Zariski closures of images of algebraic subsets under the uniformization map on finite-volume quotients of the complex unit ball

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Abstract

We prove the analogue of the Ax-Lindemann-Weierstrass Theorem for *not necessarily arithmetic* lattices of the automorphism group of the complex unit ball $\mathbb{B}^n$ using methods of several complex variables, algebraic geometry and Kähler geometry. Consider a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ and the associated uniformization map $\pi : \mathbb{B}^n \to \mathbb{B}^n/\Gamma =: X_\Gamma$. Given an algebraic subset $S \subset \mathbb{B}^n$ and writing Z for the Zariski closure of $\pi(S)$ in X_Γ (which is equipped with a canonical quasi-projective structure), in some precise sense we realize Z as a variety uniruled by images of algebraic subsets under the uniformization map, and study the asymptotic geometry of an irreducible component $\widetilde{Z}$ of $\pi^{-1}(Z)$ as $\widetilde{Z}$ exits the boundary $\partial\mathbb{B}^n$ by exploiting the strict pseudoconvexity of $\mathbb{B}^n$, culminating in the proof that $\widetilde{Z} \subset \mathbb{B}^n$ is totally geodesic. Our methodology sets the stage for tackling problems in functional transcendence theory for arbitrary lattices of $\mathrm{Aut}(\Omega)$ for (possibly reducible) bounded symmetric domains Ω.

Let (X, ds^2_X) be a complex hyperbolic space form of finite volume, i.e., $X = \mathbb{B}^n/\Gamma$ is the quotient of the complex unit ball $\mathbb{B}^n$ by a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$, and ds^2_X be a canonical Kähler-Einstein metric on X. In Mok [Mo2] (2010), motivated by the study of the Gauss map on subvarieties of X we considered the question of determining Zariski closures of images of totally geodesic complex submanifolds $S \subset \mathbb{B}^n$ under the universal covering map $\pi : \mathbb{B}^n \to X$. We proved that the Zariski closure $Z = \overline{\pi(S)}^{\mathscr{Z}ar} \subset X$ must be a totally geodesic subset.

In place of $X = \mathbb{B}^n/\Gamma$ one can more generally consider $\Omega \Subset \mathbb{C}^N \subset M$ a bounded symmetric domain in the Harish-Chandra realization $\Omega \Subset \mathbb{C}^N$ and in the Borel embedding $\Omega \subset M$ into its dual Hermitian symmetric manifold M of the compact type, $\Gamma \subset \mathrm{Aut}(\Omega)$ a torsion-free lattice, and, in place of a totally geodesic complex submanifold one may consider $S \subset \Omega$ an irreducible algebraic subset, by which we mean an irreducible component of $V \cap \Omega$, where $V \subset M$ is a projective subvariety, $\dim(S) > 0$. In recent years, the question of finding Zariski closures $Z = \overline{\pi(S)}^{\mathscr{Z}ar}$ was posed in the area of functional transcendence theory in the form of the hyperbolic Ax-Lindemann-Weierstrass conjecture, when $\Gamma \subset \mathrm{Aut}(\Omega)$ is an *arithmetic* lattice. The conjecture was formulated by Pila in [Pi] (2011) in relation to the André-Oort conjecture in number theory using the method of Pila-Zannier (cf. [PZ] (2008)), and it is one of the two

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components for an *unconditional* affirmative resolution of the latter conjecture. By means of the method of o-minimality from model theory in mathematical logic in combination with other methods, the hyperbolic Ax-Lindemann-Weierstrass conjecture was resolved in the affirmative in the cocompact and arithmetic case by Ullmo-Yafaev [UY] (2014), for the moduli space $X = \mathcal{A}_g$ of principally polarized Abelian varieties by Pila-Tsimerman [PT] (2014), and in the general arithmetic case by Klingler-Ullmo-Yafaev [KUY] (2016). All these proofs relied heavily on the arithmeticity of the lattices being considered.

In relation to Mok [Mo2], the author was led to consider the same problem in complex differential geometry for *arbitrary* lattices. When Ω is irreducible, in view of the arithmeticity theorem of Margulis [Ma] (1984) only the complex unit ball $\mathbb{B}^n$ admits non-arithmetic finite-volume quotients. In this article we resolve in the affirmative the analogue of the hyperbolic Ax-Lindemann-Weierstrass conjecture in the rank-1 case for arbitrary lattices $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$. It serves as a starting point for tackling the analogous problem and other related problems (e.g. Ax-Schanuel) for arbitrary and possibly reducible lattices (cf. Remark after Lemma 4.1).

We consider the problem from a completely different perspective using methods of several complex variables, algebraic geometry and Kähler geometry. In the case where $S \subset \mathbb{B}^n$ is totally geodesic, it was proved in [Mo2, *loc. cit.*] that the Zariski closure $Z = \overline{\pi(S)}^{\mathscr{Z}ar}$ is necessarily "uniruled" by pieces of totally geodesic complex submanifolds, and a lifting $\widetilde{Z}$ of Z to $\mathbb{B}^n$ was shown to be totally geodesic from its asymptotic geometric behavior as $\widetilde{Z}$ exits $\partial\mathbb{B}^n$. For an arbitrary irreducible algebraic subvariety $S \subset \mathbb{B}^n$ we make use of the universal family $\rho : \mathscr{U} \to \mathcal{K}$ over an irreducible component $\mathcal{K} \subset \mathrm{Chow}(\mathbb{P}^n)$, the Chow space of $\mathbb{P}^n$, to construct by restriction and by descent a locally homogeneous holomorphic fiber bundle of projective varieties $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ equipped with a tautological meromorphic foliation $\mathscr{F}$. In the case of compact ball quotients, by embedding $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ as a locally homogeneous holomorphic fiber subbundle of some locally homogeneous projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$, which is necessarily projective algebraic, we are led to the study of the foliation $\mathscr{F}$ on the projective variety $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$.

For noncompact complex ball quotients $X_\Gamma = \mathbb{B}^n/\Gamma$ we make use of the existence of the minimal compactification even in the *non-arithmetic* case by the works of Siu-Yau [SY] (1982), which were shown to be projective in Mok [Mo3] (2012), and the methods of compactification of complete Kähler manifolds of finite volume of Mok-Zhong [MZ$_2$] (1989) applied to $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ to show that $\mathscr{P}_\Gamma$ and hence $\mathscr{U}_\Gamma$ is quasi-projective, and to show that the tautological foliation obtained by descent from the universal family $\rho : \mathscr{U} \to \mathcal{K}$ extends meromorphically to a compactification $\overline{\mathscr{U}_\Gamma}$. To complete the proof of Main Theorem we introduce a rescaling argument for Kähler submanifolds exiting $\partial\mathbb{B}^n$ by means of a result of Klembeck [Kl] (1978) according to which a certain standard complete Kähler metric on a strictly pseudoconvex domain is asymptotically of constant holomorphic sectional curvature. As a by-product of our proof of Main Theorem for arbitrary lattices $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ we show that the Zariski closure $Z \subset X_\Gamma$ of $\pi(S)$ is *uniruled* by subvarieties belonging to $\mathcal{K}$ (in the precise sense of Definition 3.1).

After earlier works the author's interest in further pursuing the analytic and geometric approach to functional transcendence theory was rekindled after discussions with Jacob Tsimerman in Spring 2016; it transpired that a combination of our approach from the perspective of complex geometry with works of Pila-Tsimerman on the hyperbolic Ax-Schanuel conjecture from the perspective of o-minimal geometry might shed light on the not necessarily arithmetic analogue of the latter conjecture. Ax-Schanuel Theorem for Shimura varieties has now been established by Mok-Pila-Tsimerman [MPT] (2019).

1. Introduction, background materials and statement of Main Theorem

Let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a torsion-free lattice and denote by $X := \mathbb{B}^n/\Gamma$ the quotient manifold. When $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ is an arithmetic lattice, by Satake [Sa] (1960) and Baily-Borel [BB] (1966) X admits a minimal compactification $\overline{X}_{\min}$ by adjoining a finite number of normal isolated singularities. In general $X = \mathbb{B}^n/\Gamma$ is of finite volume with respect to the canonical Kähler-Einstein metric, and by Siu-Yau [SY] (1982), X admits a compactification $X \subset \overline{X}_{\min}$ by adding a finite number of normal isolated singularities, where $\overline{X}_{\min}$ is exactly the minimal compactification of Satake-Baily-Borel in the arithmetic case. The methods of Siu-Yau [SY] are transcendental, and they apply to complete Kähler manifolds of finite volume and of pinched negative sectional curvature. When $X = \mathbb{B}^n/\Gamma$, Mok [Mo3] (2012) proved that $\overline{X}_{\min}$ is projective. For finite-volume complex ball quotients, by [Sa], [BB], [SY] and [Mo3] we have

Theorem 1.1. *Let $n \geqslant 2$ and denote by $\mathbb{B}^n \Subset \mathbb{C}^n$ the complex unit ball equipped with the canonical Kähler-Einstein metric $ds^2_{\mathbb{B}^n}$. Let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a torsion-free lattice. Denote by $X := \mathbb{B}^n/\Gamma$ the quotient manifold, of finite volume with respect to the canonical Kähler-Einstein metric ds^2_X induced from $ds^2_{\mathbb{B}^n}$. Then, there exists a projective variety $\overline{X}_{\min}$ such that $X = \overline{X}_{\min} - \{p_1, \cdots, p_m\}$, where each p_i, $1 \leqslant i \leqslant m$, is a normal isolated singularity of $\overline{X}_{\min}$.*

Here and henceforth, without loss of generality we choose the canonical Kähler-Einstein metric so that $(\mathbb{B}^n, ds^2_{\mathbb{B}^n})$ and hence (X, ds^2_X) are of constant holomorphic sectional curvature -2. From now on $X = \mathbb{B}^n/\Gamma$ will be equipped with the quasi-projective structure inherited from the projective variety $\overline{X}_{\min}$. To emphasize the dependence on the lattice Γ we will now write X_Γ for X, and $\overline{X_\Gamma}$ for $\overline{X}_{\min}$. In Mok [Mo2] we consider Zariski closures of totally geodesic complex submanifolds on $X_\Gamma = \mathbb{B}^n/\Gamma$, and we proved

Theorem 1.2. **(Mok [Mo2, Main Theorem])** *Let $X_\Gamma = \mathbb{B}^n/\Gamma$ be a complex ball quotient of finite volume with respect to ds^2_X, as in Theorem 1.1, and denote by $\pi : \mathbb{B}^n \to X_\Gamma$ the universal covering map. Let $S \subset \mathbb{B}^n$ be a totally geodesic complex submanifold in $\big(\mathbb{B}^n, ds^2_{\mathbb{B}^n}\big)$. Then, the Zariski closure Z of $\pi(S)$ in X_Γ is a totally geodesic subset.*

Denoting by $\widetilde{Z}$ an irreducible component of $\pi^{-1}(Z) \subset \mathbb{B}^n$, in the above $Z \subset X$ is said to be a totally geodesic subset if and only if $\widetilde{Z} \subset \mathbb{B}^n$ is a totally geodesic (complex) submanifold with respect to the canonical Kähler-Einstein metric.

In the current article we prove Ax-Lindemann-Weierstrass Theorem for not necessarily arithmetic finite-volume quotients of $\mathbb{B}^n$, $n \geqslant 2$. A subvariety $S \subset \mathbb{B}^n$ is said to be an irreducible algebraic subset if and only if it is an irreducible component of the intersection $V \cap \mathbb{B}^n$ for some (irreducible) projective subvariety $V \subset \mathbb{P}^n$, and an algebraic subset $S \subset \mathbb{B}^n$ is the union of a finite number of irreducible algebraic subsets. Note that a totally geodesic complex submanifold of $\mathbb{B}^n$ is precisely a non-empty intersection of the form $V \cap \mathbb{B}^n$, where $V \subset \mathbb{P}^n$ is a projective linear subspace of $\mathbb{P}^n$. We have

Main Theorem. *Let $n \geqslant 2$ and denote by $\mathbb{B}^n \Subset \mathbb{C}^n$ the complex unit ball equipped with the canonical Kähler-Einstein metric $ds^2_{\mathbb{B}^n}$. Let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a torsion-free lattice. Denote by $X_\Gamma := \mathbb{B}^n/\Gamma$ the quotient manifold, of finite volume with respect to the canonical Kähler-Einstein* *metric $g_\Gamma := ds^2_{X_\Gamma}$ induced from $ds^2_{\mathbb{B}^n}$ and equipped with the structure of a quasi-projective manifold from $X_\Gamma \subset \overline{X_\Gamma}$. Let $\pi : \mathbb{B}^n \to X_\Gamma$ be the universal covering map and denote by $S \subset \mathbb{B}^n$ an irreducible algebraic subset. Then, the Zariski closure $Z \subset X_\Gamma$ of $\pi(S)$ in X_Γ is a totally geodesic subset.*

For the proof of Main Theorem we will consider the universal family $\rho : \mathscr{U} \to \mathcal{K}$, $\mu : \mathscr{U} \to \mathbb{P}^n$ over some irreducible component $\mathcal{K}$ of the Chow space of $\mathbb{P}^n$ and obtain by restriction to $\mathbb{B}^n \subset \mathbb{P}^n$ and by descent to X_Γ a locally homogeneous holomorphic fiber bundle $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ equipped with a tautological foliation. $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ will be embedded as a locally homogeneous fiber subbundle of projective varieties in a certain locally homogeneous projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$, which implies in the compact case that $\mathscr{U}_\Gamma$ is projective. For the finite-volume and noncompact case we will need to embed the total space $\mathscr{P}_\Gamma$ onto a quasi-projective variety. For this purpose we will need the following compactification result of Mok-Zhong [MZ$_2$] (1989) from Kähler geometry.

Theorem 1.3. **(Mok-Zhong [MZ$_2$])** *Let (X, g) be a complete Kähler manifold of finite volume, bounded sectional curvature and finite topological type, and denote by ω the Kähler form of (X, g). Suppose there exists on (X, g) a Hermitian holomorphic line bundle (L, h) whose curvature form $\Theta(L, h)$ satisfies $a\omega \leqslant \Theta(L, h) \leqslant b\omega$ for some real constants $a, b > 0$. For an integer $k \geqslant 0$ and for a holomorphic section $s \in \Gamma(X, L^k)$, we say that s is of the Nevanlinna class if and only if $\log^+ \|s\|_{h^k} := \max(\log \|s\|_{h^k}, 0)$ is integrable on X. Then, denoting by $\mathcal{N}(X, L^k) \subset \Gamma(X, L^k)$ the vector subspace consisting of holomorphic sections of the Nevanlinna class, we have $\dim(\mathcal{N}(X, L^k)) < \infty$. Furthermore, there exists some $\ell > 0$ such that $\mathcal{N}(X, L^\ell)$ has no base points and defines a holomorphic embedding $\Phi_\ell : X \hookrightarrow \mathbb{P}(\mathcal{N}(X, L^\ell)^*)$ onto a dense Zariski open subset of some projective variety. In particular, X is biholomorphic to a quasi-projective manifold.*

For the study of $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ we will need the following basic result on Chow spaces of $\mathbb{P}^n$ from Chow-van der Waerden [CvdW] (1937). More details will be given in §2.

