ABSOLUTE VALUES OF L-FUNCTIONS FOR $GL(n,\mathbb{R})$ AT THE POINT 1

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Abstract. We study the values of $|L(1,F)|$ for Hecke-Maass cusp forms F on $SL(n,\mathbb{Z})$ ($n \geq 3$) of large Langlands parameters. New unconditional results on the extreme values and conditional results on the size range are derived, which determine precisely the order of magnitude of $L(1,F)$. In addition, we enhance the new average estimate toward the Ramanujan Conjecture due to Matz and Templier. An application of the Hecke multiplicativity to the Littlewood-Richardson rule for a product of two Schur polynomials is cultivated.

Contents

1. Introduction

In the past two decades, there are many great advances in the study of the size of $L(1,f)$ for L-functions over a family of f where the point $s=1$ is on the boundary of the critical strip. The initial research may be attributed to the work of Littlewood [17] and Chowla [3] almost a century ago, investigating the extreme values of the Dirichlet L-functions $L(1,\chi)$ at 1. Later there were works on the distribution of $L(1,\chi_d)$ over a family of quadratic characters by Chowla and Erdös, Elliott, etc. In 1999, Montogmery and Vaughan [24] formulated, based on a probabilistic model, three conjectures about the

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proportion of exceptional χ_d for which $L(1,\chi_d)$ lies outside a certain threshold. Granville and Soundararjan [9] considered a modified probabilistic model and, together with many new ideas and techniques, computed the distribution function

$$\Phi_x(\tau) = \exp\Big(-\frac{e^{\tau-C_1}}{\tau}\Big(1+O\Big(\frac{1}{\tau}\Big)\Big)\Big) \tag{1.1}$$

uniformly for $\tau \le \log_2 x$, where $\Phi_x(\tau)$ denotes the proportion of fundamental discriminants $d \in [-x,x]$ for which $L(1,\chi_d) > e^{\gamma}\tau$, and $C_1 = 0.8187....$

In addition to the Dirichlet L-function (of $GL(1)$ case), the investigation was extended to the L-functions associated to $GL(2)$ automorphic forms as well as their symmetric powers in a series of papers, including Luo [19], Royer [29], Cogdell & Michel [4], Royer & Wu [30], Liu-Royer-Wu [18], Lamzouri [13], etc. Analogues of the extreme value results of Littlewood and Chowla were derived, and some weaken form of the Montgomery-Vaughan conjecture could also be settled. The probabilistic model follows the Sato-Tate law arising from the equidistribution of satake parameters (or eigenvalues) over a family of $GL(2)$ Hecke eigenforms. The equidistribution is ultimately boiled down to the Petersson or Kuznetsov trace formula. Let us review some results in this case.

Suppose f is a Hecke-Maass form for $SL(2,\mathbb{Z})$ of type ν, so its Laplacian eigenvalue and Langlands parameter are $\lambda_f = \nu(1-\nu)$ and $\mu_f = (\nu-\frac12, \frac12-\nu)$. Then $L(1,\mathrm{sym}^m f)$ is the value of its mth symmetric power L-function at 1 where $1 \le m \le 4$. Conditionally (on GRC and GRH)[‡1], there exist positive constants $\mathrm{A}_m^{\pm}$ such that

$$(\log_2 \lambda_f)^{-\mathrm{A}_m^-} \ll L(1,\mathrm{sym}^m f) \ll (\log_2 \lambda_f)^{\mathrm{A}_m^+}$$

as $\lambda_f \to \infty$, and unconditionally there are infinitely many forms $f^{\pm}$ with $\lambda_{f^{\pm}} \to \infty$ for which

$$L(1,\mathrm{sym}^m f^-) \ll (\log_2 \lambda_{f^-})^{-\mathrm{A}_m^-} \quad , \quad L(1,\mathrm{sym}^m f^+) \gg (\log_2 \lambda_{f^+})^{\mathrm{A}_m^+}$$

where $\log_r \cdot$ means $\log(\log_{r-1} \cdot)$. It is computed that for $m = 1,2,3,4$,

$$(\mathrm{A}_m^-, \mathrm{A}_m^+) = (2,2),\ (1,3),\ (4,4),\ (\frac{5}{4}, 5) \tag{1.2}$$

respectively. The figures show an asymmetry in the order of magnitude of $L(1,\mathrm{sym}^m f)^{-1}$ and $L(1,\mathrm{sym}^m f)$ when m is even. Note that $L(s,\mathrm{sym}^m f)$ is an L-function for $GL(m+1)$. Later we shall see that the phenomenon of asymmetry appears in the whole family of L-function for $GL(n)$ (Hecke-Maass forms) when n is odd. Moreover, the harmonic weighted distribution function $\Phi_x(\mathrm{sym}^m,\tau)$ (cf. [13], [35], [34]) is shown to satisfy a formula like (1.1) for τ in a slightly shorter range.

The value of an L-function at 1 may encode properties of other objects. The Dirichlet class number formula expresses the class number $h(d)$ of a quadratic number field in terms of $L(1,\chi_d)$, hence the distribution of class numbers can be understood. Also, the Hecke-Maass form f for $SL_2(\mathbb{Z})$ (considered above) is an eigenfunction of the Laplacian on the hyperbolic manifold $SL_2(\mathbb{Z}) \setminus \mathbb{H}$. Suppose f is normalized with its first Fourier coefficients equal to 1. Then $2\|f\|^2\cosh(\pi|\nu|) = L(1,\mathrm{sym}^2 f)$ where $\|\cdot\|$ is the L^2-norm.

[‡1] GRC and GRH abbreviate Generalized Ramanujan Conjecture and Grand Riemann Hypothesis respectively.

The proportion $\Psi_x^+(\tau)$ (resp. $\Psi_x^-(\tau)$) of $\lambda_f \le x$ for which $\|f\|^2 e^{\pi\sqrt{\lambda_f}} > \tau$ (resp. $< \tau^{-1}$) satisfies

$$\Psi_x^+(\tau) = \exp\left(-\big(c_1 + o(1)\big)\frac{e^{c_2\tau^{1/3}}}{\tau^{1/3}}\right)$$

(resp. $\Psi_x^-(\tau) = \exp(-(c_3 + o(1))e^{c_4\tau}/\tau))$), uniformly for $\tau \le c_5(\log_2 x)^{1/3}$, where $c_i > 0$, $1 \le i \le 5$, are constants. These examples arouse the interest in the values of L-functions at 1 beyond the low rank case of Dirichlet characters or $GL(2)$ automorphic forms.

In this paper, we are concerned with the L-functions for $GL(n)$ associated to the Hecke-Maass cusp forms for $SL(n,\mathbb{Z})$ ($n \ge 3$). Let us follow the set-up in Goldfeld's book [7], and write $\mathcal{H}^\natural$ for the set of all Hecke-Maass cusp forms for $SL(n,\mathbb{Z})$ of L^2-norm 1. Suppose $\phi \in \mathcal{H}^\natural$ is of type $\nu \in \mathbb{C}^{n-1}$. Then its Langlands parameter $\mu_\phi \in \mathbb{C}^n$ is determined (as in (3.1) below) and its associated Laplacian eigenvalue is $\lambda_\phi = \frac{1}{2}(\|\rho\|_2^2 - \|\mu_\phi + \rho\|_2^2)$ where $\rho := (\frac{n+1}{2} - 1, \cdots, \frac{n+1}{2} - n)$ is the half sum of positive roots and $\|\cdot\|_2$ is the Euclidean norm.[‡2] Here we establish new results in the general context – for $|L(1,\phi)|$ with $\phi \in \mathcal{H}^\natural$ – generalizing Littlewood [17] and Chowla [3], Luo [19] and Cogdell & Michel [4].

Briefly speaking, the *exact* order of magnitudes $A_n^\pm$ of the size range are determined for almost all $\phi \in \mathcal{H}^\natural$ whose $\mu_\phi \in \mathtt{i}\mathbb{R}^n$ (i.e. whose λ_ϕ is not in the complementary cuspidal spectrum). Then it reveals that a generic L-function and the symmetric lift $L(s, \mathrm{sym}^{n-1} f)$ (both of which are L-functions for $GL(n)$) attain different extreme small value at 1 for *odd* n. We also compute the distribution functions $\Phi_T^\pm(\mathcal{K},\tau)$, and a result analogous to (1.1) can be generalized to the $GL(n)$ case. To its end, we introduce a probabilistic model

$$L(1,\rho_{\mathrm{St}}) = \textstyle\prod_p \det(I - \rho_{\mathrm{St}}(\underline{\theta}_p)p^{-1})^{-1}$$

where $\underline{\theta}_p$ is a random vector distributed over the subset of the n-torus governed by $\theta_1 + \cdots + \theta_n \equiv 0 \pmod{2\pi}$ according to the p-adic Plancherel measure $d\mu_p$, and the random vectors $\underline{\theta}_p$ (where p runs through all primes) are independent (see §8).

The machinery for the proof comprises many techniques in analytic number theory and probability from the aforementioned papers, with necessary adaption to the more sophisticated case of $GL(n)$. As GRC is open, a more delicate analysis (in comparison with the holomorphic cusp form case) is undertaken; then we may use an average bound towards the Ramanujan Conjecture in lieu. To build a passage from the moments of $|L(1,\phi)|^z$ to the probability moment $\mathbb{E}[|L(1,\rho_{\mathrm{St}})|^z]$, we need an equidistribution result for the satake parameters associated to ϕ. The recent work of Matz & Templier [23] provides the key ingredient in this regard. However there are still obstacles: (i) their bounds towards the Ramanujan Conjecture falls short of our purpose and (ii) the computation of the main term as in Gross [10] seems not effective enough to reveal its arithmetic properties. We provide different treatments and get new findings (in § 7.5-7.8), which is a novelty in our work in addition to the results for $L(1,\phi)$.

For (i), we show the zero density of $\phi \in \mathcal{H}$ failing *marginally* the Ramanujan Conjecture at a fixed non-archimedean place. This refines [23, Corollary 1.6] to a more intimate generalization of Sarnak's result [32], see Theorem 7.3. For (ii), the main term of the unweighted trace formula $\sum_\phi A_\phi(m_1, \cdots, m_{n-1})$ will *vanish* if the product

[‡2] See [11, VIII,§1].

$m_1^{n-1}m_2^{n-2}\cdots m_{n-1}$ is *not* an nth power (see Proposition 7.5), enriching the observation in [23] – the main term is zero when $m_1\cdots m_{n-1}>1$ is squarefree. A consequence is: Specialized to $\sum_{\phi\in\mathcal{H}_T}A_\phi(m,1,\cdots,1)$, the main term is simply $m^{-(n-1)/2}\mathbb{1}_{\boxed{n}}(m)\Lambda(T)$ where $\mathbb{1}_{\boxed{n}}$ is the characteristic function of the set of the nth powers of positive integers and $\Lambda(T)\asymp T^{n(n+1)/2-1}$. Note it is known for the case $n=2$, cf. [15] and [23]. Another consequence is a dainty *product formula* in the spirit of Pieri's rule for *Schur polynomials*, see Corollary 7.8.

2. Statement of results

The set $\mathcal{H}^\natural$ of Hecke-Maass cusp forms ϕ on $GL(n)$ (defined in §3.1) can be separated into two parts according as the Langlands parameter $\mu_\phi\in \mathtt{i}\mathfrak{a}^*\subset \mathtt{i}\mathbb{R}^n$ or not (cf. §7.3). Denote the subset of $\phi\in\mathcal{H}^\natural$ whose $\mu_\phi\in\mathtt{i}\mathfrak{a}^*$ by $\mathcal{H}$. For $T\ge 10^2$, we let

$$\mathcal{H}_T^\natural := \{\phi\in\mathcal{H}^\natural:\ \|\mu_\phi\|_2\le T\}\quad,\quad \mathcal{H}_T:=\{\phi\in\mathcal{H}_T^\natural:\ \mu_\phi\in\mathtt{i}\mathfrak{a}^*\}.$$

The case of $\mu_\phi\notin\mathtt{i}\mathbb{R}^n$ corresponds to the complementary cuspidal spectrum, which is asserted in GRC to be vacant but not yet settled. Nevertheless, Matz and Templier [23, (1.1)] showed that $\#(\mathcal{H}_T)\gg T^d$ where $d=n(n+1)/2-1$. Also $\#(\mathcal{H}_T^\natural)\ll T^d$, cf. [26].

Throughout we let the integer $n\ge 3$ and T be any sufficiently large number. Define

$$A_n^+=n \qquad\text{and}\qquad A_n^-=\begin{cases} n & \text{if } n \text{ is even,}\\ n\cos(\pi/n) & \text{if } n \text{ is odd.}\end{cases}$$

and let $B_n^\pm$ be the positive constants in Lemma 5.3.

Theorem 2.1. *(1) There exists a subset $\mathcal{K}\subset\mathcal{H}$ of full density such that for every $\phi\in\mathcal{K}_T:=\mathcal{K}\cap\mathcal{H}_T$,*

$$(\log_2 T)^{-A_n^-}\ll_n |L(1,\phi)|\ll_n (\log_2 T)^{A_n^+}\ ;$$

the exceptional set satisfies $\#(\mathcal{H}_T\setminus\mathcal{K}_T)\le\#(\mathcal{H}_T)\exp(-(\log T)/(\log_2 T)^{1+o(1)})$.

(2) Assume GRC (so $\mathcal{H}_T^\natural=\mathcal{H}_T$) and GRH. For all $\phi\in\mathcal{H}_T^\natural$, we have

$$\{1+o(1)\}(2B_n^-\log_2 T)^{-A_n^-}\le |L(1,\phi)|\le\{1+o(1)\}(2B_n^+\log_2 T)^{A_n^+}.$$

Theorem 2.2. *There exist $\phi^\pm\in\mathcal{H}_T$ such that*

$$|L(1,\phi^-)|\le\{1+o(1)\}(B_n^-\log_2 T)^{-A_n^-}\quad,\quad |L(1,\phi^+)|\ge\{1+o(1)\}(B_n^+\log_2 T)^{A_n^+}.$$

The proportion of such exceptional $\phi^\pm$ in $\mathcal{H}_T$ is at least $\exp\big(-(\log T)/(\log_2 T)^{3+o(1)}\big)$.

Remark 2.1. 1. For odd $n\ge3$, $A_n^-=n\cos(\pi/n)$ is likely different from A_{n-1}^- in (1.2), which is true for $n=3,5$: $(A_3^-,\mathrm{A}_2^-)=(\frac32,1),(A_5^-,\mathrm{A}_4^-)=(4.045\cdots,\frac54)$.

2. Like the low rank cases, the (conditional) bounds and the extreme value results in Theorems 2.1 and 2.2 differ by a factor of 2 (in front of $B_n^\pm$).

3. Alongside the Montgomery-Vaughan conjecture (cf. Conjecture 1 in [9]), one would predict the proportion of $\phi^\pm$ in $\mathcal{H}_T$ satisfying $|L(1,\phi^\pm)|^{\pm1}\ge(B_n^\pm\log_2 T)^{A_n^\pm}$ respectively is $>\exp(-C\log T/\log_2 T)$ and $<\exp(-c\log T/\log_2 T)$ for some constants $C>c>0$. Theorem 2.2 provides a (better) lower bound after sacrificing an $o(1)$ factor.

Let $\mathcal{K} \subset \mathcal{H}$ and $\mathcal{K}_T = \mathcal{K} \cap \mathcal{H}_T$. Define for $\tau > 0$,

$$\Phi_T^+(\mathcal{K}, \tau) = \frac{\#\{\phi \in \mathcal{K}_T : |L(1,\phi)| > (B_n^+\tau)^{A_n^+}\}}{\#(\mathcal{K}_T)}$$

and

$$\Phi_T^-(\mathcal{K}, \tau) = \frac{\#\{\phi \in \mathcal{K}_T : |L(1,\phi)| < (B_n^-\tau)^{-A_n^-}\}}{\#(\mathcal{K}_T)}.$$

Theorem 2.3. *There exist a subset $\mathcal{K} \subset \mathcal{H}$ of full density and constants $C_n^\pm$ such that*

$$\Phi_T^\pm(\mathcal{K}, \tau) = \exp\Big(- \frac{e^{\tau - C_n^\pm}}{\tau}\Big(1 + O\Big(\frac{1}{\tau}\Big)\Big)\Big)$$

uniformly for $\tau \le \log_2 T - (2 + o(1))\log_3 T$.

Theorems 2.2 and 2.3 are consequences of the complex moment result, Theorem 2.4, which links to the moment of the probabilistic model L-function $L(1, \rho_{\mathrm{St}})$ defined in (8.4).

Theorem 2.4. *There exists a subset $\mathcal{K} \subset \mathcal{H}$ of full density such that for $\mathcal{K}_T := \mathcal{K} \cap \mathcal{H}_T$,*

$$\frac{1}{\#(\mathcal{K}_T)} \sum_{\phi \in \mathcal{K}_T} |L(1,\phi)|^{2z} = \mathbb{E}[|L(1,\rho_{\mathrm{St}})|^{2z}] + O\big(\exp\big(- (\log T)/(\log_2 T)^{2+o(1)}\big)\big)$$

holds uniformly for $|z| \le (\log T)/(\log_2 T)^{2+o(1)}$.

Remark 2.2. 1. By (8.5), explicitly the main term is

$$\mathbb{E}[|L(1,\rho_{\mathrm{St}})|^{2z}] = \prod_p \int_{T_0/\mathfrak{S}_n} \big|\det(I - \rho_{\mathrm{St}}(\underline{\theta})p^{-1})\big|^{-2z}\, d\mu_p$$

which equals $\big((1 + o(1))(B_n^\pm \log z)^{A_n^\pm}\big)^{2z}$ for *real* $z \to \pm\infty$, by Proposition 8.3.

2. Define for $\tau > 0$,

$$\Phi^+(\rho_{\mathrm{St}}, \tau) := \mathrm{Prob}(|L(1,\rho_{\mathrm{St}})| > (B_n^+\tau)^{A_n^+})$$

and

$$\Phi^-(\rho_{\mathrm{St}}, \tau) = \mathrm{Prob}(|L(1,\rho_{\mathrm{St}})| < (B_n^-\tau)^{-A_n^-}).$$

Then we have

$$\Phi^\pm(\rho_{\mathrm{St}}, \tau) = \exp\Big(- \frac{e^{\tau - C_n^\pm}}{\tau}\Big(1 + O\Big(\frac{1}{\tau}\Big)\Big)\Big) \tag{2.1}$$

where $C_n^\pm$ are constants same as in Theorem 2.3, and $\Phi_T^\pm(\mathcal{K}, \tau) \to \Phi^\pm(\rho_{\mathrm{St}}, \tau)$ as $T \to \infty$.

Notation. The symbol p is reserved for (rational) primes, $\mathbb{N} = \{1, 2, \cdots\}$ and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We denote $\mathtt{i} = \sqrt{-1}$, $\sigma = \Re\mathrm{e}\, s$ and $\tau = \Im\mathrm{m}\, s$ for the complex variable s. (But $\tau(m)$ is the Dirichlet divisor function.) A vector is sometimes underlined or written in bold face, the former (e.g. $\underline{\theta}$) is designated to have n coordinates while the latter (e.g. $\boldsymbol{k}$) has $n-1$ coordinates. For $\boldsymbol{k} = (k_1, \cdots, k_{n-1}) \in \mathbb{N}_0^{n-1}$, we write $|\boldsymbol{k}| := \sum_j k_j$ and $\|\boldsymbol{k}\| := \sum_j (n-j)k_j$. An m-tuple $(a, \cdots, a)$ may be abbreviated as (a_m) or a_m. An unspecified positive constant is denoted by $\mathtt{c}, \mathtt{c}'$, etc, whose values may differ at each occurrence; we may write $\mathtt{c}_*$ and $\mathtt{c}'_*$ in order to emphasize the dependence on $*$. The vinogradov symbol $\ll_*$ is used whenever its dependence on $*$ would be emphasized.

Organization of the paper. Sections 3 and 4 provide the background of Hecke-Maass cusp forms and the L-functions as well as some basic facts using classical L-function techniques. Section 5 gives an approximation to $\log L(s,\phi)$ conditionally on some weak forms of GRC and GRH, then we bound $\log L(1,\phi)$ for random Satake parameters and figure out the subset $\mathcal{K}_T$ of $\mathcal{H}_T$ for unconditional results – completing most of Theorem 2.1. Section 6 evaluates the exceptional set $\mathcal{H}_T \setminus \mathcal{K}_T$ using a zero density estimate and average bounds toward the Ramanujan Conjecture. The zero density result is derived with Montgomery's zero detection method and the large sieve inequality. Section 7 gives expositions on the Schur polynomials and the Littlewood-Richardson rule for $GL(n)$ forms, Weyl's integration formulas and the Plancherel measure for $PGL_n(\mathbb{Q}_p)$, and the novel automorphic Plancherel density theorem for $GL(n)$ (of Matz and Templier). Two applications – on the Ramanujan Conjecture and a case of Pieri's formula for the Schur polynomial – will be cultivated, exploiting the Satake isomorphism, the Hall-Littlewood polynomials and the Hecke multiplicativity of Fourier coefficients. Sections 8 and 9 are based on the essence of moment methods in [4, 30, 14]. Section 8 is devoted to the probabilistic model $L(s,\rho_{\mathrm{St}})$ and study the complex moments of $L(1,\rho_{\mathrm{St}})$ using tools in Section 7. In Section 9, we relate the complex moments of $L(1,\phi)$ to the probabilistic complex moments via the (consequence of) automorphic Plancherel density theorem, and complete the proofs of Theorems 2.2-2.4.

3. Background

The main reference is Goldfeld's book [7]. Let $\Gamma := SL(n,\mathbb{Z})$, $G := GL(n,\mathbb{R})$, $K := O(n,\mathbb{R})$ and $\mathfrak{h}^n := G/(K\cdot\mathbb{R}^\times)$. Let $L^2(\Gamma\setminus\mathfrak{h}^n)$ be the Hilbert space of square integrable functions on $\Gamma\setminus\mathfrak{h}^n$. A Maass cusp form f for Γ of type ν ($\in \mathbb{C}^{n-1}$) is, cf. [7, §5.1], a smooth function $f\in L^2(\Gamma\setminus\mathfrak{h}^n)$ such that f is a common eigenfunction of the differential operators corresponding to elements in the center $\mathcal{D}^n$ of the universal enveloping algebra of $\mathfrak{gl}(n,\mathbb{R})$, and f satisfies the automorphic condition and the cuspidal condition.

