 1, 2, 3, \ldots$. Then, the sequence of germs of holomorphic embeddings $\{\widetilde{\mu}_k := \Phi_k \circ (\mu \circ \varphi_k)\}_{k=1}^{+\infty}$ at $0 \in \Delta$ into Ω (passing to some subsequence if necessary) converges to the germ $\widetilde{\mu}$ of holomorphic isometry $(\Delta, m_0 g; 0) \to (\Omega, h; \mathbf{0})$ for some integer $m_0 \geq 1$, say $\widetilde{\mu}$ is defined on $U' = B^1(0, \varepsilon')$ for some $\varepsilon' > 0$, satisfying the following properties:*

1. *$\|\widetilde{\sigma}(\widetilde{\mu}(w))\|^2 = \|\sigma(\mu(b))\|^2$ being independent of $w \in U'$, where $\widetilde{\sigma}(z)$ is the second fundamental form of $\widetilde{\mu}(U')$ in (Ω, g_Ω) at $z = \widetilde{\mu}(w)$, $w \in U'$,*
2. *the normal form of $\frac{\widetilde{\mu}'(w)}{\|\widetilde{\mu}'(w)\|_{g_\Omega}}$ is independent of $w \in U'$ and so is the rank of $\frac{\widetilde{\mu}'(w)}{\|\widetilde{\mu}'(w)\|_{g_\Omega}}$.*

Moreover, $\widetilde{\mu}$ extends to a holomorphic isometry $(\Delta, m_0 g_\Delta) \to (\Omega, g_\Omega)$ so that the property 1 actually holds true on Δ for the extension of $\widetilde{\mu}$. Furthermore, by the same kind of process, $\widetilde{\mu}$ induces a holomorphic isometry $(\Delta, m_0 g_\Delta) \to (\Omega, g_\Omega)$ satisfying the above two properties on the whole Δ. We also denote such a holomorphic isometry by $\widetilde{\mu}$.

Proof. The first assertion about convergent of subsequence of certain sequence of germs of holomorphic maps $\widetilde{\mu}_k : (\Delta; 0) \to (\Omega; \mathbf{0})$ follows from Lemma 3.5. More precisely, from Lemma 3.5, the limit is the germ of holomorphic isometry $\widetilde{\mu} : (\Delta, m_0 g_\Delta; 0) \to (\Omega, g_\Omega; \mathbf{0})$. We also denote by $\widetilde{\mu}$ the extension of $\widetilde{\mu}$ as holomorphic isometry $(\Delta, m_0 g_\Delta) \to (\Omega, g_\Omega)$ [Mk12]. It remains to show that such $\widetilde{\mu}$ satisfies the properties 1 and 2. By Weierstrass' Theorem, $\widetilde{\mu}'(w) = \lim_{k \to +\infty} \widetilde{\mu}_k'(w)$ for each $w \in U'$ and $\widetilde{\mu}'(w) \neq \mathbf{0}$ because $\widetilde{\mu}$ is a germ of holomorphic isometry $(\Delta, m_0 g_\Delta; 0) \to (\Omega, g_\Omega; \mathbf{0})$. We identify $\Omega \cong G_0/K$. Let $\widetilde{\eta}_k(w)$ (resp. $\eta(w)$) be the normal form of $\frac{\widetilde{\mu}_k'(w)}{\|\widetilde{\mu}_k'(w)\|_{g_\Omega}}$ (resp. $\frac{\mu'(w)}{\|\mu'(w)\|_{g_\Omega}}$) for $w \in U'$ (resp. $w \in U \cap \Delta$). We also let $\widetilde{\eta}(w)$ be the normal form of $\frac{\widetilde{\mu}'(w)}{\|\widetilde{\mu}'(w)\|_{g_\Omega}}$. Let $H_{\eta(w)}(\alpha, \beta) = R_{\eta(w)\overline{\eta(w)}\alpha\overline{\beta}}(\Omega, g_\Omega)$ be the Hermitian bilinear form and $\hat{H}_{\eta(w)}$ be the corresponding

Hermitian matrix. The characteristic polynomial of $\hat{H}_{\eta(w)}$ is given by $P_{\eta(w)}(\lambda) := \det(\lambda I_N - \hat{H}_{\eta(w)})$. Moreover all eigenvalues of $H_{\eta(w)}$ are lying in the interval $[-2, 0]$ by Lemma 3.7.

For the normal forms $\widetilde{\eta}_k(w)$ and $\widetilde{\eta}(w)$, we also define the Hermitian bilinear forms $H_{\widetilde{\eta}_k(w)}$, $H_{\widetilde{\eta}(w)}$ with the corresponding Hermitian matrices $\hat{H}_{\widetilde{\eta}_k(w)}$ respectively. Then, the characteristic polynomial of $\hat{H}_{\widetilde{\eta}_k(w)}$ (resp. $\hat{H}_{\widetilde{\eta}(w)}$) is given by $P_{\widetilde{\eta}_k(w)}(\lambda) := \det(\lambda I_N - \hat{H}_{\widetilde{\eta}_k(w)})$ (resp. $P_{\widetilde{\eta}(w)}(\lambda) := \det(\lambda I_N - \hat{H}_{\widetilde{\eta}(w)})$). By Lemma 3.7, all eigenvalues of $H_{\widetilde{\eta}_k(w)}$ (resp. $H_{\widetilde{\eta}(w)}$) are lying in the interval $[-2, 0]$. For simplicity, we may suppose that $\varphi_k(U') \subset U \cap \Delta$ for any $k \geq 1$. Fix an arbitrary point $w \in U'$. From the construction, $\frac{\widetilde{\mu}_k'(w)}{\|\widetilde{\mu}_k'(w)\|_{g_\Omega}}$ is equivalent to $\frac{\varphi_k'(w)}{|\varphi_k'(w)|}\frac{\mu'(\varphi_k(w))}{\|\mu'(\varphi_k(w))\|_{g_\Omega}}$ under G_0-action so that the normal form $\widetilde{\eta}_k(w)$ is equivalent to $\eta(\varphi_k(w))$ under the K-action and for $k \geq 1$. From the uniqueness of the normal form (cf. [Mk02]), we have $\widetilde{\eta}_k(w) = \eta(\varphi_k(w))$ and thus $H_{\widetilde{\eta}_k(w)} = H_{\eta(\varphi_k(w))}$ for integer $k \geq 1$. Note that $H_{\widetilde{\eta}_k(w)}$ (resp. $H_{\eta(\varphi_k(w))}$) is equivalent to the Hermitian bilinear form $H_{\frac{\widetilde{\mu}_k'(w)}{\|\widetilde{\mu}_k'(w)\|_{g_\Omega}}}$ (resp. $H_{\frac{\mu'(\varphi_k(w))}{\|\mu'(\varphi_k(w))\|_{g_\Omega}}}$) on $T_{\widetilde{\mu}_k(w)}(\Omega) \cong \mathbb{C}^N$ (resp. $T_{\mu(\varphi_k(w))}(\Omega) \cong \mathbb{C}^N$) in the sense that the corresponding Hermitian matrices are similar as matrices due to the invariance of $H_v(\alpha, \beta) = R_{v\overline{v}\alpha\overline{\beta}}(\Omega, g_\Omega)$ under the action of $\mathrm{Aut}_0(\Omega) \cong G_0$. Moreover, the corresponding eigenvalues are the same under such equivalence because the corresponding characteristic polynomial remains unchanged.

Note that the characteristic polynomial $P_{\eta(\zeta)}(\lambda)$ only depends on the eigenvalues of $H_{\eta(\zeta)}$, which are the same as those of $H_{\frac{\mu'(\zeta)}{\|\mu'(\zeta)\|_{g_\Omega}}}$ for $\zeta \in U \cap \Delta$. Since eigenvalues of $H_{\eta(\zeta)}$ are real numbers lying inside $[-2, 0] \subset \mathbb{R}$, and coefficients of $P_{\eta(\zeta)}(\lambda)$ are bounded functions of ζ on $U \cap \Delta$ and may be written as a quotient of real-valued, real-analytic functions of ζ on $U_b = B^1(b, \varepsilon_b)$. Therefore, Lemma 3.6 asserts that for a general point $b' \in U \cap \partial\Delta$, all coefficients of $P_{\eta(\zeta)}(\lambda)$ can be extended as a real-analytic function of ζ on $U_{b'} = B^1(b', \varepsilon_{b'})$ for some $\varepsilon_{b'} > 0$. We can suppose that $b \in U \cap \partial\Delta$ is the general point chosen so that all coefficients of $P_{\eta(\zeta)}(\lambda)$ can be extended as a real-analytic function of ζ around U_b, and $\varphi_k(U')$ lies inside $U_b \cap \Delta$ for k sufficiently large and shrinking U' if necessary. Thus, there is a subsequence of $\{P_{\eta(\varphi_k(w))}(\lambda)\}_{k=1}^{+\infty}$ converges to some polynomial $P_\infty(\lambda)$ of λ which is independent of $w \in U'$ by the construction, in particular the roots of $P_\infty(\lambda)$ are independent of $w \in U'$. Moreover, since $P_{\widetilde{\eta}_k(w)}(\lambda) = P_{\eta(\varphi_k(w))}(\lambda)$ and the subsequence of $\{P_{\widetilde{\eta}_k(w)}(\lambda)\}_{k=1}^{+\infty}$ converges to $P_{\widetilde{\eta}(w)}(\lambda)$, we have $P_{\widetilde{\eta}(w)}(\lambda) = P_\infty(\lambda)$ so that the eigenvalues of $H_{\widetilde{\eta}(w)}$ are independent of $w \in U'$. In particular, by computing the eigenvalues of $H_{\widetilde{\eta}(w)}$ as in the proof of Lemma 3.7, the normal form $\widetilde{\eta}(w)$ is independent of $w \in U'$ and so is the rank of $\widetilde{\eta}(w)$, i.e. $\widetilde{\mu}$ satisfies the property 2.

We suppose that the germ $\widetilde{\mu}$ is defined on $U' = B^1(0, \varepsilon')$ for some $\varepsilon' > 0$. Denote by $\widetilde{\sigma}_k(z)$ (resp. $\widetilde{\sigma}(z)$) the $(1,0)$-part of the second fundamental form of $\widetilde{\mu}_k(U')$ (resp. $\widetilde{\mu}(U')$) in (Ω, h) at $z = \widetilde{\mu}_k(w)$ (resp. $z = \widetilde{\mu}(w)$), $k = 1, 2, 3, \ldots$. Denote by $\kappa(w)$ the Gaussian curvature of $(\mu(U \cap \Delta), g_\Omega|_{\mu(U\cap\Delta)}))$ at $w \in U \cap \Delta$, then from the invariance of holomorphic sectional curvature of (Ω, g_Ω) under G_0-action, we have

$$R_{\widetilde{\eta}_k(w)\overline{\widetilde{\eta}_k(w)}\widetilde{\eta}_k(w)\overline{\widetilde{\eta}_k(w)}}(\Omega, g_\Omega) = \|\sigma(\mu(\varphi_k(w)))\|^2 + \kappa(\varphi_k(w)).$$

On the other hand, there is subsequence of $R_{\widetilde{\eta}_k(w)\overline{\widetilde{\eta}_k(w)}\widetilde{\eta}_k(w)\overline{\widetilde{\eta}_k(w)}}(\Omega, h)$ $(k = 1, 2, 3, \ldots)$ converging to $R_{\widetilde{\eta}(w)\overline{\widetilde{\eta}(w)}\widetilde{\eta}(w)\overline{\widetilde{\eta}(w)}}(\Omega, h)$ for $w \in U'$. Therefore, for $w \in U'$, we have $\|\widetilde{\sigma}(\widetilde{\mu}(w))\|^2 = \|\sigma(\mu(b))\|^2$ by the above formula and continuity of $\|\sigma(\mu(\zeta))\|^2$ as a function of $\zeta \in B^1(b, \varepsilon_b)$. Since $\widetilde{\mu}$ extends as a holomorphic isometry $\widetilde{\mu} : (\Delta, m_0 g_\Delta) \to (\Omega, g_\Omega)$ by [Mk12], by the real-analyticity of $\|\widetilde{\sigma}(\widetilde{\mu}(w))\|^2$ on Δ and the identity theorem for real-analytic functions, we have $\|\widetilde{\sigma}(\widetilde{\mu}(w))\|^2 \equiv \|\sigma(\mu(b))\|^2$ for $w \in \Delta$. Thus $\widetilde{\mu}$ satisfies the property 1.