Theorem 1.4. *On the projective space $\mathbb{P}^n$, for an integer r, $0 \leqslant r < n$, and a positive integer d, denote by $\mathcal{Q} = \mathcal{Q}(n, r, d)$ the set of pure r-dimensional cycles W on $\mathbb{P}^n$ of degree d. Then, $\mathcal{Q}$ admits canonically the structure of a projective subvariety of some projective space $\mathbb{P}^s$, where $s = s(n, r, d)$. Moreover, $\mathrm{Aut}(\mathbb{P}^n) \cong \mathbb{P}GL(n+1, \mathbb{C})$ acts canonically on $\mathbb{P}^s$ preserving the subset $\mathcal{Q} \subset \mathbb{P}^s$. Furthermore, the subset $\mathcal{W} := \big\{w = ([W], x) \in \mathcal{Q} \times \mathbb{P}^n : x \in W\big\} \subset \mathcal{Q} \times \mathbb{P}^n \subset \mathbb{P}^s \times \mathbb{P}^n$ is a projective subvariety invariant under the natural action of $\mathrm{Aut}(\mathbb{P}^n)$ on $\mathbb{P}^s \times \mathbb{P}^n$.*

Here an element $[W] \in \mathcal{Q}$ denotes an r-cycle on $\mathbb{P}^n$, which is neither necessarily irreducible nor reduced. As an r-cycle $[W] = k_1[W_1] + \cdots + k_s[W_s]$, where $s \geqslant 1$ and $W_i, 1 \leqslant i \leqslant s$, are reduced and irreducible r-dimensional subvarieties of $\mathbb{P}^n$.

In the proof of Main Theorem we will study pre-images of Zariski closures of the image of an algebraic subset on $\mathbb{B}^n$ under the universal covering map, for which we will need the following result of Klembeck [Kl] (1978) on the asymptotic behavior of complete Kähler metrics on strictly pseudoconvex domains.

Theorem 1.5. **(Klembeck [Kl])** *Let $U \subset \mathbb{C}^n$ be a domain, ρ be a smooth real function on U and b be a point on U. Suppose $\rho(b) = 0$ and $d\rho(x) \neq 0$ for any $x \in U$, and assume that ρ is strictly plurisubharmonic on U, i.e., $\sqrt{-1}\partial\overline{\partial}\rho > 0$ on U. Let $U' \subset U$ be the open subset defined by $\rho < 0$, and s be the Kähler metric on U' with Kähler form given by $\omega_s = \sqrt{-1}\partial\overline{\partial}(-\log(-\rho))$.*

Then, (U', s) is asymptotically of constant holomorphic sectional curvature -2 at b, i.e., defining $\epsilon(x) \geqslant 0$ at $x \in U'$ to be the smallest nonnegative number such that holomorphic sectional curvatures of (U', s) at x are bounded between $-2-\epsilon(x)$ and $-2+\epsilon(x)$, then $\epsilon(x) \to 0$ as $x \in U'$ approaches $b \in \partial U' \cap U$.

2. Construction of locally homogeneous projective fiber subbundles of projective bundles

Consider the standard inclusions $\mathbb{B}^n \Subset \mathbb{C}^n \subset \mathbb{P}^n$. Let $W_0 \subset \mathbb{P}^n$ be an irreducible subvariety and S be an irreducible component of $W_0 \cap \mathbb{B}^n$. Let $\mathcal{K}$ be an irreducible component of the Chow space of $\mathbb{P}^n$ to which the reduced cycle W_0 belongs, written $[W_0] \in \mathcal{K}$. $\mathrm{Aut}(\mathbb{P}^n) = \mathbb{P}GL(n+1, \mathbb{C})$ acts on $\mathcal{K}$ naturally. Let $\rho : \mathscr{U} \to \mathcal{K}$ be the universal family associated to $\mathcal{K}$, i.e., $\mathscr{U} = \{([W], x) \in \mathcal{K} \times \mathbb{P}^n : x \in W\} \subset \mathcal{K} \times \mathbb{P}^n$, and $\rho : \mathscr{U} \to \mathcal{K}$ is induced from the canonical projection from $\mathcal{K} \times \mathbb{P}^n$ onto the first factor. For $[W] \in \mathcal{K}$ we have $[W] = k_1[W_1] + \cdots + k_s[W_s]$ where $s = s(W) \geqslant 1$ and $W_i, 1 \leqslant i \leqslant s$ are the reductions of the irreducible components of the pure r-dimensional complex space $W \subset \mathbb{P}^n$. From the definition of $\mathcal{K}$ a general member $[W] \in \mathcal{K}$ is irreducible and reduced, i.e., $s = 1$ and $k_1 = 1$. The canonical projection from $\mathcal{K} \times \mathbb{P}^n$ to $\mathbb{P}^n$ restricted to $\mathscr{U}$ gives $\mu : \mathscr{U} \to \mathbb{P}^n$, called the evaluation map of the universal family $\rho : \mathscr{U} \to \mathcal{K}$, which realizes $\mathscr{U}$ as the total space of a homogeneous holomorphic fiber bundle (with respect to $\mathrm{Aut}(\mathbb{P}^n)$) with fibers isomorphic to $\mathscr{U}_0 := \mu^{-1}(0)$. Hence, the map $\rho : \mathscr{U} \to \mathcal{K}$ is a holomorphic fibration with equidimensional fibers $\rho^{-1}(\kappa)$, $\kappa = [W]$. Each fiber $\rho^{-1}(\kappa)$ projects via the evaluation map $\mu : \mathscr{U} \to \mathbb{P}^n$ onto the support $\mathrm{Supp}(W) = W_1 \cup \cdots \cup W_s \subset \mathbb{P}^n$ of the cycles $W \subset \mathbb{P}^n$ belonging to $\mathcal{K}$. (We will henceforth make no notational distinction between W and its support.) The holomorphic fibration $\rho : \mathscr{U} \to \mathcal{K}$ defines naturally a foliation $\mathscr{F}$ on $\mathscr{U}$ such that, at a smooth point $u \in \mathscr{U}$ where ρ is a submersion, $\rho(u) =: [W] \in \mathcal{K}$, the germ of leaf of $\mathscr{F}$ passing through u is just the germ of $\widehat{W} := \rho^{-1}(\rho(u))$ at u, $\widehat{W}$ being the tautological lifting of W to $\mathscr{U}$. We have $W = \mu(\widehat{W}) = \mu(\rho^{-1}(\rho(u)))$.

Write $\mathscr{U}' := \mathscr{U}|_{\mathbb{B}^n}$. $\mathrm{Aut}(\mathbb{P}^n)$ acts on $\mathscr{U}'$, hence the holomorphic fiber bundle $\mu' : \mathscr{U}' \to \mathbb{B}^n$ is homogeneous under $\mathrm{Aut}(\mathbb{B}^n)$, and it descends under the action of a torsion-free discrete subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ to X_Γ to give a complex space $\mathscr{U}_\Gamma := \mathscr{U}'/\Gamma$ equipped with the evaluation map $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$, realizing the latter as a locally homogeneous holomorphic fiber bundle with fibers isomorphic to $\mathscr{U}_0$. The fiber bundle $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ will be our major object of study. In the case where X_Γ is compact, we will show that $\mathscr{U}_\Gamma$ is a projective variety by embedding $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ into some projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$. (Projective bundles are understood to be holomorphic.) By the Kodaira Embedding Theorem, we have the following well-known result on projective bundles.

PROPOSITION 2.1. *Let Z be a projective manifold and A be an ample line bundle on Z, and let $\varpi : \mathscr{P} \to Z$ be a projective bundle over Z. Denote by T_ϖ the relative tangent bundle of $\varpi : \mathscr{P} \to Z$. Then, for a sufficiently large integer k, the holomorphic line bundle $\det(T_\varpi) \otimes \varpi^* A^k$ is ample on $\mathscr{P}$. As a consequence, $\mathscr{P}$ is a projective manifold.*

We proceed to construct a projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ which admits $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ naturally as an embedded complex subspace. In what follows we denote by $G := \mathbb{P}GL(n+1, \mathbb{C})$, which is identified with the automorphism group of $\mathbb{P}^n$, by $G_0 := \mathbb{P}U(n, 1) \subset G$ the automorphism group of $\mathbb{B}^n \Subset \mathbb{C}^n \subset \mathbb{P}^n$, and by $\Gamma \subset G_0$ a torsion-free lattice. We write $P \subset G$ for the parabolic subgroup at $0 \in \mathbb{P}^n$, and $K \subset G_0$ for the isotropy subgroup of $(\mathbb{B}^n, ds^2_{\mathbb{B}^n})$ at $0 \in \mathbb{B}^n$. Lie algebras of real or complex Lie groups will be denoted by corresponding fraktur characters, so that $\mathfrak{g}$ stands for the Lie algebra of the complex Lie group G, and $\mathfrak{g}_0$ for the Lie algebra of the real Lie group G_0, etc. On the complex unit ball $\mathbb{B}^n \Subset \mathbb{C}^n$, $K = U(n)$ acts as a compact group of linear transformations on $\mathbb{C}^n$ extending to projective linear transformations on $\mathbb{P}^n$, and we denote by $K^{\mathbb{C}}$ its complexification, i.e., $K^{\mathbb{C}} = GL(n, \mathbb{C})$, preserving $\mathbb{C}^n \subset \mathbb{P}^n$ and acting as a group of automorphisms of $\mathbb{P}^n$ which restricts to $\mathbb{C}^n$ to give the usual action of $GL(n, \mathbb{C})$ on $\mathbb{C}^n \subset \mathbb{P}^n$. The complex Lie algebra $\mathfrak{g}$, considered as the Lie algebra of holomorphic vector fields on $\mathbb{P}^n$, admits the Harish-Chandra decomposition $\mathfrak{g} = \mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$, where $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of $\mathfrak{k}$, equivalently the Lie algebra of $K^{\mathbb{C}}$, $\mathfrak{m}^- \subset \mathfrak{g}$ is the Abelian Lie subalgebra consisting of holomorphic vector fields on $\mathbb{P}^n$ vanishing to the order $\geqslant 2$ at $0 \in \mathbb{P}^n$, and $\mathfrak{m}^+ \subset \mathfrak{g}$ is the Abelian Lie subalgebra whose elements restrict to constant vector fields on $\mathbb{C}^n \subset \mathbb{P}^n$. The vector subspace $\mathfrak{p} := \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^- \subset \mathfrak{g}$ is the parabolic subalgebra at $0 \in \mathbb{P}^n$. Writing $M^+ = \exp(\mathfrak{m}^+)$ and $M^- = \exp(\mathfrak{m}^-)$, the mapping $\lambda : M^+ \times K^{\mathbb{C}} \times M^- \to G$ defined by $\lambda(m^+, k, m^-) = m^+ k m^- \in G$ maps $M^+ \times K^{\mathbb{C}} \times M^-$ biholomorphically onto a dense open subset of G. We note that $K^{\mathbb{C}} M^- = P$ is a Levi decomposition of $P \subset G$, with $K^{\mathbb{C}} \subset P$ being a Levi factor. By means of λ, and writing $e \in G$ for the identity element, the orbit of $0 = eP \in G/P$ under M^+ is identified with $M^+ \cong \mathbb{C}^n$, from which the Harish-Chandra realization $\mathbb{B}^n \Subset \mathbb{C}^n$ arises as the obvious projection of $\lambda^{-1}(G_0)$ into $M^+ \cong \mathbb{C}^n$. The description above applies in general to bounded symmetric domains, cf. Wolf [Wo] (1972) and Mok [Mo1] (1989).

Let E be a finite-dimensional complex vector space, and $\Phi : P \to \mathbb{P}GL(E)$ be a projective linear representation. Introduce an equivalence relation $\sim$ on $G \times \mathbb{P}(E)$ by declaring $(g, e) \sim (g', e')$; $g, g' \in G$, $e, e' \in E$; if and only if there exists $p \in P$ such that $g' = gp^{-1}$ and $e' = pe$, where pe means $\Phi(p)e$, and define $G \times_P \mathbb{P}(E) = (G \times \mathbb{P}(E))/\sim$. Denote by $[g, e]$ the equivalence class of (g, e) with respect to $\sim$. The natural map $\tau : G \times_P \mathbb{P}(E) \to G/P$ defined by $\tau([g, e]) = gP$ realizes $G \times_P \mathbb{P}(E)$ as the total space of a projective bundle $\varpi : \mathscr{P} \to G/P = \mathbb{P}^n$. Left multiplication on G induces a holomorphic action of G on $\mathscr{P}$ compatible with the natural transitive holomorphic action of G on $G/P = \mathbb{P}^n$. By a homogeneous projective bundle on $\mathbb{P}^n$ we will always mean $\varpi : \mathscr{P} \to \mathbb{P}^n$ arising this way. We have $K = P \cap G_0$. Defining $G_0 \times_K \mathbb{P}(E)$ analogously we have on $G_0 \times_K \mathbb{P}(E)$ the structure of a smooth projective bundle $\varpi' : \mathscr{P}' \to G_0/K = \mathbb{B}^n$. $\mathscr{P}'$ is *a priori* only a smooth projective bundle since K is a real Lie group, but embedding $G_0 \times_K \mathbb{P}(E)$ canonically into $G \times_P \mathbb{P}(E)$ one identifies $G_0 \times_K \mathbb{P}(E)$ as an open subset of $G \times_P \mathbb{P}(E)$, and hence $\mathscr{P}'$ as the restriction of $\mathscr{P}$ over $\mathbb{B}^n \subset \mathbb{P}^n$, so that $\mathscr{P}' = \mathscr{P}|_{\mathbb{B}^n}$ inherits the structure of a (holomorphic) projective bundle over $\mathbb{B}^n$. From now on we will identify $\mathscr{P}'$ with $\mathscr{P}|_{\mathbb{B}^n}$. The fiber of $\mathscr{P}$ over a point $x \in \mathbb{P}^n$ will be denoted by $\mathscr{P}_x$.

Let $\nu : \mathcal{E} \to M$ be a locally trivial holomorphic fiber bundle over a complex manifold M. By a locally trivial holomorphic fiber subbundle $\mathcal{E}' \subset \mathcal{E}$ we mean a subvariety $\mathcal{E}' \subset \mathcal{E}$ such that, writing $\mathcal{E}'_0 := \mathcal{E}_0 \cap \mathcal{E}'$, and shrinking the neighborhoods U if necessary, the trivializations $\varphi : \nu^{-1}(U) \xrightarrow{\cong} \mathcal{E}_0 \times U$ can be chosen such that $\varphi(\nu^{-1}(U) \cap \mathcal{E}') = \mathcal{E}'_0 \times U \subset \mathcal{E}_0 \times U$, endowed with the canonical projection map $\nu' : \mathcal{E}' \to M$ given by $\nu' = \nu|_{\mathcal{E}'}$. The adjective "locally trivial" will be understood and omitted in what follows. For $M = X_\Gamma$ a holomorphic fiber (sub)bundle is said to be locally homogeneous if it descends from a homogeneous holomorphic fiber (sub)bundle on $\mathbb{B}^n$ under the action of G_0.