Associated to $\nu = (\nu_1,\cdots,\nu_{n-1}) \in \mathbb{C}^{n-1}$, the Langlands parameters $\mu_1(\nu),\cdots,\mu_n(\nu)\in\mathbb{C}^n$, cf. [8, §2], (which are also the parameters introduced in [7, (11.6.15)]) are defined as

$$\mu_i(\nu) := B_{n-i}(\nu) - B_{n-i+1}(\nu) + i - \frac{n+1}{2} = B_{n-i}(\nu - \frac{1}{n}) - B_{n-i+1}(\nu-\frac{1}{n}) \tag{3.1}$$

where $B_0(\nu) = B_n(\nu) := 0$,

$$B_i(\nu) := i\sum_{1\le j\le n-i} j\nu_j + (n-i)\sum_{n-i<j\le n-1}(n-j)\nu_j \qquad (1\le i\le n-1)$$

(with empty sums meaning 0) and $\nu - \frac{1}{n} := \nu - \frac{1}{n}(1,\cdots,1)$. Clearly,

$$B_{n-i}(\nu) = (\mu_1(\nu)+\cdots+\mu_i(\nu)) + \frac{i(n-i)}{2} \quad , \quad \sum_{i=1}^n \mu_i(\nu) = 0. \tag{3.2}$$

Every Maass cusp form f has a Fourier expansion (cf. [7, (9.1.2)]). For $\epsilon_i \in \{\pm1\}$, let $\delta = \mathrm{diag}(\epsilon_1,\cdots,\epsilon_{n-1},1)$ and define the operator $T_\delta f(z) := f(\delta z\delta)$. Suppose the Maass form f is a common eigenfunction of all T_δ's. The Fourier series of f is a multiple sum over $m_1,\cdots,m_{n-1}\ge 1$ with the Fourier coefficient $\mathrm{A}_f(m_1,\cdots,m_{n-1})$. Let $w = (w_{ij}) \in \Gamma$

satisfy $w_{ij} = 0$ unless $i + j = n + 1$ and $w_{i,n-i+1} = 1$ for $i = 2, \cdots, n$. For any Maass cusp form f of type $\nu = (\nu_1, \cdots, \nu_{n-1})$, its dual Maass form $\widetilde{f}$ defined as

$$\widetilde{f}(\mathrm{z}) = \big(w \cdot f\big)(\mathrm{z}) := f(w\, {}^{\mathrm{t}}\mathrm{z}^{-1} w)$$

is a Maass cusp form of type $\widetilde{\nu} := (\nu_{n-1}, \cdots, \nu_1)$ where ${}^{\mathrm{t}}\mathrm{z}^{-1}$ is the inverse of the transpose of z. If f is a common function of all T_δ's, then so is $\widetilde{f}$. From [7, Proposition 9.2.1], we have

$$\mathrm{A}_{\widetilde{f}}(m_1, \cdots, m_{n-1}) = \mathrm{A}_f(m_{n-1}, \cdots, m_1).$$

Remark 3.1. $\langle w \cdot f, g\rangle = \langle f, w \cdot g\rangle$, thus $\langle \widetilde{f}, \widetilde{g}\rangle = \langle f, g\rangle$ where $\langle\, ,\, \rangle$ is the inner product in $L^2(\Gamma \setminus \mathfrak{h}^n)$.

3.1. **Hecke-Maass cusp forms.** Let $\mathcal{R}$ be the full Hecke ring for Γ, see [7, §9.3 & §3.10]. The adjoint operator T^* of any $T \in \mathcal{R}$ also lies in $\mathcal{R}$ and commutes with T. In fact $\mathcal{R}$ is a commutative family of normal operators which commute with $D \in \mathcal{D}^n$ and T_δ's as well. The (nonzero) Maass cusp forms that are common eigenfunctions of all $T \in \mathcal{R}$, $D \in \mathcal{D}^n$ and T_δ's are called Hecke-Maass cusp forms. A Hecke-Maass cusp form ϕ is said to even or odd according as $T_{\delta_0}\phi = \pm\phi$ where $\delta_0 = (-1, 1, \cdots, 1)$.

Let $\mathcal{H}^\natural = \{\phi_j\}$ be the orthonormal basis for $L^2_{\mathrm{cusp}}(\Gamma \setminus \mathfrak{h}^n)$ consisting of Hecke-Maass cusp forms. (See [7, p.357] for the spectral decomposition.) The eigenvalues of ϕ under the Hecke operators T_m in $\mathcal{R}$, $m \geq 1$, are scalar multiples of Fourier coefficients, cf. [7, (9.3.5) and Theorem 9.3.11]. More specifically, for all $m \geq 1$, if $T_m\phi = A_\phi(m, 1, \cdots, 1)\phi$, then $\mathrm{A}_\phi(m, 1, \cdots, 1) = c_\phi A_\phi(m, 1, \cdots, 1)$ where $c_\phi \in \mathbb{C}^\times$ is a scalar independent of m.

Let $A_\phi(m_1, \cdots, m_{n-1}) = c_\phi^{-1}\mathrm{A}_\phi(m_1, \cdots, m_{n-1})$ be the normalized Fourier coefficients. We have the following Hecke multiplicative relations:

$$A_\phi(m_1, \cdots, m_{n-1})A_\phi(m_1', \cdots, m_{n-1}') = A_\phi(m_1m_1', \cdots, m_{n-1}m_{n-1}')$$

if $(m_1 \cdots m_{n-1}, m_1' \cdots, m_{n-1}') = 1$ and

$$\begin{aligned} &A_\phi(m, 1, \cdots, 1)A_\phi(m_1, \cdots, m_{n-1}) \\ &= \sum_{\substack{\prod_{\ell=1}^n c_\ell = m \\ c_1|m_1, c_2|m_2, \cdots, c_{n-1}|m_{n-1}}} A_\phi\left(\frac{m_1c_n}{c_1}, \frac{m_2c_1}{c_2}, \cdots, \frac{m_{n-1}c_{n-2}}{c_{n-1}}\right). \end{aligned} \tag{3.3}$$

According as ϕ is even or odd, $A_\phi(m_1, \cdots, m_{n-2}, -m_{n-1}) = \pm A_\phi(m_1, \cdots, m_{n-2}, m_{n-1})$, and for odd n, all Hecke-Maass cusp forms ϕ_j's are even, cf. [7, Propositions 9.2.5-6].

The dual $\widetilde{\phi}$ of a Hecke-Maass cusp form ϕ is also a Hecke-Maass cusp form (so $\widetilde{\phi} \in \mathcal{H}^\natural$ by Strong Multiplicity One Theorem [7, Theorem 12.6.1]). In general, if f is a common eigenfunction of all $T \in \mathcal{R}$ and $Tf = \lambda_T f$, then from $T^*\widetilde{f} = \widetilde{Tf}$ for all $T \in \mathcal{R}$, so is its dual $\widetilde{f}$ and $T\widetilde{f} = \overline{\lambda_T}\widetilde{f}$. Let $A_{\widetilde{\phi}}(m_1, \cdots, m_{n-1})$ be the normalized Fourier coefficient of its dual $\widetilde{\phi}$. By the addendum after [7, Theorem 9.3.11], we have

$$A_{\widetilde{\phi}}(m_1, \cdots, m_{n-1}) = A_\phi(m_{n-1}, \cdots, m_1) = \overline{A_\phi(m_1, \cdots, m_{n-1})}. \tag{3.4}$$

3.2. **L-functions associated to Hecke-Maass cusp forms.** Let ϕ be a Hecke-Maass cusp form of type ν and $A_\phi(m,1,\cdots,1)$ be the eigenvalue under T_m. Its L-function defined as

$$L(s,\phi)=\sum_{m\geq 1}A_\phi(m,1,\cdots,1)m^{-s},$$

for $\Re e\, s>(n+1)/2$, factors into the Euler product

$$L(s,\phi)=\prod_p\prod_{i=1}^n(1-\alpha_{\phi,i}(p)p^{-s})^{-1}$$

where $\alpha_{\phi,i}(p)$, $1\leq i\leq n$, are the Satake parameters. Moreover $L(s,\phi)$ extends to an entire function and its completed L-function

$$\Lambda(s,\phi):=\pi^{-\frac{ns}{2}}\prod_{i=1}^n\Gamma\left(\frac{s+\mu_i(\nu)}{2}\right)L(s,\phi)$$

($\mu_i(\nu)$, $1\leq i\leq n$, are the Langlands parameters) satisfies the functional equation

$$\Lambda(s,\phi)=\varepsilon(\phi)\Lambda(1-s,\widetilde{\phi})$$

where $\varepsilon(\phi)$ is the root number, which is of modulus one, and $\widetilde{\phi}$ is the dual Maass form of ϕ.

Remark 3.2. (1) The Satake parameters $\alpha_{\phi,i}(p)$, abbreviated as $\alpha_{p,i}$, satisfy

(3.5)
$$\prod_{i=1}^n(1-\alpha_{p,i}X)$$
$$=1-A(p,1,\cdots,1)X+A(1,p,\cdots,1)X^2-\cdots+(-1)^{n-1}A(1,\cdots,1,p)X^{(n-1)}+(-1)^nX^n.$$

In particular, $\prod_{i=1}^n\alpha_{\phi,i}(p)=1$. For $\Re e\, s\gg 1$ (meaning sufficiently large $\Re e\, s$),

$$L(s,\phi)^{-1}=\sum_{m_1,\cdots,m_n\geq 1}A(m_1,\cdots,m_{n-1})\frac{\mu_n(m_1,\cdots,m_n)}{(m_1m_2^2\cdots m_n^n)^s}$$

where $\mu_n(m_1,\cdots,m_n):=|\mu(m_1\cdots m_n)|\mu(m_1)\mu(m_2)^2\cdots\mu(m_n)^n$. ($\mu$ is the Möbius function.)

(2) The Satake parameters in the pth local factor of $L(s,\widetilde{\phi})$ satisfy $\{\alpha_{\widetilde{\phi},i}(p):\ 1\leq i\leq n\}=\{\overline{\alpha_{\phi,i}(p)}:\ 1\leq i\leq n\}$ by (3.4) and (3.5). Suppose $\widetilde{\phi}$ is of type $\widetilde{\nu}$. For $0\leq i\leq n$, $B_i(\widetilde{\nu})=B_{n-i}(\nu)$ and thus for $1\leq i\leq n$,

$$\mu_i(\widetilde{\nu})=-\mu_{n+1-i}(\nu). \tag{3.6}$$

Given two Hecke-Maass cusp forms ϕ and ϕ'. Let $\alpha_{p,i}$ and μ_i, resp. $\alpha'_{p,i}$ and μ'_i, be the Satake and Langlands parameters of ϕ, resp. ϕ'. Define the Rankin-Selberg L-function

$$\begin{aligned}L(s,\phi\times\phi') &:= \zeta(ns)\sum_{m_1,\cdots,m_{n-1}\geq 1}\frac{A(m_1,\cdots,m_{n-1})A'(m_1,\cdots,m_{n-1})}{(m_1^{n-1}m_2^{n-2}\cdots m_{n-1})^s}\\ &= \prod_p\prod_{i=1}^n\prod_{j=1}^n\left(1-\frac{\alpha_{p,i}\alpha'_{p,j}}{p^s}\right)^{-1}\end{aligned}$$

where $\Re\mathrm{e}\, s \gg 1$. It extends to a meromorphic function with at most a simple pole of residue $c \cdot \langle \phi, \widetilde{\phi'} \rangle$ at $s = 1$, see Remark 3.3. Moreover the following functional equation holds:

$$\Lambda(s, \phi \times \phi') = \varepsilon(\phi \times \phi')\Lambda(1-s, \widetilde{\phi} \times \widetilde{\phi'})$$

where $\varepsilon(\phi \times \phi')$ denotes the root number and

$$\Lambda(s, \phi \times \phi') := \pi^{-\frac{n^2 s}{2}} \prod_{i=1}^{n} \prod_{j=1}^{n} \Gamma\left(\frac{s + \mu_i - \overline{\mu_j'}}{2}\right) L(s, \phi \times \phi').$$

Remark 3.3. (1) Here the definition of $L(s, f \times g)$ is $L_{f\times\widetilde{g}}(s)$ (not $L_{f\times g}(s)$) in [7, Def 12.1.2]. The above functional equation follows from $\langle f, \widetilde{g}E_P(\cdot, \overline{s})\rangle$ and the functional equation of $E_P(z,s)$ ([7, Prop 10.7.5]). In view of the Laurent expansion of $E_P(z,s)$ at $s = 1$ in [1, Theorem 1], the residue of $\langle f, \widetilde{g}E_P(\cdot, \overline{s})\rangle$ is $c_n\langle f, \widetilde{g}\rangle$ which is zero if there is no pole at $s = 1$. Consequently, we have

$$\operatorname*{res}_{s=1} L(s, \phi \times \phi') = c_n' \langle \phi, \widetilde{\phi'} \rangle \cdot \prod_{i=1}^{n} \prod_{j=1}^{n} \Gamma\left(\frac{1 + \mu_i - \overline{\mu_j'}}{2}\right)^{-1}.$$

Both c_n and c_n' denote some constants that depend on n only.

(2) In case $\phi' = \widetilde{\phi}$, $L(s, \phi \times \widetilde{\phi})$ has a simple pole at $s = 1$ and its Dirichlet series converges absolutely in $\Re\mathrm{e}\, s > 1$ (by [7, Remark 12.1.8]); moreover, $\Lambda(s, \phi \times \widetilde{\phi}) = \varepsilon(\phi \times \widetilde{\phi})\Lambda(1-s, \widetilde{\phi} \times \phi)$ where (with (3.6))

$$\Lambda(s, \phi \times \widetilde{\phi}) = \pi^{-\frac{n^2 s}{2}} \prod_{i=1}^{n} \prod_{j=1}^{n} \Gamma\left(\frac{s + \mu_i(\nu) + \overline{\mu_j(\nu)}}{2}\right) L(s, \phi \times \widetilde{\phi}).$$

(cf. [8, (4.2)].)

4. Basic facts for L-functions

Let $\phi, \phi' \in \mathcal{H}^\natural$ be Hecke-Maass cusp forms of type ν and ν', and denote their Satake and Langlands parameters by $\alpha_{p,i}, \mu_i := \mu_i(\nu)$ and $\alpha_{p,i}', \mu_i' := \mu_i(\nu')$, $i = 1, \cdots, n$. We define the analytic conductors $C_\phi(\tau)$ and $C_{\phi\times\phi'}(\tau)$ for the L-functions $L(s,\phi)$ and $L(s, \phi \times \phi')$ to be

$$C_\phi(\tau) = \prod_{i=1}^{n}(3 + |\tau + \mu_i|) \quad \text{and} \quad C_{\phi\times\phi'}(\tau) = \prod_{i=1}^{n}\prod_{j=1}^{n}(3 + |\tau + \mu_i + \mu_j'|).$$

As $\sum_i \mu_i = 0$, see (3.2), we have $C_\phi(\tau) \le C_{\phi\times\phi'}(\tau)$. We write C_* for $C_*(0)$. Clearly $C_\phi = C_{\widetilde{\phi}}$ and $C_{\phi\times\phi'} \ll (C_\phi C_{\phi'})^n$.

4.1. **Convexity bounds.** In [16], Li studied the size of $L(1)$ for a wide class of L-functions (whose critical strips are $0 \le \Re\mathrm{e}\, s \le 1$). In particular, his results lead to: for $1 \le \sigma \le 3$,

$$\begin{aligned} L(\sigma, \phi) &\ll \exp\left(\mathsf{c}_n \cdot \sqrt{\log C_\phi}\right), \\ (\sigma - 1)L(\sigma, \phi \times \widetilde{\phi}) &\ll \exp\left(\mathsf{c}_n' \cdot \frac{\log C_\phi}{\log_2 C_\phi}\right) \end{aligned} \tag{4.1}$$

where both constants $\mathtt{c}_n, \mathtt{c}'_n > 0$ depend on n only, and the implied constants are independent of σ. With Cauchy-Schwarz's inequality, one easily sees that $L(s, \phi \times \phi') \ll_\varepsilon \left(C_{\phi\times\widetilde{\phi}} C_{\phi'\times\widetilde{\phi}'}\right)^\varepsilon \ll C_{\phi\times\phi'}{}^{4n\varepsilon}$ and $L(s,\phi) \ll_\varepsilon C_{\phi\times\widetilde{\phi}}{}^\varepsilon \ll C_\phi{}^{n\varepsilon}$ on the line $\Re\mathrm{e}\, s = 1+\varepsilon$, for any $\varepsilon > 0$. The following proposition is derived from the standard argument with convexity principle.

Proposition 4.1. *Let $\varepsilon > 0$ be arbitrarily small. For any $-\varepsilon \le \sigma \le 1+\varepsilon$ and $\tau \in \mathbb{R}$, we have*

$$L(s,\phi) \ll_\varepsilon C_\phi{}^\varepsilon C_\phi(\tau)^{\frac{1}{2}(1-\sigma)+\varepsilon} \quad , \quad (s-1)L(s,\phi\times\phi') \ll_\varepsilon (1+|\tau|) C_{\phi\times\phi'}{}^\varepsilon C_{\phi\times\phi'}(\tau)^{\frac{1}{2}(1-\sigma)+\varepsilon}.$$

Remark 4.1. For its proof, we need (3.2) and the following estimate: Recall the Stirling formula $\Gamma(\sigma + i\tau) \asymp |\tau|^{\sigma-1/2}\exp(-\pi|\tau|/2)$ if $|\tau| \ge 1$ and $\sigma \ll 1$. Suppose $|\Re\mathrm{e}\,\alpha| < 1/2$ and $0 < \varepsilon < 1$. Then

$$\frac{\Gamma((1-s-\alpha)/2)}{\Gamma((s+\alpha)/2)} \asymp (1+|\tau+\alpha|)^{\frac{1}{2}-\sigma-\Re\mathrm{e}\,\alpha}$$

for $s = \sigma + \mathtt{i}\tau$ with $-\varepsilon < \sigma < 1+\varepsilon$ and $\tau \in \mathbb{R}$, and lying away from the poles.

4.2. **The number of zeros of** $L(s,\phi)$**.** Let $\sigma \ge 1/2$ and $H \ge 1$ be any number. A generic zero of $L(s,\phi)$ (in the critical strip) is written as $\rho = \beta + \mathtt{i}\gamma$. For any Hecke-Maass cusp form ϕ, we define

$$N(\sigma, H, \phi) := \#\{\rho = \beta + \mathtt{i}\gamma : \beta \ge \sigma, |\gamma| \le H, L(\rho,\phi) = 0\}.$$

Then $N(1, H, \phi) = 0$ for all $H \ge 1$, and we have the following typical result.

Lemma 4.2. *For any $h \ge 1$,*

$$N(\tfrac{1}{2}, h+1, \phi) - N(\tfrac{1}{2}, h, \phi) \ll \log C_\phi(h)$$

where the implied constant is absolute.

Proof. This follows from [16, (12)] with the choice $s = \frac{3}{2} + \mathtt{i}h$. □

4.3. **The logarithms** $\log L(s,\phi)$ **and** $\log L(s, \phi\times\widetilde{\phi})$**.** From the Euler products, we may write

$$\log L(s,\phi) = \sum_{m\ge 2} \frac{\Lambda_\phi(m)}{m^s \log m} \qquad \text{and} \qquad \log L(s,\phi\times\widetilde{\phi}) = \sum_{m\ge 2} \frac{\Lambda_{\phi\times\widetilde{\phi}}(m)}{m^s\log m} \tag{4.2}$$

when $\Re\mathrm{e}\, s \gg 1$. Clearly the sums run over prime powers, and we may express $\Lambda_\phi(p^j)$ and $\Lambda_{\phi\times\widetilde{\phi}}(p^j)$ in terms of the Satake parameters:

$$\frac{\Lambda_\phi(p^j)}{\log p} = \sum_{i=1}^n \alpha_{p,i}^j, \qquad \frac{\Lambda_{\phi\times\widetilde{\phi}}(p^j)}{\log p} = \left|\sum_{i=1}^n \alpha_{p,i}^j\right|^2. \tag{4.3}$$

Thus, we see that $\Lambda_{\phi\times\widetilde{\phi}}(p^j)\log p = |\Lambda_\phi(p^j)|^2$ and, in light of (4.1), both series in (4.2) are absolutely convergent in $\Re\mathrm{e}\, s > 1$. To estimate the size of $\log L(s,\phi)$ where $\Re\mathrm{e}\, s$ is close to 1, a classical approach uses Borel-Carathedory theorem and the convexity bound in Proposition 4.1. Let $\tau \in \mathbb{R}$ and $\frac{1}{2} < \sigma_0 \le 1$ be any numbers. Suppose $L(s,\phi)$ has no zero in the rectangle with corners at $1-\delta+\mathtt{i}(\tau\pm 2\sqrt{\delta})$, $1+\mathtt{i}(\tau\pm 2\sqrt{\delta})$ where $\delta=1-\sigma_0$. Then for $\sigma_0<\sigma<\frac{3}{2}$,

$$\log L(\sigma+\mathtt{i}\tau,\phi)\ll\frac{\log C_\phi}{\sigma-\sigma_0} \tag{4.4}$$

where the implied constant is independent of σ and σ_0. Apply Borel-Carathedory to the disk centred at $3/2+\mathtt{i}\tau$ with radius $1+\delta$, the same result holds for $\log L(s,\phi\times\widetilde{\phi})$.

5. Conditional bounds for $\log L(s,\phi)$ and Proof of Theorem 2.1

5.1. **Conditional results for** $\log L(s,\phi)$**.** Suppose $L(s,\phi)$ has no zero in a sufficiently large region around $s=1$. The estimate in (4.4) will be refined, but we are unable to get the expected tight bound unless some weak form of GRC for non-archimedean places is imposed (cf. (7.6)). Thus we introduce a Weak Ramanujan Conjecture condition to ϕ:

$$\mathbf{WRC}_\varrho(Y):\ \log\max_{1\le i\le n}|\alpha_{\phi,i}(p)|\le\varrho,\ \forall\ p\le Y$$

where $\varrho>0$.

Lemma 5.1. *Let $\eta\in(0,\frac{1}{10^2})$ be any fixed constant. Suppose $X\ge(\log C_\phi)^{4/\eta}$ and $0<\varrho\le\eta^{-1}$. Assume the Hecke Maass cusp form ϕ satisfies*

(1) $\mathbf{WRC}_\varrho(X^{3/2})$, and
(2) $L(s,\phi)$ has no zero in the rectangular region $1-\eta\le\sigma\le 1$, $|\tau|\le 4X$.