We now construct another holomorphic isometry by $\widetilde{\mu}$ as follows: We may choose a general point $b' \in \partial\Delta$ such that $\widetilde{\mu}$ extends holomorphically around b' and $\|\widetilde{\sigma}(\widetilde{\mu}(w))\|^2$ extends real-analytically around b' (cf. [Mk12, Mk09]). Let $\{w'_k\}_{k=1}^{+\infty}$ be some sequence of points in Δ converging to b' as $k \to +\infty$, and let $\hat{\varphi}_k \in \mathrm{Aut}(\Delta), \hat{\Phi}_k \in \mathrm{Aut}(\Omega)$ such that $\hat{\varphi}_k(0) = w'_k$, $\hat{\Phi}_k(\widetilde{\mu}(w'_k)) = \mathbf{0}$ for $k = 1, 2, 3, \ldots$. Then Montel's Theorem asserts that some subsequence of $\{\hat{\Phi}_k \circ (\widetilde{\mu} \circ \hat{\varphi}_k)\}_{k=1}^{+\infty}$ converges uniformly on any compact subsets $\hat{U}$ of Δ to some holomorphic map $\hat{\mu} : \hat{U} \to \Omega$. By the same arguments as before, $\hat{\mu} : (\Delta, m_0 g_\Delta; x_0) \to (\Omega, g_\Omega; \hat{\mu}(x_0))$ is a germ of holomorphic isometry for some $x_0 \in \Delta$ and $\hat{\mu}$ extends to a holomorphic isometry $(\Delta, m_0 g_\Delta) \to (\Omega, g_\Omega)$. Denote by $\hat{\eta}(w)$ the normal form of $\frac{\hat{\mu}'(w)}{\|\hat{\mu}'(w)\|_{g_\Omega}}$, then $\hat{\eta}(w)$ is independent of $w \in \hat{U}$ for any compact subset $\hat{U} \subset \Delta$ by the same arguments as before, say for any $\hat{U} = \overline{B^1(0, \hat{\varepsilon})}$ with $\hat{\varepsilon} \in (0, 1)$. Denote also by $\hat{\mu}$ the extension of $\hat{\mu}$ as holomorphic isometry $(\Delta, m_0 g_\Delta) \to (\Omega, g_\Omega)$ and $\hat{\eta}(w)$ the normal form of $\frac{\hat{\mu}'(w)}{\|\hat{\mu}'(w)\|_{g_\Omega}}$ for $w \in \Delta$. Then, $\hat{\eta}(w)$ is actually independent of $w \in \Delta$ and so is the rank of $\hat{\eta}(w)$. Denote by $\hat{\sigma}(z)$ the second fundamental form of $\hat{\mu}(\Delta)$ in (Ω, g_Ω) at $z = \hat{\mu}(w)$, then we have $\|\hat{\sigma}(\hat{\mu}(w))\|^2 \equiv \|\widetilde{\sigma}(\widetilde{\mu}(b'))\|^2 = \|\sigma(\mu(b))\|^2$ by the same arguments as before. For simplicity, we may replace the notation $\hat{\mu}$, $\hat{\sigma}$ by $\widetilde{\mu}$, $\widetilde{\sigma}$ respectively. $\square$

Remark.

1. The positive integer m_0 is actually the vanishing order of $\rho(\mu(w))$ as $w \to b$ and we have $-\rho(\mu(w)) = (1 - |w|^2)^{m_0} \chi(w)$ on $U_b = B^1(b, \varepsilon_b)$ for some positive smooth function χ on U_b.

2. The reason of equipping a bounded symmetric domain with the Bergman metric in the statement of Theorem 1.2 is because we need to apply the extension theorem of Mok [Mk12] for germs of holomorphic isometries of the Poincaré disk into bounded symmetric domains with respect to their Bergman metrics up to normalizing constants. Otherwise, we may consider any invariant Kähler metric g'_Ω on a bounded symmetric domain Ω so that (Ω, g'_Ω) has non-positive holomorphic bisectional curvature.

4 Proof of Theorem 1.2

We first prove the following theorem, then it could be generalized to the case where Ω is reducible and of tube type. On the other hand, we will show that given a bounded symmetric domain Ω, then the problem may be reduced to our study on the case where Ω is of tube type.

Theorem 4.9. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank $r \geq 2$ in its Harish-Chandra realization. Suppose that Ω is of tube type. Let $\mu : U = B^1(b_0, \varepsilon) \to \mathbb{C}^N$ be a holomorphic embedding such that $\mu(U \cap \Delta) \subset \Omega$ and $\mu(U \cap \partial\Delta) \subset \partial\Omega$. Denote by $\sigma(z)$ the second fundamental form of $\mu(U \cap \Delta)$ in Ω at $z = \mu(w)$, then $\lim_{w \in U \cap \Delta,\, w \to b} \|\sigma(\mu(w))\| = 0$ for general point $b \in U \cap \partial\Delta$.*

4.1 Geometry of the induced holomorphic isometric embedding

In this section, we suppose that Ω is an irreducible bounded symmetric domain of tube type and of rank ≥ 2. Recall that we have constructed a germ of holomorphic isometry $\widetilde{\mu} : (\Delta, m_0 g_\Delta; 0) \to (\Omega, g_\Omega; \mathbf{0})$ from μ and $\widetilde{\mu}$ is defined on $U' = B^1(0, \varepsilon')$ for some $\varepsilon' > 0$ such that $\|\widetilde{\sigma}(\widetilde{\mu}(w))\|^2 \equiv \|\sigma(\mu(b))\|^2$ for $w \in U'$ and $\widetilde{\mu}'(w) = d\widetilde{\mu}\left(\frac{\partial}{\partial w}\right)(w)$ is of constant rank on U', say of rank k for some k, $1 \leq k \leq r = \mathrm{rank}(\Omega)$. By Proposition 3.8, we may suppose that the following setting for the germ $\widetilde{\mu}$ at $\mathbf{0}$ should also valid for the whole holomorphic isometry $\widetilde{\mu} : (\Delta, m_0 g_\Delta) \to (\Omega, g_\Omega)$. We write $Z = \widetilde{\mu}(U')$ and $\eta(w)$ as the normal form of $\widetilde{\mu}'(w)$, which is of the form $\sum_{j=1}^k \eta_j(w) e_{\psi_j}$ with $\eta_1(w) \geq \cdots \geq \eta_k(w) > 0$, and $\Psi = \{\psi_1, \ldots, \psi_r\}$ a maximal strongly orthogonal set of non-compact positive roots [Wo72]. Then, we consider the null space $\mathcal{N}_\eta$ of the Hermitian bilinear form $H_\eta(\alpha, \beta) = R_{\eta\overline{\eta}\alpha\overline{\beta}}(\Omega, g_\Omega)$, which is of complex dimension $n_k(\Omega)$ [Mk89]. Here $n_k(\Omega)$ is the k-th null dimension of the irreducible bounded symmetric domain Ω. In case $k = r = \mathrm{rank}(\Omega)$, we simply write $n_0(\Omega) = n_{r-k}(\Omega) = \dim_{\mathbb{C}} \Omega$. For $x \in \Omega$, let Q_x be a Hermitian bilinear form on $T_x(\Omega) \otimes \overline{T_x(\Omega)}$ given by $Q(\alpha \otimes \overline{\beta}, \alpha' \otimes \overline{\beta'}) = R_{\alpha\overline{\alpha'}\beta'\overline{\beta}}(\Omega, g_\Omega)$. For $w \in U'$, we define

$$W_{\widetilde{\mu}(w)} = \left\{ v \in T_{\widetilde{\mu}(w)}\Omega : Q_{\widetilde{\mu}(w)}(v \otimes \overline{\zeta}, \cdot) \equiv 0 \;\; \forall\, \zeta \in \mathcal{N}_{\widetilde{\mu}'(w)} \right\},$$

where $\mathcal{N}_{\widetilde{\mu}'(w)} = \mathcal{N}_{\eta(w)} = \{v \in T_{\widetilde{\mu}(w)}\Omega : R_{\eta(w)\overline{\eta(w)}v\overline{v}}(\Omega, h) = 0\} = \{v \in T_{\widetilde{\mu}(w)}\Omega : \eta(w) \otimes \overline{v} \in \mathrm{Ker}(Q_{\widetilde{\mu}(w)})\}$. Then, we have $T_{\widetilde{\mu}(w)}(Z) \subset W_{\widetilde{\mu}(w)} \subset T_{\widetilde{\mu}(w)}(\Omega)$. Note that $\zeta(w) = \zeta(\widetilde{\mu}'(w)) \in \mathcal{N}_{\widetilde{\mu}'(w)}$ varies antiholomorphically with respect to w. Let

$$\mathcal{N}_k = \bigcap_{j=1}^{k} \{\varphi \in \Delta_M^+ : \varphi \neq \psi_j,\ \varphi - \psi_j \text{ is not a root}\},$$

then $\mathcal{N}_\eta = \bigoplus_{\varphi \in \mathcal{N}_k} \mathfrak{g}_\varphi$. Let $\widetilde{\mathcal{N}} = \bigcap_{\varphi \in \mathcal{N}_k} \{\psi \in \Delta_M^+ ; \psi \neq \varphi,\ \psi - \varphi \text{ is not a root}\}$, then the normal form of $W_{\mu(w)}$ is given by

$$\bigcap_{\zeta \in \mathcal{N}_\eta} \mathcal{N}_\zeta = \bigoplus_{\psi \in \widetilde{\mathcal{N}}} \mathfrak{g}_\psi.$$

Lemma 4.10. *In the above constructions, if Ω is of tube type, then for any $x \in Z$, $W_x = T_x(\Omega'_x)$ for some characteristic subdomain $\Omega'_x \subseteq \Omega$ of rank k passing through x and Ω'_x is of tube type.*

Proof. We fix an arbitrary $x \in Z$. Consider the case where $\Omega = D^{\mathrm{VI}}$. If $k = 3 = \mathrm{rank}(\Omega)$, then $W_x = T_x(\Omega)$ so that the result follows directly and $\Omega'_x = \Omega$. If $k = 1$, then $W_x = T_x(Z) = T_x(\Delta_\eta)$ with $\Delta_\eta \subset \Omega$ being the minimal disk passing through $x = \widetilde{\mu}(w)$ because $\bigcap_{\zeta \in \mathcal{N}_{\eta(w)}} \mathcal{N}_\zeta = \mathbb{C}\eta(w)$

(cf [MT92, p. 98]). Suppose that $k=2$. Note that the automorphism group of the exceptional domain D^{VI} corresponds to the Lie group E_7. From [Zh84] and [Si81, p. 868], $\Psi=\{\psi_1,\psi_2,\psi_3\}$ with $\psi_1=x_1-x_2$, $\psi_2=x_1+x_2+x_3$ and $\psi_3=\sum_{j=1}^7 x_j-x_3$, where x_j, $1\le j\le 7$, is the standard basis of $\mathbb{R}^7$. We can write $\eta(w)=\eta_1(w)e_{x_1-x_2}+\eta_2(w)e_{x_1+x_2+x_3}$, then

$$\mathcal{N}_2=\bigcap_{j=1}^{2}\{\varphi\in\Delta_M^+:\varphi\neq\psi_j,\ \varphi-\psi_j \text{ is not a root}\}=\left\{\sum_{j=1}^{7}x_j-x_3\right\}=\{\psi_3\}.$$

Actually, if $\eta(w)=\eta_1(w)e_{\psi_{j_1}}+\eta_2(w)e_{\psi_{j_2}}$ with some distinct $j_1,j_2\in\{1,2,3\}$, then $\mathcal{N}_{\eta(w)}=\mathbb{C}e_{\psi_{j_3}}$ with $j_3\in\{1,2,3\}\smallsetminus\{j_1,j_2\}$. Therefore, in any case, e_{ψ_j} is a characteristic vector so that the normal form of $W_{\widetilde{\mu}(w)}$ is $\mathcal{N}_{e_{\psi_j}}=T_{\mathbf{0}}(\Omega')$, where $\Omega'\subset\Omega=D^{\mathrm{VI}}$ is a characteristic subdomain of rank 2 (cf. Proposition 1.8. in [MT92]). From [Wo72], we have $\Omega'\cong D^{\mathrm{IV}}_{10}$. For Ω being of type-IV, if $k=1$ (resp. $k=2$), then $W_x=T_x(Z)=T_x(\Delta_\eta)$ (resp. $W_x=T_x(\Omega)$) for a unique minimal disk $\Delta_\eta\subset\Omega$ passing through $x\in Z$ and $T_x(\Delta_\eta)=\mathbb{C}\eta$ (These arguments not only work for D^{IV}_N, but also for any irreducible bounded symmetric domain of rank 2, including D^{V}).
For Ω of type I, II or III, the result follows from the use of normal form of η and computations in [Mk89]. For the case where $k=r$, we have $W_x=T_x(\Omega)$. For each $x\in Z$, we see that the normal form of W_x is the holomorphic tangent space to some characteristic symmetric subdomain $\Omega'\subset\Omega$ of rank k at $\mathbf{0}$ as follows:

1. For $\Omega=D^{\mathrm{I}}_{p,p}$, $2\le p=r$, and $1\le k\le p$, then the normal form η is given by

$$\mathrm{diag}_{p,p}(\eta_1,\ldots,\eta_k,0,\ldots,0)$$

   and it is clear that

$$\bigcap_{\zeta\in\mathcal{N}_\eta}\mathcal{N}_\zeta=\left\{\begin{bmatrix}Z' & \\ & \mathbf{0}\end{bmatrix}\in M(p,p;\mathbb{C}):Z'\in M(k,k;\mathbb{C})\right\}=T_{\mathbf{0}}(D^{\mathrm{I}}_{k,k})$$

   by [Mk89], where we identify $D^{\mathrm{I}}_{k,k}$ with its image via the standard embedding $D^{\mathrm{I}}_{k,k}\hookrightarrow D^{\mathrm{I}}_{p,p}$, $Z'\mapsto\begin{bmatrix}Z' & \\ & \mathbf{0}\end{bmatrix}$.