In order to embed $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ into a projective bundle we will embed $\mu : \mathscr{U} \to \mathbb{P}^n$ into a projective bundle $\varpi : \mathscr{P} \to \mathbb{P}^n$ in such a way that G acts canonically on $\mathscr{P}$. For this we need some basic facts about $\mathrm{Chow}(\mathbb{P}^n)$. This is provided by the Chow coordinates (Chow-van der Waerden [CvdW]). Fix an integer r, $0 < r < n$, and an integer $d > 0$. For an irreducible and reduced r-cycle W of degree d consider the set $\mathscr{F}$ of $(r+1)$-tuples $(\Pi_1, \cdots, \Pi_{r+1})$, $1 \leqslant i \leqslant r+1$, of hyperplanes $\Pi_i \subset \mathbb{P}^n$, such that $\Pi_1 \cap \cdots \cap \Pi_{r+1} \cap W \neq \emptyset$. Writing $\mathbb{P}^n = \mathbb{P}(H)$, $H \cong \mathbb{C}^{n+1}$, hyperplanes $\Pi \subset \mathbb{P}(H)$ are parametrized by the dual projective space $\check{\mathbb{P}}^n = \mathbb{P}(H^*)$. The set $\mathscr{F} \subset \check{\mathbb{P}}^n \times \cdots \times \check{\mathbb{P}}^n$ is then a hypersurface and it is the zero set of some $\sigma = \sigma_W \in \Gamma\big(\check{\mathbb{P}}^n \times \cdots \times \check{\mathbb{P}}^n, \epsilon_1^*\mathcal{O}(d) \otimes \cdots \otimes \epsilon_{r+1}^*\mathcal{O}(d)\big)$, $\check{\mathbb{P}}^n = \mathbb{P}(H^*)$, where $\epsilon_k : \check{\mathbb{P}}^n \times \cdots \times \check{\mathbb{P}}^n \to \check{\mathbb{P}}^n$ is the canonical projection onto the k-th factor. Writing $\mathcal{O}(d, \cdots, d)$ for $\epsilon_1^*\mathcal{O}(d) \otimes \cdots \otimes \epsilon_{r+1}^*\mathcal{O}(d)$, σ corresponds to a plurihomogeneous polynomial of multi-degree $(d, \cdots, d)$, which is called the Chow form of $[W]$, uniquely determined up to a non-zero multiplicative scalar. In the general case of $[W] = k_1[W_1] + \cdots + k_s[W_s]$ one defines $\sigma_W = \sigma_{W_1}^{k_1} \cdots \sigma_{W_s}^{k_s}$, which is again of multi-degree $(d, \cdots, d)$. Writing $\mathcal{Q}$ for the set of all r-cycles of degree d in $\mathbb{P}^n$, the mapping Ψ associating $[W] \in \mathcal{Q}$ to $[\sigma_W] \in \mathbb{P}(J)$, $J = \Gamma\big(\check{\mathbb{P}}^n \times \cdots \times \check{\mathbb{P}}^n, \mathcal{O}(d, \cdots, d)\big)^*$, is injective, and we have defined the structure of a projective variety on $\mathcal{Q}$ by identifying it with $\Psi(\mathcal{Q}) \subset \mathbb{P}(J)$, noting that G acts canonically on $\mathbb{P}(J)$. Restricting to an irreducible component $\mathcal{K}$ of $\mathcal{Q}$ we summarize the relevant statements in the following lemma using the notation in the above.

Lemma 2.1. *Let r and d be positive integers, $1 \leqslant r \leqslant n-1$. Let $\mathcal{K}$ be an irreducible component of* $\mathrm{Chow}(\mathbb{P}^n)$ *parametrizing r-cycles of degree d in $\mathbb{P}^n$. Then, writing $J := \Gamma\big(\check{\mathbb{P}}^n \times \cdots \times \check{\mathbb{P}}^n, \mathcal{O}(d, \cdots, d)\big)^*$, in which there are $r+1$ Cartesian factors of $\check{\mathbb{P}}^n$, the association of $[W] \in \mathcal{K}$ to $[\sigma_W] \in \mathbb{P}(J)$, where $\sigma_W \in J$ denotes the Chow form of W (which is unique up to scaling constants), identifies $\mathcal{K}$ as a projective subvariety of $\mathbb{P}(J)$. Furthermore, G leaves $\mathcal{K} \subset \mathbb{P}(J)$ invariant and acts holomorphically on $\mathcal{K}$.*

We proceed to embed $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ into a projective bundle. We have

Proposition 2.2. *Let $\rho : \mathscr{U} \to \mathcal{K}$ be the universal family for the irreducible component $\mathcal{K}$ of* $\mathrm{Chow}(\mathbb{P}^n)$*, $\Gamma \subset G_0$ be a torsion-free cocompact discrete subgroup, $X_\Gamma := \mathbb{B}^n/\Gamma$. Define $\mathscr{U}' = \mathscr{U}|_{\mathbb{B}^n}$ and write $\mu_\Gamma : \mathscr{U}_\Gamma := \mathscr{U}'/\Gamma \to X_\Gamma$ for the induced locally homogeneous holomorphic fiber bundle on X_Γ with fibers biholomorphic to $\mathscr{U}_0$. Then, there exists a locally homogeneous projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ such that $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$ is a locally homogeneous holomorphic fiber subbundle over X_Γ. Moreover, the total space $\mathscr{P}_\Gamma$ is a projective manifold, hence $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$ is a projective variety.*

Proof. It suffices to show that $\mu : \mathscr{U} \to \mathbb{P}^n$ embeds into some homogeneous projective bundle $\varpi : \mathscr{P} \to \mathbb{P}^n$ such that $\mathscr{U}$ is invariant under the action of G on $\mathscr{P}$. In this way, the restriction $\mu' : \mathscr{U}' \to \mathbb{B}^n$, $\mathscr{U}' := \mu^{-1}(\mathbb{B}^n)$, $\mu' = \mu|_{\mathscr{P}'}$ is invariant under $G_0 \subset G$, and it descends under the action of $\Gamma \subset G_0$ to give $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$, $\mathscr{U}_\Gamma = \mathscr{U}'/\Gamma$, which embeds as a locally homogeneous holomorphic fiber subbundle of the projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$. Given this, Proposition 2.1 implies the projectivity of $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$.

For the Chow component $\mathcal{K} \subset \mathrm{Chow}(\mathbb{P}^n)$ whose members are r-cycles of degree d in $\mathbb{P}^n$, by Lemma 2.1, writing $\mathbb{P}^n =: \mathbb{P}(H)$, $\mathcal{K}$ embeds canonically into the projective space $\mathbb{P}(J) :\cong \mathbb{P}^N$, where $J = \Gamma\big(\check{\mathbb{P}}^n \times \cdots \times \check{\mathbb{P}}^n, \mathcal{O}(d, \cdots, d)\big)^*$, $\check{\mathbb{P}}^n = \mathbb{P}(H^*)$. By definition $\mathscr{U} \subset \mathcal{K} \times \mathbb{P}^n \hookrightarrow \mathbb{P}^N \times \mathbb{P}^n \hookrightarrow \mathbb{P}(J \otimes H) \cong: \mathbb{P}^s$ and hence $\mathscr{U} \hookrightarrow \mathbb{P}^s$ embeds canonically into $\mathbb{P}^s$. We have $\mathscr{U}_0 = \mu^{-1}(0) \subset \mathscr{U} \subset \mathcal{K} \times \mathbb{P}^n \hookrightarrow \mathbb{P}^s$. Restricting the G-action on $\mathbb{P}^s$ to the parabolic subgroup P, written $\Phi : P \to \mathrm{Aut}(\mathbb{P}^s)$, we obtain from the discussion preceding Lemma 2.1 a homogeneous projective bundle $\varpi : \mathscr{P} \to \mathbb{P}^n$ equipped with a G-action. The restriction $\mathscr{P}' = \mathscr{P}|_{\mathbb{B}^n}$ is invariant under $G_0 \subset G$, hence it descends to $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$. The mapping $\varphi(u) = (u, \mu(u))$ embeds $\mathscr{U}$ into

$\mathscr{U} \times \mathbb{P}^n$, and its image $\mathscr{U}^\sharp$ equipped with the projection onto $\mathbb{P}^n$ realizes $\mathscr{U}^\sharp \subset \mathbb{P}^s \times \mathbb{P}^n = \mathscr{P}$ as a holomorphic fiber subbundle of $\mathscr{P}$. Identifying $\mathscr{U}$ with $\mathscr{U}^\sharp$ and $\mu : \mathscr{U} \to \mathbb{P}^n$ with the projection of $\mathscr{U}^\sharp \subset \mathbb{P}^s \times \mathbb{P}^n$ onto the second factor $\mathbb{P}^n$, we have realized $\mu : \mathscr{U} \to \mathbb{P}^n$ as a locally homogeneous holomorphic fiber subbundle of $\varpi : \mathscr{P} \to \mathbb{P}^n$. Proposition 2.2 follows. $\square$

In the case where $\Gamma \subset G_0$ is a torsion-free nonuniform lattice, we will study $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ by constructing Hermitian metrics with positive curvature (1,1)-form on a homogeneous holomorphic line bundle on $\mathscr{P}_\Gamma$ and prove quasi-projectivity by means of techniques of compactification of complete Kähler manifolds of finite volume.

As a preparation we consider the general situation of a representation $\Phi : P \to \mathbb{P}GL(E)$ and the associated homogeneous projective bundle $\varpi : \mathscr{P} \to \mathbb{P}^n$.

Writing $E :\cong \mathbb{C}^{m+1}$ we have $\mathbb{P}GL(E) \cong \mathbb{P}GL(m+1, \mathbb{C}) \cong SL(m+1, \mathbb{C})/Z$, where $Z = \{\lambda I_{m+1} : \lambda^{m+1} = 1\}$. Recall that $K \cong U(n)$ is the isotropy subgroup of $(\mathbb{B}^n, ds^2_{\mathbb{B}^n})$ at $0 \in \mathbb{B}^n$, $K = P \cap G_0$. Denoting by $\alpha : SL(m+1, \mathbb{C}) \to SL(m+1, \mathbb{C})/Z \cong \mathbb{P}GL(E)$ the quotient map, let $Q \subset SL(m+1, \mathbb{C})$ be the subgroup $\alpha^{-1}(\Phi(K))$. Let now ζ be a Hermitian Euclidean metric on the complex vector space E which is invariant under the compact subgroup $Q \subset GL(E)$. The metric ζ induces on $\mathbb{P}(E)$ a Fubini-Study metric g_c with Kähler form ω_c such that, with respect to the canonical projection $\beta : E - \{0\} \to \mathbb{P}(E)$, $\beta^*(\omega_c) = \sqrt{-1}\partial\overline{\partial} \log \|w\|^2_\zeta$, where $w = (w_1, \cdots, w_{m+1})$ are Euclidean coordinates on E and $\|w\|_\zeta$ denotes the norm of w measured in terms of ζ.

Restricting Φ to $G_0 \cap P = K$, from $\Phi|_K : K \to \mathbb{P}GL(E)$ we obtain $\varpi' : \mathscr{P}' = G_0 \times_K \mathbb{P}(E) \to G/K = \mathbb{B}^n$ as a homogeneous projective bundle over $\mathbb{B}^n$ such that $\mathscr{P}'$ is identified with $\mathscr{P}|_{\mathbb{B}^n}$. Since g_c is invariant under Q, identifying $\mathbb{P}(E)$ with the fiber $\mathscr{P}_0 = \varpi^{-1}(0)$ we have correspondingly a Fubini-Study metric g_0 on $\mathscr{P}_0$. By the Q-invariance of g_0, the latter metric can be transported by means of G_0-action to fibers $\mathscr{P}_x$ over $x \in \mathbb{B}^n$, yielding thereby a G_0-invariant Hermitian metric g on the relative tangent bundle $T_{\varpi'}$. Denoting by L the determinant line bundle $\det(T_{\varpi'})$ and writing h for $\det(g)$, we have a Hermitian holomorphic line bundle (L, h) on $\mathscr{P}'$ which is invariant under the G_0-action described. The curvature form $\Theta(L, h)$, as a closed (1,1)-form on $\mathscr{P}'$, is G_0-invariant and positive when restricted to each of the fibers $\mathscr{P}_x$, $x \in \mathbb{B}^n$. Let (A, t) be a positive homogeneous Hermitian holomorphic line bundle on $\mathbb{B}^n$. By the G_0-invariance of θ and the compactness of the fibers $\mathscr{P}_x \cong \mathbb{P}(E)$, for a sufficiently large integer k, $\theta := \Theta\big(L \otimes (\varpi'^* A)^k, h \otimes (\varpi'^*(t))^k\big) > 0$. Since θ is G_0-invariant, for a torsion-free lattice $\Gamma \subset G_0$, $\varpi_\Gamma : \mathscr{P}_\Gamma = \mathscr{P}/\Gamma \to X_\Gamma = \mathbb{B}^n/\Gamma$, the Hermitian holomorphic line bundle $(\Lambda, s) := \big(L \otimes (\varpi'^* A)^k, h \otimes (\varpi'^*(t))^k\big)$ descends to a positive line bundle on $\mathscr{P}_\Gamma$.

We summarize the discussion above to the following result which will be used in the case of finite-volume noncompact quotient manifolds.

PROPOSITION 2.3. *Let E be a finite-dimensional complex vector space, $\Phi : P \to \mathbb{P}GL(E)$ be a representation, $\varpi : \mathscr{P} \to \mathbb{P}^n$ be the associated homogeneous projective bundle on $\mathbb{P}^n$, and write $\mathscr{P}' := \mathscr{P}|_{\mathbb{B}^n}$. Then, there exists a G_0-invariant Hermitian holomorphic line bundle (Λ, s) on $\mathscr{P}'$ whose curvature form θ is positive definite. Hence, given any torsion-free discrete subgroup $\Gamma \subset G_0$, (Λ, s) descends to a locally homogeneous Hermitian holomorphic line bundle $(\Lambda_\Gamma, s_\Gamma)$ with positive curvature form θ_Γ.*

We prove now an embedding theorem for $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$.

PROPOSITION 2.4. *For a torsion-free nonuniform lattice $\Gamma \subset G_0$ the total space $\mathscr{P}_\Gamma$ of the locally* *homogeneous projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ is biholomorphic to a dense Zariski open subset of some projective subvariety $V \subset \mathbb{P}^\ell$ of a projective space $\mathbb{P}^\ell$.*

Proof. By Proposition 2.3, $\mathscr{P}' = \mathscr{P}|_{\mathbb{B}^n}$ is equipped with a homogeneous Hermitian holomorphic line bundle (Λ, s) whose curvature form θ is positive definite. We can now equip $\mathscr{P}'$ with the Kähler form θ, which descends to a Kähler form θ_Γ on $\mathscr{P}_\Gamma$. From the definition we have

$$\begin{aligned}\theta := \Theta\big(L \otimes (\varpi'^* A)^k, h \otimes (\varpi'^*(t))^k\big) &= \Theta(L,h) + k\varpi'^*(\Theta(K_{\mathbb{B}^n}, t)) \\ &= \Theta(L,h) + k(n+1)\varpi'^*(\omega_\Gamma) > 0\,.\end{aligned}$$

Recall that ω_Γ is the Kähler form of the complete Kähler metric on X_Γ with constant holomorphic sectional curvature -2. Replacing k by $k+1$ we may assume furthermore that $\theta > (n+1)\varpi'^*(\omega_\Gamma) > \varpi'^*(\omega_\Gamma)$. It follows that the length of any smooth curve $\gamma : [0,1] \to \mathscr{P}_\Gamma$ with respect to θ_Γ dominates the length of the smooth curve $\gamma \circ \varpi_\Gamma : [0,1] \to X_\Gamma$ with respect to ω_Γ. Since $(X_\Gamma, \omega_\Gamma)$ is a complete Kähler manifold, we conclude that $(\mathscr{P}_\Gamma, \theta_\Gamma)$ is also a complete Kähler manifold.