Uniformly for $\sigma\ge 1-\frac{1}{2}\eta$ and $|\tau|\le X$, we have

$$\log L(s,\phi)\ll_\eta\frac{X^{1-\sigma}-1}{(1-\sigma)\log_2 X}+\log_2 X$$

where the implied constant depends on η only.

Proof. Let $\sigma_0=1-\frac{3}{4}\eta$. Using the Mellin transform pair e^{-x} and $\Gamma(s)$, we deduce that

$$\log L(s,\phi)=\sum_{m=2}^{\infty}\frac{\Lambda_\phi(m)}{m^s\log m}e^{-m/X}+O_\eta(X^{-(\sigma-\sigma_0)}\log C_\phi) \tag{5.1}$$

for $1-\frac{1}{2}\eta\le\sigma\le\frac{3}{2}$ and $|\tau|\le X$. (See the proof of [14, Proposition 3.4] for details.)

Under $\mathbf{WRC}_\varrho(X^{3/2})$, we have $|\Lambda_\phi(p^j)|\le ne^{j\varrho}\log p$ for $p\le X^{3/2}$, otherwise, we apply the trivial bound $|\Lambda_\phi(p^j)|\le np^{j/2}$. Thus the result follows as the sum over m in (5.1) is

$$\ll\sum_p p^{-\sigma}e^{-p/X}\ll\frac{X^{1-\sigma}-1}{(1-\sigma)\log X}+\log_2 X.$$

For $\sigma\ge 1$ the last inequality is obvious, and for $\sigma\le 1$, it is handled in [14, (3.20)-(3.22)] with

$$\sum_{p\le y}\frac{1}{p^\sigma}\ll\frac{y^{1-\sigma}-1}{(1-\sigma)\log y}+\log_2 y. \tag{5.2}$$

$\square$

Lemma 5.2. *Under the same assumptions in Lemma 5.1, we have*

$$\log L(1,\phi) = -\sum_{p\le X}\log\prod_{i=1}^{n}\Big(1-\frac{\alpha_{\phi,i}(p)}{p}\Big)+O(\frac{1}{\log_2 C_\phi}). \tag{5.3}$$

Proof. It follows from (5.1) with $X=(\log C_\phi)^{4/\eta}$ that

$$\log L(1,\phi)=\sum_{m=2}^{\infty}\frac{\Lambda_\phi(m)}{m\log m}e^{-m/X}+O_\eta((\log C_\phi)^{-2})$$

With $\mathbf{WRC}_\varrho(X^{3/2})$ (yielding $\Lambda_\phi(m)\ll\Lambda(m)$ for $m\le X^{3/2}$) or the trivial bound for the Satake parameters, we infer that

$$\sum_{m\ge X}\frac{|\Lambda_\phi(m)|}{m\log m}e^{-m/X}\ll\sum_{\substack{p,j\\ p^j>X}}\frac{e^{-p^j/X}}{jp^j}+X^{-1}\ll\sum_{p>X}\frac{e^{-p/X}}{p}+X^{-1}\ll\frac{1}{\log X}.$$

Replacing $e^{-p^j/X}$ with $e^{-jp/X}$ for $j\ge 2$ and noting $e^{-p^j/X}-e^{-jp/X}\ll\min(1,p^j/X)$, this results in

$$\log L(1,\phi)\ =\ \sum_{p\le X}\sum_{j\ge1}\frac{\Lambda_\phi(p^j)}{p^j\log p^j}e^{-jp/X}+O(\frac{1}{\log X}).$$

Using (4.3), one finds that

$$\begin{aligned}\sum_{j\ge1}\frac{\Lambda_\phi(p^j)}{p^j\log p^j}e^{-jp/X}\ &=\ \sum_{j\ge1}\frac{(pe^{p/X})^{-j}}{j}\sum_{i=1}^{n}\alpha_{\phi,i}(p)^j=\log\prod_{i=1}^{n}\Big(1-\frac{\alpha_{\phi,i}(p)}{pe^{p/X}}\Big)^{-1}\\ &=\ -\log\prod_{i=1}^{n}\Big(1-\frac{\alpha_{\phi,i}(p)}{p}\Big)+O\Big(\frac{1-e^{-p/X}}{p}\Big).\end{aligned}$$

Thus the sum of the O-terms over $p\le X$ is at most $O(1/(\log X))$. □

Remark 5.1. Suppose both GRC and GRH are true. Lemma 5.1 implies $\log L(s,\phi)\ll\log_3 C_\phi$ on $\Re e\, s=1$. Write $\delta=1/(2\log_2 C_\phi)$. Taking $\sigma_0=\frac12+\delta$ and $\sigma=\frac12+2\delta$ in (4.4), we obtain from the convexity principle that

$$\log L(\sigma+\mathtt{i}\tau,\phi)\ll_\varepsilon(\log C_\phi)^{2(1-\sigma)}\log_2^2 C_\phi$$

uniformly for $\frac12+\delta\le\sigma\le1$ and $\tau\in\mathbb{R}$, and then (5.1) holds for all $X>2$ with the O-term replaced by $O(X^{1/2+\delta-\sigma}(\log C_\phi)^{1-2\delta}\log_2^2 C_\phi)$. Then (5.3) holds for $X=(\log C_\phi)^2\log_2^8 C_\phi=(\log C_\phi)^{2+o(1)}$.

5.2. **Bounding a short sum.** In view of (5.3), it remains to deal with the sum over $p\le X$. If GRC holds, we shall have $|\alpha_{\phi,j}(p)|=1$ and thus model $\alpha_{\phi,j}(p)$ with $e^{\mathtt{i}\theta_j}$ where $\theta_j\in[0,2\pi)$. As $\prod_{j=1}^n\alpha_{\phi,j}(p)=1$, we shall estimate for the case of random θ_j's subject to $\sum_{j=1}^n\theta_j=0$. Write $\mathbb{T}^r=[0,2\pi)^r$ for a r-torus ($r=n-1$ or n). For $\underline{\theta}\in\mathbb{T}^n$, we set

$$g(\underline{\theta})=g(\theta_1,\cdots,\theta_n):=\sum_{j=1}^{n}\cos\theta_j$$

and

$$f_p(\underline{\theta}):=\log\prod_{j=1}^{n}\Big|1-\frac{e^{\mathtt{i}\theta_j}}{p}\Big|^{-2}=-\sum_{j=1}^{n}\log(1-2p^{-1}\cos\theta_j+p^{-2}).$$

Let $\underline{\theta}^{\pm}$ and $\underline{\varsigma}_p^{\pm} \in \mathbb{T}^n$ be points at which g and f_p attain their maximum/minimum, i.e.

$$g(\underline{\theta}^{\pm}) = \max_{\substack{\underline{\theta}\in\mathbb{T}^n \\ \theta_1+\cdots+\theta_n=0}} / \min \, g(\underline{\theta}), \quad f_p(\underline{\varsigma}_p^{\pm}) = \max_{\substack{\underline{\theta}\in\mathbb{T}^n \\ \theta_1+\cdots+\theta_n=0}} / \min \, f_p(\underline{\theta}).$$

Lemma 5.3. *Let $X > 3$ be any real number. Then*

$$\max_{\substack{\underline{\theta}\in\mathbb{T}^n \\ \theta_1+\cdots+\theta_n=0}} \Big(\pm \sum_{p\le X} f_p(\underline{\theta}) \Big) = 2A_n^{\pm} \log\big(B_n^{\pm} \log X\big) + O\Big(\frac{1}{\log^3 X}\Big)$$

where $A_n^+ = n$ for all $n \ge 3$, $A_n^- = n$ or $n\cos(\pi/n)$ for even or odd $n \ge 3$ respectively, and $B_n^{\pm}$ are some positive constants.

Proof. Subject to the constraint $\theta_1 + \cdots + \theta_n = 0$, the two functions g and f_p on $\mathbb{T}^n$ may be viewed as functions on $\mathbb{T}^{n-1}$. Suppose $\boldsymbol{\theta}^{\pm} \in \mathbb{T}^{n-1}$ is projected to $\underline{\theta}^{\pm}$. We may take, by inspection, $\boldsymbol{\theta}^+$ to be $(0, \cdots, 0)$, and for even n, $\boldsymbol{\theta}^- = (\pi, \cdots, \pi)$. But for odd n, since $\boldsymbol{\theta}^-$ must be a critical point, one gets $\sin\theta_j = \sin\theta_n$, $\forall\, j$, where $\sum_{j=1}^n \theta_j = 0$. Among all the candidates, it is seen that $\boldsymbol{\theta}^- = ((n-1)\pi/n, \cdots, (n-1)\pi/n) \in \mathbb{T}^{n-1}$ yields the minimum. So $g(\underline{\theta}^+) = n$ and $g(\underline{\theta}^-) = -n$ or $-n\cos(\pi/n)$ according as n is even or odd.

Clearly, $2n\log(1+p^{-1})^{-1} \le f_p(\underline{\theta}) \le 2n\log(1-p^{-1})^{-1}$, we hence observe that $\underline{\varsigma}_p^+ = \underline{\theta}^+$ for all p, and if $2|n$, we have $\underline{\varsigma}_p^- = \underline{\theta}^-$ as well. For the remaining case, we write

$$f(\underline{\theta}) = \frac{2g(\underline{\theta})}{p} - h_p(\underline{\theta})$$

where

$$h_p(\underline{\theta}) = \sum_{j=1}^n \big(\log(1 - 2p^{-1}\cos\theta_j + p^{-2}) + 2p^{-1}\cos\theta_j \big) \ll_n p^{-2}.$$

The inequality

$$\frac{2g(\underline{\theta}^-)}{p} + O(p^{-2}) = f(\underline{\theta}^-) \ge f(\underline{\varsigma}_p^-) = \frac{2g(\underline{\varsigma}_p^-)}{p} + O(p^{-2}),$$

implies

$$g(\underline{\varsigma}_p^-) - O(p^{-1}) \le g(\underline{\theta}^-) \le g(\underline{\varsigma}_p^-)$$

(whose second inequality follows from the definition of $\underline{\theta}^-$), i.e. $g(\underline{\varsigma}_p^-) - g(\underline{\theta}^-) \ll p^{-1}$. Thus,

$$f_p(\underline{\varsigma}_p^-) = \frac{2g(\underline{\theta}^-)}{p} + \frac{2}{p}\big(g(\underline{\varsigma}_p^+) - g(\underline{\theta}^-)\big) - h_p(\underline{\varsigma}_p^-)$$

where the last two summands are $\ll p^{-2}$. In summary,

$$f_p(\underline{\varsigma}_p^{\pm}) = \frac{2g(\underline{\theta}^{\pm})}{p} + h_p^{\pm} \tag{5.4}$$

where $h_p^{\pm} \ll p^{-2}$.

Consequently, letting $\mathbb{P}$ be the set of all primes, we obtain for some constant b,

$$\pm \sum_{p\le X} f_p(\underline{\varsigma}_p^{\pm}) = \pm 2g(\underline{\theta}^{\pm})\big(\log_2 X + b\big) + \sum_{p\in\mathbb{P}} h_p^{\pm} + O\big(e^{-\mathsf{c}\sqrt{\log X}}\big),$$

cf. [25, p.182] for $\sum_{p\le X} p^{-1}$ and $\sum_{p>X} h_p^{\pm} \ll \sum_{p>X} p^{-2} \ll 1/(X\log X)$. This completes the proof with $A_n^{\pm} = \pm g(\underline{\theta}^{\pm})$ and $B_n^{\pm} = e^b \exp\big(\sum_{p\in\mathbb{P}} h_p^{\pm}/A_n^{\pm}\big)$. $\square$

5.3. **Construction of $\mathcal{K}_T$ and Proof of Theorem 2.1.** Suppose $\phi \in \mathcal{H}_T^\natural$. Then $\log C_\phi \ll \log T$ and we shall take $X = (\log T)^\beta$ for some β specified later. Define $\theta_{\phi,i}(p) \in \mathbb{C}$ such that $\alpha_{\phi,i}(p) = e^{\mathrm{i}\theta_{\phi,i}(p)}$, $\forall\ p$, and $\underline{\theta}_\phi(p) = (\theta_{\phi,1}(p), \cdots, \theta_{\phi,n}(p))$. If ϕ satisfies the condition $\mathbf{WRC}_\varrho(X)$, then $f_p(\underline{\theta}_\phi(p)) = f_p(\Re\mathrm{e}\, \underline{\theta}_\phi(p)) + O(\varrho/p)$ for $p \le X$. The summation of $O(\varrho/p)$ over $p \le X$ is $\ll \varrho \log_3 T$. Thus we construct the set $\mathcal{K}_T$ as follows (and take $\mathcal{K} = \bigcup_T \mathcal{K}_T$).

Let $\eta \in (0, 10^{-2})$ and $T \ge 10^2$. Define

$$\mathcal{H}_T(\eta) := \{\phi \in \mathcal{H}_T :\ L(s,\phi)L(s,\widetilde{\phi}) \neq 0,\ \forall\ s \in \mathcal{S}\} \tag{5.5}$$

where $\mathcal{S} := \mathcal{S}_{\eta,T} = \{s :\ \sigma \ge 1-\eta,\ |\tau| \le T^\eta\} \cup \{s :\ \sigma \ge 1\}$ and $\widetilde{\phi}$ is the dual Maass form of ϕ, and

$$\mathcal{K}_T = \mathcal{K}_T(\eta) := \left\{\phi \in \mathcal{H}_T(\eta) :\ \begin{array}{l} \phi \text{ satisfies } \mathbf{WRC}_\varrho(Y) \text{ where} \\ \varrho = 1/(\log_3 T)^{1+\eta} \text{ and } Y = (\log T)^{6/\eta} \end{array}\right\}. \tag{5.6}$$

If $\phi \in \mathcal{K}_T$, then from Lemma 5.2, we infer that

$$\log |L(1,\phi)|^2 = \sum_{p \le X} f_p(\Re\mathrm{e}\, \underline{\theta}_\phi(p)) + o(1) \tag{5.7}$$

holds for $X = (\log T)^\beta$ with $\beta = 4/\eta$. Under GRH and GRC, (5.7) holds for $\beta = 2 + o(1)$ by Remark 5.1. Appealing to Lemma 5.3, we get

$$-2A_n^- \log(B_n^- \log X) + o(1) \le \log |L(1,\phi)|^2 \le 2A_n^+ \log(B_n^+ \log X) + o(1),$$

i.e. $(1+o(1))(\beta B_n^- \log T)^{-A_n^-} \le |L(1,\phi)| \le (1+o(1))(\beta B_n^+ \log T)^{A_n^+}$, which holds unconditionally for $\phi \in \mathcal{K}_T$ with $\beta = 4/\eta$, or conditionally for $\phi \in \mathcal{H}_T^\natural$ with $\beta = 2 + o(1)$. The proof will be complete after the evaluation of $\#(\mathcal{H}_T \setminus \mathcal{K}_T)$, which is left to the next section.

6. The size of $\mathcal{H}_T \setminus \mathcal{K}_T$ and a zero density theorem

Now we explain the evaluation of the exceptional set $\mathcal{H}_T \setminus \mathcal{K}_T$. Our result is

$$\#\big(\mathcal{H}_T \setminus \mathcal{K}_T\big) \ll \#\big(\mathcal{H}_T\big) \exp\bigg(-\mathsf{c}\frac{\varrho \log T}{\log_2 T}\bigg). \tag{6.1}$$

6.1. **The size of $\mathcal{H}_T \setminus \mathcal{K}_T$.** Let $\mathcal{W}_T(\eta)$ be the set of $\phi \in \mathcal{H}_T$ that does not satisfy $\mathbf{WRC}_\varrho(Y)$ for $\varrho = 1/(\log_3 T)^{1+\eta}$ and $Y = (\log T)^{6/\eta}$. With (5.5), it is clear that

$$\#\big(\mathcal{H}_T \setminus \mathcal{K}_T\big) \le \#\big(\mathcal{H}_T \setminus \mathcal{H}_T(\eta)\big) + \#\big(\mathcal{W}_T(\eta)\big).$$

To treat $\#\big(\mathcal{W}_T(\eta)\big)$, we appeal to the quantitative bounds towards Ramanujan in Theorem 7.3 of the next section, which yields

$$\begin{aligned} \#\big(\mathcal{W}_{T,p}\big) &:= \#\bigg\{\phi \in \mathcal{H}_T :\ \log \max_{1\le i\le n} |\alpha_{\phi,i}(p)| > \varrho\bigg\} \\ &\ll T^d \exp\bigg(-\mathsf{c}\frac{\varrho \log T}{\log p}\bigg) \end{aligned}$$

for some constant $\mathsf{c} > 0$ and the implied constant depends on n only. Consequently,

$$\#\big(\mathcal{W}_T(\eta)\big) \le \sum_{p\le(\log T)^{6/\eta}} \#\big(\mathcal{W}_{T,p}\big) \ll T^d \exp\bigg(-\mathsf{c}\frac{\varrho\log T}{\log_2 T}\bigg).$$

Next $\#(\mathcal{H}_T\backslash\mathcal{H}_T(\eta))$ does not exceed the number of $\phi\in\mathcal{H}_T^\natural$ that satisfies $L(s,\phi)L(s,\widetilde{\phi}) = 0$ for some $s\in\mathcal{S}$, see (5.5). As $\phi\in\mathcal{H}_T^\natural$ implies $\widetilde{\phi}\in\mathcal{H}_T^\natural$ (cf. §3.1 and (3.6)), it boils down to count $\{\phi\in\mathcal{H}_T^\natural :\ L(s,\phi)=0 \text{ for some } s\in\mathcal{S}\}$. The zero density estimate, Theorem 6.2 below, implies

$$\#\big(\mathcal{H}_T\setminus\mathcal{H}_T(\eta)\big) \le \sum_{\phi\in\mathcal{H}_T^\natural} N(1-\eta,T^\eta,\phi) \ll T^{9n^2\eta}$$

and (6.1) hence follows.

We devote the remains of this section to prove Theorems 6.2 following quite closely the argument in [14, Sections 4-5], hence the proofs are sketchy as the details are available in [14].

6.2. **A large sieve inequality.** We need the following auxiliary tool, which is a large sieve inequality derived from the Rankin-Selberg L-functions.

Proposition 6.1. *For any $\{a(\boldsymbol{m}) : \boldsymbol{m}\in\mathbb{N}^{n-1}\}\subset\mathbb{C}$ and any numbers $T,L\ge 1$, we have*

$$\sum_{\phi\in\mathcal{H}_T^\natural}\bigg|\sum_{\det \mathrm{y}(\boldsymbol{m})\le L} a(\boldsymbol{m})A_\phi(\boldsymbol{m})\bigg|^2 \ll_\varepsilon T^\varepsilon\big(T^{d+\frac{n(n-1)}{2}}L^{\frac{1}{n}+\varepsilon}+L\big)\sum_{\det \mathrm{y}(\boldsymbol{m})\le L}|a(\boldsymbol{m})|^2$$

where the inner sum on the left side runs over $\boldsymbol{m}=(m_1,\cdots,m_{n-1})$ and $\mathrm{y}(\boldsymbol{m})$ is defined as $\mathrm{y}(\boldsymbol{m}) := \operatorname{diag}\big(y_1y_2\cdots y_{n-1}\quad y_1y_2\cdots y_{n-2}\quad\cdots\quad y_1\quad 1\big)$. Note that $d+n(n-1)/2 = n^2-1$.

Proof. By duality principle, we are led to the bilinear form

$$\mathcal{B} := \sum b_{\phi_1}\overline{b_{\phi_2}}I(\phi_1,\phi_2;L)$$

where the sum runs over $\phi_1,\phi_2\in\mathcal{H}^\natural$, $\{b_\phi\}_{\mathcal{H}^\natural}$ is any arbitrary set indexed by $\phi\in\mathcal{H}^\natural$ with $\|\mu_\phi\|_2$ and

$$I(\phi_1,\phi_2;L) := \frac{1}{2\pi\mathtt{i}}\int_{(2)}\frac{L(s,\phi_1\times\widetilde{\phi}_2)}{\zeta(ns)}\Gamma(s)L^s\,ds.$$

As $\|\mu_\phi\|_2\le T$, the Langlands parameters of ϕ are $\ll T$. Thus we infer with Proposition 4.1 that

$$L(s,\phi_1\times\widetilde{\phi}_2) \ll (T+|\tau|)^{\frac{n^2}{2}(1-\sigma)+n^2\varepsilon}$$

where $s=\sigma+i\tau$ satisfies $-\varepsilon\le\sigma\le 1+\varepsilon$ and $|s-1|\ge\varepsilon$. Shifting the line of integration to $\Re\mathrm{e}\, s = 1/n+\varepsilon$, the term from the pole at $s=1$ is $\ll T^\varepsilon L$ by (4.1), which appears when $\phi_1=\phi_2$, and the integral over the line $\Re\mathrm{e}\, s = 1/n+\varepsilon$ is $\ll T^{n(n-1)/2+\varepsilon}L^{1/n+\varepsilon}$. Together with the bound $\#(\mathcal{H}_T^\natural)\ll T^d$ (which implies $\sum|b_{\phi_1}b_{\phi_2}|\ll T^d\sum|b_\phi|^2$), we infer that

$$\mathcal{B} \ll (T^\varepsilon L + T^{d+n(n-1)/2+\varepsilon}L^{1/n+\varepsilon})\sum|b_\phi|^2.$$

$\square$

6.3. **Zero density estimates.** Let $\sigma \geq 1/2$ and $H \geq 1$ be any number. We are concerned with the number of zeros $\rho = \beta + \mathtt{i}\gamma$ of $L(s, \phi)$ in the box $\sigma \pm \mathtt{i}H, 1 \pm \mathtt{i}H$, i.e. $N(\sigma, H, \phi)$, for the family $\mathcal{H}_T^\natural$.

Theorem 6.2. *Let $T, H \geq 2$ be any numbers. Define $E = d + n(n + \frac{3}{2}) + 2n(n+1)\vartheta$ (see (7.6) for ϑ). Then for any $\varepsilon > 0$ and any $\frac{1}{2} + \varepsilon \leq \alpha \leq 1$, we have*

$$\sum_{\phi \in \mathcal{H}_T^\natural} N(\alpha, H, \phi) \ll_\varepsilon H k^{E(1-\alpha)/(3-2\alpha)+\varepsilon}$$

where $k = \max(T, H)$. The implied constant depends on n and ε only.