2. For $\Omega=D^{\mathrm{III}}_r$, the normal form η is given by $\mathrm{diag}_{p,p}(\eta_1,\ldots,\eta_k,0,\ldots,0)$, then it is clear that

$$\bigcap_{\zeta\in\mathcal{N}_\eta}\mathcal{N}_\zeta=\left\{\begin{bmatrix}Z' & \\ & \mathbf{0}\end{bmatrix}\in M_s(r;\mathbb{C}):Z'\in M_s(k;\mathbb{C})\right\}=T_{\mathbf{0}}(D^{\mathrm{III}}_k)$$

   by [Mk89], where we identify D^{III}_k with its image via the standard embedding $D^{\mathrm{III}}_k\hookrightarrow D^{\mathrm{III}}_r$, $Z'\mapsto\begin{bmatrix}Z' & \\ & \mathbf{0}\end{bmatrix}$.

3. For $\Omega = D^{\mathrm{II}}_{2r}$, we have the normal form

$$\eta = \begin{bmatrix} \eta_1 J_1 & & & \\ & \ddots & & \\ & & \eta_k J_1 & \\ & & & \mathbf{0} \end{bmatrix}, \quad J_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

then it is clear that

$$\bigcap_{\zeta \in \mathcal{N}_\eta} \mathcal{N}_\zeta = \left\{ \begin{bmatrix} Z' & \\ & \mathbf{0} \end{bmatrix} \in M_a(2r;\mathbb{C}) : Z' \in M_a(2k;\mathbb{C}) \right\} = T_{\mathbf{0}}(D^{\mathrm{II}}_{2k})$$

by [Mk89], where D^{II}_{2k} is identified with its image via the standard embedding $D^{\mathrm{II}}_{2k} \hookrightarrow D^{\mathrm{II}}_{2r}$, $Z' \mapsto \begin{bmatrix} Z' & \\ & \mathbf{0} \end{bmatrix}$.

For each of the above case, from classification of boundary components of irreducible bounded symmetric domain and the notion of characteristic subdomain in [Wo72] and [MT92], we see that $\Omega' \subset \Omega$ is a characteristic subdomain of rank k. Then, by using G_0-action and the fact that Ω' is invariant geodesic submanifold of Ω, we see that $W_x = T_x(\Omega'_x)$ for some characteristic subdomain $\Omega'_x \subseteq \Omega$ of rank k. Since Ω is of tube type, all its characteristic subdomains are of tube type (cf [Wo72]). $\square$

Remark. When Ω is an arbitrary irreducible bounded symmetric domain (not necessarily of tube type) of rank $r \geq 2$ and $T_x(Z)$ is spanned by a rank k vector $\eta_x \in T_x(\Omega)$ for each $x \in Z$ with $k < r$. Then it follows that for any $x \in Z$, $W_x = T_x(\Omega'_x)$ for some invariant geodesic submanifold $\Omega'_x \subseteq \Omega$ passing through x such that Ω'_x is an irreducible bounded symmetric domain of rank k and of tube type.

Lemma 4.11. *In the above construction, $W \subset T_\Omega|_Z$ is a holomorphic vector subbundle.*

Proof. We may write $W_x = \{\gamma \in T_x\Omega : Q(\gamma \otimes \overline{\zeta}, \cdot) \equiv 0,\ \forall\, \zeta \in \mathcal{N}_\eta\}$ for $x \in Z = \widetilde{\mu}(U')$. Note that ζ is antiholomorphic, where $\zeta(w) \in \Gamma_{\mathrm{loc},x}(Z, \mathcal{N}')$ with $\mathcal{N}' := \bigcup_{w \in U'} \mathcal{N}_{\eta(w)}$ is an antiholomorphic vector subbundle of $T_\Omega|_Z$. For $(1,0)$ tangent vector v tangent to Z at x, for any $\alpha, \beta \in \Gamma_{loc,x}(Z, T_\Omega|_Z)$ and $\gamma \in \Gamma_{\mathrm{loc},x}(Z, W)$ a local smooth section, then we have

$$0 = \nabla_{\overline{v}}(Q(\gamma \otimes \overline{\zeta}, \alpha \otimes \overline{\beta})) = Q(\nabla_{\overline{v}}\gamma \otimes \overline{\zeta}, \alpha \otimes \overline{\beta})$$

because ζ is antiholomorphic, so $(\nabla_{\overline{v}}\gamma)(x) \in W_x$. Hence $W \subset T_\Omega|_Z$ is a holomorphic vector subbundle. $\square$

Lemma 4.12. *Define the $(1,0)$-part of the second fundamental form $\tau : T_Z \otimes W \to T_\Omega|_Z / W$ of the holomorphic vector subbundle $(W, g_\Omega|_W) \subset (T_\Omega|_Z, g_\Omega)$ by $\tau_x(\eta \otimes \gamma) = (\nabla_\eta \gamma)(x) \mod W_x$ for each $x \in Z$, $\eta \in T_x(Z)$ and $\gamma \in W_x$, then τ is holomorphic.*

Proof. We need to show that for local holomorphic sections $\eta, \beta \in \Gamma_{\mathrm{loc},x}(Z, T_Z)$ and $\gamma \in \Gamma_{\mathrm{loc},x}(Z, W)$, $\nabla_{\overline{\beta}}(\nabla_\eta \gamma)(x) \in W_x$ for any $x \in Z$ so that projecting to the quotient bundle $T_\Omega|_Z/W$ would imply $\nabla_{\overline{\beta}}(\tau(\eta \otimes \gamma)) = 0$, i.e. τ is holomorphic. Note that $R(\eta, \overline{\beta})\gamma = -\nabla_{\overline{\beta}}(\nabla_\eta \gamma)$, so it suffices to show that $R_{\eta\overline{\beta}\gamma\overline{\xi}}(\Omega, g_\Omega) = 0$ for any ξ orthogonal to W, equivalently $R(\eta, \overline{\beta})\gamma$ takes values in W. For each $x \in Z$, $W_x = T_x(\Omega'_x)$ for some characteristic subdomain $\Omega'_x \subset \Omega$ of rank k containing x. Note that $\Omega'_x \subset \Omega$ is an invariantly geodesic submanifold, we can regard x as a base point o of Ω and thus

$$[[\mathfrak{m}^-, W_x], W_x] \subset W_x$$

by Lemma 4.3 in [Ts93]. This shows that $(R(\eta, \overline{\beta})\gamma)(x) = [[\overline{\beta(x)}, \eta(x)], \gamma(x)] \in W_x$ because $\eta(x) \in T_x(S) \subset W_x$ and $\gamma(x) \in W_x$. This shows that $-\nabla_{\overline{\beta}}(\nabla_\eta \gamma) = R(\eta, \overline{\beta})\gamma$ takes value in W so that τ is holomorphic. Moreover, we can regard $\tau \in \Gamma(Z, T_Z^* \otimes W^* \otimes (T_\Omega|_Z/W))$ as a holomorphic section. $\square$

Lemma 4.13. *Under the above assumptions, for any $x \in Z$ and $\eta, \beta \in \Gamma_{\mathrm{loc},x}(Z, T_Z)$, we have $\tau_x(\eta(x) \otimes \beta(x)) = 0$, i.e. $(\nabla_\eta \beta)(x) \in W_x$, equivalently $\tau|_{T_Z \otimes T_Z} \equiv 0$.*

Proof. By Lemma 4.12, $\tau|_Z \in \Gamma(Z, S^2T_Z^* \otimes (T_\Omega|_Z/W))$ is a holomorphic section. Let $\nu_k = \epsilon_k \mod W$ be holomorphic basis of the quotient bundle $T_\Omega|_Z/W$, namely, $\nu_k(\zeta) = \epsilon_k(\zeta) \mod W_{\widetilde{\mu}(\zeta)}$, where $\epsilon_k(\zeta) = \frac{\partial}{\partial z_k}\big|_{z=\widetilde{\mu}(\zeta)}$. We write $\eta(\zeta) = \widetilde{\mu}'(\zeta) = d\widetilde{\mu}\left(\frac{\partial}{\partial \zeta}\right)(\zeta)$ for simplicity. Note that $\widetilde{\mu}|_{U'}$ can be extended as a holomorphic isometry $\widetilde{\mu} : (\Delta, m_0 g_\Delta) \to (\Omega, g_\Omega)$, so we can also extend Z as a complex submanifold $Z' = \widetilde{\mu}(\Delta)$ of Ω by [Mk12], and we also denote the extension by $\widetilde{\mu}$. By Proposition 3.8, Lemma 4.10, Lemma 4.11 and Lemma 4.12, we can extend the domain of definition of τ and the holomorphic vector bundle W so that $T_{Z'} \subset W \subset T_\Omega|_{Z'}$. We also extend $\tau|_Z$ to $\tau|_{Z'} \in \Gamma(Z', S^2T_{Z'}^* \otimes T_\Omega|_{Z'}/W)$. We may write

$$\tau|_{Z'}(\zeta) = \sum_k \tau_{11}^k(\zeta) d\zeta \otimes d\zeta \otimes \nu_k(\zeta)$$

so that $\tau_{\eta\eta}^k(\zeta) = \tau_{11}^k(\zeta)$. Then, we have

$$\|\tau|_{Z'}(\zeta)\| \le \sum_k |\tau_{11}^k(\zeta)| \|d\zeta\|^2 \|\nu_k(\zeta)\|.$$

We write $\hat{\tau} = \tau|_{Z'}$ for simplicity. Note that

$$\|d\zeta\| \le C'' \cdot \delta(\zeta) \quad (C'' > 0 \text{ is a real constant})$$

with $\delta(\zeta) = 1 - |\zeta|$ by using the fact that $\widetilde{\mu}$ is a holomorphic isometry and actually $\|\widetilde{\mu}'(\zeta)\|_{g_\Omega}^2 = \left\|\frac{\partial}{\partial \zeta}\right\|_{m_0 g_\Delta}^2 = \frac{m_0}{(1-|\zeta|^2)^2}$. We also have

$$\tau|_{T_{Z'} \otimes T_{Z'}}(\zeta) = \frac{\tau(\eta(\zeta) \otimes \eta(\zeta))}{\|\eta(\zeta)\|^2} = \frac{1}{m_0}(1 - |\zeta|^2)^2 \sum_k \tau_{11}^k(\zeta)\nu_k(\zeta)$$

so that $\|\tau|_{T_{Z'} \otimes T_{Z'}}(\zeta)\| \le \frac{4}{m_0} \sum_k |\tau_{11}^k(\zeta)| \cdot \delta(\zeta)^2 \|\nu_k(\zeta)\|$. Note that $\|\tau|_{T_{Z'} \otimes T_{Z'}}(\zeta)\|^2$ can be extended as a real-analytic function around a general point $b' \in \partial\Delta$, say on $U_{b'} = B^1(b', \varepsilon_{b'})$, and that all

$\tau^k_{11}(\zeta)$ are holomorphic functions on a neighborhood of $\overline{U_{b'}}$. Note that $\|\nu_k(\zeta)\| \leq \|\epsilon_k(\zeta)\|_{g_\Omega}$ (cf [Mk10]).