We also equip $(\mathscr{P}_\Gamma, \theta_\Gamma)$ with the Hermitian-Einstein positive line bundle $(\Lambda_\Gamma, s_\Gamma)$. Since the Kähler form θ on $\mathscr{P}'$ is G_0-invariant, for any open subset $\mathcal{O} \Subset \mathbb{B}^n$ and any $\varphi \in G_0$, $\mathrm{Volume}\big(\varpi^{-1}(\mathcal{O}), \theta\big) = \mathrm{Volume}\big(\varpi^{-1}(\varphi(\mathcal{O})), \theta\big) < \infty$. Recall that $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ is of fiber dimension s. Since both (1,1)-forms θ and $\varpi^*(\omega)$ are invariant under G_0, both θ^{s+n} and $\theta^s \wedge (\varpi^*(\omega))^n$ are G_0-invariant volume forms on $\mathscr{P}'$, and the function $h : \mathscr{P}' \to \mathbb{R}$ defined by $\theta^{s+n}(p) = h(p)\theta^s \wedge (\varpi^*(\omega))^n$ is a G_0-invariant function on $\mathscr{P}'$. Since any point $p \in \mathscr{P}'$ is equivalent under the action of G_0 to some point on $\mathscr{P}_0$, which is compact, we observe that there exists a constant $C_0 > 0$ such that $0 < h(p) \leqslant C_0$ for any $p \in \mathscr{P}'$. For any open subset $\mathcal{O} \subset X_\Gamma$, by Fubini's Theorem we have

$$\int_{\varpi_\Gamma^{-1}(\mathcal{O})} \theta_\Gamma^s \wedge \big(\varpi_\Gamma^*(\omega_\Gamma)\big)^n = \int_{\mathscr{P}_0} \theta^s \times \int_{\mathcal{O}} \omega_\Gamma^n\,;$$

$$\begin{aligned}\mathrm{Volume}\big(\varpi_\Gamma^{-1}(\mathcal{O}), \theta_\Gamma\big) &= \frac{1}{(s+n)!}\int_{\varpi_\Gamma^{-1}(\mathcal{O})} \theta_\Gamma^{s+n} \leqslant \frac{C_0}{(s+n)!}\int_{\varpi_\Gamma^{-1}(\mathcal{O})} \theta_\Gamma^s \wedge \big(\varpi_\Gamma^*(\omega_\Gamma)\big)^n \\ &= \frac{n!}{(s+n)!}\cdot C_0 \int_{\mathscr{P}_0} \theta^s \times \int_{\mathcal{O}} \frac{\omega_\Gamma^n}{n!} = C \cdot \mathrm{Volume}(\mathcal{O}, \omega_\Gamma)\,,\end{aligned}$$

where $C = \dfrac{n!}{(s+n)!}\cdot C_0 \displaystyle\int_{\mathscr{P}_0} \theta^s$. In particular,

$$\mathrm{Volume}(\mathscr{P}_\Gamma, \theta_\Gamma) \leqslant C \cdot \mathrm{Volume}(X_\Gamma, \omega_\Gamma) < \infty\,.$$

From the G_0-invariance of $(\mathscr{P}', \theta)$ and the compactness of the fibers of $\varpi' : \mathscr{P}' \to \mathbb{B}^n$, it follows that $(\mathscr{P}', \theta)$ is of bounded sectional curvature. Since X_Γ is a quasi-projective manifold (as in Theorem 1.1), it is of finite topological type. Hence, $\mathscr{P}_\Gamma$ is also of finite topological type. As a consequence, one can apply the embedding result on complete Kähler manifolds of finite volume given by Theorem 1.3 here (from Mok-Zhong [MZ$_2$]) to complete the proof of the proposition. □

Without loss of generality we may assume V to be normal, and we also write $\mathscr{P}_\Gamma \subset \overline{\mathscr{P}_\Gamma} := V$ for the compactification of $\mathscr{P}_\Gamma$ as a normal projective variety. Since $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ is a proper holomorphic map, by the Riemann extension theorem and the normality of $\overline{\mathscr{P}_\Gamma}$, it extends holomorphically to $\overline{\mathscr{P}_\Gamma}$, to be denoted as $\varpi_\Gamma^\sharp : \overline{\mathscr{P}_\Gamma} \to \overline{X_\Gamma}$.

We now return to our study of the locally homogeneous holomorphic fiber subbundle $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ of $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$. We have

Proposition 2.5. *For a torsion-free lattice $\Gamma \subset G_0$, identifying the total space of the locally homogeneous holomorphic projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ as a quasi-projective variety by means of Proposition 2.4 and embedding $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ as a locally homogeneous holomorphic fiber subbundle of $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$, $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$ is a quasi-projective subvariety.*

Proof. Let $\Psi : \mathscr{P}_\Gamma \xrightarrow{\cong} W \subset V \subset \mathbb{P}^\ell$ be a biholomorphism of $\mathscr{P}_\Gamma$ onto a Zariski open subset W of a projective subvariety $V \subset \mathbb{P}^\ell$, where V is the topological closure of W in $\mathbb{P}^\ell$, and identify $\mathscr{P}_\Gamma$ with $W = \Psi(\mathscr{P}_\Gamma)$, and write $\overline{\mathscr{P}_\Gamma} := V$. Subsets of $\mathscr{P}_\Gamma$ will likewise be identified with their images under Ψ in W. Recall that $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ is a projective bundle, and $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ is a holomorphic fiber subbundle with fibers $\mu_\Gamma^{-1}(x) =: \mathscr{U}_x$. Write $q := \dim(\mathscr{U}_x)$ for $x \in X_\Gamma$, and denote by d the degree of $\mathscr{U}_x \subset \mathscr{P}_x \subset \mathscr{P}_\Gamma \subset \mathbb{P}^\ell$ as a projective subvariety. Denote by $\widetilde{\mathcal{H}}$ the subset of $\mathrm{Chow}(\mathbb{P}^\ell)$ consisting of all pure q-dimensional cycles in $\mathbb{P}^\ell$ of degree d. The continuous mapping $\varphi : X_\Gamma \to \widetilde{\mathcal{H}}$ defined by $\varphi(x) = [\mathscr{U}_x] \in \widetilde{\mathcal{H}}$ is a holomorphic mapping from the complex manifold X_Γ into some irreducible component $\mathcal{H} \subset \widetilde{\mathcal{H}}$. Recall that $\overline{X_\Gamma}$ is normal and $\overline{X_\Gamma} - X_\Gamma$ is a finite set. Since X_Γ is of complex dimension $n \geqslant 2$, by Hartogs extension the holomorphic map $\varphi : X_\Gamma \to \mathcal{H}$ extends *meromorphically* to $\varphi^\sharp : \overline{X_\Gamma} \to \mathcal{H}$.

Let $\mathcal{Z} \subset \overline{X_\Gamma} \times \mathcal{H}$ be the graph of the meromorphic map $\varphi^\sharp$, i.e., $\mathcal{Z}$ is the topological closure of $\mathrm{Graph}(\varphi)$ in $\overline{X_\Gamma} \times \mathcal{H}$. Denote by $\alpha : \mathcal{V} \to \mathcal{H}$ the universal family over $\mathcal{H} \subset \mathrm{Chow}(\mathbb{P}^\ell)$, $\mathcal{V} \subset \mathcal{H} \times \mathbb{P}^\ell$. Let now $\mathcal{W} \subset \mathcal{Z} \times \mathbb{P}^\ell$ be the total space of the pull-back of the universal family $\alpha : \mathcal{V} \to \mathcal{H}$ by the canonical projection map $\beta : \mathcal{Z} \to \mathcal{H}$, i.e., writing C_η for the q-cycle in $\mathbb{P}^\ell$ represented by $\eta \in \mathcal{H}$ we have $\mathcal{W} = \big\{(x, \eta, y) \in \overline{X_\Gamma} \times \mathcal{H} \times \mathbb{P}^\ell : (x, \eta) \in \mathcal{Z}, y \in C_\eta\big\}$. Then, denoting by $\gamma : \overline{X_\Gamma} \times \mathcal{H} \times \mathbb{P}^\ell \to \overline{X_\Gamma} \times \mathbb{P}^\ell$ the canonical projection, by the proper mapping theorem and Chow's Theorem, $\mathcal{Q} := \gamma(\mathcal{W}) \subset \overline{X_\Gamma} \times \mathbb{P}^\ell$ is projective. Consider the canonical projections $\delta : \mathcal{Q} \to \overline{X_\Gamma}$ and $\lambda : \overline{X_\Gamma} \times \mathbb{P}^\ell \to \mathbb{P}^\ell$. Then, by the theorems above $\mathcal{E} := \lambda(\mathcal{Q}) \subset \overline{\mathscr{P}_\Gamma} \subset \mathbb{P}^\ell$ is projective, and it contains $\lambda(\delta^{-1}(X_\Gamma)) = \mathscr{U}_\Gamma$. Finally it follows from $\mathscr{U}_\Gamma = \mathcal{E} \cap \mathscr{P}_\Gamma$ that $\mathscr{U}_\Gamma \subset \mathcal{E}$ is a dense Zariski open subset, hence $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$ is a quasi-projective subvariety, as desired. $\square$

3. Uniruling on Zariski closures of images of algebraic subsets under the uniformization map

Let $\Gamma \subset G_0$ be a torsion-free lattice, $X_\Gamma := \mathbb{B}^n/\Gamma$, $n \geqslant 2$, $\overline{X_\Gamma}$ be its minimal compactification, and $\pi : \mathbb{B}^n \to X_\Gamma$ be the universal covering map. We consider an irreducible algebraic subset $S \subset \mathbb{B}^n$ of positive dimension and define $Z \subset X_\Gamma$ to be the Zariski closure of $\pi(S)$ in X_Γ. To characterize Z we will resort to studying meromorphic foliations on holomorphic fiber bundles over X_Γ and on compactifications of total spaces of such fiber bundles. We start with some generalities about complex spaces (cf. Grauert-Peternell-Remmert [GPR, p.100ff.]) and about meromorphic foliations on them.

Let $(Y, \mathcal{O}_Y)$ be a reduced irreducible complex space, assumed to be embedded as a subvariety of a complex manifold M. Let $\Omega_M = \mathcal{O}(T_M^*)$ be the cotangent sheaf on M, and $\mathscr{I}_Y \subset \mathcal{O}_M$ be the ideal sheaf of $Y \subset M$. Define now $\mathscr{S} \subset \Omega_M$ to be the subsheaf spanned by $\mathscr{I}_Y \Omega_M$ and $\{df : f \in \mathscr{I}_Y\}$. Then, $\mathscr{S} \subset \Omega_M$ is a coherent subsheaf and $\Omega_Y := \Omega_M/\mathscr{S}$ is called the cotangent sheaf of Y. The tangent sheaf $\mathcal{T}_Y$ is by definition the coherent sheaf $\mathscr{H}om_{\mathcal{O}_Y}(\Omega_Y, \mathcal{O}_Y)$, which is naturally identified with a coherent subsheaf of $\mathcal{T}_M|_Y$. The tangent sheaf $\mathcal{T}_Y$ on Y thus defined is unique up to isomorphisms independent of the embedding $Y \subset M$. When the assumption $Y \subset M$ is dropped, the tangent sheaf $\mathcal{T}_Y$ is defined using an atlas $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ on Y consisting of subvarieties of coordinate open sets by gluing together the tangent sheaves of U_α.

A meromorphic foliation $\mathscr{F}$ on a reduced irreducible complex space Y is by definition given by a coherent subsheaf $\mathcal{T}_{\mathscr{F}}$ of the tangent sheaf $\mathcal{T}_Y$ such that, outside some subvariety $A \subsetneq Y$, $A \supset \mathrm{Sing}(Y)$, $\mathcal{T}_{\mathscr{F}}|_{Y-A}$ is a locally free subsheaf and, writing $\mathcal{T}_{\mathscr{F}}|_{Y-A} := \mathcal{O}(F)$ for a holomorphic distribution $F \subset T_{Y-A}$, F satisfies the involutive property $[F, F] = F$. The tangent subsheaf of the meromorphic foliation $\mathcal{T}_{\mathscr{F}} \subset \mathcal{T}_Y$ is also uniquely determined subject to the conditions that $\mathcal{T}_{\mathscr{F}}$ agrees with $\mathcal{O}(F)$ over $Y - A$ and that $\mathcal{T}_Y/\mathcal{T}_{\mathscr{F}}$ is torsion-free, in which case $\mathcal{T}_{\mathscr{F}} \subset \mathcal{T}_Y$ is said to be saturated. In the sequel $\mathcal{T}_{\mathscr{F}} \subset \mathcal{T}_Y$ is always assumed saturated. The singular locus $\mathrm{Sing}(\mathscr{F}) \subset Y$ is the union of $\mathrm{Sing}(Y)$ and the locus on $\mathrm{Reg}(Y)$ over which $\mathscr{F}$ fails to be a locally free subsheaf. $\mathrm{Sing}(\mathscr{F}) \subsetneq Y$ is a subvariety, and we write $\mathrm{Reg}(\mathscr{F}) := Y - \mathrm{Sing}(\mathscr{F})$. Given an irreducible complex-analytic subvariety $Z \subset Y$ such that $Z \cap \mathrm{Reg}(\mathscr{F}) \neq \emptyset$, we say that Z is saturated with respect to $\mathscr{F}$ to mean that for any point $z_0 \in \mathrm{Reg}(Z) \cap \mathrm{Reg}(\mathscr{F}) \neq \emptyset$, the leaf $\mathcal{L}(z_0)$ of $\mathscr{F}$ passing through z_0 must necessarily lie on Z. When Y is projective, we have the following result concerning Zariski closures of leaves of $\mathscr{F}$.

Proposition 3.1. *Let Y be a reduced irreducible projective variety. Let $\mathscr{F}$ be a meromorphic foliation on Y, $\mathrm{Sing}(\mathscr{F}) \supset \mathrm{Sing}(Y)$ be the singular locus of $\mathscr{F}$, and write $\mathrm{Reg}(\mathscr{F}) := Y - \mathrm{Sing}(\mathscr{F})$. Denote by $\mathcal{T}_Y$ the tangent sheaf of Y, and by $\mathcal{T}_{\mathscr{F}} \subset \mathcal{T}_Y$ the tangent subsheaf of $\mathscr{F}$. Let now $y_0 \in \mathrm{Reg}(\mathscr{F})$, and $\mathcal{L} \subset \mathrm{Reg}(\mathscr{F})$ be the leaf of $\mathscr{F}|_{\mathrm{Reg}(\mathscr{F})}$ passing through y_0. Denote by $Z \subset Y$ the Zariski closure of $\mathcal{L}$. Then, Z is saturated with respect to the meromorphic foliation $\mathscr{F}$.*

Proof. Note first of all that $\mathcal{L} \not\subset \mathrm{Sing}(Z)$, otherwise the Zariski closure of $\mathcal{L}$ in Y would be contained in $\mathrm{Sing}(Z) \subsetneq Z$, contradicting the assumption $Z = \overline{\mathcal{L}}^{\mathscr{Z}ar}$. On the projective subvariety $Z \subset Y$ consider the coherent subsheaf $\mathscr{E} := \mathcal{T}_{\mathscr{F}}|_Z + \mathcal{T}_Z \subset \mathcal{T}_Y|_Z$, where $\mathcal{T}_{\mathscr{F}}|_Z := \mathcal{T}_{\mathscr{F}} \otimes_{\mathcal{O}_Y} \mathcal{O}_Z, \mathcal{T}_Y|_Z := \mathcal{T}_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Z$. Let $\Sigma \subset Z$ be the union of $\mathrm{Sing}(Z)$, $\mathrm{Sing}(\mathscr{F}) \cap Z$, and the locus of points $x \in \mathrm{Reg}(Z) \cap \mathrm{Reg}(Y)$ where $\mathscr{E} \subset \mathcal{T}_Y|_Z$ fails to be a locally free subsheaf. Then, $\Sigma \subsetneq Z$ is a projective subvariety and we must have $\mathcal{L} \not\subset \Sigma$, otherwise the Zariski closure of $\mathcal{L}$ in Y would be contained in $\Sigma \subsetneq Z$, and we reach the same contradiction. Thus, there exists $y_1 \in \mathcal{L} \cap \mathrm{Reg}(Z) \cap \mathrm{Reg}(\mathscr{F})$ such that the coherent subsheaf $\mathscr{E} \subset \mathcal{T}_Y|_Z$ is a locally free subsheaf at y_1. On $Z - \Sigma \subset \mathrm{Reg}(Z) \cap \mathrm{Reg}(\mathscr{F}) \subset \mathrm{Reg}(Z) \cap \mathrm{Reg}(Y)$ we have $\mathscr{E}|_{Z-\Sigma} = \mathcal{O}(E)$ for some holomorphic vector subbundle $E \subset T_{Z-\Sigma}$. Since $\mathcal{T}_{\mathscr{F},y_1} \subset \mathcal{T}_{Z,y_1}$ we must have $E_{y_1} = T_{Z,y_1}$, so that $\mathrm{rank}(E) = \dim(E_{y_1}) = \dim(Z)$. It follows that $E = T_{Z-\Sigma}$. Writing $\mathcal{T}_{\mathscr{F}}|_{\mathrm{Reg}(\mathscr{F})} = \mathcal{O}(F)$ for some $F \subset T_Y|_{\mathrm{Reg}(\mathscr{F})}$ we must have $F_y + T_{Z,y} = T_{Z,y}$ for all $y \in Z - \Sigma$, i.e., $F|_{Z-\Sigma} \subset T_{Z-\Sigma}$. As a consequence, Z is saturated with respect to the meromorphic foliation $\mathscr{F}$, as desired. □

Let M be a complex manifold, $\pi : \widetilde{M} \to M$ be a covering map, and $Y \subset M$ be an irreducible subvariety. Let $\mathcal{H}$ be a reduced irreducible complex space and $\mathcal{R} \subset \mathcal{H} \times \widetilde{M}$ be a subvariety such that the canonical projection $\sigma : \mathcal{R} \to \mathcal{H}$ is surjective and the fibers $\sigma^{-1}(t) =: \{t\} \times R_t$ are equidimensional. Denote by $\nu : \mathcal{H} \times \widetilde{M} \to \widetilde{M}$ the canonical projection. We have the following notion of uniruling of subvarieties $Y \subset M$. In what follows openness and denseness of subsets are defined in terms of the complex topology.