Now we are ready to prove Theorem 6.2, using Montgomery's zero detection method. Set

$$M_X(s, \phi) = \sum_{\substack{m_1, \cdots, m_n \geq 1 \\ m_1 m_2^2 \cdots m_n^n \leq X}} A(m_1, \cdots, m_{n-1}) \frac{\mu_n(m_1, \cdots, m_n)}{(m_1 m_2^2 \cdots m_n^n)^s} \tag{6.2}$$

where $\mu_n(m_1, \cdots, m_n) = |\mu(m_1 \cdots m_n)| \mu(m_1)\mu(m_2)^2 \cdots \mu(m_n)^n$. i.e. $M_X(s, \phi)$ is an initial section of $L(s, \phi)^{-1}$, cf. Remark 3.2 (1). Immediately it follows from $|\alpha_{\phi,i}(p)| \leq p^\vartheta$ (see (7.6)) that for $\Re\mathrm{e}\, s \geq 1/2$,

$$M_X(s, \phi) \ll X^{1/2+\vartheta+\varepsilon} \tag{6.3}$$

and from $a \ll 1 + |a|^2$, (3.4) and (4.1) that for $\Re\mathrm{e}\, s = 1 + \varepsilon$,

$$\begin{aligned} L(s, \phi)^{-1} - M_X(s, \phi) &\ll \sum_{m_1 m_2^2 \cdots m_n^n > X} \frac{|A(m_{n-1}, \cdots, m_1)|^2 + 1}{(m_{n-1}^{n-1} \cdots m_1)^{1+\varepsilon} m_n^{n(1+\varepsilon)}} \\ &\ll_\varepsilon X^{-\varepsilon/4}(1 + |L(1+\varepsilon, \phi \times \widetilde{\phi})|) \\ &\ll T^{n^2\varepsilon} X^{-\varepsilon/4}. \end{aligned} \tag{6.4}$$

Take

$$x = T^{n(n+1)} \qquad \text{and} \qquad y = \left(T^d k^{n/2} x^{1+2\vartheta}\right)^{1/(3-2\alpha)}$$

the values of which will be explained in the proof. (Recall $k = \max(T, H)$.) Let $h \in [-H, H]$ and Box(h) be the rectangular box in $\mathbb{C}$ whose corners are $\alpha + \mathtt{i}(h \pm 1), 1 + \mathtt{i}(h \pm 1)$. The key ingredient of the zero detection is that if $\rho = \beta + i\gamma \in \mathrm{Box}(h)$ is a zero of $L(s, \phi)$, then a count of ρ will be reflected by the sum

$$\begin{aligned} &k^\varepsilon y^{2(1-\alpha)} \int_{h-K}^{h+K} \left|1 - L(1+\varepsilon+\mathtt{i}\tau, \phi) M_x(1+\varepsilon+\mathtt{i}\tau, \phi)\right|^2 d\tau \\ &+ y^{1/2-\alpha} \int_{h-K}^{h+K} \left|L(\tfrac{1}{2}+\mathtt{i}\tau, \phi) M_x(\tfrac{1}{2}+\mathtt{i}\tau, \phi)\right| d\tau \end{aligned}$$

where $K = \log^2 k$. Using Lemma 4.2 and taking the multiplicity (at most $O(k^\varepsilon)$) into account, we infer that

$$\begin{aligned} N(\alpha, H, \phi) \ll\; & k^\varepsilon y^{1/2-\alpha} \int_{-H}^{H} \left|L(\tfrac{1}{2}+\mathtt{i}\tau, \phi) M_x(\tfrac{1}{2}+\mathtt{i}\tau, \phi)\right| d\tau \\ & + k^{2\varepsilon} y^{2(1-\alpha)} \int_{-H}^{H} \left|1 - L(1+\varepsilon+\mathtt{i}\tau, \phi) M_x(1+\varepsilon+\mathtt{i}\tau, \phi)\right|^2 d\tau. \end{aligned}$$

The first integral is clearly $\ll Hk^{n/4}x^{1/2+\vartheta}$ by (6.3) and the convexity bound (cf. Proposition 4.1). As $1 = L(s,\phi)L(s,\phi)^{-1}$ and $L(1+\varepsilon+\mathtt{i}\tau,\phi) \ll k^{n\varepsilon}$, the second integral is bounded above by

$$\begin{aligned} &k^\varepsilon \int_{-H}^{H} \left| L(1+\varepsilon+\mathtt{i}\tau,\phi)^{-1} - M_{x'}(1+\varepsilon+\mathtt{i}\tau,\phi)\right|^2 d\tau \\ &+k^\varepsilon \int_{-H}^{H} \left| M_{x'}(1+\varepsilon+\mathtt{i}\tau,\phi) - M_{x}(1+\varepsilon+\mathtt{i}\tau,\phi)\right|^2 d\tau \end{aligned}$$

where $x' := k^{8n^2/\varepsilon}$. With (6.2) and (6.4), we deduce that

$$\begin{aligned} &\sum_{\phi\in\mathcal{H}_T^\natural} N(\alpha,H,\phi) \\ \ll\ & T^d y^{1/2-\alpha} H k^{n/4+\varepsilon} x^{1/2+\vartheta} \\ &+k^\varepsilon y^{2(1-\alpha)} \int_{-H}^{H} \sum_{m\le x'^{1/n}} \frac{1}{m^n} \sum_{\mu_\phi\in\mathcal{H}_T^\natural} \left| \sum_{x/m^n<\det \mathrm{y}(\boldsymbol{m})\le x'/m^n} a_{m,\tau}(\boldsymbol{m}) A_\phi(\boldsymbol{m})\right|^2 d\tau \end{aligned}$$

where

$$a_{m,\tau}(\boldsymbol{m}) = |\mu(m_{n-1}\cdots m_1\cdot m)| \frac{\mu(m_{n-1})\mu(m_{n-2})^2\cdots\mu(m_1)^{n-1}}{(m_{n-1}\cdots m_1^{n-1})^{1+\varepsilon-\mathtt{i}\tau}}.$$

Note $\det \mathrm{y}(\boldsymbol{m}) = m_{n-1}\cdots m_1^{n-1}$. Using the large sieve inequality (Proposition 6.1), the integral $\int_{-H}^{H}$ is $\ll H(Tx)^\varepsilon(T^{d+n(n-1)/2}x^{1/n-1}+1) \ll H(Tx)^\varepsilon$ which is the condition for the choice of x. Thus

$$\sum_{\phi\in\mathcal{H}_T^\natural} N(\alpha,H,\phi) \ll k^\varepsilon H\big(y^{1/2-\alpha}T^d k^{n/4}x^{1/2+\vartheta} + y^{2(1-\alpha)}\big) \ll k^\varepsilon H y^{2(1-\alpha)}$$

subject to our selection of y.

7. Consequences of Automorphic Plancherel density theorem for $GL(n)$

For computation we need the Schur polynomial s_λ. Indeed we shall later see that the Hecke eigenvalues are the Schur polynomials at the Satake parameters. As $\prod_{i=1}^n \alpha_{\phi,i}(p) = 1$, we adopt the degenerate Schur polynomial $S_{\boldsymbol{k}}$ which is indexed as in Goldfeld's book [7].

7.1. **The Schur polynomial and the Littlewood-Richardson rule.**

Let $\lambda = (\lambda_1,\cdots,\lambda_n) \in \mathbb{N}_0^n$ with $\lambda_1\ge\cdots\ge\lambda_n$ be a partition and $\boldsymbol{k} = (k_1,\cdots,k_{n-1}) \in \mathbb{N}_0^{n-1}$. Define

(7.1)
$$s_\lambda(x_1,\cdots,x_n) := \frac{\det\left(x_j^{\lambda_i+n-i}\right)_{1\le i,j\le n}}{\det\left(x_j^{n-i}\right)_{1\le i,j\le n}}, \quad S_{\boldsymbol{k}}(x_1,x_2,\cdots,x_n) := \frac{\det\left(x_j^{\sum_{l=1}^{n-i}(k_l+1)}\right)_{1\le i,j\le n}}{\det\left(x_j^{\sum_{l=1}^{n-i}1}\right)_{1\le i,j\le n}}$$

(where the bottom rows of the two matrices in $S_{\boldsymbol{k}}$ consist of 1's). The Schur polynomials are commonly referred to s_λ, but clearly, s_λ and $S_{\boldsymbol{k}}$ are related as follows:

(I) Given $\boldsymbol{k} = (k_1, \cdots, k_{n-1}) \in \mathbb{N}_0^{n-1}$, and set $\imath(\boldsymbol{k}) = (k_1 + \cdots + k_{n-1}, k_1 + \cdots + k_{n-2}, \cdots, k_1, 0)$. Then $S_{\boldsymbol{k}}(x_1, \cdots, x_n) = s_\lambda(x_1, \cdots, x_n)$ where $\lambda = \imath(\boldsymbol{k})$. Note $|\lambda| = \|\boldsymbol{k}\|$, i.e. $\sum \lambda_i = \sum (n-i)k_i$.

(II) Given $\lambda = (\lambda_1, \cdots, \lambda_n)$ with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$. Set $\jmath(\lambda) = (\lambda_{n-1} - \lambda_n, \cdots, \lambda_1 - \lambda_2)$. Then $s_\lambda(x_1, \cdots, x_n) = (x_1 \cdots x_n)^{\lambda_n} S_{\boldsymbol{k}}(x_1, \cdots, x_n)$ if $\boldsymbol{k} = \jmath(\lambda)$. Note $\|\boldsymbol{k}\| = |\lambda| - n\lambda_n$.

Remark 7.1. (a) $S_{\boldsymbol{0}} = s_0 = 1$.

(b) $\imath$ is injective while the fibre of $\jmath$ is contained in $\lambda + \mathbb{Z} \cdot (1_n)$.

(c) The collection of Schur polynomials $s_\lambda(x_1, \cdots, x_n)$, $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, form a basis for the space $\mathbb{C}[x_1, \cdots, x_n]^{\mathfrak{S}_n}$ of symmetric polynomials in $x_1, \cdots, x_n$ (which is in fact an algebra), see [22, I.3]. Thus if $x_1 \cdots x_n = 1$, then $S_{\boldsymbol{k}}$, $\boldsymbol{k} \in \mathbb{N}_0^{n-1}$, span $\mathbb{C}[x_1, \cdots, x_n]_0^{\mathfrak{S}_n}$ consisting of polynomials in $\mathbb{C}[x_1, \cdots, x_n]^{\mathfrak{S}_n}$ together with the constraint $x_1 \cdots x_n = 1$. (In practice, we view $\mathbb{C}[x_1, \cdots, x_n]_0^{\mathfrak{S}_n}$ as the vector space $\mathrm{Span}_{\mathbb{C}}\big\{ \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(n-1)}^{\lambda_{n-1}} : \lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0 \big\}$.)

Lemma 7.1. *(1) Define $S_{\boldsymbol{k}}(1, \cdots, 1)$ by taking $x_i \to 1$. For any $X \geq 1$ and $\boldsymbol{k} \in \mathbb{N}_0^{n-1}$,*

$$\max_{|x_i| \leq X, \forall i} |S_{\boldsymbol{k}}(x_1, \cdots, x_n)| \leq X^{\|\boldsymbol{k}\|} S_{\boldsymbol{k}}(1, \cdots, 1) \leq X^{\|\boldsymbol{k}\|}(1 + |\boldsymbol{k}|)^{n^2-n}.$$

(2) (Littlewood-Richardson's rule) *Let $\boldsymbol{k}, \boldsymbol{k}' \in \mathbb{N}_0^{n-1}$. Assume $x_1 \cdots x_n = 1$. Then*

$$S_{\boldsymbol{k}} \cdot S_{\boldsymbol{k}'} = \sum_{\boldsymbol{\xi}} d_{\boldsymbol{k}\boldsymbol{k}'}^{\boldsymbol{\xi}} S_{\boldsymbol{\xi}}$$

where the sum is over $\boldsymbol{\xi} \in \mathbb{N}_0^{n-1}$ satisfying $\|\boldsymbol{\xi}\| \leq \|\boldsymbol{k}\| + \|\boldsymbol{k}'\|$ and $\|\boldsymbol{\xi}\| \equiv \|\boldsymbol{k}\| + \|\boldsymbol{k}'\|$ mod n. The coefficients $d_{\boldsymbol{k}\boldsymbol{k}'}^{\boldsymbol{\xi}}$ are nonnegative integers.

Recall $|\boldsymbol{k}| = \sum_i k_i$ and $\|\boldsymbol{k}\| = \sum_i (n-i)k_i$ for $\boldsymbol{k} = (k_1, \cdots, k_{n-1})$.

Proof. (1) By [5, (A.19)], we may express $s_\lambda = \sum_\lambda K_{\lambda\mu} M_\mu$ where $K_{\lambda\mu} \in \mathbb{N}_0$ is the Kosta number and $M_\mu = \sum_{\sigma \in \mathfrak{S}_n} x_{\sigma(1)}^{\mu_1} \cdots x_{\sigma(n)}^{\mu_n}$. This implies

$$|s_\lambda(x_1, \cdots, x_n)| \leq s_\lambda(1, \cdots, 1)$$

if $|x_i| \leq 1$, $\forall\ i$. Now we take $u_i = x_i/X$, then by definition and the properties of determinants, $S_{\boldsymbol{k}}(x_1, \cdots, x_n) = X^{\|\boldsymbol{k}\|} S_{\boldsymbol{k}}(u_1, \cdots, u_n) = X^{\|\boldsymbol{k}\|} s_\lambda(u_1, \cdots, u_n)$ where $\lambda = \imath(\boldsymbol{k})$ defined as in (I).

Next, from [5, Exercise A.30 (ii)][‡3], we have

$$S_{\boldsymbol{k}}(1, \cdots, 1) = \prod_{1 \leq i < j \leq n} \frac{k_{n-i} + \cdots + k_{n-j+1} + j - i}{j - i} \leq \prod_{1 \leq i < j \leq n} \frac{|\boldsymbol{k}| + j - i}{j - i} \leq (1 + |\boldsymbol{k}|)^{n^2-n}.$$

(2) This follows from the Littlewood-Richardson rule for s_λ, [5, p.456]:

$$s_\lambda \cdot s_\nu = \sum_\mu c_{\lambda\nu}^\mu s_\mu \tag{7.2}$$

[‡3] There is a typo in Part (i) of this exercise: The exponent k of x should be $\sum_{j=1}^k (j-1)\lambda_j$.

where $c^{\mu}_{\lambda\nu} \in \mathbb{N}_0$ and the sum runs over partitions $\mu = (\mu_1, \cdots, \mu_n)$ with $|\mu| = |\lambda| + |\nu|$. (We have confined to the case of n variables and $\lambda, \mu \in \mathbb{N}_0^n$.) Thus, let $\lambda = \imath(\boldsymbol{k})$ and $\lambda' = \imath(\boldsymbol{k}')$,

$$S_{\boldsymbol{k}} \cdot S_{\boldsymbol{k}'} = \sum_{\mu} c^{\mu}_{\lambda\lambda'} s_{\mu}$$

where $\mu = (\mu_1, \cdots, \mu_n)$ satisfies $|\mu| = \|\boldsymbol{k}\| + \|\boldsymbol{k}'\|$. By (II), under the condition $x_1 \cdots x_n = 1$, s_μ is reduced to $S_{\boldsymbol{\xi}}$ (for the same $\boldsymbol{\xi}$) whenever $\boldsymbol{\xi} = \jmath(\mu)$. Note that $|\mu| = \|\boldsymbol{\xi}\| + n\mu_n$, which implies $\|\boldsymbol{\xi}\| + n\mu_n = \|\boldsymbol{k}\| + \|\boldsymbol{k}'\|$. (Recall $(1_n) = (1, \cdots, 1)$.) Then,

$$S_{\boldsymbol{k}} \cdot S_{\boldsymbol{k}'} = \sum_{\boldsymbol{\xi}} c^{\imath(\boldsymbol{\xi})+\ell\cdot(1_n)}_{\lambda\lambda'} S_{\boldsymbol{\xi}} \qquad \left(\ell := \frac{1}{n}(\|\boldsymbol{k}\| + \|\boldsymbol{k}'\| - \|\boldsymbol{\xi}\|)\right) \tag{7.3}$$

where the summation over $\boldsymbol{\xi}$ is constrainted as stated. Plainly $d^{\boldsymbol{\xi}}_{\boldsymbol{k},\boldsymbol{k}'} \in \mathbb{N}_0$. □

7.2. **Weyl's integration formulas for $U(n)$ and $SU(n)$.** We are concerned with the pushforward measure on the space of conjugacy classes.[‡4]

Case 1. Let $U(n)$ be the group of $n \times n$ unitary matrices and $U(n)^\sharp$ be the space of conjugacy classes. Elements in $U(n)^\sharp$ may be represented by $(e^{\mathtt{i}\theta_1}, \cdots, e^{\mathtt{i}\theta_n})$ where $\theta_1, \cdots, \theta_n \in [0, 2\pi]$. In fact, $U(n)^\sharp \cong {S^1}^n/\mathfrak{S}_n$. A function f on $U(n)^\sharp$ is (regarded as) a class function on $U(n)$, i.e. constant on each conjugacy class. Let $p : U(n) \to U(n)^\sharp$ be the natural projection. Then

$$\int_{U(n)^\sharp} f = \int_{U(n)} f \circ p = \frac{1}{n!(2\pi)^n} \int_{[0,2\pi]^n} f(\theta_1, \cdots, \theta_n) \prod_{1 \le i < j \le n} |e^{\mathtt{i}\theta_i} - e^{\mathtt{i}\theta_j}|^2 \, d\theta_1 \cdots d\theta_n,$$

cf. [27] for a detailed discussion. Any $f \in \mathbb{C}[e^{\mathtt{i}\theta_1}, \cdots, e^{\mathtt{i}\theta_n}]^{\mathfrak{S}_n}$ is a function on ${S^1}^n/\mathfrak{S}_n$ and thus a function on $U(n)^\sharp$. The above measure on $U(n)^\sharp$ induces an inner product $(\ ,\ )$ on $\mathbb{C}[e^{\mathtt{i}\theta_1}, \cdots, e^{\mathtt{i}\theta_n}]^{\mathfrak{S}_n}$. Let $\mathrm{s}_\lambda(\underline{\theta}) = \mathrm{s}_\lambda(\theta_1, \cdots, \theta_n)$ be the restriction of the Schur polynomial s_λ on ${S^1}^n$, i.e. $\mathrm{s}_\lambda(\underline{\theta}) = s_\lambda(e^{\mathtt{i}\theta_1}, \cdots, e^{\mathtt{i}\theta_n})$. Then $\{\mathrm{s}_\lambda : \lambda_1 \ge \cdots \ge \lambda_n \ge 0\}$ is an orthonormal basis for $\mathbb{C}[e^{\mathtt{i}\theta_1}, \cdots, e^{\mathtt{i}\theta_n}]^{\mathfrak{S}_n}$ by Lemma 7.2 (b) below.

Lemma 7.2. *Let $\alpha = (\alpha_1, \cdots, \alpha_n), \beta = (\beta_1, \cdots, \beta_n) \in \mathbb{Z}^n$. (a) We have*

$$I(\alpha, \beta) := \frac{1}{n!(2\pi)^n} \int_{[0,2\pi]^n} \det\left(e^{\mathtt{i}\alpha_i\theta_j}\right) \det\left(e^{-\mathtt{i}\beta_i\theta_j}\right) d\theta_1 \cdots d\theta_n = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ (\beta_{\sigma(1)}, \cdots, \beta_{\sigma(n)}) = (\alpha_1, \cdots, \alpha_n)}} \mathrm{sgn}(\sigma).$$

(b) Moreover, suppose $\alpha_1 > \cdots > \alpha_n$ and $\beta_1 \ge \cdots \ge \beta_n$ (or vice versa). Then $I(\alpha, \beta) = 1$ if $\alpha = \beta$, and 0 otherwise.

(c) The integral

$$J(\alpha, \beta) \ := \ \int_{[0,2\pi]^n} \det\left(e^{\mathtt{i}(n-i+\alpha_j)\theta_j}\right) \det\left(e^{-\mathtt{i}(n-i+\beta_i)\theta_j}\right) d\theta_1 \cdots d\theta_n$$

will vanish if there is no permutation $\sigma, \pi \in \mathfrak{S}_n$ such that

$$(n - \sigma(1) + \alpha_1, \cdots, n - \sigma(n) + \alpha_n) = (n - \pi(1) + \beta_{\pi(1)}, \cdots, n - \pi(n) + \beta_{\pi(n)}).$$

This condition will occur if $|\alpha| \neq |\beta|$.

[‡4] This measure for $U(n)^\sharp$ is solely used in the proof of Lemma 8.1 (2).