We need to obtain an estimate of $\|\epsilon_k(\zeta)\|_{g_\Omega}$ as in [Mk10] and we claim that

$$\|\epsilon_k(\zeta)\|_{g_\Omega} \leq C' \frac{1}{\delta(\zeta)}$$

for some positive real constant C'. The idea is to use Kobayashi pseudo-distance, Kobayashi pseudo-metric on Ω, and convexity of Ω. Denote by $d_\Delta(\cdot,\cdot)$ (resp. $d_\Omega(\cdot,\cdot)$) the Kobayashi pseudo-distance on Δ (resp. Ω) with $d_\Delta(0,\zeta) = \log\frac{1+|\zeta|}{1-|\zeta|}$ and $d_\Delta(\cdot,\cdot)$ is defined by using the Bergman metric ds^2_Δ on Δ (cf [Ko98]). From [Ko98], for a complex manifold M, we define the Kobayashi pseudo-metric by

$$F_M(v) = \inf\left\{\|\hat{v}\|_{ds^2_\Delta} : \hat{v} \in T_0(\Delta),\ f \in \mathrm{Hol}(\Delta, M),\ f_*\hat{v} = v\right\}$$

for $v \in T_x(M)$, $x \in M$. Since $\Omega \Subset \mathbb{C}^N$ is convex, the Carathéodory pseudo-metric on Ω coincide with the Kobayashi pseudo-metric F_Ω ([Ko98], p.220). For $x \in \Omega$, let $\delta_\Omega(x) = \delta(x, \partial\Omega)$ be the Euclidean distance from x to the boundary $\partial\Omega$. Note that $\frac{1}{\sqrt{2}}F_{\mathbb{B}^N}(\xi) = \|\xi\|_{g_{\mathbb{B}^N}}$. Fix some $x \in \Omega$. By definition of $\delta_\Omega(x) = \delta(x, \partial\Omega)$, we have $B^N(x, \delta_\Omega(x)) \subseteq \Omega$ and thus we have a holomorphic map $f : \mathbb{B}^N \to \Omega$ given by $f(w) = \delta_\Omega(x)w + x$. Then, f maps $\mathbb{B}^N$ biholomorphically onto $B^N(x, \delta_\Omega(x))$ and $df_{\mathbf{0}}\left(\frac{1}{\delta_\Omega(x)}\frac{\partial}{\partial w_j}\big|_{\mathbf{0}}\right) = \frac{\partial}{\partial z_j}\big|_x$. For $v = \epsilon_j(\zeta) = \frac{\partial}{\partial z_j}\big|_{\widetilde{\mu}(\zeta)} \in T_{\widetilde{\mu}(\zeta)}(\Omega)$, and by [Az85] and [Ko98], p.90, there is positive real constant C_2' (independent of the choice of tangent vector to Ω) such that

$$\begin{aligned}\|v\|_h &\leq \sqrt{C_2'}F_\Omega(v) \leq \sqrt{C_2'}F_{\mathbb{B}^N}\left(\frac{1}{\delta_\Omega(x)}\frac{\partial}{\partial w_j}\bigg|_{\mathbf{0}}\right)\\ &= \sqrt{2C_2'}\left\|\frac{1}{\delta_\Omega(x)}\frac{\partial}{\partial w_j}\bigg|_{\mathbf{0}}\right\|_{g_{\mathbb{B}^N}} = \sqrt{2C_2'}\frac{1}{\delta_\Omega(x)},\end{aligned}$$

where $x = \widetilde{\mu}(\zeta)$. In particular, there is a positive real constant C such that $\|\epsilon_j(\zeta)\|_{g_\Omega} \leq C\frac{1}{\delta_\Omega(\widetilde{\mu}(\zeta))}$ for $1 \leq j \leq N$ and $\zeta \in \Delta$. Since $\Omega \Subset \mathbb{C}^N$ is convex, it follows from [Me93, Proposition 2.4.] that there is $C_1 \in \mathbb{R}$ such that $C_1 - \frac{1}{2}\log\delta_\Omega(z) \leq \frac{1}{2}d_\Omega(0,z)$ for any $z \in \Omega$. From our definition of the Kobayashi pseudo-distance $d_\Omega(\cdot,\cdot)$ and that $k_\Omega(\cdot,\cdot)$ in [Me93], we have $k_\Omega(\cdot,\cdot) = \frac{1}{2}d_\Omega(\cdot,\cdot)$. Then, we have $e^{-2C_1}\delta_\Omega(z) \geq e^{-d_\Omega(0,z)}$ so that

$$\delta(\zeta) \leq 2\cdot e^{-d_\Delta(0,\zeta)} \leq 2\cdot e^{-d_\Omega(\mathbf{0},\widetilde{\mu}(\zeta))} \leq 2e^{-2C_1}\cdot\delta_\Omega(\widetilde{\mu}(\zeta)).$$

It follows that for $1 \leq j \leq N$, $\|\epsilon_j(\zeta)\|_{g_\Omega} \leq C\frac{1}{\delta_\Omega(\widetilde{\mu}(\zeta))} \leq C'\frac{1}{\delta(\zeta)}$ for $\zeta \in \Delta$, where C' is some positive real constant. The claim is proven. Then, we have

$$\|\hat{\tau}(\zeta)\| \leq \hat{C}\,\delta(\zeta)\cdot\sum_k|\tau^k_{11}(\zeta)|$$

on $U_{b'} \cap \Delta$ for some positive real constant $\hat{C}$. The summation in the above inequality is a finite sum. For a general point $b' \in \partial\Delta$, $\|\hat{\tau}(\zeta)\|^2$ can be extended as a real-analytic function in an open neighborhood $U_{b'}$ of b' in $\mathbb{C}$ (by Lemma 3.6) and each $\tau^k_{11}(\zeta)$ can be extended as a holomorphic function on some neighborhood of $\overline{U_{b'}}$, then each $|\tau^k_{11}(\zeta)|$ is bounded above by a uniform positive real constant on $U_{b'}$ so that $\|\hat{\tau}(\zeta)\| \to 0$ as $\zeta \to b''$ for any $b'' \in U_{b'} \cap \partial\Delta$. Actually, the above arguments show that $\|\hat{\tau}(\zeta)\| \to 0$ as $\zeta \in b'$ for general point $b' \in \partial\Delta$. Note that $\|\hat{\tau}(\zeta)\|^2$ depends only on normal form of the tangent vector $\widetilde{\mu}'(\zeta)$, i.e. $\|\hat{\tau}(\zeta)\|^2 = \frac{\|\tau(\widetilde{\eta}(\zeta)\otimes\widetilde{\eta}(\zeta))\|^2}{\|\widetilde{\eta}(\zeta)\|^4}$. From the construction, $\widetilde{\eta}(\zeta)$ is actually independent of $\zeta \in \Delta$ so that $\|\hat{\tau}(\zeta)\|^2$ is constant on Δ. But then $\|\hat{\tau}(\zeta)\|^2 \to 0$ as $\zeta \to b'$ for general point $b' \in \partial\Delta$ implies that $\|\hat{\tau}(\zeta)\| \equiv 0$ on Δ, i.e. $\tau|_{T_{Z'}\otimes T_{Z'}}(\zeta) \equiv 0$ on Δ. The result follows. □

Lemma 4.14. *In the above construction, we have $\tau \equiv 0$.*

Proof. By the Lemma 4.13, we have $\tau|_{T_Z\otimes T_Z} \equiv 0$, i.e. $(\nabla_\eta\hat{\eta})(x) \in W_x$ for any $\eta, \hat{\eta} \in \Gamma_{\mathrm{loc},x}(Z, T_Z)$ and $x \in Z$. Note that $R_{\eta\overline{\zeta}\alpha\overline{\beta}} = 0$ for $\eta \in \Gamma_{\mathrm{loc},x}(Z, T_Z)$, $\zeta \in \mathcal{N}_\eta$, and any $\alpha, \beta \in T_x(\Omega)$, where $x \in Z$. From the definition of W, we have $R(\nabla_\eta\eta, \overline{\zeta}, \alpha, \overline{\beta}) = 0$, because $\gamma \in \Gamma(Z, W)$ if and only if $R_{\gamma\overline{\zeta}\alpha\overline{\beta}} = 0$ for any $\alpha, \beta \in \Gamma_{\mathrm{loc}}(Z, T_\Omega|_Z)$ and any $\zeta \in \mathcal{N}_\eta$, where $\eta \in \Gamma_{\mathrm{loc}}(Z, T_Z)$. Thus we have $R(\eta, \overline{\nabla_{\overline{\eta}}\zeta}, \alpha, \overline{\beta}) = 0$ for any $\alpha, \beta \in \Gamma_{\mathrm{loc}}(Z, T_\Omega|_Z)$. In particular, $(\nabla_{\overline{\eta}}\zeta)(\widetilde{\mu}(w)) \in \mathcal{N}_{\eta(w)}$. For any $\gamma \in \Gamma_{\mathrm{loc}}(Z, W)$, $\zeta \in \mathcal{N}_\eta$ and any $\alpha, \beta \in \Gamma_{\mathrm{loc}}(Z, T_\Omega|_Z)$, we have $R_{\gamma\overline{\zeta}\alpha\overline{\beta}} = 0$ so that

$$R(\nabla_\eta\gamma, \overline{\zeta}, \alpha, \overline{\beta}) + R(\gamma, \overline{\nabla_{\overline{\eta}}\zeta}, \alpha, \overline{\beta}) = 0.$$

Since $(\nabla_{\overline{\eta}}\zeta)(\widetilde{\mu}(w)) \in \mathcal{N}_{\eta(w)}$, we have

$$R((\nabla_\eta\gamma)(\widetilde{\mu}(w)), \overline{\zeta(w)}, \alpha(\widetilde{\mu}(w)), \overline{\beta(\widetilde{\mu}(w)}) = 0$$

for arbitrary $\zeta \in \mathcal{N}_\eta$, $\alpha, \beta \in \Gamma_{\mathrm{loc}}(Z, T_\Omega|_Z)$. Therefore, $(\nabla_\eta\gamma)(\widetilde{\mu}(w)) \in W_{\widetilde{\mu}(w)}$ for arbitrary $w \in U'$, i.e. $\tau \equiv 0$. This shows that if $\tau|_{T_Z\otimes T_Z} \equiv 0$, then $\tau \equiv 0$. □

Lemma 4.15. *In the above construction, we have $Z = \widetilde{\mu}(U') \subset \Omega'$ for some characteristic subdomain $\Omega' \subseteq \Omega$ of rank k.*

Proof. From the above constructions, $T_x(Z)$ is spanned by a rank k vector $\eta(w)$ at any $x = \widetilde{\mu}(w) \in Z$ $(w \in U')$ and there is a holomorphic vector subbundle $W \subset T_\Omega|_Z$ with $T_Z \subset W \subset T_\Omega|_Z$. By Lemma 4.14, we have $\tau \equiv 0$. We first show that there is a characteristic subdomain $\Omega' \subset \Omega$ of rank k such that Z is tangent to Ω' to the order at least 2 at some point $\mu(w_0)$ $(w_0 \in U')$ and $T_{\mu(w_0)}\Omega' = W_{\mu(w_0)}$. By considering the normal form of $W_{\mu(w_0)}$, it is clear that there is a characteristic subdomain $\Omega' \subset \Omega$ of rank k such that $\mu(w_0) \in \Omega'$ and $T_{\mu(w_0)}\Omega' = W_{\mu(w_0)}$. Moreover, for fixed w_0, such Ω' is unique because if there is characteristic subdomain $\Omega'' \subset \Omega$ such that $\mu(w_0) \in \Omega''$ and $T_{\mu(w_0)}\Omega'' = W_{\mu(w_0)}$, then by using some $\Phi \in \mathrm{Aut}(\Omega)$ with $\Phi(\mu(w_0)) = \mathbf{0}$, both $\Phi(\Omega')$ and $\Phi(\Omega'')$ are linear sections by complex vector subspaces in $\mathbb{C}^N \cong \mathfrak{m}^+$, but then their tangent spaces at $\mathbf{0}$ are coincide to each other so that $\Phi(\Omega') = \Phi(\Omega'')$, i.e. $\Omega' = \Omega''$. From the assumption that $\tau \equiv 0$, we have $(\nabla_\eta\gamma)(\mu(w)) \in W_{\mu(w)}$ for any $w \in U'$, where $\eta \in \Gamma_{\mathrm{loc},\mu(w)}(Z, T_Z), \gamma \in \Gamma_{\mathrm{loc},\mu(w)}(Z, W)$ are local holomorphic sections.