Definition 3.1. *We say that $Y \subset M$ is uniruled by subvarieties belonging to $\mathcal{H}$ if and only if there exists a dense open subset $\mathcal{O} \subset \mathrm{Reg}(Y)$ for which the following statement ($\sharp$) holds for every point $x_0 \in \mathcal{O}$. ($\sharp$) There exist an open neighborhood $U_{x_0} \subset \mathcal{O}$ of x_0 and a point $t_0 \in \mathrm{Reg}(\mathcal{H})$*

satisfying $(t_0, x_0) \in \mathrm{Reg}(\mathcal{R})$*, a complex submanifold* $\mathcal{S} \subset \mathcal{G} \subset \mathcal{H}$ *of some open subset* $\mathcal{G} \subset \mathcal{H}$ *and a smooth neighborhood* $\mathscr{U}_{t_0,x_0} \subset \sigma^{-1}(\mathcal{S})$ *of* (t_0, x_0) *such that* $\sigma|_{\mathscr{U}_{t_0,x_0}} : \mathscr{U}_{t_0,x_0} \to \mathcal{S}$ *is a holomorphic submersion and such that* $\pi \circ \nu|_{\mathscr{U}_{t_0,x_0}} : \mathscr{U}_{t_0,x_0} \to M$ *maps* $\mathscr{U}_{t_0,x_0}$ *biholomorphically onto* $U_{x_0} \subset Y \subset M$*.*

We will apply the notion of uniruling to the case where $M = \mathbb{B}^n/\Gamma = X_\Gamma$ is a complex ball quotient by a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ equipped with the quasi-projective structure given by $X_\Gamma \subset \overline{X_\Gamma}$, $\mathcal{H}$ is some G-invariant irreducible subvariety of $\mathrm{Chow}(\mathbb{P}^n)$, and where $Y \subset X_\Gamma$ is an algebraic subvariety. We start with some preparation relating meromorphic foliations to meromorphic sections of Grassmann bundles.

Let M be a complex manifold, $Y \subset M$ be an irreducible subvariety, and $\mathcal{T}_Y \subset \mathcal{T}_M|_Y$ be the tangent sheaf of Y. Let $\mathcal{T}_{\mathscr{F}} \subset \mathcal{T}_Y$ be the tangent sheaf of a meromorphic foliation $\mathscr{F}$ of leaf dimension r, $0 < r < \dim(M)$, defined on Y. Write $B := \mathrm{Sing}(\mathscr{F})$, and write $F \subset T_{Y-B}$ for the involutive holomorphic distribution such that $T_{\mathscr{F}} = \mathcal{O}(F)$. Define the holomorphic section $\varphi : Y - B \to \mathrm{Gr}(r, T_M)$ by setting $\varphi(y) = [F_y] \in \mathrm{Gr}(r, T_{Y,y}) \subset \mathrm{Gr}(r, T_{M,y})$. Since $\mathscr{F}$ is locally finitely generated, at every point $y_0 \in Y$, there exist r holomorphic sections $\chi_1, \cdots, \chi_r \in \Gamma(U, \mathcal{T}|_{\mathcal{F}})$ defined on some neighborhood U of y_0, such that $\chi_1(y'), \cdots, \chi_r(y')$ span $\mathcal{T}_{\mathscr{F},y'}$ at a general point $y' \in U - B$. Then, by using Plücker coordinates, the formula $\Phi(y) = [\chi_1(y) \wedge \cdots \wedge \chi_r(y)] \in \mathrm{Gr}(r, T_{M,y}) \subset \mathbb{P}(\Lambda^r T_{M,y})$ determines a unique meromorphic section over U of $\mathrm{Gr}(r, T_M)$ extending $\varphi|_{U-B}$. From uniqueness it follows that φ extends meromorphically to $\varphi^\flat : Y \dashrightarrow \mathrm{Gr}(r, T_M)$. We call $\varphi^\flat$ the meromorphic section of $\mathrm{Gr}(r, T_M)$ associated to $(Y, \mathscr{F})$. We have

Lemma 3.1. *Let* M *be a complex manifold,* $Y \subset M$ *be an irreducible normal complex-analytic subvariety and* $A \subset Y$ *be a complex-analytic subvariety containing* $\mathrm{Sing}(Y)$*. Let* $\mathscr{F}$ *be a meromorphic foliation on* $Y - A$ *and* $T_{\mathscr{F}} \subset T_{Y-A}$ *be its tangent sheaf. Let* $\varphi^\flat : Y - A \dashrightarrow \mathrm{Gr}(r, T_M)$ *be the meromorphic section associated to* $(Y - A, \mathscr{F})$*. Then, the foliation* $\mathscr{F}$ *on* $Y - A$ *admits a meromorphic extension to* Y *if and only if* $\varphi^\flat$ *extends meromorphically from* $Y - A$ *to* Y*.*

Proof. The forward implication has been established (without assuming Y normal). Conversely, assume that $\varphi^\flat$ admits a meromorphic extension to $\varphi^\sharp : Y \dashrightarrow \mathrm{Gr}(r, T_M)$. Let $H \subset Y$ be the subvariety of codimension $\geqslant 2$ over which either Y is singular or $\varphi^\sharp$ fails to be holomorphic. Write $B := \mathrm{Sing}(\mathscr{F}) \subset Y - A$, $\psi := \varphi^\sharp|_{Y-H}$, and $\alpha : \mathrm{Gr}(r, T_M) \to M$ for the canonical projection. Let $F \subset T_{Y-H}$ be the distribution such that $T_{\mathscr{F}}|_{Y-H} = \mathcal{O}(F)$ and define $Y' \subset \mathrm{Gr}(r, T_M)$ to be the topological closure of $Y^0 := \psi(Y - H)$. The universal rank-r bundle on $\mathrm{Gr}(r, T_M)$ restricts to Y' to give a rank-r holomorphic vector bundle F' over Y' such that $F'|_{Y^0}$ is the tautological lifting of the holomorphic rank-r vector subbundle $F \subset T_{Y-H}$. By the Direct Image Theorem, $\alpha_*(\mathcal{O}(F')) =: \mathcal{E}$ is a coherent sheaf on Y such that $\mathcal{E}|_{Y-H} = \mathcal{O}(F)$ and such that $\mathcal{E}|_{Y-A-B} = \mathcal{T}_{\mathscr{F}}|_{Y-A-B}$. Since Y is normal, for any open subset $U \subset Y$ and any $\sigma \in \Gamma(U, \mathcal{E})$, $\sigma|_{U-H} \in \Gamma(U - H, \mathcal{O}(F))$ extends holomorphically to give $\sigma' \in \Gamma(U, \mathcal{T}_Y)$. $\mathcal{E}$ can thus be identified as a coherent subsheaf of $\mathcal{T}_Y$ and its double dual $\mathcal{E}^{**}$ is mapped canonically onto a coherent subsheaf $\mathscr{S} \subset \mathcal{T}_Y$ extending $\mathcal{O}(F)$ such that $\mathcal{T}_Y/\mathscr{S}$ is torsion-free and $\mathscr{S}|_{Y-A} = \mathcal{T}_{\mathscr{F}}$. $\mathscr{S}$ is the tangent sheaf of a meromorphic foliation $\mathscr{F}^\sharp$ on Y extending $\mathscr{F}$, as desired. □

We return now to the study of Zariski closures of images of algebraic subsets $S \subset \mathbb{B}^n$, $n \geqslant 2$, under the universal covering map $\pi : \mathbb{B}^n \to X_\Gamma$ for a torsion-free lattice $\Gamma \subset G_0$, $X_\Gamma := \mathbb{B}^n$. When $\Gamma \subset G_0$ is not cocompact, we embed X_Γ into its minimal compactification $\overline{X_\Gamma}$ by adding a finite number of normal isolated singularities. Recall that $\overline{X_\Gamma}$ is projective and $X_\Gamma \subset \overline{X_\Gamma}$ inherits a canonical quasi-projective structure.

Recall that the irreducible algebraic subset $S \subset \mathbb{B}^n$ in the statement of Main Theorem is an irreducible component of $W_0 \cap \mathbb{B}^n$ for some irreducible projective subvariety $W_0 \subset \mathbb{P}^n$. Consider the universal family $\rho : \mathscr{U} \to \mathcal{K}$, where $\mathcal{K}$ is an irreducible component of the Chow space of $\mathbb{P}^n$ containing $[W_0]$. By means of the stratification given by singular loci we are going to define a G-invariant irreducible subvariety $\mathcal{H} \subset \mathcal{K}$, $[W_0] \in \mathcal{H}$, such that $[W_0]$ is a smooth point of $\mathcal{H}$, and consider the restriction over $\mathcal{H}$ of the universal family $\rho : \mathscr{U} \to \mathcal{K}$, as follows. Write $\mathcal{K}_1 := \mathcal{K}$, and consider the Zariski open subset $\mathcal{K}_1^0 \subset \mathcal{K}_1$ consisting of smooth points $[F] \in \mathcal{K}$ such that $F \subset \mathbb{P}^n$ is irreducible and reduced. In case $[W_0] \in \mathcal{K}_1^0$ we define $\mathcal{H} = \mathcal{K}_1 = \mathcal{K}$. Otherwise $[W_0] \in \mathcal{K}_1 - \mathcal{K}_1^0$. Let now $\mathcal{K}_2 \subsetneq \mathcal{K}_1$ be an irreducible component of $\mathcal{K}_1 - \mathcal{K}_1^0$ containing the point $[W_0]$. Since $G = \mathbb{P}GL(n+1, \mathbb{C})$ is connected, the irreducible component $\mathcal{K}_2$ of the G-invariant subvariety $\mathcal{K}_1 - \mathcal{K}_1^0$ is also G-invariant. Define now $\mathcal{K}_2^0 \subset \mathcal{K}_2$ to consist of all $[F] \in \mathcal{K}_2$ such that F is irreducible and reduced. Iterating the process, we obtain a finite sequence of G-invariant irreducible subvarieties $\mathcal{K}_m \subsetneq \cdots \subsetneq \mathcal{K}_1 = \mathcal{K} \subset \mathrm{Chow}(\mathbb{P}^n)$ such that $[W_0]$ is a smooth point on $\mathcal{K}_m$. We define now $\mathcal{H} = \mathcal{K}_m$ and $\mathscr{V} := \rho^{-1}(\mathcal{K}_m)$. Clearly $\mathcal{H}$ is also G-invariant. We denote by $\sigma : \mathscr{V} \to \mathcal{H}$ the restriction of the universal family $\rho : \mathscr{U} \to \mathcal{K}$ over $\mathcal{H} \subset \mathcal{K}$.

A torsion-free lattice $\Gamma \subset G_0$ acts properly discontinuously on $\mathscr{V}' = \mathscr{V}|_{\mathbb{B}^n}$ without fixed points, and we have a quotient space $\mathscr{V}_\Gamma := \mathscr{V}'/\Gamma$ equipped with a map $\nu_\Gamma : \mathscr{V}_\Gamma \to X_\Gamma = \mathbb{B}^n/\Gamma$, $\nu_\Gamma = \mu_\Gamma|_{\mathscr{V}_\Gamma}$, realizing $\mathscr{V}_\Gamma$ as a locally homogeneous holomorphic fiber subbundle of $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$. The foliation $\mathscr{F}$ on $\mathscr{U}$ restricts to $\mathscr{V} = \rho^{-1}(\mathcal{H})$, and we will use the same notation $\mathscr{F}$ for the restriction to $\mathscr{V}$. $\mathscr{F}$ descends to a foliation $\mathscr{F}_\Gamma$ on $\mathscr{V}_\Gamma$.

In the case of torsion-free nonuniform lattices $\Gamma \subset G_0$ we will be dealing with quasi-projective varieties X_Γ and we have to consider their minimal compactifications $X_\Gamma \subset \overline{X_\Gamma}$ together with projective compactifications of various holomorphic fiber bundles over X_Γ. Recall that $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ is embedded as a locally homogeneous holomorphic fiber subbundle into $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$, $\mathscr{P}_\Gamma$ is quasi-projective by Proposition 2.4 and $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$ is quasi-projective by Proposition 2.5. In fact, identifying $\mathscr{U}_\Gamma$ as a subset of $\mathscr{P}_\Gamma$, compactifying $\mathscr{P}_\Gamma$ to a projective variety $\overline{\mathscr{P}_\Gamma} \subset \mathbb{P}^\ell$ and extending the canonical projection $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ to a holomorphic map $\varpi_\Gamma^\sharp : \overline{\mathscr{P}_\Gamma} \to \overline{X_\Gamma}$, the topological closure $\overline{\mathscr{U}_\Gamma} \subset \overline{\mathscr{P}_\Gamma} \subset \mathbb{P}^\ell$ is a projective subvariety. The same proof as in Proposition 2.5 shows that for the holomorphic fiber bundle $\nu_\Gamma : \mathscr{V}_\Gamma \to X_\Gamma$, the topological closure $\overline{\mathscr{V}_\Gamma} \subset \overline{\mathscr{U}_\Gamma} \subset \overline{\mathscr{P}_\Gamma} \subset \mathbb{P}^\ell$ is a projective subvariety, and we have $\nu_\Gamma^\sharp : \overline{\mathscr{V}_\Gamma} \to \overline{X_\Gamma}$, $\nu_\Gamma^\sharp := \varpi_\Gamma^\sharp|_{\overline{\mathscr{V}_\Gamma}}$. We need now to extend $\mathscr{F}_\Gamma$ on $\mathscr{V}_\Gamma$ to a meromorphic foliation $\mathscr{F}_\Gamma^\sharp$ on $\overline{\mathscr{V}_\Gamma}$. Denoting the normalization of a reduced irreducible complex space Y by $Y^{\mathfrak{n}}$, $\mathscr{F}_\Gamma$ induces a meromorphic foliation $\mathscr{F}_\Gamma^{\mathfrak{n}}$ on $\mathscr{V}_\Gamma^{\mathfrak{n}}$, and the aforementioned extension problem on $\mathscr{F}_\Gamma$ is equivalently the problem of extending $\mathscr{F}_\Gamma^{\mathfrak{n}}$ meromorphically from $\mathscr{V}_\Gamma^{\mathfrak{n}}$ to $\overline{\mathscr{V}_\Gamma}^{\mathfrak{n}}$, hence we may apply Lemma 3.1. For the purpose of proving the meromorphic extension of $\mathscr{F}_\Gamma^{\mathfrak{n}}$ to $\overline{\mathscr{V}_\Gamma}^{\mathfrak{n}}$ we will need the following Hartogs-type extension theorem for meromorphic functions in Mok-Zhang [MZ$_1$, Lemma 7.4].