Proof. Using Laplace expansion for determinants, the integral becomes

$$\frac{1}{n!(2\pi)^n}\sum_{\sigma,\pi}\operatorname{sgn}(\sigma\pi)\int_{[0,2\pi]^n} e^{\mathrm{i}(\alpha_{\sigma(1)}-\beta_{\pi(1)})\theta_1}\cdots e^{\mathrm{i}(\alpha_{\sigma(n)}-\beta_{\pi(n)})\theta_n}\,d\theta_1\cdots d\theta_n$$

$$= \frac{1}{n!}\sum_{\substack{\sigma,\pi\\(\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(n)})=(\beta_{\pi(1)},\cdots,\beta_{\pi(n)})}}\operatorname{sgn}(\sigma\pi) = \frac{1}{n!}\sum_{\pi}\sum_{\substack{\sigma,\pi\\(\alpha_{\sigma(1)},\cdots,\alpha_{\sigma(n)})=(\beta_{\pi(1)},\cdots,\beta_{\pi(n)})}}\operatorname{sgn}(\sigma)$$

which is equivalent to the result. Part (b) is obvious and Part (c) is proved similarly to (a). $\square$

Case 2. Consider the space of conjugacy classes in $SU(n)$, denoted by $SU(n)^\sharp$, which is isomorphic to $T_0/\mathfrak{S}_n$. Express $t \in T_0/\mathfrak{S}_n$ as $t = (e^{\mathrm{i}\theta_1},\cdots,e^{\mathrm{i}\theta_n})$ where $\theta_1+\cdots+\theta_n = 0$ with $\theta_i \in \mathbb{R}$. By the Weyl integration formula in [12, §VIII.5], the push-forward measure on $SU(n)^\sharp$ is

$$\int_{SU(n)^\sharp} f = \frac{1}{n!(2\pi)^{n-1}}\int_{[0,2\pi]^{n-1}} f(\theta_1,\cdots,\theta_n)\prod_{1\le i<j\le n}|e^{\mathrm{i}\theta_i}-e^{\mathrm{i}\theta_j}|^2\,d\theta_1\cdots d\theta_{n-1} \tag{7.4}$$

where $\sum_{i=1}^n \theta_i = 0$. On $T_0/\mathfrak{S}_n$, this measure is known as the Sato-Tate measure $d\mu_{\mathrm{ST}}$. Now we consider the restrictions of symmetric polynomials $f \in \mathbb{C}[x_1,\cdots,x_n]^{\mathfrak{S}_n}$ on $T_0/\mathfrak{S}_n$, the collection of which is denoted by $\mathbb{C}[e^{\mathrm{i}\theta_1},\cdots,e^{\mathrm{i}\theta_n}]_0^{\mathfrak{S}_n}$. i.e., A function $f \in \mathbb{C}[e^{\mathrm{i}\theta_1},\cdots,e^{\mathrm{i}\theta_n}]_0^{\mathfrak{S}_n}$ is given by $f(\theta_1,\cdots,\theta_n) = F(e^{\mathrm{i}\theta_1},\cdots,e^{\mathrm{i}\theta_n})$ where $F \in \mathbb{C}[x_1,\cdots,x_n]_0^{\mathfrak{S}_n}$ (see Remark 7.1 (c)). We use the same notation $\mathrm{s}_\lambda(\underline{\theta})$ to denote the Schur polynomial s_λ on $T_0/\mathfrak{S}_n$. Remark 7.1 explains that $\mathrm{S}_{\boldsymbol{k}}(\underline{\theta}) := S_{\boldsymbol{k}}(e^{\mathrm{i}\theta_1},\cdots,e^{\mathrm{i}\theta_n})$, $\boldsymbol{k} \in \mathbb{N}_0^{n-1}$, span $\mathbb{C}[e^{\mathrm{i}\theta_1},\cdots,e^{\mathrm{i}\theta_n}]_0^{\mathfrak{S}_n}$. Moreover they form an orthonormal basis with respect to the inner product $\langle\,,\rangle$ induced from the measure on $SU(n)^\sharp$. This follows from Weyl's theory on the compact Lie group $SU(n)$, cf. [12, (8.63a)-(8.63b)],

$$\begin{aligned}\langle \mathrm{S}_{\boldsymbol{k}},\mathrm{S}_{\boldsymbol{k}'}\rangle &= \frac{1}{n!(2\pi)^{n-1}}\int_{[0,2\pi]^{n-1}}\mathrm{S}_{\boldsymbol{k}}(\underline{\theta})\overline{\mathrm{S}_{\boldsymbol{k}'}(\underline{\theta})}\prod_{1\le i<j\le n}|e^{\mathrm{i}\theta_i}-e^{\mathrm{i}\theta_j}|^2\,d\theta_1\cdots d\theta_{n-1}\\ &= \delta_{\boldsymbol{k}=\boldsymbol{k}'}.\end{aligned}$$

7.3. **Automorphic representations associated to Hecke-Maass cusp forms.** We follow the excellent exposition in [23]. Let $G = GL(n)$, $\mathbb{A}$ be the ring of adeles of $\mathbb{Q}$ and $G(\mathbb{A})^1 := \{g \in G(\mathbb{A}) : |\det g|_{\mathbb{A}} = 1\}$. The set of irreducible unitary representations π in the cuspidal part of $L^2(G(\mathbb{Q})\setminus G(\mathbb{A})^1)$ is denoted by $\Pi_{\mathrm{cusp}}(G(\mathbb{A})^1)$. A Hecke-Maass cusp form $\phi \in \mathcal{H}^\natural$ is uniquely associated to an unramified representation π in $\Pi_{\mathrm{cusp}}(G(\mathbb{A})^1)$ and vice versa (for which one may see from the Strong Multiplicity One Theorem). Let $\mathfrak{a} = \{(x_1,\cdots,x_n)\in\mathbb{R}^n : \sum x_i = 0\}$ be the Lie algebra of the subgroup $A \subset G(\mathbb{R})^1$ of diagonal matrices with positive entries, and W be the Weyl group which is isomorphic to the symmetric group $\mathfrak{S}_n$. The infinitesimal character of the archimedean component π_∞ of $\pi \in \Pi_{\mathrm{cusp}}(G(\mathbb{A})^1)$ is parametrized by $\lambda_\pi \in \mathfrak{a}_{\mathbb{C}}^*/\mathfrak{S}_n$, where $\mathfrak{a}_{\mathbb{C}}^*$ is the complexification of the dual $\mathfrak{a}^*$ of $\mathfrak{a}$.

Suppose $\pi \in \Pi_{\mathrm{cusp}}(G(\mathbb{A})^1)$ is associated to the Hecke-Maass cusp form ϕ whose Langlands parameters are $\mu_{\phi,i}$, $1 \le i \le n$. Then its infinitesimal character $\lambda_\pi \in \mathfrak{a}_{\mathbb{C}}^*/\mathfrak{S}_n$ may be represented as $\mu_\phi := (\mu_{\phi,1},\cdots,\mu_{\phi,n})$ in $\mathfrak{a}_{\mathbb{C}}^*$. Let $A_\phi(m_1,\cdots,m_{n-1})$ denote the normalized Fourier coefficient of ϕ (see §3.1). For each rational prime p, the Satake parameters $\alpha_{\phi,i}(p)$, $1 \le i \le n$, are associated to the non-archimedean component π_p via Satake isomorphism, such that $\prod_{i=1}^n \alpha_{\phi,i}(p) = 1$ and (cf. [37, Proposition 5.1] as well)

$$A_\phi(p^{k_1}, p^{k_2}, \cdots, p^{k_{n-1}}) = S_{\boldsymbol{k}}(\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \cdots, \alpha_{\phi,n}(p))$$

where $S_{\boldsymbol{k}} \in \mathbb{C}[x_1, \cdots, x_n]^{\mathfrak{S}_n}$ with $\boldsymbol{k} := (k_1, \cdots, k_{n-1})$ is the degenerate Schur polynomial, see (7.1). The Satake parameter $\alpha_\phi(p) = (\alpha_{\phi,1}(p), \cdots, \alpha_{\phi,n}(p))$ is (viewed as) an element in $\mathbb{C}^{\times n}/\mathfrak{S}_n$ and, moreover, satisfies the unitary condition (cf. [23, §13.1]), saying that

$$\left\{\overline{\alpha_{\phi,1}(p)}, \cdots, \overline{\alpha_{\phi,n}(p)}\right\} = \left\{\alpha_{\phi,1}(p)^{-1}, \cdots, \alpha_{\phi,n}(p)^{-1}\right\} \quad \text{(as multisets)} \tag{7.5}$$

for all primes p, cf. Remark 3.2 (2).

The (unsettled) Generalized Ramanujan Conjecture asserts that $\mu_\phi \in \mathtt{i}\mathfrak{a}^*$ and $\alpha_\phi(p) \in T_0/\mathfrak{S}_n \subset S^{1^n}/\mathfrak{S}_n$ for all p, where S^1 is the unit circle in $\mathbb{C}$ and $T_0 = \{(t_1, \cdots, t_n) \in S^{1^n} : \prod t_i = 1\}$. The bound towards the conjecture due to [20, Theorem 1.2] is:

$$|\Re\mathrm{e}\, \mu_i(\nu)| \le \vartheta \quad \text{and} \quad |\alpha_{\phi,i}(p)| \le p^\vartheta \tag{7.6}$$

where $\vartheta = 1/2 - 1/(n^2+1)$.

7.4. **Matz-Templier's Automorphic Plancherel Density Theorem.** We parametrize $t \in T_0$ by $(e^{\mathtt{i}\theta_1}, \cdots, e^{\mathtt{i}\theta_n})$ where $\theta_n := -\sum_{i=1}^{n-1} \theta_i$. The Sato-Tate measure $d\mu_{\mathrm{ST}}$ on $T_0/\mathfrak{S}_n$ is the measure given by the integration formula (7.4), see also [32] and [37, Section 3]. In [23], Matz and Templier established an automorphic Plancherel density theorem with error term for $GL(n)$ – equidistribution law for $\alpha_\phi(p)$ with respect to the Plancherel measure $d\mu_p$ on $S^{1^n}/\mathfrak{S}_n$. The measure $d\mu_p$ is supported on $T_0/\mathfrak{S}_n$; moreover from [21, Theorem (5.1.2) and p.52], the integration formula for $d\mu_p$ is

$$\frac{1}{n!}\prod_{i=2}^n \frac{1-p^{-i}}{1-p^{-1}} \cdot \prod_{1\le i<j\le n} \left|\frac{e^{\mathtt{i}\theta_i} - p^{-1}e^{\mathtt{i}\theta_j}}{e^{\mathtt{i}\theta_i} - e^{\mathtt{i}\theta_j}}\right|^{-2} \cdot \frac{1}{(2\pi)^{n-1}}\, d\theta_1 \cdots \theta_{n-1} \tag{7.7}$$

under the same parametrization. Thus, we get a relation between the two measures on $T_0/\mathfrak{S}_n$:

$$d\mu_p = \prod_{i=2}^n (1-p^{-i}) \cdot \det(I - \rho_{\mathrm{Ad}}(t)p^{-1})^{-1}\, d\mu_{\mathrm{ST}} \tag{7.8}$$

where $\det(I - \rho_{\mathrm{Ad}}(t)p^{-1})^{-1} = (1-p^{-1})\prod_{1\le i,j\le n}(1-p^{-1}e^{\mathtt{i}(\theta_j-\theta_i)})^{-1}$[‡5] if t is parameterized as above by $(e^{\mathtt{i}\theta_1}, \cdots, e^{\mathtt{i}\theta_n})$. The work in [23] yields the following vital tool Theorem M-T for our study, which follows from [23, Theorem 1.3] and its proof.

Theorem M-T. (Matz-Templier) Let $d = n(n+1)/2 - 1$, $\Omega \subset \mathtt{i}\mathfrak{a}^*$ be a W-invariant domain with piecewise C^2-boundary. For some constant $A > 0$, we have for any $t \ge 1$,

$$\sum_{\substack{\phi\in\mathcal{H}^\natural \\ \mu_\phi \in t\Omega}} A_\phi(m_1, \cdots, m_{n-1}) = \Lambda_\Omega(t) \prod_p \int_{S^{1^n}/\mathfrak{S}_n} S_{\boldsymbol{k}_p}\, d\mu_p + O((m_1\cdots m_{n-1})^A t^{d-1/2})$$

where $\Lambda_\Omega(t) \asymp t^d$ as $t \to \infty$ (with the implied constants in $\asymp$ depending on n and Ω), and for each prime p, the tuple $\boldsymbol{k}_p = (k_1, \cdots, k_{n-1})$ is determined by $p^{k_i} \| m_i$, $1 \le i \le n-1$.

‡5 See §8.1 for the notation $\det(I - \rho_{\mathrm{Ad}}(t)X)$.

7.5. **Application I: Average bounds toward the Ramanujan Conjecture.** Amongst many, a significant application of the Matz-Templier theorem is an estimate of the number of forms ϕ that fail the Ramanujan Conjecture at a fixed prime p, see [23, Corollary 1.6]. For our purpose, we have to peel off the factor $\omega/\log p$ in the exponent of the bound in this corollary. Theorem 7.3 below, which remains non-trivial for very small ϱ, is a refinement based on the argument in the proof of [23, Corollary 1.6] and new ingredients arising from the computational aspects of Satake isomorphism and the Hall-Littlewood polynomials.

Theorem 7.3. *There are constants $\mathsf{c}, T_0 > 0$ (depending only on n) such that for any prime p and any $\varrho > 0$, the cardinality*

$$\#\left\{\phi \in \mathcal{H}_T : \ \log \max_{1\le i\le n} |\alpha_{\phi,i}(p)| > \varrho\right\} \ll T^{d-\mathsf{c}\varrho/\log p}$$

holds for all $T \ge T_0$, where the implied constant depends on n only.

Proof. Let h be a large integer ($\ll \log T$) whose value will be specified later. By [23, Lemma 13.1], for $0 \le j \le n$, the polynomials

$$\varphi_{h,j}(x_1, \cdots, x_n) := 2^j \sum_{\sigma\in\mathfrak{S}_n} x_{\sigma(1)}^h \cdots x_{\sigma(j)}^h \in \mathbb{C}[x_1, \cdots, x_n]^{\mathfrak{S}_n}$$

satisfies

$$\max_{0\le j\le n} |\varphi_{h,j}(\alpha)| \ge |\alpha|_\infty^h := \max_{1\le i\le n} |\alpha_i|^h$$

for all $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{C}^n$. Define $\varphi_{h,j}^\vee(x_1, \cdots, x_n) := \varphi_{h,j}(x_1^{-1}, \cdots, x_n^{-1})$. Now we repeat the proof of Corollary 1.6 (Section 13.1) in [23] with

$$\varphi_h := \sum_{j=0}^n \varphi_{h,j}\varphi_{h,j}^\vee.$$

Note that $\varphi_h(\alpha) \ge |\alpha|^{2h}$ when α satisfies the unitary condition (7.5). Denote the cardinality in Theorem 7.3 by N. Thus with [23, Theorem 1.1],

$$e^{2h\varrho}N \le \sum_{\phi\in\mathcal{H}_T} \varphi(\alpha_\phi(p)) \ll_n T^d \int_{S^{1^n}/\mathfrak{S}_n} \varphi_h\mu_p + T^{d-1/2}\|\tau_h\|_{L^1(GL_n(\mathbb{Q}_p))}^A \tag{7.9}$$

where the implied constant and A depend at most on n, and the function τ_h is a smooth compactly support bi-K-invariant functions on $GL_n(\mathbb{Q}_p)$ with $K = GL_n(\mathbb{Z}_p)$ such that τ_h corresponds to φ_h under the Satake correspondence. Our choice of φ_h is $O(1)$ (independent of h) on S^{1^n}, saving the exponential factor ($e^{\omega k}$ in [23]), and the trade-off is a further analysis on $\|\tau_h\|_{L^1(GL_n(\mathbb{Q}_p))}$.

To its end, we invoke the following facts: Let $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ with $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$, and f_λ be the characteristic function on the double coset $K\mathrm{diag}(p^{\lambda_1}, \cdots, p^{\lambda_n})K$.

- The Satake isomorphism b sends f_λ to

$$b(f_\lambda) = p^{\langle\rho,\lambda\rangle}P_\lambda(x_1, \cdots, x_n; p^{-1}) \in \mathbb{C}[x_1, \cdots, x_n]^{\mathfrak{S}_n} \tag{7.10}$$

  where $\rho = \frac{1}{2}(n-1, n-3, \cdots, 1-n)$ and P_λ is the Hall-Littlewood polynomial corresponding to λ. (cf. [28, Lemma 3.1].)

- The polynomials $P_\lambda(x;\xi) = P_\lambda(x_1,\cdots,x_n;\xi)$ are homogeneous (in x) of degree $|\lambda|$ and form a $\mathbb{Z}[\xi]$-basis for the ring $\Lambda_\xi := \mathbb{Z}[\xi][x_1,\cdots,x_n]^{\mathfrak{S}_n}$, cf. [22, III (2.7)]. Moreover, let Λ_ξ^k be the space of all homogeneous polynomials of degree k in Λ_ξ, together with the zero polynomial. Then $P_\lambda(x;\xi)$, $|\lambda| = k$, form a basis for Λ_ξ^k because by [22, I (3.2)-(3.3)], the Schur polynomials s_λ, $|\lambda| = k$, form a $\mathbb{Z}$-basis for Λ_0^k and by [22, III p.209], $P_\lambda = \sum_{\substack{\mu\preceq\lambda\\|\mu|=|\lambda|}} w_{\lambda\mu}(\xi)s_\mu$ where $w_{\lambda\mu}(\xi)\in\mathbb{Z}[\xi]$ and $w_{\lambda\lambda}(\xi)=1$. The partition ordering $\mu\preceq\lambda$ means $\sum_{1\le i\le j}\mu_i \le \sum_{1\le i\le j}\lambda_i$ for all $j\ge 1$, see [22, I p.7].
- The Hall-Littlewood polynomials satisfy a sort of orthogonality with respect to the measure $\Delta(x;\xi)\,dx$ on the torus ${S^1}^n$. Indeed, [33, Theorem 2] shows that for any two partitions λ and μ,

$$\int_{{S^1}^n} P_\lambda(x_1,\cdots,x_n;\xi)P_\mu(x_1^{-1},\cdots,x_n^{-1};\xi)\Delta(x,\xi)\,dx = \delta_{\lambda=\mu}\frac{n!}{v_\mu(\xi)}$$

  where under the parametrization, $(x_1,\cdots,x_n) = (e^{\mathrm{i}\theta_1},\cdots,e^{\mathrm{i}\theta_n})$ with $(\theta_1,\cdots,\theta_n)\in[0,2\pi]^n$, the measure is given by

$$\Delta(x,\xi)\,dx := \prod_{1\le i\ne j\le n}\frac{x_j-x_i}{x_j-\xi x_i}\,dx = \frac{1}{(2\pi)^n}\prod_{1\le i\ne j\le n}\frac{e^{\mathrm{i}\theta_j}-e^{\mathrm{i}\theta_i}}{e^{\mathrm{i}\theta_j}-\xi e^{\mathrm{i}\theta_i}}\,d\theta_1\cdots d\theta_n,$$

  $\delta_{\lambda=\mu}$ is the Kronecker delta and the function v_μ fulfills $1\le v_\mu(\xi)\le(1+\xi)^n$ for all $0\le\xi\le\frac{1}{2}$.
- By [10, Proposition 7.4] and [22, V (p.297-298)], the norm $\|f_\lambda\|_{L^1(GL_n(\mathbb{Q}_p))}$ of the characteristic function f_λ equals the number of K-cosets in $K\mathrm{diag}(p^{\lambda_1},\cdots,p^{\lambda_n})K$, denoted by $\deg f_\lambda$, and

$$\deg f_\lambda = \frac{p^{2\langle\rho,\lambda\rangle}}{v_\lambda(p^{-1})}\prod_{i=1}^n\frac{1-p^{-i}}{1-p^{-1}} \asymp p^{2\langle\rho,\lambda\rangle}. \tag{7.11}$$

We are now ready to evaluate $\|\tau_h\|$ where the subscript $L^1(GL_n(\mathbb{Q}_p))$ is suppressed for simplicity. Suppose $\tau_{h,j}$ (resp. $\tau_{h,j}^\vee$) corresponds to $\varphi_{h,j}$ (resp. $\varphi_{h,j}^\vee$) under the Satake isomorphism. Then, $\tau_h = \sum_{0\le j\le n}\tau_{h,j}*\tau_{h,j}^\vee$ and

$$\|\tau_h\| \le \sum_{j=0}^n \|\tau_{h,j}\|\|\tau_{h,j}^\vee\| \tag{7.12}$$

because the Satake isomorphism sends convolutions to products (i.e. $b(f*g) = b(f)b(g)$) and $\|f*g\|\le\|f\|\|g\|$.

In view of [22, III (2.8) & I (2.2)], the polynomial $\varphi_{h,j} = 2^j j!\sum_{1\le i_1<\cdots<i_j\le n} x_{i_1}^h\cdots x_{i_j}^h\in\Lambda_{1/p}^{jh}$ may be written as (noting $(1_j) = (1,\cdots,1)$ is a j-tuple)

$$\varphi_{h,j} \quad = \quad 2^j j! P_{(1_j)}(x_1^h,\cdots,x_n^h;p^{-1}) = \sum_\mu a_\mu P_\mu(x_1,\cdots,x_n;p^{-1})$$

where the sum is restricted by $|\mu| := \mu_1+\cdots+\mu_n = jh$ and

$$a_\mu = \frac{2^j j!}{n!}v_\mu(p^{-1})\int_{{S^1}^n}\varphi_{h,j}(x_1,\cdots,x_n)P_\mu(x_1^{-1},\cdots,x_n^{-1};p^{-1})\Delta(x,p^{-1})\,dx \ll_n 1$$

uniformly for all primes $p \geq 2$ and integers $h \geq 1$. Here we have used, by [22, III (2.1)],

$$|P_\lambda(x; p^{-1})| \leq \frac{1}{v_\lambda(p^{-1})} \sum_{\sigma \in \mathfrak{S}_n} \prod_{i<j} \left| \frac{x_{\sigma(i)} - p^{-1} x_{\sigma(j)}}{x_{\sigma(i)} - x_{\sigma(j)}} \right| \qquad (x \in S^{1^n}).$$

This follows that $\tau_{h,j} = \sum_{|\mu|=jh} a_\mu p^{-\langle \rho, \mu \rangle} f_\mu$ by (7.10) and thus, with (7.11), for some positive constant c (depending on n only),

$$\|\tau_{h,j}\| \leq \sum_{|\mu|=jh} p^{-\langle \rho,\mu\rangle} |a_\mu| \|f_\mu\| \ll_n \sum_{|\mu|=jh} p^{\langle \rho,\mu\rangle} \ll h^n p^{\mathsf{c}h}. \tag{7.13}$$

From [22, V (p.297 & 295)], we see that $p^{-n(n-1)/2} x_1 \cdots x_n$ corresponds to the characteristic function $f_{(1_n)}$ under the Satake isomorphism and $f_\lambda * f_{(r_n)} = f_{\lambda + (r_n)}$ for any λ and any integer r. As $\varphi_{h,j}^\vee = 2^{2j-n}(x_1 \cdots x_n)^{-h} \varphi_{h,n-j}$, we infer that

$$\tau_{h,j}^\vee = 2^{2j-n} \tau_{h,n-j} * \left(f_{((-1)_n)}{}^{*h} \right) = 2^{2j-n} \sum_{|\mu|=(n-j)h} a'_\mu p^{-\langle \rho,\mu\rangle} f_\mu * \left(f_{((-1)_n)}{}^{*h} \right)$$

where $a'_\mu \ll 1$. Note $\|f * g\| = \|f\| \|g\|$ for characteristic functions f and g (in fact, $\|f\| = \mu(f)$ in [22, p.297] if f is a characteristic function). Thus $\|f_\mu * \left(f_{((-1)_n)}{}^{*h} \right)\| = \|f_\mu\| / \|f_{(1_n)}\|^h$ and consequently, $\|\tau_{h,j}^\vee\| \ll h^n p^{\mathsf{c}h}$ as well.

Set $h = \left[\delta_0 \frac{\log T}{\log p} \right] + 1$ for some suitable small constant $\delta_0 > 0$ so $\|\tau_h\| \leq T^{1/(4A)}$ in view of (7.12) and (7.13). Theorem 7.3 follows readily from (7.9). □

7.6. **Integration of Schur polynomials with respect to Plancherel measures.** The Schur polynomials are orthogonal to each other with respect to $d\mu_{\mathrm{ST}}$. In the scenario of Plancherel measure, we retrieve with Cauchy's identity and (7.8): $d\mu_p = c_p \det(I - \rho_{\mathrm{Ad}}(t) p^{-1})^{-1} \, d\mu_{\mathrm{ST}}$ on $T_0/\mathfrak{S}_n$ where $c_p = \prod_{i=2}^n (1 - p^{-i})$ and $\det(I - \rho_{\mathrm{Ad}}(t) p^{-1})^{-1} = (1 - p^{-1}) \prod_{1 \leq i,j \leq n} (1 - p^{-1} e^{\mathrm{i}(\theta_j - \theta_i)})^{-1}$. The results will be applied in later sections.