Denote by $\pi : \mathbb{G}(T_\Omega, n_{r-k}(\Omega)) \to \Omega$ the Grassmann bundle, where $\mathbb{G}(T_x(\Omega), n_{r-k}(\Omega))$ is the Grassmannian of the complex $n_{r-k}(\Omega)$-dimensional vector subspaces of $T_x(\Omega)$ for each $x \in \Omega$. From [MT92, p. 99], we can let $\mathcal{NS}_{r-k}(\Omega)$ be the collection of all $n_{r-k}(\Omega)$-planes which are holomorphic tangent spaces to the $(r-k)$-th characteristic subdomains of Ω, then $\mathcal{NS}_{r-k}(\Omega)$ lies in the Grassmann bundle $\mathbb{G}(T_\Omega, n_{r-k}(\Omega))$ and is a holomorphic fiber bundle over Ω with $\mathcal{NS}_{r-k}(\Omega) \cong \mathcal{NS}_{r-k,\mathbf{0}}(\Omega) \times \Omega$. For each $x \in \Omega$ and each $(r-k)$-th characteristic subdomain $\Omega'_x \subset \Omega$ containing x, we can lift Ω'_x to $\mathcal{NS}_{r-k}(\Omega)$ as

$$\widehat{\Omega'_x} = \{[T_y(\Omega')] \in \mathcal{NS}_{r-k,y}(\Omega) : y \in \Omega'_x\}.$$

Such lifting of $(r-k)$-th characteristic subdomains of Ω forms a tautological foliation $\mathscr{F}$ on $\mathcal{NS}_{r-k}(\Omega)$ with $n_{r-k}(\Omega)$-dimensional leaves $\widehat{\Omega'_x}$. Then, we let $\hat{Z}$ be the tautological lifting of S to $\mathcal{NS}_{r-k}(\Omega)$ defined by

$$\hat{Z} = \{[W_x] \in \mathcal{NS}_{r-k,x}(\Omega) : x \in Z\}.$$

Then $\hat{Z}$ is tangent to $\widehat{\Omega'}$ at $[W_{\mu(w_0)}]$ because of $(\nabla_\eta\gamma)(\mu(w_0)) \in W_{\mu(w_0)}$. Actually, since $(\nabla_\eta\gamma)(x) \in W_x$ for any $x \in Z$, $\hat{Z}$ is tangent to the leaf $\widehat{\Omega'_x}$ of $\mathscr{F}$ at $[W_x]$ for any $x \in Z$, where $\Omega'_x \subset \Omega$ is the characteristic subdomain of rank k at x satisfying $T_x(\Omega'_x) = W_x$. Therefore, $\hat{Z}$ is an integral curve of the integrable distribution defined by the foliation $\mathscr{F}$. From the general theory of foliation, such integral curve of the distribution induced from $\mathscr{F}$ must lie inside the single leaf $\widehat{\Omega'}$ of $\mathscr{F}$, which is also the maximal integral submanifold of the induced integrable distribution. Actually, any smooth real curve γ passing through $\mu(w_0)$ on $\hat{Z}$ should lie inside the single leaf $\widehat{\Omega'}$ of $\mathscr{F}$ so that $\hat{Z}$ itself should lie inside the leaf $\widehat{\Omega'}$ of the foliation $\mathscr{F}$ because $\hat{Z}$ is path connected. Note that Z is the image of $\hat{Z}$ under the canonical projection $\mathbb{G}(T_\Omega, n_{r-k}(\Omega)) \to \Omega$. But then the above argument shows that Z should lie in Ω' because $\hat{Z} \subset \widehat{\Omega'}$.

□

Remark. After proving Lemma 4.15, the first author realizes that Tsai [Ts93, p. 144] has also used a similar technique in which he considered invariant geodesic submanifolds of an irreducible compact Hermitian symmetric space. Notice that same kind of technique could be also used for reducible bounded symmetric domains (or reducible compact Hermitian symmetric spaces). Using the notations in [Ts93, p. 144], the requirement for $\hat{Z}$ lying inside a single leaf of $\mathscr{F}$ is that $\partial_w[W_{\mu(w)}](x) \subset T_x(\Omega'_x) = W_x$ for each $x \in Z$, which is equivalent to that for any local holomorphic sections $\gamma \in \Gamma_{\mathrm{loc},x}(Z, W)$ and $\eta \in \Gamma_{\mathrm{loc},x}(Z, T_Z)$, $(\nabla_\eta\gamma)(x) \in W_x$. Of course this is actually equivalent to the assumption $\tau \equiv 0$.

From the above constructions and Lemmas, we can complete the proof of Theorem 4.9 as follows:

Proof of Theorem 4.9 . From the holomorphic embedding $\mu : U \to \mathbb{C}^N$ and choosing an arbitrary general point $b \in U \cap \partial\Delta$, we have constructed a germ of holomorphic isometry $\widetilde{\mu} : (\Delta, m_0 g_\Delta; 0) \to$

$(\Omega, g_\Omega; \mathbf{0})$ satisfying the two properties mentioned in Proposition 3.8, say $\widetilde{\mu}$ is defined on $U' = B^1(0, \varepsilon')$. Denote also by $\widetilde{\mu} : (\Delta, m_0 g_\Delta) \to (\Delta, g_\Omega)$ the extension of $\widetilde{\mu}$ as a holomorphic isometry, the two properties are precisely (1) $\|\widetilde{\sigma}(\widetilde{\mu}(w))\|^2 \equiv \|\sigma(\mu(b))\|^2$ and (2) the normal form of $\frac{\widetilde{\mu}'(w)}{\|\widetilde{\mu}'(w)\|_{g_\Omega}}$ is independent of $w \in \Delta$ and of rank k, where k is some integer satisfying $1 \le k \le r$. By Lemma 4.15, $Z = \widetilde{\mu}(U')$ lies inside a characteristic subdomain $\Omega' \subseteq \Omega$ of rank k. In case $k = r = \mathrm{rank}(\Omega)$, then we have $\Omega' = \Omega$. Note that Ω is of tube type, so Ω' is also of tube type. Denote by $\sigma'(x)$ the second fundamental form of $(Z, g_{\Omega'}|_Z)$ in $(\Omega', g_{\Omega'})$ at $x \in Z$, where the Kähler metric $g_{\Omega'} = g_\Omega|_{\Omega'}$ on Ω' is precisely the restriction of g_Ω to Ω'. We write $\Omega' = G_0'/K'$ and automorphisms of Ω' can be extended as automorphism of Ω. Fix an arbitrary point $w \in U'$. If $\widetilde{\mu}'(w)$ is a rank k' vector in $T_{\widetilde{\mu}(w)}\Omega'$, then applying the K'-action would imply that the normal form of $\widetilde{\mu}'(w)$ is tangent to some totally geodesic polydisk $\Pi_{k'} \cong \Delta^{k'}$ in the maximal polydisk $\Pi_k \cong \Delta^k$ of Ω', which also lies in $\Delta^r \cong \Pi \subset \Omega$. This also implies that the normal form of $\widetilde{\mu}'(w)$ as a tangent vector in $T_{\widetilde{\mu}(w)}\Omega$ is of rank k'. Therefore $k = k'$ and $\widetilde{\mu}'(w)$ is a generic vector in $T_{\widetilde{\mu}(w)}(\Omega')$ for $w \in U'$. The idea is to consider a certain holomorphic line bundle over the projectivized tangent bundle $\mathbb{P}T_{\Omega'}$, and make use of the Poincaré-Lelong equation as an analogue of the arguments in [Mk02] to the local holomorphic curve $\widetilde{\mu}(U') \subset \Omega'$ such that the tangent space to $Z := \widetilde{\mu}(U')$ at $\widetilde{\mu}(w)$ is spanned by generic vector (i.e. a rank k vector) because Ω' is of tube type. From [Mk02], we have

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\|s\|_o^2 = mc_1(L, \widehat{g_{\Omega'}}) - lc_1(\pi^*E, \pi^*g_o) + [\mathcal{S}_{k-1}(\Omega')],$$

with $s \in \Gamma(\mathbb{P}T_{\Omega'}, L^{-m} \otimes \pi^*E^l)$, $E = \mathcal{O}(1)|_{\Omega'}$, $L \to \mathbb{P}T_{\Omega'}$ the tautological line bundle. Denote by ω the Kähler form of $(\Omega', g_{\Omega'})$. Since $\widetilde{\mu} : (\Delta, m_0 g_\Delta; \mathbf{0}) \to (\Omega, g_\Omega; \mathbf{0})$ is a germ of holomorphic isometry and $\widetilde{\mu}(U') \subset \Omega'$, we may regard $\widetilde{\mu} : (\Delta, m_0 g_\Delta; \mathbf{0}) \to (\Omega', g_{\Omega'}; \mathbf{0})$ as a germ of holomorphic isometry. Let

$$\hat{Z} = \{[\alpha] \in \mathbb{P}(T_x\Omega') : x \in Z,\ T_x(Z) = \mathbb{C}\alpha\}$$

be the tautological lifting of Z to $\mathbb{P}T_{\Omega'}$. Note that $\hat{Z}$ is a complex manifold without boundary so that $\int_{\hat{Z}} \sqrt{-1}\partial\overline{\partial}\log\|s\|_o = 0$ by Stokes' Theorem. Moreover, $\int_Z \omega|_Z$ is finite due to

$$\begin{aligned}\int_Z \omega|_Z = \int_{U'} \frac{m_0}{(1-|w|^2)^2}\sqrt{-1}dw \wedge d\overline{w} &= \int_{U'} \left(\frac{m_0}{(1-x^2-y^2)^2}\right) 2dxdy \\ &\le C\int_{U'} dxdy = C\mathrm{Vol}(U') < +\infty,\end{aligned}$$

where $w = x + \sqrt{-1}y$, $\mathrm{Vol}(U')$ is the Euclidean volume of U', C is the uniform upper bound of $2\left(\frac{m_0}{(1-x^2-y^2)^2}\right)$ on U' since $\frac{1}{1-x^2-y^2} \le \frac{1}{1-\varepsilon'^2}$ on $B^1(0, \varepsilon') = U'$. Since $\mathcal{S}_{k-1}(\Omega') \cap \hat{Z} = \varnothing$, we have

$$\int_{\hat{Z}} (mc_1(L, \widehat{g_{\Omega'}}) - lc_1(\pi^*E, \pi^*g_o)) = 0,$$

$$\int_Z (kc_1(T_Z, g_{\Omega'}|_Z) - 2c_1(E, g_o)) = 0.$$

Note that $c_1(T_Z, g_{\Omega'}|_Z) = \frac{1}{2\pi}\kappa_Z\omega|_Z$ by formula of the Gaussian curvarture κ_Z of $(Z, g_{\Omega'}|_Z)$ and [Mk89], p. 36. Moreover, $m = k$, $l = 2$ by [Mk02]. Then $\int_Z k\kappa_Z\omega = -c\int_Z \omega$ for some $c > 0$.

Denote by Δ_k the holomorphic disk of maximal Gaussian curvature $-\frac{2}{k}$, i.e. of diagonal type in the maximal polydisk $\Delta^k \cong \Pi_k \subset \Omega'$. Actually, $-k\kappa_{\Delta_k} \equiv c$ and $\kappa_{\Delta_k} \equiv -\frac{2}{k}$ so that $c = 2$. But then the equality $-2\int_Z \omega = \int_Z k\kappa_Z\omega$ and the inequality $\int_Z k\kappa_Z\omega \leq -2\int_Z \omega$ implies that $\kappa_Z \equiv -\frac{2}{k}$. Then we have $\|\sigma'(\widetilde{\mu}(w))\|^2 \leq -\frac{2}{k} + \frac{2}{k} = 0$ so that $\|\sigma'(\widetilde{\mu}(w))\|^2 \equiv 0$ on U', i.e. $(Z, g_\Omega|_Z) \subset (\Omega', g_\Omega|_{\Omega'})$ is totally geodesic. But then $(\Omega', g_\Omega|_{\Omega'}) \subseteq (\Omega, g_\Omega)$ is totally geodesic so that $(Z, g_\Omega|_Z) \subset (\Omega, g_\Omega)$ is totally geodesic and thus $\|\widetilde{\sigma}(\widetilde{\mu}(w))\|^2 \equiv 0$ on U'. In particular, $\|\sigma(\mu(b))\|^2 = \|\widetilde{\sigma}(\widetilde{\mu}(w))\|^2 = 0$. Since $b \in U \cap \partial\Delta$ is an arbitrary general point, we see that $\|\sigma(\mu(w))\|^2 \to 0$ as $w \to b$ for general point $b \in U \cap \partial\Delta$. $\square$

4.2 Complete proof of Theorem 1.2

In Section 3, we have construct a holomorphic isometry $(\Delta, m_0 g_\Delta) \to (\Omega, g_\Omega)$ into an irreducible bounded symmetric domain with certain properties. The following shows that our study on such a holomorphic isometry may be reduced to the case where Ω is of tube type.