Lemma 3.2. *Let B be an irreducible projective variety, and $E \subset B$ be a subvariety of codimension $\geqslant 2$. Let $\mathcal{X}$ be an irreducible projective variety and $\alpha : \mathcal{X} \to B$ be a surjective morphism. Let $\Omega \subset B$ be an open subset in the complex topology and f be a meromorphic function on $\mathcal{X}|_{\Omega - E} := \alpha^{-1}(\Omega - E)$. Then, f extends to a meromorphic function on $\mathcal{X}|_\Omega = \alpha^{-1}(\Omega)$.*

Remark The main point of Lemma 3.2 is that $\alpha^{-1}(E) \subset \mathcal{X}$ may have irreducible components of codimension 1 even though $E \subset B$ is of codimension $\geqslant 2$. Realizing $\mathcal{X} \subset \mathbb{P}^m$ as a projective subvariety and writing $X_t := \alpha^{-1}(t)$, we define $\varphi : \Omega - E \dashrightarrow \mathcal{H}$ by $\varphi(t) = [\mathrm{Graph}(f|_{X_t})] \in \mathcal{H}$ for a general point $t \in \Omega - E$, where $\mathcal{H} \subset \mathrm{Chow}(\mathbb{P}^m \times \mathbb{P}^1)$ is some irreducible component. By

Hartogs extension, φ extends meromorphically to $\varphi^\sharp : \Omega \dashrightarrow \mathcal{H}$, and the meromorphic extension $f^\sharp : \mathcal{X}|_\Omega \dashrightarrow \mathbb{P}^1$ of $f : \mathcal{X}|_{\Omega-E} \dashrightarrow \mathbb{P}^1$ is defined by pulling back the universal family over $\mathcal{H}$ by $\varphi^\sharp$ and projecting to the $\mathbb{P}^1$ factor of $\mathbb{P}^m \times \mathbb{P}^1$. For details cf. [MZ$_1$, Lemma 7.4].

We deduce from Lemma 3.2 the following extension theorem for meromorphic foliations. We will apply the setting of Lemma 3.1 to the case where $\Omega = B$ and where B and $\mathcal{X}$ are normal.

Lemma 3.3. *Let B be a normal irreducible projective variety, and $E \subset B$ be a subvariety of codimension $\geqslant 2$. Let $\mathcal{X}$ be a normal irreducible projective variety and $\alpha : \mathcal{X} \to B$ be a surjective morphism. Let $\mathscr{L}$ be a meromorphic foliation on $\mathcal{X}|_{B-E} := \alpha^{-1}(B-E)$. Then, $\mathscr{L}$ extends to a meromorphic foliation on $\mathcal{X}$.*

Proof. Let r be the leaf dimension of the meromorphic foliation $\mathscr{L}$ on $\mathcal{X}|_{B-E}$. Identify the total space $\mathcal{X} \subset \mathbb{P}^N$ as a projective subvariety of some projective space $\mathbb{P}^N$. The tangent sheaf $\mathcal{T}_\mathcal{X}$ is a saturated coherent subsheaf of $\mathcal{T}_{\mathbb{P}^N}|_\mathcal{X} = \mathcal{O}(T_{\mathbb{P}^N}|_\mathcal{X})$. The meromorphic foliation $\mathscr{L}$ on $\mathcal{X}|_{B-E}$ corresponds to a saturated coherent subsheaf $\mathcal{T}_\mathscr{L} \subset \mathcal{T}_{\mathcal{X}|_{B-E}}$ of rank r such that, outside a complex-analytic subvariety $\mathcal{A} \subset \mathcal{X}|_{B-E}$ of codimension $\geqslant 2$ containing the singularity set $\mathrm{Sing}(\mathcal{X}|_{B-E})$ of the normal projective variety $\mathcal{X}|_{B-E}$, $\mathcal{T}_\mathscr{L}|_{\mathcal{X}|_{B-E}-\mathcal{A}} = \mathcal{O}(L)$ for some rank-r integrable holomorphic distribution $L \subset T_{\mathcal{X}|_{B-E}-\mathcal{A}}$. The holomorphic distribution L then defines a holomorphic section ψ of the Grassmann bundle $\mathrm{Gr}(r, T_{\mathcal{X}|_{B-E}-\mathcal{A}}) \subset \mathrm{Gr}(r, T_{\mathbb{P}^N}|_{\mathcal{X}|_{B-E}-\mathcal{A}})$ over $\mathcal{X}|_{B-E} - \mathcal{A}$. Given a point x on the complex manifold $\mathcal{X}|_{B-E} - \mathcal{A}$, $\psi(x) = [V(x)]$ for some r-dimensional linear subspace $V(x) \subset T_{\mathbb{P}^N,x} \cong \mathbb{C}^N$. Denote by $\Psi(x)$ the unique r-dimensional projective linear subspace in $\mathbb{P}^N$ passing through x such that $T_x(\Psi(x)) = V(x)$. Then, $\Psi : \mathcal{X}|_{B-E} - \mathcal{A} \to \mathscr{G}$, where $\mathscr{G}$ is the Grassmannian of r-dimensional projective linear subspaces in $\mathbb{P}^N$; $\mathscr{G} \cong \mathrm{Gr}(r+1, \mathbb{C}^{N+1})$.

We may identify $\mathscr{G}$ as a projective subvariety of $\mathbb{P}\big(\Lambda^{r+1}(\mathbb{C}^{N+1})\big)$ by means of the Plücker embedding. Write $d := \dim_\mathbb{C}\big(\Lambda^{r+1}(\mathbb{C}^{N+1})\big)$. Make now a generic choice of homogeneous coordinates $[w_0, \cdots, w_d]$ on $\mathbb{P}\big(\Lambda^{r+1}(\mathbb{C}^{N+1})\big)$ so that $\Psi(\mathcal{X}|_{B-E} - \mathcal{A})$ does not entirely lie on the hyperplane $\{w_0 = 0\}$ of $\mathbb{P}\big(\Lambda^{r+1}(\mathbb{C}^{N+1})\big) \cong \mathbb{P}^d$. Then, writing $z_k := \frac{w_k}{w_0}$ for $1 \leqslant k \leqslant d$, so that $(z_1, \cdots, z_d)$ constitutes a system of inhomogeneous coordinates on $\mathbb{P}^d$, we have $\Psi(x) = (\Psi_1(x), \cdots, \Psi_d(x))$, where each Ψ_k is a meromorphic function on $\mathcal{X}|_{B-E} - \mathcal{A}$. By Hartogs extension each $\Psi_k, 1 \leqslant k \leqslant d$, extends meromorphically to $\mathcal{X}|_{B-E}$. Denote by $\Psi_k^\dagger, 1 \leqslant k \leqslant d$, the meromorphic extension of Ψ_k to $\mathcal{X}|_{B-E}$. Then, by Lemma 3.2 each $\Psi_k^\dagger, 1 \leqslant k \leqslant d$, extends to a meromorphic function $\Psi_k^\sharp$ on $\mathcal{X}$, yielding a meromorphic extension $\Psi^\sharp = (\Psi_1^\sharp, \cdots, \Psi_d^\sharp)$, $\Psi^\sharp : \mathcal{X} \dashrightarrow \mathbb{P}^d$. Since the graph of $\Psi^\sharp$ is obtained by taking the topological closure of the graph of Ψ, and since $\Psi(\mathcal{X}|_{B-E} - \mathcal{A}) \subset \mathscr{G} \subset \mathbb{P}^d$, we have also $\Psi^\sharp : \mathcal{X} \dashrightarrow \mathscr{G}$.

Finally, to apply Lemma 3.1 we need to extend the holomorphic section ψ of $\mathrm{Gr}(r, T_{\mathbb{P}^N})$ over $\mathcal{X}|_{B-E} - \mathcal{A}$ to a meromorphic section $\psi^\sharp$ of $\mathrm{Gr}(r, T_{\mathbb{P}^N})$ over $\mathcal{X}$. Since $\Psi(x)$ passes through x whenever x belongs to the dense open subset $\mathcal{X}|_{B-E} - \mathcal{A} \subset \mathcal{X}$ in the complex topology, it follows by continuity that at a point $y \in \mathcal{X}$ where $\Psi^\sharp(y)$ is holomorphic, $\Psi^\sharp(y)$ (which is well-defined) must pass through y, hence $\psi^\sharp$ can be defined at y by $\psi^\sharp(y) = [T_y(\Psi^\sharp(y))] \in \mathrm{Gr}(r, T_y(\mathbb{P}^N)$. It follows that $\psi^\sharp(y)$ is defined and holomorphic outside of a normal projective subvariety $\mathcal{S}$ of $\mathcal{X}$ of codimension $\geqslant 2$ and hence by Hartogs extension $\psi^\sharp$ is defined as a meromorphic section of $\mathrm{Gr}(r, T_{\mathbb{P}^N})$ over $\mathcal{X}$ in such a way that $\psi^\sharp|_{\mathcal{X}|_{B-E}-\mathcal{A}} \equiv \psi$. By Lemma 3.1 the meromorphic foliation $\mathscr{L}$ on $\mathcal{X}|_{B-E}$ extends meromorphically to $\mathcal{X}$. $\square$

We conclude this section with the following result on $\nu_\Gamma : \mathscr{V}_\Gamma \to X_\Gamma$ and on the compactification $\nu_\Gamma^\sharp : \overline{\mathscr{V}_\Gamma} \to \overline{X_\Gamma}$ for torsion-free lattices $\Gamma \subset G_0$.

Proposition 3.2. *Let $\mathcal{K}$ be an irreducible component of* $\mathrm{Chow}(\mathbb{P}^n)$, $[W_0] \in \mathcal{K}$, $S \subset \mathbb{B}^n$ *be an irreducible component of* $W_0 \cap \mathbb{B}^n$, *and* $\mathscr{S} \subset \mathscr{U}$ *be the tautological lifting of* S *to* $\mathscr{U}$. *Let* $\mathcal{H} \subset \mathcal{K}$ *be a* G*-invariant subvariety such that* $[W_0] \in \mathcal{H}$ *and such that, denoting by* $\sigma : \mathscr{V} \to \mathcal{H}$, $\nu : \mathscr{V} \to \mathbb{P}^n$ *the restriction of the universal family* $\rho : \mathscr{U} \to \mathcal{K}$, $\mu : \mathscr{U} \to \mathbb{P}^n$ *from* $\mathscr{U}$ *to* $\mathscr{V}$, *a general point of* $\mathscr{S}$ *is a smooth point of* $\mathscr{V}$. *Then, denoting* $\nu|_{\mathbb{B}^n}$ *by* $\nu' : \mathscr{V}' \to \mathbb{B}^n$, *and by* $\nu_\Gamma : \mathscr{V}_\Gamma \to X_\Gamma$ *the locally homogeneous fiber bundle obtained from* $\nu' : \mathscr{V}' \to \mathbb{B}^n$ *by taking quotients with respect to* Γ, *the tautological foliation* $\mathscr{F}$ *on* $\mathscr{V}$ *descends to a meromorphic foliation* $\mathscr{F}_\Gamma$ *on* $\mathscr{V}_\Gamma$. *Moreover, when* $\Gamma \subset G_0$ *is a torsion-free nonuniform lattice the topological closure* $\overline{\mathscr{V}_\Gamma} \subset \overline{\mathscr{P}_\Gamma}$ *is projective, and* $\mathscr{F}_\Gamma$ *extends to a meromorphic foliation* $\mathscr{F}_\Gamma^\sharp$ *on* $\overline{\mathscr{V}_\Gamma}$. *As a consequence, for any torsion-free lattice* $\Gamma \subset G_0$ *the Zariski closure* Z *of* $\pi(S)$ *in* X_Γ *is uniruled by subvarieties belonging to* $\mathcal{H}$.

Proof of Proposition 3.2. As in the second last paragraph preceding Lemma 3.2, we have seen that the tautological foliation $\mathscr{F}$ on $\mathscr{V}$ descends to a meromorphic foliation $\mathscr{F}_\Gamma$ on $\mathscr{V}_\Gamma$. When $\Gamma \subset G_0$ is a torsion-free nonuniform lattice, by the last paragraph preceding Lemma 3.2, the topological closure $\overline{\mathscr{V}_\Gamma} \subset \overline{\mathscr{P}_\Gamma}$ is projective. By Lemma 3.3 and passing to normalizations (as in the paragraph preceding Lemma 3.2) the meromorphic foliation $\mathscr{F}_\Gamma$ on $\mathscr{V}_\Gamma$ extends to a meromorphic foliation $\mathscr{F}_\Gamma^\sharp$ on $\overline{\mathscr{V}_\Gamma}$.

For an arbitrary torsion-free lattice $\Gamma \subset G_0$ it remains to establish the concluding sentence on the uniruling of Z by subvarieties belonging to $\mathcal{H}$. Write $\widetilde{\pi} : \mathscr{V} \to \mathscr{V}_\Gamma$ for the covering map induced from the universal covering map $\pi : \mathbb{B}^n \to X_\Gamma$, and denote by $\mathscr{Z} \subset \mathscr{V}_\Gamma$ the Zariski closure of $\widetilde{\pi}(\mathscr{S})$ in $\mathscr{V}_\Gamma$. We observe that $\nu_\Gamma(\mathscr{Z}) = Z$. In the notation as in §2 denote by $\overline{\mathscr{Z}} \subset \overline{\mathscr{P}_\Gamma}$ the topological closure of $\mathscr{Z}$ in the projective compactification $\overline{\mathscr{P}_\Gamma}$ of the projective bundle $\mathscr{P}_\Gamma$. $\overline{\mathscr{Z}} \subset \overline{\mathscr{P}_\Gamma}$ is projective. Denote by $\overline{\nu_\Gamma} : \overline{\mathscr{Z}} \to \overline{X_\Gamma}$ the restriction of $\varpi_\Gamma^\sharp : \overline{\mathscr{P}_\Gamma} \to \overline{X_\Gamma}$ to $\overline{\mathscr{Z}}$. By the proper mapping theorem $\nu_\Gamma(\mathscr{Z}) = \overline{\nu_\Gamma}(\overline{\mathscr{Z}}) \cap X_\Gamma \subset X_\Gamma$ is a quasi-projective subvariety containing $\pi(S)$, hence $\nu_\Gamma(\mathscr{Z}) \supset Z$. Suppose $\nu_\Gamma(\mathscr{Z}) \supsetneq Z$. Then, $\nu_\Gamma^{-1}(Z) \cap \mathscr{Z} \subsetneq \mathscr{Z}$ is a quasi-projective subvariety of $\mathscr{V}_\Gamma$ containing $\widetilde{\pi}(\mathscr{S})$, contradicting with $\mathscr{Z} := \overline{\widetilde{\pi}(\mathscr{S})}^{\mathscr{Z}ar}$. Hence, $\nu_\Gamma(\mathscr{Z}) = Z$ by contradiction, as observed.