Proposition 7.4. *Let $\boldsymbol{k} \in \mathbb{N}_0^{n-1}$. (1) If $\|\boldsymbol{k}\| \not\equiv 0 \bmod n$, then $\int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}} \, d\mu_p = 0$. Otherwise, we have*

$$\int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}} \, d\mu_p = \prod_{i=1}^{n-1} (1 - p^{-i}) \cdot \sum_{\boldsymbol{\eta} \in \mathbb{N}_0^{n-1}} d_{\boldsymbol{k}\boldsymbol{\eta}}^{\boldsymbol{\eta}} \cdot p^{-\|\boldsymbol{\eta}\|}$$

where $d_{\boldsymbol{k}\boldsymbol{\eta}}^{\boldsymbol{\eta}} = c_{\lambda\mu}^{\mu + \ell \cdot (1_n)}$ with $\lambda = \imath(\boldsymbol{k})$, $\mu = \imath(\boldsymbol{\eta})$, and $\ell = \|\boldsymbol{k}\|/n$. (See (7.2) for $c_{\lambda\mu}^\nu$.) The summation over $\boldsymbol{\eta}$ is supported on $|\boldsymbol{\eta}| \geq \|\boldsymbol{k}\|/n$, see Remark 7.2 (3).

(2) Moreover, if $\|\boldsymbol{k}\| \not\equiv \|\boldsymbol{k}'\| \bmod n$ where $\boldsymbol{k}' \in \mathbb{N}_0^{n-1}$, then $\int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}} \overline{S_{\boldsymbol{k}'}} \, d\mu_p = 0$.

Proof. (1) By Cauchy's identity in [7, p.233], we have

$$\det(I - \rho_{\mathrm{Ad}}(\underline{\theta}) p^{-1})^{-1} = \frac{1 - p^{-1}}{1 - p^{-n}} \sum_{\boldsymbol{\eta} \in \mathbb{N}_0^{n-1}} \mathrm{S}_{\boldsymbol{\eta}}(\underline{\theta}) \overline{\mathrm{S}_{\boldsymbol{\eta}}(\underline{\theta})} p^{-\|\boldsymbol{\eta}\|}.$$

Let $c_p' = c_p(1-p^{-1})/(1-p^{-n}) = \prod_{j=1}^{n-1}(1-p^{-j})$. Thus

$$\int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}}\, d\mu_p = c_p' \sum_{\boldsymbol{\eta}\in\mathbb{N}_0^{n-1}} p^{-\|\boldsymbol{\eta}\|} \int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}} S_{\boldsymbol{\eta}} \overline{S_{\boldsymbol{\eta}}}\, d\mu_{\mathrm{ST}}$$

Applying Lemma 7.1 (2) to $S_{\boldsymbol{k}}S_{\boldsymbol{\eta}}$ and the orthogonality, the integral is expressed into

$$\sum_{\boldsymbol{\xi}} d^{\boldsymbol{\xi}}_{\boldsymbol{k}\boldsymbol{\eta}} \int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{\xi}} \overline{S_{\boldsymbol{\eta}}}\, d\mu_{\mathrm{ST}} = d^{\boldsymbol{\eta}}_{\boldsymbol{k}\boldsymbol{\eta}}$$

where $\|\boldsymbol{\eta}\| \equiv \|\boldsymbol{k}\| + \|\boldsymbol{\eta}\| \bmod n$ in light of the constraint in the summation over $\boldsymbol{\xi}$. This is void if $n \nmid \|\boldsymbol{k}\|$, hence the result follows. Write $\mu = \imath(\boldsymbol{\eta})$. Indeed by (7.3), we have $d^{\boldsymbol{\eta}}_{\boldsymbol{k}\boldsymbol{\eta}} = c^{\mu+\ell\cdot(1_n)}_{\lambda\mu}$ where $\lambda = \imath(\boldsymbol{k})$, $\mu = \imath(\boldsymbol{\eta})$ and $\ell = \|\boldsymbol{k}\|/n$. The second assertion follows plainly.

(2) Write $\boldsymbol{k}'' = (k'_{n-1}, \cdots, k'_1)$ for $\boldsymbol{k}' = (k'_1, \cdots, k'_{n-1})$, and observe

$$S_{(k_1,\cdots,k_{n-1})}(x_1,\cdots,x_n) = (x_1\cdots x_n)^{\sum_i k_i} S_{(k_{n-1},\cdots,k_1)}(x_1^{-1},\cdots,x_n^{-1}). \tag{7.14}$$

Applying the Littlewood-Richardson rule for $S_{\boldsymbol{k}}$ (i.e. Lemma 7.1 (2)) to $S_{\boldsymbol{k}}S_{\boldsymbol{k}''} = S_{\boldsymbol{k}}\overline{S_{\boldsymbol{k}'}}$ on T_0, we have

$$\int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}} \overline{S_{\boldsymbol{k}'}}\, d\mu_p = \sum_{\boldsymbol{\xi}} d^{\boldsymbol{\xi}}_{\boldsymbol{k}\boldsymbol{k}''} \int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{\xi}}\, d\mu_p.$$

Each $\boldsymbol{\xi}$ is required to satisfy $\|\boldsymbol{\xi}\| \equiv \|\boldsymbol{k}\| + \|\boldsymbol{k}''\| \equiv \|\boldsymbol{k}\| - \|\boldsymbol{k}'\| \bmod n$ (by Lemma 7.1 and $\|\boldsymbol{k}''\| = n|\boldsymbol{k}'| - \|\boldsymbol{k}'\|$). Part (1) imposes the condition $n | \|\boldsymbol{\xi}\|$ for non-vanishing integrals on the right-side. This implies $\|\boldsymbol{k}\| \equiv \|\boldsymbol{k}'\| \bmod n$. □

Remark 7.2. (1) For the case $S_{\boldsymbol{0}} = 1$ one sees readily $\mu_p(T_0/\mathfrak{S}_n) = 1$, for $\sum_{\boldsymbol{\eta}\in\mathbb{N}_0^{n-1}} p^{-\|\boldsymbol{\eta}\|} = \prod_{j=1}^{n-1}(1-p^{-j})^{-1}$.

(2) For $\boldsymbol{k} = (k, 0_{n-3}, k)$ and $\boldsymbol{\eta} = (0_{n-2}, \eta)$ in $\mathbb{N}_0^{n-1}$ where $\eta \geq k$, we take correspondingly $\lambda = (2k, k, \cdots, k, 0)$, $\mu = (\eta, 0_{n-1})$ and ($\ell = k$ so) $\nu := \mu + \ell\cdot(1_n) = (k+\eta, k, \cdots, k)$; the skew diagram $\mu\backslash\lambda$ is a horizontal strip and $d^{\boldsymbol{\eta}}_{\boldsymbol{k}\boldsymbol{\eta}} = c^{(k+\eta,k,\cdots,k)}_{(2k,k,\cdots,k,0)(\eta,0_{n-1})} = 1$ by the (third) Pieri formula (cf. [5, (A.7)] or [2, Pieri's formula]). For such $\boldsymbol{k}$'s, $\int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}}\, d\mu_p > 0$ which provides non-vanishing examples in addition to Corollary 7.7 below.

(3) The choice of $\eta \geq k$ in Part (2) underlies a somewhat sophisticated condition of *Horn's inequalities*, that is, the inequalities $(*_{IJK})$ in [6]. We refer to Fulton [6], particularly Theorem 11, for details. Indeed, specialized to $r = 1$ (amongst all inequalities) in [6, Theorem 11 (p.222)] (noting $T_1^n = U_1^n$), one infers that $c^{\nu}_{\lambda\mu} = 0$ if $\nu_k > \lambda_i + \mu_j$ for some $i + j = k + 1$. Now, associated to $\boldsymbol{k}$ and $\boldsymbol{\eta}$, we set $\lambda = \imath(\boldsymbol{k})$, $\mu = \imath(\boldsymbol{\eta})$ and $\nu = \mu + \ell\cdot(1_n)$. If $\nu_n > \lambda_n + \mu_1$ (i.e. $\ell > |\boldsymbol{\eta}|$), then $d^{\boldsymbol{\eta}}_{\boldsymbol{k}\boldsymbol{\eta}} = c^{\nu}_{\lambda\mu} = 0$. Indeed one may deduce more conditions from other i, j, k.

7.7. **An asymptotic unweighted trace formula of Petersson type.** The Casselman-Shalika formula $A_\phi(p^{k_1}, \cdots, p^{k_{n-1}}) = S_{\boldsymbol{k}}(\alpha_\phi(p))$ holds in general even if $\alpha_\phi(p) \in \mathbb{C}^{\times n}$ does not belong to S^{1n}. (Still $\prod_i \alpha_{\phi,i}(p) = 1$.) Suppose $|\alpha_{\phi,i}(p)| \leq p^{\vartheta}$, $\forall\ i$. Set $\underline{\theta}_\phi(p) := (\theta_{\phi,1}(p), \cdots, \theta_{\phi,n}(p)) \in \mathbb{C}^n$ such that $\alpha_{\phi,i}(p) = e^{\mathrm{i}\theta_{\phi,i}(p)}$, $\forall\ p$. Apparently from

Lemma 7.1,

$$|A_\phi(p^{k_1},\cdots,p^{k_{n-1}})| = |\mathrm{S}_{\boldsymbol{k}}(\underline{\theta}_\phi(p))| = |S_{\boldsymbol{k}}(\alpha_\phi(p))| \le p^{\vartheta\|\boldsymbol{k}\|}S_{\boldsymbol{k}}(1,\cdots,1). \tag{7.15}$$

Moreover the product of two coefficients may be described with the Littlewood-Richardson rule, and as outlined in [23], one obtains the following (unweighted trace formula) result. Define $\Lambda(t) := \Lambda_B(t)$ in Theorem M-T with Ω equal to the unit ball $B\subset i\mathfrak{a}^*$.

Proposition 7.5. *Let $m_1,\cdots,m_{n-1},m_1',\cdots,m_{n-1}'\in\mathbb{N}$. For each prime p, the tuple $\boldsymbol{k}_p=(k_{p,1},\cdots,k_{p,n-1})$ with $p^{k_{p,j}}\|m_j$ is associated to $(m_1,\cdots,m_{n-1})$ and similarly $\boldsymbol{k}_p'$ is associated to $(m_1',\cdots,m_{n-1}')$. Then there is a constant $L>0$ such that for any $T\ge 1$,*

$$\begin{aligned}&\sum_{\phi\in\mathcal{H}_T} A_\phi(m_1,\cdots,m_{n-1})\overline{A_\phi(m_1',\cdots,m_{n-1}')}\\ &= \Lambda(T)\prod_p\int_{S^{1^n}/\mathfrak{S}_n} S_{\boldsymbol{k}_p}\overline{S_{\boldsymbol{k}_p'}}\,d\mu_p + O(T^{d-1/2}(mm')^L)\end{aligned}$$

where $\Lambda(T)\asymp T^d$ ($d=\frac12 n(n+1)-1$), $m=\prod_{i=1}^{n-1}m_i$ and $m'=\prod_{i=1}^{n-1}m_i'$.

If $(m_1m_{n-1}')^{n-1}\cdots(m_{n-1}m_1')\neq h^n$ for any $h\in\mathbb{N}$, then the main term will vanish.

Remark 7.3. We may reformulate the unweighted trace formula into

$$\sum_{\phi\in\mathcal{H}_T}\prod_p S_{\boldsymbol{k}_p}(\alpha_\phi(p))S_{\boldsymbol{k}_p'}(\overline{\alpha_\phi(p)}) = \Lambda(T)\prod_p\int_{S^{1^n}/\mathfrak{S}_n} S_{\boldsymbol{k}_p}\overline{S_{\boldsymbol{k}_p'}}\,d\mu_p + O(T^{d-1/2}(mm')^L).$$

Proof. As $\overline{A_\phi(m_1',\cdots,m_{n-1}')} = A_\phi(m_{n-1}',\cdots,m_1')$, we write $\boldsymbol{k}''=(k_{n-1}',\cdots,k_1')$ for $\boldsymbol{k}'=(k_1',\cdots,k_{n-1}')$. Assume mm' has ω distinct prime factors, so $mm'\big|\prod_{j=1}^{\omega}p_j^\infty$. Then

$$A_\phi(m_1,\cdots,m_{n-1})\overline{A_\phi(m_1',\cdots,m_{n-1}')} = \prod_{j=1}^{\omega} S_{\boldsymbol{k}_j}(\alpha_\phi(p_j))S_{\boldsymbol{k}_j''}(\alpha_\phi(p_j)). \tag{7.16}$$

Here we write $\boldsymbol{k}_j$ for $\boldsymbol{k}_p$, etc, when $p=p_j$. With the Littlewood-Richardson rule, the left-side of (7.16) equals

$$\sum_{\boldsymbol{\xi}_1,\cdots,\boldsymbol{\xi}_\omega}\prod_{j=1}^{\omega} d_{\boldsymbol{k}_j\boldsymbol{k}_j''}^{\boldsymbol{\xi}_j}\cdot\prod_{j=1}^{\omega}S_{\boldsymbol{\xi}_j}(\alpha_\phi(p_j)).$$

Summing over ϕ and invoking Theorem M-T, we get

$$\sum_{\substack{\phi\in\mathcal{H}\\ \|\mu_\phi\|_2\le T}} A_\phi(m_1,\cdots,m_{n-1})\overline{A_\phi(m_1',\cdots,m_{n-1}')} = [\text{Main}]+[\text{Error}]$$

where (noting $d_{\boldsymbol{k}\boldsymbol{k}'}^{\boldsymbol{\xi}}\ge 0$)

$$\begin{aligned}[\text{Main}] &= \Lambda(T)\sum_{\boldsymbol{\xi}_1,\cdots,\boldsymbol{\xi}_\omega}\prod_{j=1}^{\omega} d_{\boldsymbol{k}_j\boldsymbol{k}_j''}^{\boldsymbol{\xi}_j}\cdot\prod_{j=1}^{\omega}\int_{S^{1^n}/\mathfrak{S}_n} S_{\boldsymbol{\xi}_j}\,d\mu_j,\\ [\text{Error}] &\ll T^{d-1/2}\sum_{\boldsymbol{\xi}_1,\cdots,\boldsymbol{\xi}_\omega}\prod_{j=1}^{\omega} d_{\boldsymbol{k}_j\boldsymbol{k}_j''}^{\boldsymbol{\xi}_j}\cdot\Big(\prod_{j=1}^{\omega}p_j^{|\boldsymbol{\xi}_j|}\Big)^A.\end{aligned}$$

Interchanging the summation and integration, the sum over $\boldsymbol{\xi}_j$'s in [Main] is restored with the Littlewood-Richardson rule to

$$\prod_{j=1}^{\omega} S_{\boldsymbol{k}_j} S_{\boldsymbol{k}''_j} = \prod_{j=1}^{\omega} S_{\boldsymbol{k}_j} \overline{S_{\boldsymbol{k}'_j}}$$

by (7.14) and that the variables all lie on S^{1^n}. By Proposition 7.4 (2), the main term will vanish if $\|\boldsymbol{k}_j\| \not\equiv \|\boldsymbol{k}'_j\| \bmod n$ for some j, which is equivalent to $(m_1 m'_{n-1})^{n-1} \cdots (m_{n-1} m'_1)$ is not an nth power.

The product of prime powers in [Error] is $\ll (mm')^{n-1}$ as $|\boldsymbol{\xi}_j| \le \|\boldsymbol{\xi}_j\| \le \|\boldsymbol{k}_j\| + \|\boldsymbol{k}'_j\| \le (n-1)(|\boldsymbol{k}_j| + |\boldsymbol{k}'_j|)$. Also $d^{\boldsymbol{\xi}}_{\boldsymbol{k}\boldsymbol{k}''} \le (d^{\boldsymbol{\xi}}_{\boldsymbol{k}\boldsymbol{k}''})^2$, and

$$\begin{aligned}
\sum\nolimits_{\boldsymbol{\xi}} {d^{\boldsymbol{\xi}}_{\boldsymbol{k}\boldsymbol{k}'}}^2 &= (\mathrm{S}_{\boldsymbol{k}}\mathrm{S}_{\boldsymbol{k}'}, \mathrm{S}_{\boldsymbol{k}}\mathrm{S}_{\boldsymbol{k}'}) \\
&\le S_{\boldsymbol{k}}(1,\cdots,1) S_{\boldsymbol{k}'}(1,\cdots,1) (\mathrm{S}_{\boldsymbol{k}}, \mathrm{S}_{\boldsymbol{k}})^{1/2} (\mathrm{S}_{\boldsymbol{k}'}, \mathrm{S}_{\boldsymbol{k}'})^{1/2} \\
&\le \big((1+|\boldsymbol{k}|)(1+|\boldsymbol{k}'|)\big)^{n^2-n}
\end{aligned}$$

by Lemma 7.1 (1). Observing $\tau(m) = \prod_{j=1}^{\omega}(1+|\boldsymbol{k}_j|)$, we deduce that

$$[\text{Error}] \ll T^{d-1/2} (mm')^{A(n-1)} \big(\tau(m)\tau(m')\big)^{n^2-n}$$

and hence the result. □

7.8. **Application II: Generalization of a case of Pieri's formula.** [‡6] The value of Littlewood-Richardson coefficient $c^{\nu}_{\lambda\mu}$ in (7.2) is mysterious, Pieri's formula describes explicitly the cases of λ (or μ) equal to (m) or (1_m), $1 \le m \le n$. The next lemma is a realization of the multiplicativity of the Fourier coefficients as a case of Littlewood-Richardson rule. Consequently a (probably new) formula for $\lambda = (\alpha_{n-1})$ is derived in Corollary 7.8 below, which reduces to a case of Pieri's formula for the Schur polynomials in n variables when $\alpha = 1$.

Lemma 7.6. *Assume $x_1 \cdots x_n = 1$. For any $\alpha \in \mathbb{N}$ and any $\boldsymbol{k} \in \mathbb{N}_0^{n-1}$, we have*

$$S_{(\alpha,0_{n-2})} S_{\boldsymbol{k}} = \sum_{\boldsymbol{\xi} \in \mathcal{V}_\alpha(\boldsymbol{k})} S_{\boldsymbol{\xi}}$$

where $\mathcal{V}_\alpha(\boldsymbol{k}) := \boldsymbol{k} + \{(\ell_n - \ell_1, \ell_1 - \ell_2, \cdots, \ell_{n-2} - \ell_{n-1}) : \sum_{i=1}^n \ell_i = \alpha,\ 0 \le \ell_i \le k_i,\ \forall\, 1 \le i < n\}$. (Note $\|\boldsymbol{\xi}\| = \|\boldsymbol{k}\| + n\ell_n - \alpha$ for $\boldsymbol{\xi} \in \mathcal{V}_\alpha(\boldsymbol{k})$.)

Proof. Let $\boldsymbol{h} = (\alpha, 0_{n-2}) \in \mathbb{N}_0^{n-1}$. In view of Lemma 7.1 (2), it remains to show $d^{\boldsymbol{\xi}}_{\boldsymbol{h}\boldsymbol{k}} = 1$ for $\boldsymbol{\xi} \in \mathcal{V} := \mathcal{V}_\alpha(\boldsymbol{k})$, or 0 otherwise. The value of $d^{\boldsymbol{\xi}}_{\boldsymbol{h}\boldsymbol{k}}$ equals $\int S_{\boldsymbol{h}} S_{\boldsymbol{k}} \overline{S_{\boldsymbol{\xi}}}\, d\mu_{\mathrm{ST}}$. Note $A_\phi(p^\alpha, 1\cdots,1) = \mathrm{S}_{\boldsymbol{h}}(\underline{\theta}_\phi(p))$ and $A_\phi(p^{\eta_1}, \cdots, p^{\eta_{n-1}}) = \mathrm{S}_{\boldsymbol{\eta}}(\underline{\theta}_\phi(p))$ for any $\boldsymbol{\eta} = (\eta_1, \cdots, \eta_{n-1}) \in \mathbb{N}_0^{n-1}$. Thus, the Hecke multiplicative relation (3.3) gives

$$\mathrm{S}_{\boldsymbol{h}}(\underline{\theta}_\phi(p)) \mathrm{S}_{\boldsymbol{k}}(\underline{\theta}_\phi(p)) = \sum_{\boldsymbol{\eta} \in \mathcal{V}} \mathrm{S}_{\boldsymbol{\eta}}(\underline{\theta}_\phi(p)). \tag{7.17}$$

[‡6] The content of this section will not be applied in other sections.

Consider

$$M_T \ := \sum_{\phi\in\mathcal{H}_T} \mathrm{S}_{\boldsymbol{h}}(\underline{\theta}_\phi(p))\mathrm{S}_{\boldsymbol{k}}(\underline{\theta}_\phi(p))\overline{\mathrm{S}_{\boldsymbol{\xi}}(\underline{\theta}_\phi(p))}\det(I-\rho_{\mathrm{Ad}}(\underline{\theta}_\phi(p))p^{-1}). \tag{7.18}$$

By Theorem M-T and (7.8), we see that

$$M_T \xrightarrow[T\to\infty]{} c_p\int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{h}}S_{\boldsymbol{k}}\overline{S_{\boldsymbol{\xi}}}\,d\mu_{\mathrm{ST}} = c_p d^{\boldsymbol{\xi}}_{\boldsymbol{hk}}$$

where $c_p=\prod_{i=2}^n(1-p^{-i})$. Next, we replace $\mathrm{S}_{\boldsymbol{h}}\mathrm{S}_{\boldsymbol{k}}$ in (7.18) with (7.17) to get the limit value $c_p\delta_{\boldsymbol{\xi}\in\mathcal{V}}$ where $\delta_{\boldsymbol{\xi}\in\mathcal{V}}=1$ if $\boldsymbol{\xi}\in\mathcal{V}$, or 0 otherwise. The proof is done. $\square$

Corollary 7.7. *Let $m\in\mathbb{N}$ and T be sufficiently large. Then*

$$\Lambda(T)^{-1}\sum_{\phi\in\mathcal{H}_T} A_\phi(m,1,\cdots,1) = \frac{1}{m^{(n-1)/2}}\mathbb{1}_{\boxed{n}}(m) + O(T^{-1/2}m^A)$$

where $\mathbb{1}_{\boxed{n}}(m)=1$ if $m=h^n$ for some $h\in\mathbb{N}$ and 0 otherwise.