Proposition 4.16. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank k and let $\widetilde{\mu} : (\Delta, \lambda g_\Delta) \to (\Omega, g_\Omega)$ be a holomorphic isometry such that the tangent space $T_x(Z)$ of $Z := \widetilde{\mu}(\Delta)$ is $\mathrm{Aut}(\Omega)$-equivalent and spanned by a rank k vector η_x in $T_x(\Omega)$. Then, there exists an invariant geodesic submanifold $\Omega' \subset \Omega$ containing Z such that Ω' is an irreducible bounded symmetric domain of rank k and of tube type. In particular, $Z \subset (\Omega, g_\Omega)$ is totally geodesic.*

Remark. Note that if $T_x(Z)$ is spanned by a rank k vector in $T_x(\Omega)$ with $k < r = \mathrm{rank}(\Omega)$, then the construction in Section 4.1 is also valid for a bounded symmetric domain of non-tube type and one construct an invariant geodesic submanifold Ω'' of Ω which contains Z and Ω'' is a bounded symmetric domain of rank k and of tube type. In particular, we may suppose that $T_x(Z)$ is spanned by a generic vector in $T_x(\Omega)$ and Ω is of rank k.

Proof. If Ω is of tube type, then the result follows from the proof of Theorem 4.9. Thus it suffices to consider the case where Ω is of non-tube type. From the classification of irreducible bounded symmetric domain, Ω is biholomorphic to either $D^{\mathrm{I}}_{p,q}$ $(p < q)$, D^{II}_{2n+1} $(n \geq 2)$ or D^{V}. Define $P : T_\Omega \otimes T_\Omega \to T_\Omega \otimes T_\Omega$ by $g(P(\alpha \otimes \beta), \overline{\gamma} \otimes \overline{\delta}) = R_{\alpha\overline{\gamma}\beta\overline{\delta}}(\Omega, ds^2_\Omega)$. Here $g_x(\cdot,\cdot)$ is a natural Hermitian pairing of the basis for $S^2T_x(\Omega)$, i.e. $g_x(e_i \cdot e_j, \overline{e_s} \cdot \overline{e_l}) = 1$ (resp. 0) if $\{i,j\} = \{s,l\}$ (resp. $\{i,j\} \neq \{s,l\}$). Then P is parallel because $\nabla R \equiv 0$. We define $\rho : (T_\Omega \otimes T_\Omega) \otimes T^*_\Omega \to T_\Omega$ so that for each $x \in \Omega$,

$$\rho_x : (T_x(\Omega) \otimes T_x(\Omega)) \otimes T^*_x(\Omega) \to T_x(\Omega)$$

is a multi-linear map given by $\rho_x(\mu \otimes \nu)(\omega^*) = \omega^*(\nu)\mu$ for decomposable elements $(\mu \otimes \nu) \otimes \omega^* \in (T_x(\Omega) \otimes T_x(\Omega)) \otimes T^*_x(\Omega)$. We have $P(\alpha \otimes \alpha) = \sum_{\varphi,\varphi' \in \Delta^+_M} R_{\alpha\overline{e_\varphi}\alpha\overline{e_{\varphi'}}}(\Omega, g_\Omega) e_\varphi \otimes e'_\varphi$ and $\rho(P(\alpha \otimes \alpha) \otimes e^*_\phi) = \sum_{\varphi \in \Delta^+_M} R_{\alpha\overline{e_\varphi}\alpha\overline{e_\phi}}(\Omega, g_\Omega) e_\varphi$. Define the vector subbundle $V := \rho(P(\eta \otimes \eta) \otimes T^*_\Omega) \subset T_\Omega|_Z$, where η is a non-zero holomorphic vector field on $Z = \widetilde{\mu}(\Delta) \subset \Omega$.

By using the normal form $\eta(w) \in T_{\mathbf{0}}(\Omega)$ of $\frac{\widetilde{\mu}'(w)}{\|\widetilde{\mu}'(w)\|_{g_\Omega}}$, if Ω is of the classical type, then it follows from direct computation of the Riemannian curvature of (Ω, g_Ω) that the normal form of V_x $(x \in Z)$ as a complex vector subspace of $T_{\mathbf{0}}(\Omega)$ is exactly $M(p,p;\mathbb{C}) = T_{\mathbf{0}}(D^{\mathrm{I}}_{p,p})$ (resp. $M_a(2n;\mathbb{C}) = T_{\mathbf{0}}(D^{\mathrm{II}}_{2n})$) if $\Omega \cong D^{\mathrm{I}}_{p,q}$ $(p < q)$ (resp. D^{II}_{2n+1} $(n \geq 2)$). In the case where $\Omega \cong D^{\mathrm{V}}$, it follows from the computation of Tsai [Ts93, pp. 149-151] and $R(v, \overline{w})v' = -[[v, \overline{w}], v']$ that the normal form of V_x $(x \in Z)$ as a complex vector subspace of $T_{\mathbf{0}}(\Omega)$ is exactly $T_{\mathbf{0}}(\Omega')$ for some invariant geodesic submanifold $\Omega' \subset \Omega$ satisfying $\Omega' \cong D^{\mathrm{IV}}_8$. Actually, we may write the normal form $\eta(w) = \eta_1(w)e_{x_1-x_2} + \eta_2(w)e_{x_1+x_2+x_3}$, then we compute $R(\eta(w), \overline{e_\varphi})\eta(w) = [[e_{-\varphi}, \eta(w)], \eta(w)]$ for each non-compact positive root φ. It then follows from Tsai [Ts93, pp. 149-151] that the normal form of V_x is $\rho(P(\eta(w), \eta(w)) \otimes T^*_{\mathbf{0}}(\Omega))$, which is spanned by $e_{x_1-x_i}$, $4 \leq i \leq 6$; $e_{x_1+x_3+x_i}$, $4 \leq i \leq 6$; $e_{x_1-x_2}$ and $e_{x_1+x_2+x_3}$. Here $\eta(w) = \eta_{\widetilde{\mu}(w)}$ for $w \in \Delta$. In particular, the normal form of V_x is exactly $T_o(Q^8) = T_{\mathbf{0}}(D^{\mathrm{IV}}_8)$. It is then obvious that $\mathrm{Span}_{\mathbb{C}}\{e_{\psi_j}(x) : j = 1, \ldots, k\} \subset V_x$ and $\eta_x \in V_x$ for each $x \in Z$ for each $x \in Z$. By similar arguments as in the proof of Lemma 4.11, $V \subset T_\Omega|_Z$ is a holomorphic vector subbundle with $T_Z \subset V$. Let $\tau : T_Z \otimes V \to T_\Omega|_Z/V$ be $\tau(\eta \otimes \gamma) = \nabla_\eta \gamma$ mod V. Then it follows from the arguments in the proof of Lemma 4.12 that τ is holomorphic since $V_x = T_x(\Omega'_x)$ for some invariant geodesic submanifold $\Omega'_x \subset \Omega$. It follows from arguments in the proof of Lemma 4.13 that $\tau|_{T_Z \otimes T_Z} \equiv 0$. From the definition of $V \subset T_\Omega|_Z$ and the fact that $(\nabla_\eta \hat{\eta})(x) \in V_x$ for any $x \in Z$ and $\eta, \hat{\eta} \in \Gamma_{\mathrm{loc},x}(Z, T_Z)$, we have $\tau \equiv 0$. Actually, ρ is a contraction and thus for $\hat{\eta}, \eta \in \Gamma_{\mathrm{loc},x}(Z, T_Z)$, we have

$$
\begin{aligned}
\nabla_{\hat{\eta}}(\rho(P(\eta \otimes \eta) \otimes \omega^*))(x) =& \rho(\nabla_{\hat{\eta}}(P(\eta \otimes \eta)) \otimes \omega^*))(x) + \rho(P(\eta \otimes \eta) \otimes (\nabla_{\hat{\eta}}\omega^*))(x) \\
=& \rho(P((\nabla_{\hat{\eta}}\eta)(x) \otimes \eta(x)) \otimes \omega^*(x)) + \rho(P(\eta(x) \otimes (\nabla_{\hat{\eta}}\eta)(x)) \otimes \omega^*(x)) \\
&+ \rho(P(\eta(x) \otimes \eta(x)) \otimes (\nabla_{\hat{\eta}}\omega^*)(x)),
\end{aligned}
$$

which lies in V_x because $(\nabla_{\hat{\eta}}\eta)(x) \in V_x$ and $[[\mathfrak{m}^-, V_x], V_x] \subset V_x$ (cf. Tsai [Ts93, Lemma 4.3.]). In other words, V is parallel on Z. By applying the foliation technique as in the proof of Lemma 4.15, there is an invariant geodesic submanifold $\Omega' \subset \Omega$ such that $Z \subset \Omega'$ and $T_x(\Omega') = V_x$ for any $x \in Z$. In addition, such a submanifold Ω' is irreducible and of tube type as a Hermitian symmetric space of the non-compact type. If Ω is of tube type, then it follows from the above construction that $\Omega' = \Omega$. If Ω' is of non-tube type, then Ω is biholomorphic to either $D^{\mathrm{I}}_{p,q}$ $(p < q)$, D^{II}_{2n+1} $(n \geq 2)$ or D^{V} so that we have the following:

(i) If $\Omega \cong D^{\mathrm{I}}_{p,q}$ $(p < q)$ (resp. $\Omega \cong D^{\mathrm{II}}_{2n+1}$ $(n \geq 2)$), then $\Omega' \cong D^{\mathrm{I}}_{p,p}$ (resp. $\Omega' \cong D^{\mathrm{II}}_{2n}$).

(ii) If $\Omega \cong D^{\mathrm{V}}$, then $\Omega' \cong D^{\mathrm{IV}}_8$.

From the arguments in the proof of Theorem 4.9, $Z = \widetilde{\mu}(\Delta) \subset (\Omega', g_\Omega|_{\Omega'})$ is totally geodesic and thus $Z \subset (\Omega, g_\Omega)$ is totally geodesic. □

Indeed, the proof of Theorem 1.2 (under the assumption that the bounded symmetric domain Ω is irreducible) already follows from Proposition 4.16 and the proof of Theorem 4.9. Now, it remains to consider the bounded symmetric domain Ω being reducible. The idea is to generalize the methods to the case where Ω is reducible throughout sections 3, 4.1 and that in Proposition 4.16, then this would complete the proof of Theorem 1.2.

Now, We may write $\Omega = \Omega_1 \times \cdots \times \Omega_m \Subset \mathbb{C}^{N_1} \times \cdots \times \mathbb{C}^{N_m} = \mathbb{C}^N$ for some integer $m \geq 1$, where $\Omega_j \Subset \mathbb{C}^{N_j}$ is an irreducible bounded symmetric domain in its Harish-Chandra realizations for $j = 1, \ldots, m$. Equipping Ω (resp. Δ) with the Bergman metric ds^2_Ω (resp. ds^2_Δ), then by slight modifications we may obtain analogues of Lemma 2.4, Lemma 3.5, Lemma 3.7, Proposition 3.8 and the results in Section 4.1 in the case where Ω is reducible. Recall that $\mu : U = B^1(b_0, \varepsilon) \to \mathbb{C}^{N_1} \times \cdots \times \mathbb{C}^{N_m} = \mathbb{C}^N$ is a holomorphic embedding such that $\mu(U \cap \Delta) \subset \Omega$ and $\mu(U \cap \partial\Delta) \subset \partial\Omega$. Writing $\mu = (\mu_1, \ldots, \mu_m)$ with $\mu_j : U \to \mathbb{C}^{N_j}$ being a holomorphic map, $j = 1, \ldots, m$.