Recall that $\dim(S) =: r$ and define $d := \dim(Z)$, $e := \dim(\mathscr{Z})$. By Proposition 3.1, $\mathscr{F}_\Gamma$ on $\mathscr{V}_\Gamma$ restricts to a meromorphic foliation on $\mathscr{Z}$ which we denote by $\mathscr{E}$. The singular locus $\mathrm{Sing}(\mathscr{E}) \subsetneq \mathscr{Z}$ of $\mathscr{E}$ is a subvariety containing $\mathrm{Sing}(\mathscr{Z})$. Write $\mathrm{Reg}(\mathscr{E}) := \mathscr{Z} - \mathrm{Sing}(\mathscr{E}) \subset \mathrm{Reg}(\mathscr{Z})$ and consider the open subset $\Omega := \widetilde{\pi}^{-1}(\mathrm{Reg}(\mathscr{E})) \subset \mathscr{P}'$. Let $\mathfrak{A} \subset \Omega$ be the subvariety over which $\mathrm{rank}(d\sigma) < e - r$. Then $\mathfrak{A}$ is invariant under $\pi_1(Z)$ and it descends to a subvariety $\mathscr{A} \subset \mathrm{Reg}(\mathscr{E})$. Write $\lambda := \nu_\Gamma|_{\mathscr{Z}}$. Define now $\mathscr{B} \subset \mathscr{Z}$ to be the locus of points $v \in \mathscr{Z}$ where (a) $v \in \mathrm{Sing}(\mathscr{Z})$, or (b) $\lambda(v) \in \mathrm{Sing}(Z)$, or (c) $v \in \mathrm{Reg}(\mathscr{Z}), \lambda(v) \in \mathrm{Reg}(Z)$ but $\lambda|_{\mathrm{Reg}(\mathscr{Z}) \cap \nu_\Gamma^{-1}(\mathrm{Reg}(Z))} : \mathrm{Reg}(\mathscr{Z}) \cap \nu_\Gamma^{-1}(\mathrm{Reg}(Z)) \to \mathrm{Reg}(Z)$ fails to be a submersion at v. Then, $\mathscr{B} \subset \mathscr{Z}$ is a quasi-projective subvariety.

Consider now $\mathscr{W} := \mathrm{Reg}(\mathscr{E}) - \mathscr{A} - \mathscr{B} \subset \mathscr{Z}$, which is dense in $\mathscr{Z}$ in the complex topology. Let v be a point on $\mathscr{W} \subset \mathscr{Z}$, $\lambda(v) =: x$. By the definition of universal families the restriction of λ to each local leaf $\mathcal{L} \subset \mathrm{Reg}(\mathscr{E})$ of $\mathscr{E}$ is an immersion into X_Γ. Since $\lambda : \mathscr{W} \to \mathrm{Reg}(Z)$ is a submersion, there exists a $(d-r)$-dimensional complex submanifold $\mathscr{D} \subset \mathscr{W}_0$ of some open neighborhood $\mathscr{W}_0$ of v in $\mathscr{W}$ such that $\mathscr{D}$ is biholomorphic to Δ^{d-r} and $\lambda|_{\mathscr{D}} : \mathscr{D} \to \mathrm{Reg}(Z)$ is a holomorphic embedding onto a locally closed complex submanifold $N \subset \mathrm{Reg}(Z)$. Lifting $\mathscr{D}$ to $\widetilde{\mathscr{D}}$ on any connected component of Ω and noting that $\widetilde{\mathscr{D}} \cap \mathfrak{A} = \emptyset$, shrinking $\widetilde{\mathscr{D}}$ if necessary we may assume that σ maps $\widetilde{\mathscr{D}}$ biholomorphically onto some $(d-r)$-dimensional complex submanifold $\mathcal{S}$ of some open subset $\mathcal{G}$ of $\mathcal{H}$. Clearly, there exists a contractible open neighborhood $\widetilde{\mathcal{O}}$ of $\widetilde{\mathscr{D}}$ in

$\sigma^{-1}(\mathcal{S})$ such that $\widetilde{\pi}$ maps $\widetilde{\mathcal{O}}$ biholomorphically onto a neighborhood $\mathcal{O}$ of v on $\mathrm{Reg}(\mathscr{E})$ and $\mathcal{O}$ is the disjoint union of local leaves of $\mathscr{E} = \mathscr{F}_\Gamma|_{\mathscr{X}}$ passing through points on $\mathscr{D}$, and such that $\nu_\Gamma \circ \widetilde{\pi}$ maps $\widetilde{\mathcal{O}}$ biholomorphically onto a neighborhood U of $x = \lambda(v)$. Here we make take N to be a complex submanifold of U and U to be a disjoint union of r-dimensional complex submanifolds $L(y) \subset U$ passing through $y \in N$ which are images under $\pi \circ \nu = \nu_\Gamma \circ \widetilde{\pi}$ of connected smooth open subsets of $S_\eta \cap \widetilde{\mathcal{O}}$, where $S_\eta := W_\eta \cap \mathbb{B}^n$ and W_η belongs to $\mathcal{S} \subset \mathcal{H}$, and we have a uniruling of $\mathscr{X}$ by subvarieties belonging to $\mathcal{H}$ (cf. Definition 3.1), as desired. $\square$

The uniruling of Z by subvarieties belonging to $\mathcal{H}$ is the key statement in Proposition 3.2 which will allow us to establish Main Theorem in §4 by analytic methods.

4. Proof of the Main Theorem and generalizations

Proof of Main Theorem in the compact case. Recall that $g = ds^2_{\mathbb{B}^n}$ is the canonical Kähler-Einstein metric with Kähler form $\omega_g = \sqrt{-1}\partial\overline{\partial}(-\log(1-\|z\|^2))$, which is of constant holomorphic sectional curvature -2, and X_Γ is endowed with the quotient metric $g_\Gamma = ds^2_{X_\Gamma}$. Recall that $\dim(S) = r$, and write $\dim(Z) = d$. When $\Gamma \subset G_0 = \mathrm{Aut}(\mathbb{B}^n)$ is cocompact, the proof that $Z = \overline{\pi(S)}^{\mathscr{Z}ar} \subset X_\Gamma$ is a totally geodesic subset will now be deduced from Proposition 3.1, as follows. A general point $x \in S$ must be a smooth point on $\pi^{-1}(Z)$, otherwise $Z = \overline{\pi(S)}^{\mathscr{Z}ar} \subset X \subset \mathrm{Sing}(Z) \subsetneq Z$, a plain contradiction. In particular, there is a unique irreducible component $\widetilde{Z}$ of $\pi^{-1}(Z)$ which contains S.

Recall that $\mathscr{S} \subset \mathscr{V}$ is the tautological lifting of S to the total space $\mathscr{V}$ of the G-invariant subfamily $\sigma : \mathscr{V} \to \mathcal{H}$ of the universal family $\rho : \mathscr{U} \to \mathcal{K}$, and that $\mathscr{X} := \overline{\widetilde{\pi}(\mathscr{S})}^{\mathscr{Z}ar} \subset \mathscr{V}_\Gamma$. By Proposition 3.1, $\mathscr{X}$ is saturated with respect to the tautological foliation $\mathscr{F}$ on $\mathscr{V}_\Gamma$. By Proposition 3.2, $Z = \nu_\Gamma(\mathscr{X})$ is uniruled by subvarieties belonging to $\mathcal{H}$ in the sense of Definition 3.1. Consider the germ at some point $b \in \partial\widetilde{Z}$ of a complex submanifold Σ which is the union of nonempty open subsets of a certain family of subvarieties $W_\eta \subset \mathbb{P}^n$ belonging to $\mathcal{H}$ and which exits the boundary of $\mathbb{B}^n$ near $b \in \partial\widetilde{Z} \subset \partial\mathbb{B}^n$. We are going to prove that $Z \subset X_\Gamma$ is a totally geodesic subset by exploiting the asymptotic total geodesy of $\big(\Sigma \cap \widetilde{Z}, g|_{\Sigma\cap\widetilde{Z}}\big)$ at $b \in \partial\mathbb{B}^n \cap \Sigma$ using asymptotic curvature properties resulting from Klembeck [Kl] (Theorem 1.5 here).

By Proposition 3.2, $\mathscr{X}$ is uniruled by subvarieties belonging to $\mathcal{H}$. In the notation there consider now the mapping $\beta := \nu|_{\sigma^{-1}(\mathcal{S})} : \sigma^{-1}(\mathcal{S}) \to \mathbb{P}^n$. Note that $\dim(\sigma^{-1}(\mathcal{S})) = (d-r)+r = d$ and that the image $\beta(\sigma^{-1}(\mathcal{S}))$ contains the open set $\widetilde{\mathcal{O}} \subset \widetilde{Z}$. Since by construction β is a local biholomorphism into $\widetilde{Z}$ at $\widetilde{v} \in \widetilde{Z}$, β must necessarily be an immersion into $\mathbb{P}^n$ at a general smooth point of $\sigma^{-1}(\mathcal{S})$.

Observe that singularities of $\sigma^{-1}(\mathcal{S})$ are of complex codimension $\geqslant 1$ while $\beta^{-1}(\partial\mathbb{B}^n) = \nu^{-1}(\partial\mathbb{B}^n) \cap \sigma^{-1}(\mathcal{S})$ is a real hypersurface in $\sigma^{-1}(\mathcal{S})$. It follows that for a general point $\widehat{b} \in \beta^{-1}(\partial\mathbb{B}^n)$, $\widehat{b}$ is a smooth point of $\sigma^{-1}(\mathcal{S})$ and β is an immersion at $\widehat{b}$. Thus, for some open neighborhood $\mathcal{W}$ of $\widehat{b}$ in $\sigma^{-1}(\mathcal{S})$, $\beta|_{\mathcal{W}} : \mathcal{W} \xrightarrow{\cong} \Sigma \subset \mathbb{C}^n$ maps $\mathcal{W}$ biholomorphically onto some complex submanifold $\Sigma \subset U$ of some open set $U \subset \mathbb{C}^n$, $b \in U$, such that $\Sigma \cap \mathbb{B}^n$ is a nonempty open subset of $\widetilde{Z}$. One may say that $\widetilde{Z}$ is analytically continued by grafting Σ to $\widetilde{Z}$ at $\widehat{b}$. For convenience we may assume that both Σ and $\Sigma\cap\mathbb{B}^n \subset \Sigma$ are connected, and that $\Sigma\cap\mathbb{B}^n = \widetilde{Z}\cap U$.

For easy reference we include a proof of the following well-known lemma.

Lemma 4.1. *At a general point $b \in \sigma^{-1}(\mathcal{S}) \cap \partial\mathbb{B}^n$ the function $\varphi := \|z\|^2 - 1$ of $\partial\mathbb{B}^n = \{\varphi = 0\}$ must necessarily vanish exactly to the order 1.*

Proof. Otherwise φ vanishes to the order $k \geqslant 2$ on a neighborhood N of a general point b in $W_\eta \cap \partial\mathbb{B}^n$ and $-\varphi|_{W_\eta} = \theta^k$ such that $\theta \geqslant 0$ and $d\theta(p) \neq 0$ for $p \in N$. We have

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}\varphi &= -\sqrt{-1}\partial\overline{\partial}\theta^k \\ &= -k\theta^{k-1}\sqrt{-1}\partial\overline{\partial}\theta - k(k-1)\theta^{k-2}\sqrt{-1}\partial\theta \wedge \overline{\partial}\theta \\ &= -k\theta^{k-2}\big(\theta\sqrt{-1}\partial\overline{\partial}\theta + (k-1)\sqrt{-1}\partial\theta \wedge \overline{\partial}\theta\big)\end{aligned}$$

At $p \in N$ we have $\theta(p) = 0$ while $\mu := \sqrt{-1}\partial\theta(p) \wedge \overline{\partial\theta(p)} \geqslant 0$ and $\mu\big(\frac{1}{\sqrt{-1}}\xi \wedge \overline{\xi}\big) > 0$ whenever $\xi \in T_p^{1,0}(\mathbb{B}^n)$ and $\partial\theta(\xi) \neq 0$, hence $\sqrt{-1}\partial\overline{\partial}\varphi\big(\frac{1}{\sqrt{-1}}\xi \wedge \overline{\xi}\big) < 0$. Thus, for $x \in W_\eta \cap \mathbb{B}^n$ sufficiently close to p, by continuity there also exists $\xi \in T_x^{1,0}(\mathbb{B}^n)$ such that $\sqrt{-1}\partial\overline{\partial}\varphi\big(\frac{1}{\sqrt{-1}}\xi \wedge \overline{\xi}\big) < 0$, contradicting the plurisubharmonicity of φ, as desired. $\square$

Proof of Main Theorem in the compact case continued. By Lemma 4.1 the function $\varphi|_\Sigma$ vanishes on $\Sigma \cap \partial\mathbb{B}^n$ exactly to the order 1 at a general point $p \in \Sigma \cap \partial\mathbb{B}^n$. Thus, shrinking Σ if necessary, $\Sigma \cap \mathbb{B}^n = \{x \in \Sigma : \varphi < 0\} = \widetilde{Z} \cap U \subset \Sigma$ and φ is a defining function of $\Sigma \cap \mathbb{B}^n \subset \Sigma$ along $\Sigma \cap \partial\mathbb{B}^n = \partial\widetilde{Z} \cap U$. It follows from Klembeck [Kl] (Theorem 1.5 here) that $\big(\Sigma \cap \mathbb{B}^n, g|_{\Sigma\cap\mathbb{B}^n}\big)$, where the Kähler form ω_g of $(\mathbb{B}^n, g)$ given by $\omega_g = \sqrt{-1}\partial\overline{\partial}(-\log(-\varphi))$, is asymptotically of constant holomorphic sectional curvature -2 at any boundary point $p \in \Sigma \cap \partial\mathbb{B}^n$. This implies that $\big(\Sigma \cap \mathbb{B}^n, g|_{\Sigma\cap\mathbb{B}^n}\big)$ is asymptotically totally geodesic along $\Sigma \cap \partial\mathbb{B}^n$.

Finally we are going to deduce the total geodesy of $\widetilde{Z} \subset \mathbb{B}^n$ and hence that $Z \subset X_\Gamma$ is a totally geodesic subset. Choose $R < \infty$ exceeding the diameter of the compact ball quotient X_Γ. Then, denoting by $B(a; r) \subset \mathbb{B}^n$ the geodesic ball with respect to the canonical Kähler-Einstein metric g of radius $r > 0$ centered at $a \in \mathbb{B}^n$, we have $\pi(B(y; R)) = X_\Gamma$ for any point y in $\mathbb{B}^n$. Take now any point $x \in \widetilde{Z}$. Let $x_k, k \geqslant 0$, be a sequence of points on $\widetilde{Z}$ such that x_k converges to $b \in \Sigma \cap \partial\mathbb{B}^n$. For any $k \geqslant 0$, it follows from $\pi(B(x_k; R)) = X_\Gamma$ that there exists a point $y_k \in B(x_k; R)$ such that $\pi(y_k) = \pi(x)$. In other words, there exists $\gamma_k \in \Gamma$ such that $\gamma_k(x) = y_k$. Denoting by $d(\cdot;\cdot)$ the distance function with respect to g, we have $d(x_k; y_k) < R$. Comparing the Kähler form ω_g of $(\mathbb{B}^n, g)$ with the Kähler form $\omega_{g_e} = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\|z\|^2$ of the Euclidean metric g_e we have

$$\begin{aligned}\omega_g = \sqrt{-1}\partial\overline{\partial}(-\log(1 - \|z\|^2)) &= \frac{\sqrt{-1}\partial\overline{\partial}\|z\|^2}{1 - \|z\|^2} + \frac{\sqrt{-1}\partial\|z\|^2 \wedge \overline{\partial}\|z\|^2}{(1 - \|z\|^2)^2} \\ &\geqslant \frac{\sqrt{-1}\partial\overline{\partial}\|z\|^2}{1 - \|z\|^2} = \frac{2\omega_{g_e}}{(1 + \|z\|)(1 - \|z\|)}\ .\end{aligned}$$

Thus, $g \geqslant \frac{g_e}{1 - \|z\|}$ on $\mathbb{B}^n$. Since x_k converges in $\mathbb{C}^n$ to b, from $d(x_k; y_k) < R$ we conclude that there exists a constant $C > 0$ such that we have

$$\|y_k - x_k\| \leqslant C\sqrt{1 - \|x_k\|} \to 0$$

as k tends to ∞, so that y_k also converges in $\mathbb{C}^n$ to b. On the other hand, by the invariance of g under $\Gamma \subset G_0$, holomorphic sectional curvatures at x are the same as holomorphic sectional curvatures at $y_k = \gamma_k(x)$. From the last paragraph $\big(\widetilde{Z}, g|_{\widetilde{Z}}\big)$ is of constant holomorphic sectional curvature -2 at x. Denoting by τ the second fundamental form of $\widetilde{Z}$ in $(\mathbb{B}^n, g)$, by the Gauss equation we have

$$-2 = R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(\widetilde{Z}, g|_{\widetilde{Z}}) = R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(\mathbb{B}^n, g) - \|\tau(\alpha,\alpha)\| = -2 - \|\tau(\alpha,\alpha)\|\,,$$

where $\alpha \in T_x^{1,0}(\widetilde{Z})$ and $\|\cdot\|$ stands for norms measured with respect to Hermitian metrics induced from g. It follows that $\tau(\alpha,\alpha) = 0$ at x for all $\alpha \in T_x^{1,0}$, hence $\tau(\alpha',\alpha'') = 0$ for all $\alpha',\alpha'' \in T_x^{1,0}$ by polarization. Since $x \in \widetilde{Z}$ is arbitrary, $\tau \equiv 0$, and we conclude that $\widetilde{Z} \subset \mathbb{B}^n$ is totally geodesic with respect to g, as desired. The proof of Main Theorem for the case of cocompact torsion-free lattices $\Gamma \subset G_0$ is complete. $\square$

Remark Replacing $\mathbb{B}^n$ by a possibly reducible bounded symmetric domain Ω of higher rank, in the same setup we have a complex-analytic subvariety $\widetilde{Z} \subset \Omega$ exiting as a germ of complex submanifold at points $b \in \partial\Omega$ belonging to some G_0-invariant subset $E \subset \partial\Omega$. If the analytic continuation $\widetilde{Z} \cup U$ by grafting a coordinate chart U at b happens to be transversal to the foliation of E by maximal complex submanifolds (cf. Wolf [Wo]), then Z is strictly pseudoconvex at b and the same rescaling argument shows that $\widetilde{Z}$ is the image of a holomorphic isometry of some $\mathbb{B}^m$ and it is hence algebraic by Mok [Mo4]. In general the rescaling argument shows that $\widetilde{Z}$ decomposes near b into a union of algebraic subsets which are images of holomorphic isometric embeddings of some $\mathbb{B}^m$, which furnishes a starting point of our approach under preparation to Ax-Lindemann-Weierstrass for bounded symmetric domains applicable to non-arithmetic lattices.