Proof. Following from Theorem M-T and Proposition 7.4 (1), it remains to evaluate

$$\prod_{p^\ell\| m} c_p'\sum_{\boldsymbol{\eta}\in\mathbb{N}_0^{n-1}} d^{\boldsymbol{\eta}}_{(\ell,0_{n-2})\boldsymbol{\eta}}\cdot p^{-\|\boldsymbol{\eta}\|}$$

where $c_p'=\prod_{i=1}^{n-1}(1-p^{-i})$. By Lemma 7.6, $\boldsymbol{\eta}\in\mathcal{V}_\ell(\boldsymbol{\eta})$ if and only if $n|\ell$ and $\eta_i\ge\ell/n$ for all i. Thus the sum over $\boldsymbol{\eta}$ gives $p^{-\ell(n-1)/2}{c_p'}^{-1}\mathbb{1}_{\boxed{n}}(p^\ell)$ and hence the result. $\square$

Corollary 7.8. *For any $(\alpha_{n-1}):=(\alpha,\cdots,\alpha)\in\mathbb{N}^{n-1}$ and partition $\lambda=(\lambda_1,\cdots,\lambda_n)$, we have*

$$s_{(\alpha_{n-1})}(x_1,\cdots,x_n)\cdot s_\lambda(x_1,\cdots,x_n)=\sum_{\nu\in\mathcal{U}} s_\nu(x_1,\cdots,x_n)$$

where $\mathcal{U}=\lambda+(\alpha_n)-\{(l_1,\cdots,l_n):\ \sum_{j=1}^n l_j=\alpha,\ 0\le l_j\le\lambda_j-\lambda_{j+1},\ \forall\ 1\le j\le n-1\}$. (Note: $s_{(\alpha_{n-1})}=s_{(\alpha_{n-1},0)}$.)

Proof. By the Littlewood-Richardson rule, $s_{(\alpha,\cdots,\alpha)}s_\lambda=\sum_\nu c_\nu s_\nu$ where $|\nu|=|\lambda|+(n-1)\alpha$. Impose the extra condition $x_1\cdots x_n=1$. As $\jmath(\alpha_{n-1},0)=(\alpha,0_{n-2})$, we have $s_{(\alpha_{n-1})}=S_{(\alpha,0_{n-2})}$ and $s_\lambda=S_{\boldsymbol{k}}$ if $\boldsymbol{k}=\jmath(\lambda)$. Set $\boldsymbol{\xi}=\jmath(\nu)$. From Lemma 7.6 we infer that c_ν is either 0 or 1. Moreover, $\nu\in\mathcal{U}$ if and only if $|\nu|=|\lambda|+(n-1)\alpha$ and $\boldsymbol{\xi}\in\mathcal{V}_\alpha(\boldsymbol{k})$, i.e.

$$\left\{\begin{array}{rcl}\nu_{n-1}-\nu_n &=& \lambda_{n-1}-\lambda_n+\ell_n-\ell_1\\ \nu_{n-2}-\nu_{n-1} &=& \lambda_{n-2}-\lambda_{n-1}+\ell_1-\ell_2\\ &\vdots& \\ \nu_1-\nu_2 &=& \lambda_1-\lambda_2+\ell_{n-2}-\ell_{n-1}\end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl}\nu_1-\nu_n &=& \lambda_1-\lambda_n+\ell_n-\ell_{n-1}\\ \nu_2-\nu_n &=& \lambda_2-\lambda_n+\ell_n-\ell_{n-2}\\ &\vdots& \\ \nu_{n-1}-\nu_n &=& \lambda_{n-1}-\lambda_n+\ell_n-\ell_1\end{array}\right.$$

where $(\lambda_{n-1}-\lambda_n,\lambda_{n-2}-\lambda_{n-1},\cdots,\lambda_1-\lambda_2)+(\ell_n-\ell_1,\ell_1-\ell_2,\cdots,\ell_{n-2}-\ell_{n-1})\in\mathcal{V}_\alpha(\boldsymbol{k})$. Given the linear system, the condition $|\nu|=|\lambda|+(n-1)\alpha$ is equivalent to $\nu_n=\lambda_n+\alpha-\ell_n$. Equivalently, $\nu=\lambda+(\alpha,\cdots,\alpha)-(l_1,\cdots,l_n)$ for some $(l_1,\cdots,l_n)$ satisfying $\sum l_j=\alpha$ and $l_j\le k_{n-j}$ $(1\le j\le n-1)$. Plainly this ends the proof. $\square$

Remark 7.4. When $\alpha=1$, it is a case of Pieri's formula asserting that $s_{(1_{n-1})}s_\lambda=\sum_\pi s_\pi$ where the sum is over all partitions π whose Young diagram can be obtained from that of λ by adding $n-1$ boxes with no two in any row. (cf. [5, p.462] or [2].) The formula in

Corollary 7.8 may be phrased as: The sum is over all partitions ν whose Young diagram is obtained by deleting $r \leq \alpha$ boxes from the (horizontal) excessive boxes, then adding α boxes to each non-bottom row and adding r boxes to the bottom row. For example, consider $s_{(4,4,4)}(x_1,x_2,x_3,x_4)s_{(6,4,3,1)}(x_1,x_2,x_3,x_4)$. Figure 1 is the Young diagram of $(6,4,3,1)$ where the excessive boxes are shaded. Figure 2 shows a deletion of 3 excessive boxes (one from rows 1,2,3 respectively). Figure 3 is the Young diagram after adding 4 boxes to each non-bottom row and 3 boxes in the bottom row. The partition ν is $(9,7,6,4)$.

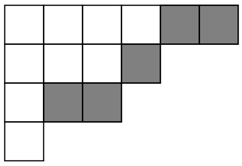

Figure 1

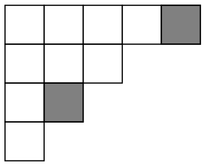

Figure 2

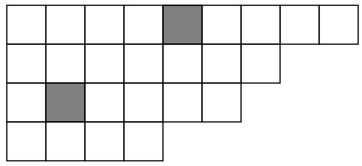

Figure 3

8. A probabilistic model for $L(s,\phi)$

We have seen that the Satake parameters of ϕ in the family $\mathcal{H}$ follow an equidistribution law given by the Plancherel measures. Now we consider the probabilistic models for L-functions and study the expectation of the complex moments, which is an initial step to understand the value distribution of $\{L(1,\phi)\}_{\phi\in\mathcal{H}}$, cf. [9, 4, 30, 14].

8.1. **The characteristic polynomials of representations on $SU(n)$.** For a representation ρ on $SU(n)$, its characteristic polynomial $\det(I-\rho(g)X)$ with indeterminate X ($g \in SU(n)$) is a class function and hence descends to a function on $T_0/\mathfrak{S}_n$. Let $e^{\mathrm{i}\theta_1},\cdots,e^{\mathrm{i}\theta_n}$ be the eigenvalues of g. The characteristic polynomial is abbreviated as $\det(I-\rho(\underline{\theta})X)$ where $\underline{\theta}=(\theta_1,\cdots,\theta_n)$.

Let ρ_{St} be the standard representation of $SU(n)$ on $\mathbb{C}^n$, and $\check{\rho}_{\mathrm{St}}$ denote the contragredient representation of ρ_{St}. We have the decomposition $\rho_{\mathrm{St}}\otimes\check{\rho}_{\mathrm{St}} \cong \mathrm{id}\oplus\rho_{\mathrm{Ad}}$ and the following characteristic polynomials:

$$\begin{aligned}
\det(I-\rho_{\mathrm{St}}(\underline{\theta})X) &= \textstyle\prod_{i=1}^n(1-e^{\mathrm{i}\theta_i}X),\\
\det(I-\check{\rho}_{\mathrm{St}}(\underline{\theta})X) &= \det(I-{}^{\mathrm{t}}\rho_{\mathrm{St}}(\underline{\theta})^{-1}X)=\det(I-\rho_{\mathrm{St}}(-\underline{\theta})X),\\
\det(I-\rho_{\mathrm{St}}\otimes\check{\rho}_{\mathrm{St}}(\underline{\theta})X) &= \textstyle\prod_{1\leq i,j\leq n}(1-e^{\mathrm{i}(\theta_i-\theta_j)}X)\\
&= (1-X)^n\textstyle\prod_{1\leq i<j\leq n}|1-e^{\mathrm{i}(\theta_i-\theta_j)}X|^2 \quad (\text{if } X\in\mathbb{R}),\\
\det(I-\rho_{\mathrm{Ad}}(\underline{\theta})X) &= \det(I-\rho_{\mathrm{St}}\otimes\check{\rho}_{\mathrm{St}}(\underline{\theta})X)/(1-X).
\end{aligned}$$

8.2. **The complex moments of a characteristic polynomial.** The characteristic polynomial $\det(I-\rho_{\mathrm{St}}(\underline{\theta})X)$ may be regarded as the characteristic polynomial of the standard representation of $U(n)$ on $\mathbb{C}^n$ together with the condition $\sum_i\theta_i=0$. We shall study the power series expansion of its complex moments with the analysis on $U(n)^\sharp$.

Define for $z \in \mathbb{C}$,

$$\det(I - \rho_{\mathrm{St}}(\underline{\theta})X)^{-z} =: \sum_{\ell \geq 0} \lambda^{z,\ell}(\underline{\theta}) X^{\ell} \qquad \text{for } |X| < 1. \tag{8.1}$$

In fact, as $\det(I - \rho_{\mathrm{St}}(\underline{\theta})X)^{-z} = \prod_{i=1}^{n}(1 - e^{\mathrm{i}\theta_i}X)^{-z}$, we obtain by binomial expansion that

$$\lambda^{z,\ell}(\underline{\theta}) = \sum_{\substack{\underline{\ell} \in \mathbb{N}_0^n \\ |\underline{\ell}| = \ell}} \prod_{1 \leq j \leq n} C(z, \ell_j) e^{\mathrm{i}(\ell_1\theta_1 + \cdots + \ell_n\theta_n)} \tag{8.2}$$

where $C(x,r)$ is the rth coefficient in the binomial expansion of $(1-X)^{-x}$, i.e. $C(x,0) = 1$ and for $r \geq 1$,

$$C(x,r) = \binom{x+r-1}{r} = \frac{1}{r!}\prod_{j=0}^{r-1}(x+j).$$

Clearly $\lambda^{z,\ell}(\underline{\theta}) \in \mathbb{C}[e^{\mathrm{i}\theta_1}, \cdots, e^{\mathrm{i}\theta_n}]^{\mathfrak{S}_n}$, so with (II) in §7.1,

$$\lambda^{z,\ell}(\underline{\theta}) = \sum_{\nu} \mu_{\nu}^{z,\ell} \mathrm{s}_{\nu}(\underline{\theta}) = \sum_{\boldsymbol{k}} \mathrm{S}_{\boldsymbol{k}}(\underline{\theta}) \sum_{\nu:\, \boldsymbol{j}(\nu) = \boldsymbol{k}} \mu_{\nu}^{z,\ell} e^{\mathrm{i}\nu_n \sum_{i=1}^{n}\theta_i}$$

where $\mu_{\nu}^{z,\ell} \in \mathbb{C}$ and $\nu = (\nu_1, \cdots, \nu_n)$. Write $\mu_{\boldsymbol{k}}^{z,\ell} = \sum_{\nu:\, \boldsymbol{j}(\nu)=\boldsymbol{k}} \mu_{\nu}^{z,\ell}$. When $\sum \theta_i = 0$,

$$\lambda^{z,\ell}(\underline{\theta}) = \sum_{\boldsymbol{k}} \mu_{\boldsymbol{k}}^{z,\ell} \mathrm{S}_{\boldsymbol{k}}(\underline{\theta}) \tag{8.3}$$

lies in $\mathbb{C}[e^{\mathrm{i}\theta_1}, \cdots, e^{\mathrm{i}\theta_n}]_0^{\mathfrak{S}_n}$. Since $\{\mathrm{S}_{\boldsymbol{k}}\}$ is an orthonormal basis for $\mathbb{C}[e^{\mathrm{i}\theta_1}, \cdots, e^{\mathrm{i}\theta_n}]_0^{\mathfrak{S}_n}$, we have $\mu_{\boldsymbol{k}}^{z,\ell} = \langle \lambda^{z,\ell}, \mathrm{S}_{\boldsymbol{k}} \rangle$.

Lemma 8.1. *(1) $\mu_{\boldsymbol{k}}^{z,1} = z$ and $\lambda^{z,1}(\underline{\theta}) = z\mathrm{S}_{(0,\cdots,0,1)}(\underline{\theta})$.*

(2) $\mu_{\boldsymbol{k}}^{z,\ell} = 0$ if $\|\boldsymbol{k}\| > \ell$ or $\|\boldsymbol{k}\| \not\equiv \ell$ mod n.

(3) $\sum_{\boldsymbol{k}} |\mu_{\boldsymbol{k}}^{z,\ell}|^2 \leq \max_{\underline{\theta}} |\lambda^{z,\ell}(\underline{\theta})|^2 \leq C(n|z|, \ell)^2$.

Proof. (1) When $\ell = 1$, we have $\lambda^{z,1}(\underline{\theta}) = z\sum_{1 \leq i \leq n} e^{\mathrm{i}\theta_i} = z\mathrm{S}_{(0_{n-2},1)}(\underline{\theta})$ (which may be seen from the definition with a little calculation).

(2) As $\mu_{\nu}^{z,\ell} = (\lambda^{z,\ell}, \mathrm{s}_{\nu})$, we obtain that

$$\begin{aligned} \mu_{\nu}^{z,\ell} &= \frac{1}{n!(2\pi)^n} \sum_{\substack{\underline{\ell} \in \mathbb{N}_0^n \\ |\underline{\ell}| = \ell}} \prod_{1 \leq j \leq n} C(z, \ell_j) \\ &\quad \times \int_{[0,2\pi]^n} e^{\mathrm{i}(\ell_1\theta_1 + \cdots + \ell_n\theta_n)} \det\left(e^{-\mathrm{i}(\nu_i + n - i)\theta_j}\right) \prod_{1 \leq i < j \leq n} (e^{\mathrm{i}\theta_i} - e^{\mathrm{i}\theta_j})\, d\theta_1 \cdots d\theta_n. \end{aligned}$$

The last integral equals

$$\int_{[0,2\pi]^n} \det\left(e^{\mathrm{i}(n-i+\ell_j)\theta_j}\right) \det\left(e^{-\mathrm{i}(\nu_i+n-i)\theta_j}\right)\, d\theta_1 \cdots d\theta_n,$$

i.e. $J(\ell_1, \cdots, \ell_n, \nu_1, \cdots, \nu_n)$ in Lemma 7.2, that will vanish if $\sum \ell_j \neq \sum \nu_i$. Thus $\mu_{\nu}^{z,\ell} \neq 0$ implies $|\nu| = \ell$, consequently $\mu_{\boldsymbol{k}}^{z,\ell} \neq 0$ implies $\|\boldsymbol{k}\| = \ell - n\nu_n$ for some $\nu_n \geq 0$ (see (II)).

(3) This follows from $\sum_{\boldsymbol{k}} |\mu_{\boldsymbol{k}}^{z,\ell}|^2 = \langle \lambda^{z,\ell}, \lambda^{z,\ell}\rangle \leq \max_{\underline{\theta}} |\lambda^{z,\ell}(\underline{\theta})|^2$; clearly

$$\max_{\underline{\theta}} |\lambda^{z,\ell}(\underline{\theta})| \leq \sum_{\substack{\underline{\ell}\in\mathbb{N}_0^n \\ |\underline{\ell}|=\ell}} \prod_{1\leq j\leq n} C(|z|,\ell_j) \leq C(n|z|,\ell)$$

by the analogous inequality in the proof of [14, Lemma 6.1 (p.461)]. □

8.3. **The expectation L-function $\mathbb{E}[L(s,\rho_{\mathrm{St}})^z L(s,\check{\rho}_{\mathrm{St}})^z]$.** For a prime p, we define

$$L_p(s,\rho_{\mathrm{St}}) = \det(I - \rho_{\mathrm{St}}(\underline{\theta}_p)p^{-s})^{-1}$$

where $\underline{\theta}_p$ denotes a random vector distributed on $T_0/\mathfrak{S}_n$ according to the Plancherel measure $d\mu_p$. Also we assume the random vectors $\{\underline{\theta}_p\}_p$ are independent. Introduce

$$L(s,\rho_{\mathrm{St}}) = \prod_p L_p(s,\rho_{\mathrm{St}}) \tag{8.4}$$

to model $L(s,\phi)$, and in fact, for $\Re\mathrm{e}\, s \gg 1$, $L(s,\phi) = L(s,\rho_{\mathrm{St}})|_{\underline{\theta}_p=\underline{\theta}_\phi(p),\forall p}$ where $\underline{\theta}_\phi(p) = (\theta_{\phi,1}(p),\cdots,\theta_{\phi,n}(p))$ is the tuple associated to Satake parameters, cf. §7.7. We also consider $L(s,\check{\rho}_{\mathrm{St}})^z$. Recall that $\det(I-\check{\rho}_{\mathrm{St}}(\underline{\theta})X) = \det(I-\rho_{\mathrm{St}}(-\underline{\theta})X)$ when $\underline{\theta}\in\mathbb{R}^n$. We define

$$\lambda^z(m) := \prod_{p^\ell\|m} \lambda^{z,\ell}(\underline{\theta}) \quad \text{and} \quad \check{\lambda}^z(m) := \prod_{p^\ell\|m} \lambda^{z,\ell}(-\underline{\theta})$$

where the product runs over all primes (with $\ell = 0$ for all but finitely many ℓ's), thus

$$L(s,\rho_{\mathrm{St}})^z = \sum_{m\geq 1} \lambda^z(m)m^{-s} \quad \text{and} \quad L(s,\check{\rho}_{\mathrm{St}})^z = \sum_{m\geq 1} \check{\lambda}^z(m)m^{-s}$$

for $\Re\mathrm{e}\, s \gg 1$.

For $\Re\mathrm{e}\, s \gg 1$, we have (the expectation L-function)

$$\begin{aligned} & \mathbb{E}[L(s,\rho_{\mathrm{St}})^z L(s,\check{\rho}_{\mathrm{St}})^z] \\ = & \prod_p \mathbb{E}[L_p(s,\rho_{\mathrm{St}})^z L_p(s,\check{\rho}_{\mathrm{St}})^z] \\ = & \prod_p \int_{T_0/\mathfrak{S}_n} \det(I-\rho_{\mathrm{St}}(\underline{\theta})p^{-s})^{-z} \det(I-\check{\rho}_{\mathrm{St}}(\underline{\theta})p^{-s})^{-z}\, d\mu_p, \end{aligned} \tag{8.5}$$

so for real $s \gg 1$, $\mathbb{E}[|L_p(s,\rho_{\mathrm{St}})|^{2z}] = \int_{T_0/\mathfrak{S}_n} |\det(I-\rho_{\mathrm{St}}(\underline{\theta})p^{-s})|^{-2z}\, d\mu_p$. Here and in the sequel, we express a function $f(t)$ on $T_0/\mathfrak{S}_n$ as $\mathrm{f}(\underline{\theta}) = f(e^{\mathrm{i}\theta_1},\cdots,e^{\mathrm{i}\theta_n})$ to apply the integration formula for $d\mu_p$, and for simplicity, we abuse the notation so that $\int_{T_0/\mathfrak{S}_n}\mathrm{f}(\underline{\theta})$ means $\int_{T_0/\mathfrak{S}_n} f(t)$.

Let $*$ be the Dirichlet convolution. Then

$$\mathbb{E}[\lambda^z * \check{\lambda}^z(m)] = \sum_{m=ab} \mathbb{E}[\lambda^z(a)\check{\lambda}^z(b)]$$

where

$$\mathbb{E}[\lambda^z(a)\check{\lambda}^z(b)] = \prod_{\substack{p^\ell\|a \\ p^{\ell'}\|b}} \int_{T_0/\mathfrak{S}_n} \lambda^{z,\ell}(\underline{\theta})\lambda^{z,\ell'}(-\underline{\theta})\, d\mu_p.$$

By (8.1), it follows

$$\mathbb{E}[L(s,\rho_{\mathrm{St}})^z L(s,\check{\rho}_{\mathrm{St}})^z] = \sum_{m\ge 1}\mathbb{E}[\lambda^z * \check{\lambda}^z(m)]m^{-s} = \prod_p \sum_{\ell\ge 0}\mathbb{E}[\lambda^z * \check{\lambda}^z(p^\ell)]p^{-\ell s} \tag{8.6}$$

(for $\Re\mathrm{e}\, s \gg 1$), but in fact this series is absolutely convergent in $\Re\mathrm{e}\, s > 1/2$ because the product in (8.6) is $\prod_p (1+O(p^{-2\sigma}))$ by Lemma 8.2 below.

Lemma 8.2. *Let $z\in\mathbb{C}$. (1) The multiplicative function $\mathbb{E}[\lambda^z * \check{\lambda}^z(m)]$ is supported on squarefull numbers, i.e. $\mathbb{E}[\lambda^z * \check{\lambda}^z(p)] = 0$ for all primes p.*

(2) For all primes p, $|\mathbb{E}[\lambda^z(p^u)\check{\lambda}^z(p^v)]| \le C(n|z|,u)C(n|z|,v)$ and for $\sigma>0$,

$$|\mathbb{E}[L_p(s,\rho_{St})^z L_p(s,\check{\rho}_{St})^z]| \le \sum_{\ell\ge 0}\big|\mathbb{E}[\lambda^z * \check{\lambda}^z(p^\ell)]\big|p^{-\ell\sigma} \le \Big(1-\frac{1}{p^\sigma}\Big)^{-2n|z|} - \frac{2n|z|}{p^\sigma}$$

which is $\ll \exp((n|z|)^2 p^{-2\sigma})$ if $p^\sigma > 2n|z|$.

(3) For any $\varepsilon>0$, there is a constant $\mathsf{c} = \mathsf{c}_{n,\varepsilon} > 0$ such that

$$\begin{aligned}\big|\mathbb{E}[L(s,\rho_{St})^z L(s,\check{\rho}_{St})^z]\big| &\le \sum_{m\ge 1}\big|\mathbb{E}[\lambda^z * \check{\lambda}^z(m)]\big|m^{-\sigma}\\ &\le \exp\Big\{\mathsf{c}z'\Big(\log_2 z' + \frac{z'^{(1-\sigma)/\sigma}-1}{(1-\sigma)\log z'}\Big)\Big\}\end{aligned}$$

holds uniformly for $\sigma\in[\frac12+\varepsilon,1]$ and $z\in\mathbb{C}$, where $z' = 2n|z|+3$.