4.2.1 Basic settings

We may write the Bergman kernel $K_\Omega(z,\xi) = \frac{1}{Q_\Omega(z,\xi)}$ for some real constant $C'_\Omega > 0$ and some polynomial $Q_\Omega(z,\xi)$ in $(z,\overline{\xi})$, then $\omega_{ds^2_\Omega} = -\sqrt{-1}\partial\overline{\partial}\log Q_\Omega(z,z)$. In the case where $\Omega = \Delta$, we have $Q_\Delta(z,\xi) = \pi \cdot (1 - z\overline{\xi})^2$ for $z, \xi \in \mathbb{C}$. For the construction of a germ of holomorphic isometry $\widetilde{\mu}$ in Lemma 3.5 and Proposition 3.8, for general point $b \in U \cap \partial\Delta$, there is an open neighborhood U_b of b in $U \subset \mathbb{C}$ such that

$$Q_\Omega(\mu(w), \mu(w)) = \chi(w)(1 - |w|^2)^{\lambda'} = \frac{\chi(w)}{\pi^{\frac{\lambda'}{2}}} Q_\Delta(w,w)^{\frac{\lambda'}{2}}$$

on U_b for some non-vanishing smooth function χ on a neighborhood of $\overline{U_b}$ and some positive integer λ'. Then we may construct the sequence $\{\widetilde{\mu}_j = \Phi_j \circ \mu \circ \varphi_j\}_{j=1}^{+\infty}$ as in Section 3 such that

$$\widetilde{\mu}_j^* \omega_{ds^2_\Omega} = \frac{\lambda'}{2}\omega_{ds^2_\Delta} - \sqrt{-1}\partial\overline{\partial}\log\chi(\varphi_j(\zeta)),$$

then we obtain a germ of holomorphic isometry $\widetilde{\mu} : \left(\Delta, \frac{\lambda'}{2} ds^2_\Delta; 0\right) \to \left(\Omega, ds^2_\Omega; \mathbf{0}\right)$ by taking limit of some subsequence of $\{\widetilde{\mu}_j\}_{j=1}^{+\infty}$. Note that such a germ $\widetilde{\mu}$ could be extended to a holomorphic isometry $\left(\Delta, \frac{\lambda'}{2} ds^2_\Delta\right) \to \left(\Omega, ds^2_\Omega\right)$ by the extension theorem of Mok [Mk12]. Then we may generalize Proposition 3.8 to the case where Ω is reducible. Indeed, by decomposing $T_x(\Omega) = T_{x_1}(\Omega_1) \oplus \cdots \oplus T_{x_m}(\Omega_m)$ for $x = (x_1, \ldots, x_m) \in \Omega_1 \times \cdots \times \Omega_m$, we may decompose the normal form $\eta(w) = \eta_1(w) + \ldots + \eta_m(w) \in T_{\mathbf{0}}(\Omega_1) \oplus \cdots \oplus T_{\mathbf{0}}(\Omega_m)$ of $\frac{\widetilde{\mu}'(w)}{\|\widetilde{\mu}'(w)\|_{ds^2_\Omega}}$.

4.2.2 First step

The first step is to show that since $Z := \widetilde{\mu}(\Delta)$ has $\mathrm{Aut}(\Omega')$-equivalent tangent space $T_x(Z)$ spanned by a rank k vector η_x of $T_x(\Omega)$, then Z lies inside an invariant geodesic submanifold $\Omega' \subset \Omega$ of rank k and of tube type as a bounded symmetric domain.

Tube type: We first consider the case where Ω is of tube type (equivalently all Ω_j's are of tube type). For $x \in \Omega$, let Q_x be a Hermitian bilinear form on $T_x(\Omega) \otimes \overline{T_x(\Omega)}$ given by $Q(\alpha \otimes \overline{\beta}, \alpha' \otimes \overline{\beta'}) =$

$R_{\alpha\overline{\alpha'}\beta'\overline{\beta}}(\Omega, ds^2_\Omega)$. For $x_j \in \Omega_j$, we also let $Q^{(j)}_{x_j}$ be a Hermitian bilinear form on $T_{x_j}(\Omega_j) \otimes \overline{T_{x_j}(\Omega_j)}$ by $Q^{(j)}_{x_j}(\alpha\otimes\overline{\beta}, \alpha'\otimes\overline{\beta'}) = R_{\alpha\overline{\alpha'}\beta'\overline{\beta}}(\Omega_j, ds^2_{\Omega_j})$ and let $\mathcal{N}^{(j)}_{\alpha_j}$ be the null space of the Hermitian bilinear form $H^{(j)}_{\alpha_j}(v, v') := R_{\alpha_j\overline{\alpha_j}v\overline{v'}}(\Omega_j, ds^2_{\Omega_j})$ for $\alpha_j \in T_{x_j}(\Omega_j)$.

For $w \in U'$, we define $W_{\widetilde{\mu}(w)} = \left\{ v \in T_{\widetilde{\mu}(w)}\Omega : Q_{\widetilde{\mu}(w)}(v \otimes \overline{\zeta}, \cdot) \equiv 0 \;\; \forall\, \zeta \in \mathcal{N}_{\widetilde{\mu}'(w)} \right\}$, then we have $W_{\widetilde{\mu}(w)} = \bigoplus_{j=1}^m W^{(j)}_{\widetilde{\mu}_j(w)}$, where

$$W^{(j)}_{\widetilde{\mu}_j(w)} = \left\{ v_j \in T_{\mu_j(w)}(\Omega_j) : Q^{(j)}_{\widetilde{\mu}_j(w)}(v_j \otimes \overline{\zeta}, \cdot) \equiv 0 \;\; \forall\, \zeta \in \mathcal{N}^{(j)}_{\widetilde{\mu}'_j(w)} \right\},\; j = 1, \ldots, m.$$

Up to permuting the irreducible factors Ω_j's of Ω we may assume that $\eta(w) = \eta_1(w) + \ldots + \eta_m(w) \in T_{\mathbf{0}}(\Omega) = T_{\mathbf{0}}(\Omega_1) \oplus \cdots \oplus T_{\mathbf{0}}(\Omega_m)$ is of rank $k = \sum_{j=1}^m k_j$ and each $\eta_j(w) \in T_{\mathbf{0}}(\Omega_j)$ is of rank k_j. Here we may suppose $k_l > 0$ for $l = 1, \ldots, m'$, $k_j = 0$, $\eta_j(w) = 0$ and $\widetilde{\mu}_j(w) \equiv x'_j$ is a constant map for $m' + 1 \le j \le m$ provided that $m' < m$. For $x = (x_1, \ldots, x_m) \in Z \subset \Omega = \Omega_1 \times \cdots \times \Omega_m$, we have

$$W_x = \bigoplus_{j=1}^m W^{(j)}_x = \begin{cases} T_{x_1}(\Omega'_{1,x_1}) \oplus \cdots \oplus T_{x_{m'}}(\Omega'_{1,x_{m'}}) \oplus \{\mathbf{0}\} \oplus \cdots \oplus \{\mathbf{0}\} & \text{if } m' < m \\ T_{x_1}(\Omega'_{1,x_1}) \oplus \cdots \oplus T_{x_m}(\Omega'_{1,x_m}) & \text{if } m' = m \end{cases}$$

for some characteristic subdomain $\Omega'_{j,x_j} \subseteq \Omega_j$, $j = 1, \ldots, m'$. Notice that it is possible that $\Omega'_{i,x_i} = \Omega_i$ for some i. The rest of the results obtained in Section 4.1 may be generalized in the case where Ω (resp. Ω') is reducible. It follows from the arguments in Section 4.1 that there is a characteristic subdomain of Ω containing the Poincaré disk $Z = \widetilde{\mu}(\Delta)$ which is of the form $\Omega'_1 \times \cdots \times \Omega'_{m'} \times \{x_{m'+1}\} \times \cdots \times \{x_m\} =: \Omega'$ (resp. $\Omega'_1 \times \cdots \times \Omega'_m =: \Omega'$) if $m' < m$ (resp. $m' = m$), where $\Omega'_j \subset \Omega_j$ is a characteristic subdomain of rank k_j, $1 \le j \le m'$. Notice that each Ω'_j is of tube type and each $\eta_j(w) \in T_{\mathbf{0}}(\Omega'_j)$ is of rank $k_j = \mathrm{rank}(\Omega'_j)$ for $j = 1, \ldots, m'$.

Non-tube type: Suppose that $\Omega = \Omega_1 \times \cdots \times \Omega_m$ is of non-tube type. We may suppose that $T_x(Z)$ is spanned by a generic vector in $T_x(\Omega)$, otherwise we are done by using the same method in the case where Ω being of tube type. Similar to the case in which we considered the holomorphic vector subbundle $W \subset T_\Omega|_Z$, one may generalize the method in the proof of Proposition 4.16 to the case where Ω is reducible and equipped with the Bergman metric ds^2_Ω. The key point is that our construction of the holomorphic vector subbundle $V \subset T_\Omega|_Z$ comes from the Riemannian curvature tensor of (Ω, ds^2_Ω), which is decomposed into sum of Riemannian curvature tensors of $(\Omega_j, ds^2_{\Omega_j})$ in some sense, $j = 1, \ldots, m$. Then, it follows that there is an invariant geodesic submanifold $\Omega'_j \subseteq \Omega_j$ of rank equal to that of Ω_j and of tube type for $j = 1, \ldots, m$ such that $Z \subset \Omega' := \Omega'_1 \times \cdots \times \Omega'_m$. Here $\Omega' \subset \Omega$ is an invariant geodesic submanifold which is of tube type and of rank equal to that of Ω.

In any case, given a bounded symmetric domain Ω of rank r, the Poincaré disk Z lies inside an invariant geodesic submanifold $\Omega' \subset \Omega$ of rank k and of tube type, $T_x(Z)$ is spanned by a generic vector in $T_x(\Omega')$ and is $\mathrm{Aut}(\Omega')$-equivalent. This completes the first step.

4.2.3 Second step

Notice that the method of using Poincaré-Lelong equation as in the proof of Theorem 4.9 may be extended to the case where the bounded symmetric domain Ω' is reducible.

Proposition 4.17. *Let $\Omega' = \Omega'_1 \times \cdots \times \Omega'_{m'}$ be a bounded symmetric domain of tube type and of rank k equipped with a Kähler metric $g'_{\Omega'} = \bigoplus_{j=1}^{m'} \mathrm{Pr}_j^* g'_{\Omega'_j}$ on Ω', where m' is some positive integer, $g'_{\Omega'_j} = \lambda_j g_{\Omega'_j}$ for some positive integer λ_j and $\mathrm{Pr}_j : \Omega' \to \Omega'_j$ is the projection onto the j-th irreducible factor of Ω', $j = 1, \ldots, m'$. We also let $Z \subset \Omega'$ be the local holomorphic curve, i.e. Z is the image of a germ of holomorphic isometry $\widetilde{\mu} : (\Delta, \lambda ds^2_\Delta; 0) \to (\Omega', g'_{\Omega'}; \mathbf{0})$ for some positive real constant $\lambda > 0$, such that $T_x(Z)$ is spanned by a rank k vector $\eta_x \in T_x(\Omega')$. Then $(Z, g'_{\Omega'}|_Z) \subset (\Omega', g'_{\Omega'})$ is totally geodesic.*

Proof. If Ω' is irreducible, then we are done by the proof of Theorem 4.9. Consider the case where $\Omega' = \Omega'_1 \times \cdots \times \Omega'_{m'}$ is reducible and of tube type, where each Ω'_j is an irreducible bounded symmetric domain of rank k_j and $m' \geq 2$ is some integer. Under the assumptions, we have $k = \sum_{j=1}^{m'} k_j$ and each Ω'_j is of tube type. We only need to apply the method in the proof of Theorem 4.9 and that in [Mk02], and we generalize the settings to the case where Ω' is reducible. Denote by $S^{(j)}_{l,x_j}(\Omega'_j)$ the l-th characteristic variety for Ω'_j at $x_j \in \Omega'_j$, $j = 1, \ldots, m'$. Then $\mathcal{S}_{k-1,x}(\Omega')$ is indeed a union of m' hypersurfaces of $\mathbb{P}(T_x(\Omega'))$ and thus is a divisor of $\mathbb{P}(T_x(\Omega'))$ for each $x \in \Omega'$. In particular $\mathcal{S}_{k-1}(\Omega')$ still defines a divisor line bundle $[\mathcal{S}_{k-1}(\Omega')] \subset \mathbb{P}T_{\Omega'}$. For $x = (x_1, \ldots, x_{m'}) \in \Omega'$, denote by

$$\mathcal{S}^j_{k-1,x}(\Omega') = \left\{ [v_1 \oplus \cdots \oplus v_{m'}] \in \mathbb{P}\left(T_{x_1}(\Omega'_1) \oplus \cdots \oplus T_{x_{m'}}(\Omega'_{m'})\right) : v_j \in \widehat{S}^{(j)}_{k_j-1,x_j}(\Omega'_j) \right\},$$