For the proof of Main Theorem it remains to consider torsion-free nonuniform lattices $\Gamma \subset G_0$, i.e, the case where $(X_\Gamma, \omega_\Gamma)$ is (necessarily) of finite volume but noncompact. In this case, in the notation of the proof of Main Theorem in the cocompact case, taking sequences of points (x_k) on $\widetilde{Z}$ converging to points on $\widetilde{Z} \cap \partial\mathbb{B}^n$, we have to take care of the possibility that $\pi(x_k) \in X_\Gamma$ escapes to infinity. To deal with this situation we prove

Proposition 4.1. *In the notation of Main Theorem let $S \subset \mathbb{B}^n$ be an irreducible algebraic subset and write $Z := \overline{\pi(S)}^{\mathscr{Z}ar} \subset X_\Gamma$. Let $\widetilde{Z}$ be an irreducible component of $\pi^{-1}(Z)$. Suppose $b \in \partial\mathbb{B}^n$ and let $U \Subset \mathbb{C}^n$ be an open neighborhood of b for which there exists a complex submanifold $\Sigma \subset U$ of dimension $d := \dim(Z)$ such that* (a) *$\Sigma \cap \widetilde{Z}$ is a nonempty connected open subset of $\widetilde{Z}$ and $\partial\mathbb{B}^n \cap \Sigma$ is connected;* (b) *Σ is transversal to $\partial\mathbb{B}^n$ at every point $p \in \partial\mathbb{B}^n \cap \Sigma$. Let $\{U_k : 0 \leqslant k < \infty\}$ be a sequence of open neighborhoods of b in $\mathbb{C}^n$, $U_0 = U$, such that for each $k \geqslant 0$, $U_{k+1} \Subset U_k$ and $\Sigma \cap \widetilde{Z} \cap U_k$ is connected, and such that $\bigcap_{0 \leqslant k < \infty} U_k = \overline{\mathcal{O}} \subset \partial\mathbb{B}^n$ for some open neighborhood $\mathcal{O}$ of b in $\partial\mathbb{B}^n$. Then, there exists a compact subset $Q \subset Z$ and a sequence of points $x_k \in U_k$ such that $\pi(x_k) \in Q$ for any $k \geqslant 0$. As a consequence, $Z \subset X_\Gamma$ is a totally geodesic subset.*

For the proof of Proposition 4.1 we will need the following well-known statement about holomorphic functions on the unit ball for which we include a proof for easy reference.

Lemma 4.2. *Let $n \geqslant 1$, $b \in \partial\mathbb{B}^n$, and $U \Subset \mathbb{C}^n$ be a neighborhood of b. Suppose $f : \overline{U} \cap \overline{\mathbb{B}^n} \to \mathbb{C}$ is a continuous function which is holomorphic on $U \cap \mathbb{B}^n$ and vanishes on $\overline{U} \cap \partial\mathbb{B}^n$. Then, $f \equiv 0$ on $\overline{U} \cap \overline{\mathbb{B}^n}$.*

Proof. Slicing by complex lines transversal to $\partial\mathbb{B}^n$ at points $p \in U \cap \partial B^n$ one reduces the problem to the case of $n = 1$. By the Riemann mapping theorem it suffices to show that, for a continuous function $g : \overline{\Delta} \to \mathbb{C}$ which is holomorphic on Δ and vanishes on an open arc of $\partial\Delta$, we must have $g \equiv 0$. To see this, replacing g by $g \circ \varphi$ for some $\varphi \in \mathrm{Aut}(\Delta)$ we may assume that $g(\zeta) = 0$ for $\zeta \in \partial\Delta$ satisfying $\mathrm{Re}(\zeta) \geqslant 0$. Then, the continuous function $h : \overline{\Delta} \to \mathbb{C}$ defined by

$h(z) := g(z)g(-z)$ vanishes on $\partial\Delta$ and it is holomorphic on Δ. By the maximum principle $h \equiv 0$ and hence $g \equiv 0$, as desired. $\square$

Proof of Proposition 4.1. Denote by $\{q_1, \cdots, q_m\}$ the finite set of normal isolated singularities of $\overline{X_\Gamma}$. For each q_i, $1 \leqslant i \leqslant m$, let V_i be an open neighborhood of q_i in $\overline{X_\Gamma}$ such that there is a biholomorphism $\lambda_i : V_i \xrightarrow{\cong} E_i \subset \mathbb{B}^{N_i}$ onto a subvariety E_i of the complex unit ball $\mathbb{B}^{N_i}$ in $\mathbb{C}^{N_i}$, $\nu(q_i) = 0$. We assume without loss of generality that $\overline{V_1}, \cdots, \overline{V_m}$ are disjoint. Let $\{U_k : 0 \leqslant k < \infty\}$ be the sequence of open neighborhoods of b in $\mathbb{C}^n$ as in the statement of the proposition. Arguing by contradiction assume that there exists no compact subset $Q \subset Z$ with the desired property as stated in the proposition. Let $C \subset Z$ be the compact subset such that $Z - C$ is the disjoint union of the open subsets $V_i \cap \widetilde{Z}, 1 \leqslant i \leqslant m$. By assumption, for k sufficiently large, say $k \geqslant \ell$, $\pi(\Sigma \cap \widetilde{Z} \cap U_k) \subset Z - C = (V_1 \cap \widetilde{Z}) \cup \cdots \cup (V_m \cap \widetilde{Z})$. Since $\Sigma \cap \widetilde{Z} \cap U_k$ is connected for each $k \geqslant 0$, we may assume that $\pi(\Sigma \cap \widetilde{Z} \cap U_k) \subset V_i$ for one of the disjoint open sets V_i, say $i = 1$, whenever $k \geqslant \ell$. Since $\lambda_1 : V_1 \xrightarrow{\cong} E_1 \subset \mathbb{B}^{N_1}$ there exists a holomorphic function h on $\mathbb{B}^{N_1}$ such that $h(0) = 0$ and such that the holomorphic function $f : \Sigma \cap \widetilde{Z} \cap U_\ell \to \mathbb{C}$ defined by $f := h \circ \lambda_1 \circ \pi$ is nonconstant. By assumption, for any sequence of points $x_k \in \Sigma \cap \widetilde{Z} \cap U_k$, $\pi(x_k)$ converges to $q_1 \in \overline{X_\Gamma}$. It follows that $f(x)$ converges to 0 as $x \in \Sigma \cap \widetilde{Z} \cap U_\ell$ converges to some boundary point $p \in \mathcal{O} \subset \partial\mathbb{B}^n \cap \Sigma$. By Lemma 4.2, such a holomorphic function must necessarily be identically zero, a plain contradiction. We have thus proven by contradiction that there is some compact subset $Q \subset Z$ for which there exists some sequence of points $x_k \in \Sigma \cap \widetilde{Z} \cap U_k$ such that $\pi(x_k) \in Q$ for each k. By the proof of Main Theorem in the compact case and by Proposition 4.1 this forces $\widetilde{Z}$ to be totally geodesic on an open neighborhood of some point $x \in Z$, hence $Z = \pi(\widetilde{Z})$ is a totally geodesic subset in X_Γ, as desired. $\square$

Proof of Main Theorem continued. Main Theorem was first proved for the case where X_Γ is compact, and the case where X_Γ is noncompact has been incorporated in Proposition 4.1. The proof of Main Theorem is now complete. $\square$

The proof of Main Theorem also yields readily the following result about the Zariski closure of unions of totally geodesic subsets of $\overline{X_\Gamma}$.

THEOREM 4.1. *Let A be any set of indices and $\Sigma_\alpha \subset X_\Gamma \subset \overline{X_\Gamma}$, $\alpha \in A$, be a family of closed totally geodesic subsets of X_Γ of positive dimension. Write $E := \bigcup \big\{\Sigma_\alpha : \alpha \in A\big\}$. Then, the Zariski closure of E in X_Γ is a union of finitely many totally geodesic subsets.*

Proof. Since the minimal compactification $X_\Gamma \subset \overline{X_\Gamma}$ is obtained by adding a finite number of normal isolated singularities, by the Remmert-Stein extension theorem the topological closure in $\overline{X_\Gamma}$ of each $\Sigma_\alpha \subset X_\Gamma$, $\alpha \in A$, is necessarily a projective subvariety of $\overline{X_\Gamma}$. Hence for any $\alpha \in A$, $\Sigma_\alpha \subset X_\Gamma$ is a quasi-projective subvariety. Denote by Z the Zariski closure of $E = \bigcup \big\{\Sigma_\alpha : \alpha \in A\big\}$. Write $Z = Z_1 \cup \cdots \cup Z_m$ for the decomposition of Z into irreducible components. For $1 \leqslant k \leqslant m$ we proceed to prove that $Z_k \subset X_\Gamma$ is a totally geodesic subset. For each dimension d, $1 \leqslant d \leqslant \dim(Z_k)$, denote by $A(k,d) \subset A$ the set of indices $\alpha \in A$ such that $\Sigma_\alpha \subset Z_k$ and $\dim(\Sigma_\alpha) = d$. Let $E(k,d)$ be the union of $\big\{\Sigma_\alpha : \alpha \in A(k,d)\big\}$ and denote by $Z(k,d)$ the Zariski closure of $E(k,d)$. Then, there exists some d_0, $1 \leqslant d_0 \leqslant \dim(Z_k)$, such that $Z(k,d_0) = Z_k$. Thus, replacing Z by Z_k and A by $A(k,d_0)$, for the proof of Theorem 4.1 without loss of generality we may assume that Z is irreducible and that all $\Sigma_\alpha, \alpha \in A$, are of the same complex dimension d_0.

Consider now the Chow space $\mathcal{K}$ of all d_0-dimensional projective subspaces in $\mathbb{P}^n$ and the universal family $\rho : \mathscr{U} \to \mathcal{K}$, $\mu : \mathscr{U} \to \mathbb{P}^n$. (In this case $\mu : \mathscr{U} \to \mathbb{P}^n$ can be identified with the

Grassmann bundle $\mathrm{Gr}(d_0, T_{\mathbb{P}^n})$.) Denote by $\mathcal{L}_\alpha \subset \mathscr{U}_\Gamma$ the tautological lifting of $Z_\alpha \subset X_\Gamma$ to $\mathscr{U}_\Gamma$. Let $\mathscr{Z} \subset \mathscr{U}_\Gamma$ be the Zariski closure of $\bigcup_{\alpha\in A} \mathcal{L}_\alpha$. Writing $\mathscr{Z} = \mathscr{Z}_1 \cup \cdots \cup \mathscr{Z}_s$ for the decomposition of $\mathscr{Z}$ into irreducible components and recalling that Z is assumed now to be irreducible, there exists an integer ℓ, $1 \leqslant \ell \leqslant s$, such that $\mu_\Gamma(\mathscr{Z}_\ell) = Z$. Then, $\mu_\Gamma(\mathscr{Z}_\ell) \subset X_\Gamma$ is the Zariski closure Z of $E = \bigcup \big\{\Sigma_\alpha : \alpha \in A\big\}$. Denote by $\mathscr{F}$ the tautological holomorphic foliation on $\mathscr{U}$ and by $\mathscr{F}_\Gamma$ the induced holomorphic foliation on $\mathscr{U}_\Gamma$. In analogy to the proof of Proposition 3.1, there exists a point $e_\ell \in E \cap \mathrm{Reg}(\mathscr{Z}) \cap \mathscr{Z}_\ell$ such that $\mathscr{S}_\ell := \mathcal{T}_{\mathscr{F}_\Gamma}|_{\mathscr{Z}_\ell} + \mathcal{T}_{\mathscr{Z}_\ell} \subset \mathcal{T}_{\mathscr{U}_\Gamma}|_{\mathscr{Z}_\ell}$ is a locally free subsheaf at $e_\ell \in E$ (since the points to be excluded form a quasi-projective subvariety $\mathscr{Q}_\ell \subsetneq \mathscr{Z}_\ell$, which implies that $\mathcal{T}_{\mathscr{F}_\Gamma}|_{\mathscr{Z}_\ell} + \mathcal{T}_{\mathscr{Z}_\ell} = \mathcal{T}_{\mathscr{Z}_\ell}$). It follows that $\mathcal{T}_{\mathscr{F}_\Gamma}|_{\mathscr{Z}_\ell} \subset \mathcal{T}_{\mathscr{Z}_\ell}$ and Z is uniruled by subvarieties belonging to $\mathcal{K}$ in the sense of Definition 3.1, and the proof of Main Theorem shows that Z is a totally geodesic subset, as desired. □

Combining Main Theorem and Theorem 4.1 we conclude with

Corollary 4.1. *Let A be any set of indices and $S_\alpha \subset \mathbb{B}^n \subset \mathbb{P}^n$ be a family of algebraic subsets on the complex unit ball $\mathbb{B}^n$ of positive dimension. Let $\pi : \mathbb{B}^n \to X_\Gamma$ be the universal covering map and define $E := \bigcup \big\{\pi(S_\alpha) : \alpha \in A\big\} \subset X_\Gamma$. Then, the Zariski closure of E in X_Γ is a finite union of totally geodesic subsets.*

Proof. Denote by $Z \subset X_\Gamma$ the Zariski closure of E. For each $\alpha \in A$ denote by $Z_\alpha \subset X_\Gamma$ the Zariski closure of $\pi(S_\alpha) \subset X_\Gamma$. By Main Theorem, for each $\alpha \in A$, $Z_\alpha \subset X_\Gamma$ is a totally geodesic subset. Hence, Z is the Zariski closure of $E' = \bigcup \big\{Z_\alpha : \alpha \in A\big\} \subset X_\Gamma$. By Theorem 4.1, Z is a finite union of totally geodesic subsets, as desired. □

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