where $\widehat{S}^{(j)}_{k_j-1,x_j}(\Omega'_j)$ is the cone over $S^{(j)}_{k_j-1,x_j}(\Omega'_j)$ in $T_{x_j}(\Omega'_j)$, then $\mathcal{S}_{k-1,x}(\Omega') = \bigcup_{j-1}^{m'} \mathcal{S}^j_{k-1,x}(\Omega')$. In particular we have $\mathcal{S}_{k-1}(\Omega') \cong \Omega' \times \mathcal{S}_{k-1,o}(\Omega') \subset \Omega' \times \mathbb{P}(T_o(\Omega')) \cong \mathbb{P}T_{\Omega'}$. Similarly, we define $\mathcal{S}_{k-1}(X'_c) \subset \mathbb{P}T_{X'_c}$. Let $L \to \mathbb{P}T_{X'_c}$ be the tautological line bundle and $\pi : \mathbb{P}T_{X'_c} \to X'_c$ be the projectivized tangent bundle over X'_c. Writing $X'_c = X'_{c,1} \times \cdots \times X'_{c,m'}$ with each $X'_{c,j}$ being an irreducible compact dual Hermitian symmetric space of Ω'_j, then $\mathrm{Pic}(X'_c) \cong \mathrm{Pic}(X'_{c,1}) \times \cdots \times \mathrm{Pic}(X'_{c,m'})$ because each $X'_{c,j}$ is a Fano manifold. Denote by $\mathrm{Pr}_j : X'_c = X'_{c,1} \times \cdots \times X'_{c,m'} \to X'_{c,j}$ be the canonical projection onto the j-th irreducible factor of X'_c and $\pi_j := \mathrm{Pr}_j \circ \pi$, $j = 1, \ldots, m'$. Therefore, $\mathrm{Pic}(\mathbb{P}T_{X'_c})$ is generated by $\pi^*(\mathrm{Pr}_j^* \mathcal{O}_{X'_{c,j}}(1))$, $j = 1, \ldots, m'$, and L. Pulling back of a non-trivial holomorphic section of $S^{k_j} T^*_{X'_{c,j}} \otimes \mathcal{O}_{X'_{c,j}}(2)$ by the projection $\mathrm{Pr}_j : X'_c \to X'_{c,j}$ gives a non-trivial holomorphic section in the holomorphic vector bundle $S^{k_j} T^*_{X'_c} \otimes \mathrm{Pr}_j^*(\mathcal{O}_{X'_{c,j}}(2))$, which further gives a non-trivial holomorphic section in $L^{-k_j} \otimes \pi_j^* \mathcal{O}_{X'_{c,j}}(2)$. Then it follows from [Mk02, Proposition 3] that $[\mathcal{S}^j_{k-1}(X'_c)] \cong L^{-k_j} \otimes \pi_j^* \mathcal{O}_{X'_{c,j}}(2)$ provided that Ω'_j is of rank ≥ 2. If $\Omega'_j \cong \Delta$ is biholomorphic to the unit disk for some j, then we also have $[\mathcal{S}^j_{k-1}(X'_c)] \cong L^{-1} \otimes \pi_j^* \mathcal{O}_{X'_{c,j}}(2)$ with $X'_{c,j} \cong \mathbb{P}^1$. Moreover, we may simply consider the divisor line bundle $[\mathcal{S}_{k-1}(X'_c)] \cong$

$\bigotimes_{j=1}^{m'}[\mathcal{S}^j_{k-1}(X'_c)]^{\lambda_j}$ so that

$$[\mathcal{S}_{k-1}(X'_c)] \cong L^{l_0} \otimes \bigotimes_{j=1}^{m'} \pi^*(\mathrm{Pr}_j^* \mathcal{O}_{X'_{c,j}}(2\lambda_j)),$$

where $l_0 = -\sum_{j=1}^{m'} \lambda_j k_j$. We denote by $\pi : \mathbb{P}T_{\Omega'} \to \Omega'$ be the canonical projection, $\mathrm{Pr}_j : \Omega' \to \Omega'_j$ the projection onto the j-th irreducible factor of Ω' and $\pi_j = \mathrm{Pr}_j \circ \pi$ for simplicity. Let E_j be the restriction of $\mathcal{O}_{X'_{c,j}}(1)$ to Ω'_j for $j = 1, \ldots, m'$. We denote also by L the restriction of L to Ω' and $\widehat{g'_{\Omega'}}$ is the canonical Hermitian metric on $L|_{\Omega'}$ induced from the Kähler metric $g'_{\Omega'}$ on Ω'. By duality we have

$$[\mathcal{S}_{k-1}(\Omega')] \cong L^{l_0} \otimes \bigotimes_{j=1}^{m'} \pi_j^* E_j^{2\lambda_j}.$$

It follows from [Mk02] that we have the Poincaré-Lelong equation

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\|s\|_o^2 = -l_0 c_1\big(L, \widehat{g'_{\Omega'}}\big) - \sum_{j=1}^{m'} 2\lambda_j c_1\left(\pi_j^* E_j, \pi_j^* h_o^j\right) + [\mathcal{S}_{k-1}(\Omega')],$$

where s is a non-trivial holomorphic section of $L^{l_0} \otimes \bigotimes_{j=1}^{m'} \pi_j^* E_j^{2\lambda_j}$. Here the Hermitian metric h_o^j on $E_j = \mathcal{O}_{X'_{c,j}}(1)|_{\Omega'_j}$ is induced from the Kähler metric $g'_{\Omega'_j}$ on Ω'_j. Similar to the case where Ω' is irreducible, we consider the tautological lifting $\hat{Z}$ of Z to $\mathbb{P}T_{\Omega'}$, then $\hat{Z} \cap \mathcal{S}_{k-1}(\Omega') = \varnothing$. Therefore, we have

$$\int_{\hat{Z}} \left(-l_0 c_1\big(L, \widehat{g'_{\Omega'}}\big) - \sum_{j=1}^{m'} 2\lambda_j c_1\left(\pi_j^* E_j, \pi_j^* h_o^j\right) \right) = 0,$$

$$\int_Z \left(l_0 c_1(T_Z, g'_{\Omega'}|_Z) + \sum_{j=1}^{m'} 2\lambda_j c_1\left(\mathrm{Pr}_j^* E_j, \mathrm{Pr}_j^* h_o^j\right) \right) = 0.$$

Moreover, denote by Δ_k a totally geodesic holomorphic disk in $(\Omega', g'_{\Omega'})$ of constant Gaussian curvature κ_{Δ_k} which is equal to the maximal holomorphic sectional curvature of $(\Omega', g'_{\Omega'})$. Then we have $\kappa_{\Delta_k} = -\frac{2}{\sum_{j=1}^{m'} \lambda_j k_j}$, where $k_j = \mathrm{rank}(\Omega'_j)$, $j = 1, \ldots, m'$. It follows from [Mk02] that $\sum_{j=1}^{m'} 2\lambda_j c_1\left(\mathrm{Pr}_j^* E_j, \mathrm{Pr}_j^* h_o^j\right) = -2\sum_{j=1}^{m'} \lambda_j \mathrm{Pr}_j^* \omega_{g_{\Omega'_j}} = -2\omega_{g'_{\Omega'}}$. Therefore, we have

$$\int_Z l_0 \kappa_Z \omega_{g'_{\Omega'}} = 2\int_Z \omega_{g'_{\Omega'}}.$$

Notice that $\kappa_Z \le \kappa_{\Delta_k} = \frac{2}{l_0}$ by the Gauss equation for $(Z, g'_{\Omega'}|_Z) \subset (\Omega', g'_{\Omega'})$. In particular, we have $\int_Z l_0 \kappa_Z \omega_{g'_{\Omega'}} \ge \int_Z 2\,\omega_{g'_{\Omega'}}$ and equality holds true only if $\kappa_Z \equiv \kappa_{\Delta_k}$ is the maximal holomorphic sectional curvature of $(\Omega', g'_{\Omega'})$, i.e. $(Z, g'_{\Omega'}|_Z) \subset (\Omega', g'_{\Omega'})$ is totally geodesic by the Gauss equation. $\square$

4.2.4 Conclusion of the proof

From our construction and the above two steps, we may complete the proof of Theorem 1.2 as follows:

Proof of Theorem 1.2. The case where Ω being of rank 1 is obviously true by our constructions in Section 3, so we assume that Ω is of rank ≥ 2. Following the constructions of a local holomorphic curve Z throughout Sections 3 and 4 we first consider the case where Ω is of tube type. Then we have shown that $Z \subset \Omega'$ for some characteristic subdomain $\Omega' \subset \Omega$ of rank k and $T_x(Z)$ is spanned by a generic vector in $T_x(\Omega')$. Here Ω' is also of tube type. It follows from Proposition 4.17 that $(Z, ds^2_\Omega|_Z) \subset (\Omega', ds^2_\Omega|_{\Omega'})$ is totally geodesic so that $Z \subset (\Omega, ds^2_\Omega)$ is totally geodesic. From the proof of Theorem 4.9, we have $\|\sigma(\mu(w))\|^2 \to 0$ as $w \to b$ for general point $b \in U \cap \partial\Delta$. Hence, the proof is completed under the assumption that Ω is of tube type. Actually, without assuming Ω being of tube type, we still obtain an invariant geodesic submanifold $\Omega' \subset \Omega$ which is of tube type, of rank k and containing Z provided that $T_x(Z)$ is spanned by a rank k vector in $T_x(\Omega)$ for some positive integer $k < \mathrm{rank}(\Omega)$. Then the result follows in this situation.

It remains to consider the case where Ω is of non-tube type and $T_x(Z)$ is spanned by a generic vector $\eta_x \in T_x(\Omega)$. Notice that Proposition 4.16 may be generalized to the case where Ω is reducible because of Proposition 4.17, namely $Z \subset \Omega'$ for some invariant geodesic submanifold $\Omega' \subset \Omega$ such that Ω' is of tube type and of rank equal to $\mathrm{rank}(\Omega)$. We may write $\Omega' = \Omega'_1 \times \cdots \times \Omega'_m \subset \Omega = \Omega_1 \times \cdots \times \Omega_m$, then $ds^2_\Omega|_{\Omega'} = \sum_{j=1}^m (p(\Omega_j) + 2) g_{\Omega'_j}$. It follows from Proposition 4.17 that $Z \subset (\Omega', ds^2_\Omega|_{\Omega'})$ is totally geodesic and thus $Z \subset (\Omega, ds^2_\Omega)$ is totally geodesic. Similar to the case where Ω is of tube type, the rest follows from our construction. □

5 Applications

Mok [Mk11, p. 255] has given a sketch of the proof of the following theorem on holomorphic equivariant embeddings between bounded symmetric domains.

Theorem 5.18 (Theorem 3.5.2. [Mk11])**.** *Let D and Ω be bounded symmetric domains, $\Phi : \mathrm{Aut}_0(D) \to \mathrm{Aut}_0(\Omega)$ be a group homomorphism, and $F : D \to \Omega$ be a Φ-equivariant holomorphic map. Then, F is totally geodesic.*

Proof. A sketch of the proof was given in Mok [Mk11] and we explain here the details. Since F is Φ-equivariant, it suffices to consider the case where Ω is irreducible. We may write the decomposition $D = D_1 \times \cdots \times D_k$ of D into irreducible factors, where $k \geq 1$. Denote by σ the $(1,0)$-part of the second fundamental form of D in Ω. By considering the Gauss equation and the holomorphic bisectional curvature of D, it suffices to show that $\sigma(\eta_i, \eta'_i) = 0$ for any $\eta_i, \eta'_i \in T_x(D)$ tangent to the i-th irreducible factor D_i of D for $i = 1, \ldots, k$ because $\sigma(\eta_i, \eta_j) = 0$ for any $\eta_i, \eta_j \in T_x(D)$ such that η_i (resp. η_j) being tangent to D_i (resp. D_j) for distinct i, j, $1 \leq i, j \leq k$. Thus, it suffices to consider the case where D is irreducible. If D is of rank ≥ 2, then we are done. If $D \cong \mathbb{B}^n$, then we may simply restrict to any minimal disk of D by slicing the complex unit ball $D \cong \mathbb{B}^n$ with affine linear subspaces of $\mathbb{C}^n$ intersecting $D \cong \mathbb{B}^n$. This shows that the problem may be reduced to the case where $D \cong \Delta$ is the unit disk. Notice that any Φ-equivariant holomorphic map $F : \Delta \to \Omega$ is a holomorphic isometry up to a normalizing constant. It follows from Theorem 1.3 that $F : \Delta \to \Omega$ is asymptotically totally geodesic at a general point $b \in \partial\Delta$. Then the ϕ-equivariance of F implies that $\|\sigma\|^2$ is constant on the whole unit disk Δ, which implies that $\|\sigma\| \equiv 0$, i.e. $F : \Delta \to \Omega$ is totally geodesic. □

As a consequence of Theorem 5.18 we have the following characterization of compact totally geodesic subsets of quotients of bounded symmetric domains. The deduction of Theorem 5.19 from Theorem 5.18 was given in [Mk11].

Theorem 5.19 (Theorem 3.5.3 [Mk11])**.** *Let* (Ω, ds^2_Ω) *be a bounded symmetric domain equipped with the Bergman metric* ds^2_Ω*. Let* $\Gamma \subset \mathrm{Aut}_0(\Omega)$ *be a torsion-free discrete subgroup and* $X := \Omega/\Gamma$*. Denote by* h *the Kähler metric on* X *induced from* ds^2_Ω*. Suppose* $Z \subset X$ *is a compact complex-analytic subvariety and* $(\mathrm{Reg}(Z), h|_{\mathrm{Reg}(Z)})$ *is locally symmetric. Then,* $Z \subset X$ *is a totally geodesic subset.*

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Shan Tai Chan, Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong (E-mail: pmstchan@hku.hk)

Ngaiming Mok, Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong (E-mail: nmok@hku.hk)