Proof. (1) By (8.1), we have

$$\begin{aligned}&\sum_{\ell\ge 0}\mathbb{E}[\lambda^z * \check{\lambda}^z(p^\ell)]p^{-\ell s}\\ &= \int_{T_0/\mathfrak{S}_n}\det(I-\rho_{\mathrm{St}}(\underline{\theta})p^{-s})^{-z}\det(I-\rho_{\mathrm{St}}(-\underline{\theta})p^{-s})^{-z}\,d\mu_p\\ &= \int_{T_0/\mathfrak{S}_n}\sum_{\ell\ge 0}\lambda^{z,\ell}(\underline{\theta})p^{-\ell s}\cdot\sum_{\ell\ge 0}\lambda^{z,\ell}(-\underline{\theta})p^{-\ell s}\,d\mu_p\\ &= 1+\frac{2}{p^s}\int_{T_0/\mathfrak{S}_n}\lambda^{z,1}(\underline{\theta})\,d\mu_p + \sum_{\ell\ge 2}\frac{1}{p^{\ell s}}\sum_{u+v=\ell}\int_{T_0/\mathfrak{S}_n}\lambda^{z,u}(\underline{\theta})\lambda^{z,v}(\underline{\theta})\,d\mu_p.\end{aligned}$$

By Lemma 8.1 (1) and Proposition 7.4, it is seen that

$$\mathbb{E}[\lambda^z * \check{\lambda}^z(p)] = 2\int_{T_0/\mathfrak{S}_n}\lambda^{z,1}(\underline{\theta})\,d\mu_p = 0.$$

(2) Similarly, with (8.3) and Lemma 8.1 (3), we obtain

$$\big|\mathbb{E}[\lambda^z(p^u)\check{\lambda}^z(p^v)]\big| = \bigg|\int_{T_0/\mathfrak{S}_n}\lambda^{z,u}(\underline{\theta})\lambda^{z,v}(\underline{\theta})\,d\mu_p\bigg| \le C(n|z|,u)C(n|z|,v).$$

Together with (1) and $\big|\mathbb{E}[\lambda^z * \check{\lambda}^z(p^\ell)] \le \sum_{u+v=\ell}\big|\mathbb{E}[\lambda^z(p^u)\check{\lambda}^z(p^v)]\big|$, we are led to evaluate

$$1+\sum_{\ell\ge 2}\frac{1}{p^{\ell\sigma}}\sum_{u+v=\ell}C(n|z|,u)C(n|z|,v) = \Big(1+\sum_{u\ge 1}\frac{1}{p^{u\sigma}}C(n|z|,u)\Big)^2 - 2C(n|z|,1)p^{-\sigma}$$

for $\sigma>0$. The right-side equals $(1-p^{-\sigma})^{-2n|z|} - 2n|z|p^{-\sigma}$.

(3) This follows from the proof of [14, Lemma 6.4]. □

Remark 8.1. Since

$$\lambda^z(m) = \prod_{p^\ell \| m} \sum_{\boldsymbol{k}} \mu_{\boldsymbol{k}}^{z,\ell} S_{\boldsymbol{k}}(\underline{\theta}) \quad \text{and} \quad \check{\lambda}^z(m') \;=\; \prod_{p^{\ell'} \| m'} \sum_{\boldsymbol{k}} \mu_{\boldsymbol{k}}^{z,\ell} \overline{S_{\boldsymbol{k}}(\underline{\theta})},$$

we may express

$$\mathbb{E}[\lambda^z(m)\check{\lambda}^z(m')] = \prod_{\substack{p^\ell \| m \\ p^{\ell'} \| m'}} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{k}'} \mu_{\boldsymbol{k}}^{z,\ell} \mu_{\boldsymbol{k}'}^{z,\ell'} \int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}} \overline{S_{\boldsymbol{k}'}}\, d\mu_p.$$

8.4. **An asymptotic result for $\mathbb{E}[|L(1,\rho_{\mathrm{St}})|^{2r}]$.**

Proposition 8.3. *Let $r > 3$ be any real number. We have*

$$\begin{aligned}\log \mathbb{E}[|L(1,\rho_{St})|^{\pm 2r}] \;&=\; 2A_n^{\pm} r \log\left(B_n^{\pm} \log(2A_n^{\pm} r)\right) \\ &\quad + \frac{2A_n^{\pm} r}{\log(2A_n^{\pm} r)} \left\{ C_n^{\pm} - 1 + \frac{D_n^{\pm}}{\log(2A_n^{\pm} r)} + O\left(\frac{1}{\log^2 r}\right) \right\}\end{aligned}$$

where $A_n^{\pm}, B_n^{\pm}, C_n^{\pm}, D_n^{\pm}$ are constants, and the values of $A_n^{\pm}$ and $B_n^{\pm}$ are same as in Lemma 5.3, so $A_n^+ = n$, $A_n^- = n$ or $n\cos(\pi/n)$.

Remark 8.2. The distribution function (2.1) in Remark 2.2 follows from the argument in [13] and the refinement in [36, Section 7] (comparing our formula and [36, Lemma 7.2]).

Proof. Recalling Section 5.2, we have denoted $g(\underline{\theta}) = \sum_{j=1}^n \cos\theta_j$ and

$$f_p(\underline{\theta}) = -\sum_{j=1}^n \log(1 - 2p^{-1}\cos\theta_j + p^{-2}) = \log\left|\det(I - \rho_{\mathrm{St}}(\underline{\theta})p^{-1})\right|^{-2}$$

on $\mathbb{T}^n \cong T_0$ (a cover of $T_0/\mathfrak{S}_n$). Thus

$$\mathbb{E}[|L(1,\rho_{\mathrm{St}})|^{\pm 2r}] = \prod_p \int_{T_0/\mathfrak{S}_n} \left|\det(I - \rho_{\mathrm{St}}(\underline{\theta})p^{-1})\right|^{\mp 2r} d\mu_p = \prod_p \mathbb{E}[e^{\pm r f_p(\underline{\theta}_p)}].$$

Now we take advantage of *real* (positive) moments to express

$$\begin{aligned}\log \mathbb{E}[e^{r f_p(\underline{\theta}_p)}] \;&=\; r f_p(\underline{\varsigma}_p^+) + \log \mathbb{E}[e^{-r\left(f_p(\underline{\varsigma}_p^+) - f_p(\underline{\theta}_p)\right)}], \\ \log \mathbb{E}[e^{-r f_p(\underline{\theta}_p)}] \;&=\; -r f_p(\underline{\varsigma}_p^-) + \log \mathbb{E}[e^{+r\left(f_p(\underline{\varsigma}_p^-) - f_p(\underline{\theta}_p)\right)}].\end{aligned}$$

The random variables $\mp r\big(f_p(\underline{\varsigma}_p^{\pm}) - f_p(\underline{\theta}_p)\big)$ inside the expectations are both non-positive, and hence the expectations are ≤ 1.

Let $p \leq r' := 2nr + 3$, and $B_p^{\pm}$ be two balls of radius $p/(2r')$ in $\mathbb{T}^{n-1}$ that center at some preimages of $\underline{\varsigma}_p^{\pm}$ respectively, and $U_p^{\pm}$ be their images in $T_0/\mathfrak{S}_n$. Then $f_p(\underline{\theta}) = f_p(\underline{\varsigma}_p^{\pm}) + O(r^{-1})$ on $U_p^{\pm}$ by the mean value theorem and $\partial f/\partial\theta_j \ll p^{-1}$ on $\mathbb{T}^{n-1}$. Thus,

$$1 \geq \mathbb{E}[e^{\mp r\left(f_p(\underline{\varsigma}_p^{\pm}) - f_p(\underline{\theta}_p)\right)}] \geq \mu_p(U_p^{\pm})$$

and the logarithm $\log \mathbb{E}[\cdots] \ll |\log \mu_p(U_p^{\pm})|$. From the integration formulas (7.7) and (7.4), it follows that

$$d\mu_p = (1 + O_n(p^{-1}))\, d\mu_{\mathrm{ST}} \tag{8.7}$$

and thus $\log \mu_p(U_p^{\pm}) \ll p^{-1} + |\log \mu_{\mathrm{ST}}(U_p^{\pm})|$. One may check $\log \mu_{\mathrm{ST}}(U_p^{\pm}) \ll |\log(2r'/p)|$ as follows: Suppose $B_\delta := B(a,\delta) \subset \mathbb{T}^{n-1}$ is a ball of radius δ centered at a and let U_δ be its image in $T_0/\mathfrak{S}_n$. Let $\varepsilon > 0$ be chosen later. In view of the integration formula (7.4) for $d\mu_{\mathrm{ST}}$, $\mu_{\mathrm{ST}}(U_\delta) \gg_n \varepsilon^{n^2-n}\mathrm{vol}\big(B_\delta \setminus B_\varepsilon(a,\delta)\big)$ where vol is the Euclidean volume (in $\mathbb{T}^{n-1}$) and $B_\varepsilon(a,\delta) := \{\boldsymbol{\theta} \in B_\delta : |\theta_i - \theta_j| < \varepsilon \text{ for some } i \neq j\}$. Clearly $\mathrm{vol}(B_\delta) \gg \delta^{n-1}$ and $\mathrm{vol}(B_\varepsilon(a,\delta)) \ll \varepsilon\delta^{n-2}$. This implies $\mu_{\mathrm{ST}}(U_\delta) \gg \delta^{n^2-1}$ by taking $\varepsilon = \mathsf{c}\delta$ for a small enough constant $\mathsf{c} > 0$, and thus $\log \mu_{\mathrm{ST}}(U_\delta) \ll_n |\log \delta|$. Altogether, for $p \le r'$ we have

$$\log \mathbb{E}[e^{\pm r f_p(\underline{\theta}_p)}] = \pm r f_p(\underline{\varsigma}_p^{\pm}) + O(\log(2r'/p)).$$

For $p \ge r'$, it follows from Lemma 8.2 (2) with $\sigma = 1$ that $\log \mathbb{E}[e^{\pm r f_p(\underline{\theta}_p)}] \ll_n r^2p^{-2}$. But for the proposition, we need to undertake a more delicate analysis as in [13], and introduce two auxiliary functions $\mathrm{f}_\pm$ on $[0,\infty)$: let X be a random vector distributed on $T_0/\mathfrak{S}_n$ according to the Sato-Tate measure $d\mu_{\mathrm{ST}}$, define

$$\mathrm{f}_\pm(t) := \begin{cases} \log \mathbb{E}[e^{\pm 2t g(X)}] & \text{if } 0 \le t < 1, \\ \log \mathbb{E}[e^{\pm 2t(g(X) - g(\underline{\theta}^{\pm}))}] & \text{if } t \ge 1. \end{cases}$$

Explicitly, for $0 \le t < 1$, $\mathrm{f}_\pm(t) = \log \int_{T_0/\mathfrak{S}_n} e^{\pm 2t\sum_{j=1}^n \cos\theta_j}\, d\mu_{\mathrm{ST}}$ where $\theta_n = -\sum_{j=1}^{n-1}\theta_j$. The functions $\mathrm{f}_\pm$ are independent of p (and r) and differentiable in t. As in [13, Lemma 1.1], it can be shown that $\mathrm{f}_\pm'(t) \ll_n 1$ and

$$\mathrm{f}_\pm(t) = \begin{cases} O(t^2) & \text{if } 0 \le t < 1, \\ O(\log t + 1) & \text{if } t \ge 1. \end{cases}$$

As $f_p(\underline{\theta}) = 2g(\underline{\theta})p^{-1} + O(p^{-2})$, we obtain $\mathbb{E}[e^{\pm r f_p(\underline{\theta}_p)}] = \mathbb{E}[e^{\pm \frac{2r}{p} g(\underline{\theta}_p)}]\big(1 + O(r/p^2)\big)$ and as before,

$$\log \mathbb{E}[e^{\pm r f_p(\underline{\theta}_p)}] = \pm r\frac{2g(\underline{\theta}^{\pm})}{p} + \log \mathbb{E}[e^{\mp \frac{r}{p} 2(g(\underline{\theta}^{\pm}) - g(\underline{\theta}_p))}] + O\Big(\frac{r}{p^2}\Big).$$

By (5.4), we may replace $2g(\underline{\theta}^{\pm})/p$ by $f_p(\underline{\varsigma}_p^{\pm})$ subject to an error $O(p^{-2})$. By (8.7), we may replace $d\mu_p$ by $d\mu_{\mathrm{ST}}$ and $\underline{\theta}_p$ by X subject to a factor of $1 + O(p^{-1})$. Thus we infer that for $p > r$,

$$\log \mathbb{E}[e^{\pm r f_p(\underline{\theta}_p)}] = \mathrm{f}_\pm\Big(\frac{r}{p}\Big) + O(p^{-1})$$

and for $p \le r$,

$$\log \mathbb{E}[e^{\pm r f_p(\underline{\theta}_p)}] = \pm r f_p(\underline{\varsigma}_p^{\pm}) + \mathrm{f}_\pm\Big(\frac{r}{p}\Big) + O(rp^{-2}).$$

Separating into cases $p \le \sqrt{r'}$, $\sqrt{r'} < p \le r'^{3/2}$ and $r'^{3/2} < p$, we deduce from above that

$$\sum_p \log \mathbb{E}[e^{\pm r f_p(\underline{\theta}_p)}] = \pm r\sum_{p \le r} f_p(\underline{\varsigma}_p^{\pm}) + \sum_{r'^{1/2} < p \le r'^{3/2}} \mathrm{f}_\pm\Big(\frac{r}{p}\Big) + O(r^{3/4}).$$

(The exponent $3/4$ is not optimal.) A calculation with the prime number theorem shows that the second summation on the right side equals

$$r(\log r)^{-1}\int_0^\infty \frac{\mathrm{f}_\pm(t)}{t^2}\,dt + r(\log r)^{-2}\int_0^\infty \frac{\mathrm{f}_\pm(t)\log t}{t^2}\,dt + O\big(r(\log r)^{-3}\big),$$

cf. [13]. The desired formula follows with a little manipulation, completing the proof. $\square$

9. Complex moments and Proofs of Theorems 2.4, 2.2 and 2.3

We shall show that the complex moment $|L(1,\phi)|^{2z}$ converges in distribution to the probability moment $\mathbb{E}[|L(1,\rho_{\mathrm{St}})|^{2z}]$, and then extract the extreme values. Let us start with some preparatory work.

9.1. The complex moment $|L(1,\phi)|^{2z}$ and its approximation.

As explained in §8.3 (see §7.7 as well), we may express

$$L(s,\phi) = L(s,\rho_{\mathrm{St}})|_{\underline{\theta}_p=\underline{\theta}_\phi(p),\,\forall\, p}$$

and by Remark 3.2 (2) and the unitary condition (7.5),

$$L(s,\widetilde{\phi}) = L(s,\rho_{\mathrm{St}})|_{\underline{\theta}_p=-\overline{\underline{\theta}_\phi(p)},\,\forall\, p} = L(s,\check{\rho}_{\mathrm{St}})|_{\underline{\theta}_p=\underline{\theta}_\phi(p),\,\forall\, p}$$

for $\Re\mathrm{e}\, s \gg 1$. Note $L(s,\widetilde{\phi}) = \overline{L(s,\phi)}$ for $s\in\mathbb{R}$ (in fact, $L(s,\widetilde{\phi}) = \overline{L(\overline{s},\phi)}$).

Suppose $z\in\mathbb{C}$ and let

$$\lambda_\phi^z(m) := \prod_{p^\ell\| m} \lambda^{z,\ell}(\underline{\theta}_\phi(p)) \quad\text{and}\quad \lambda_{\widetilde{\phi}}^z(m) := \prod_{p^\ell\| m} \lambda^{z,\ell}(-\overline{\underline{\theta}_\phi(p)}). \tag{9.1}$$

Then for $\Re\mathrm{e}\, s\gg 1$,

$$L(s,\phi)^z L(s,\widetilde{\phi})^z = \sum_{a\geq 1}\lambda_\phi^z(a)a^{-s}\sum_{b\geq 1}\lambda_{\widetilde{\phi}}^z(b)b^{-s} = \sum_{m\geq 1}\lambda_\phi^z*\lambda_{\widetilde{\phi}}^z(m)m^{-s}.$$

where $\lambda_\phi^z*\lambda_{\widetilde{\phi}}^z$ is the Dirichlet convolution of λ_ϕ^z and $\lambda_{\widetilde{\phi}}^z$.

Lemma 9.1. *Let $z\in\mathbb{C}$ and $x\geq 10^2$. Define for $\phi\in\mathcal{H}^\natural$,*

$$\omega_\phi^z(x) = \sum_{m\geq 1}\lambda_\phi^z*\lambda_{\widetilde{\phi}}^z(m)\frac{e^{-m/x}}{m}.$$

(1) We have $\omega_\phi(x)\ll_{n,\varepsilon} x^{\vartheta+\varepsilon}\exp(\mathsf{c}_n z'(\log_2 x+\log_2 z'))$ for any $\varepsilon>0$.

(2) Suppose $\phi\in\mathcal{K}_T$ (cf. (5.6)). Then we have

$$|L(1,\phi)|^{2z} = \omega_\phi^z(x) + O(E^z(x,T))$$

where

$$E^z(x,T) \;=\; xe^{\mathsf{c}|z|\log T-(\log T)^2} + x^{-1/\log_2 T}e^{\mathsf{c}'(|z|+1)\log_3 T}. \tag{9.2}$$

Proof. (1) By (8.3), Lemma 8.1 and (7.15), we obtain the estimate

$$\begin{aligned}
\lambda_\phi^z(m) &\ll \prod_{p^\ell\| m}\sum_{\boldsymbol{k}}|\mu_{\boldsymbol{k}}^{z,\ell}||S_{\boldsymbol{k}}(\alpha_\phi(p))| \\
&\ll m^\vartheta\prod_{p^\ell\| m}C(n|z|,\ell)\sum_{\boldsymbol{k}:\ \|\boldsymbol{k}\|\leq\ell}S_{\boldsymbol{k}}(1,\cdots,1) \\
&\ll m^\vartheta\tau(m)^{n^2-1}r^{|z|}(m)
\end{aligned}$$

where $r^{|z|}(m)$ is a multiplicative function in m such that $r^{|z|}(p^\ell) = ((n-1)\ell+1)C(n|z|,\ell)$. Here we have used the crude bound $(\ell+1)^n$ for the number of $\boldsymbol{k}$ satisfying $\|\boldsymbol{k}\| \le \ell$, and $S_{\boldsymbol{k}}(1,\cdots,1) \le (1+|\boldsymbol{k}|)^{n^2-n}$. As $2uv \ge u+v$ for $u, v \in \mathbb{N}$, we infer that

$$\omega_\phi^z(x) \ll \Big(\sum_{m\ge1} r^{|z|}(m)\tau(m)^{n^2-1} m^{\vartheta-1} e^{-m/(2x)}\Big)^2.$$

As $r^{|z|}$ is the function r_m^z in [14] (with $m = n-1$), we use the estimate there [14, (6.17)]:

$$\sum_{m\le y} r^{|z|}(m) \ll_n y(\log y)^{n^2 z^*-1} e^{\mathtt{c}|z|\log_2(|z|+3)} \tag{9.3}$$

uniformly for $y \ge 3$ and $z \in \mathbb{C}$, where $z^* = \min\{n \in \mathbb{N} : \ n \ge |z|\}$. Part (1) follows readily with a partial summation.

(2) By the Mellin transform of $\Gamma(s)$, we get that

$$\omega_\phi^z(x) = \frac{1}{2\pi i}\int_{(1)} L(1+s,\phi)^z L(1+s,\widetilde{\phi})^z \Gamma(s) x^s \, ds.$$

Replace the line segment $[1-\mathtt{i}(\log T)^2, 1+\mathtt{i}(\log T)^2]$ with the path joining $1-\mathtt{i}\infty$, $1-\mathtt{i}(\log T)^2$, $-\kappa-\mathtt{i}(\log T)^2$, $-\kappa+\mathtt{i}(\log T)^2$, $1+\mathtt{i}(\log T)^2$, $1+\mathtt{i}\infty$, where $\kappa = 1/\log_2 T$. The pole at $s=0$ contributes $L(1,\phi)^z L(1,\widetilde{\phi})^z = |L(1,\phi)|^{2z}$. On the two horizontal line segments (where $\tau = \pm(\log T)^2$), we have $\Gamma(s) \ll e^{-(\log T)^2}$ and $\log L(s,\phi) \ll \log T$, the integrals over the two horizontal line segments are $\ll x\exp(|z|\log T - (\log T)^2)$. For the integral over $-\kappa-\mathtt{i}(\log T)^2$, $-\kappa+\mathtt{i}(\log T)^2$, we invoke Lemma 5.1, which gives, $\log L(s,\phi) \ll \log_3 T$ when $\phi \in \mathcal{K}_T$; moreover $\int_{(-\kappa)} |\Gamma(s)x^s|\,|ds| \ll \kappa^{-1}x^{-\kappa}$. It is absorbed in $E^z(x,T)$ as well. □

9.2. **Proof of Theorem 2.4.** Let $\mathcal{K}_T$ be defined as in (5.6). By Lemma 9.1, we have

$$\Lambda(T)^{-1}\sum_{\phi\in\mathcal{K}_T} |L(1,\phi)|^{2z} = \Lambda(T)^{-1}\sum_{\phi\in\mathcal{K}_T}\omega_\phi^z(x) + O(E^z(x,T)), \tag{9.4}$$

and with (6.1),

$$\begin{aligned}\Lambda(T)^{-1}\sum_{\phi\in\mathcal{K}_T}\omega_\phi^z(x) &= \Lambda(T)^{-1}\sum_{\phi\in\mathcal{H}_T}\omega_\phi^z(x)\\ &\quad + O\bigg(x^{\vartheta+\varepsilon}\exp\Big(\mathtt{c}z'(\log_2 x+\log_2 z') - \mathtt{c}'\varrho\frac{\log T}{\log_2 T}\Big)\bigg).\end{aligned} \tag{9.5}$$

Resolve $\omega_\phi^z(x)$ with (9.1) and (8.3) into

$$\begin{aligned}\omega_\phi^z(x) &= \sum_{m,m'} \lambda_\phi^z(m)\lambda_{\widetilde{\phi}}^z(m')\frac{e^{-mm'/x}}{mm'}\\ &= \sum_{m,m'}\frac{e^{-mm'/x}}{mm'}\prod_{\substack{p^\ell\|m\\ p^{\ell'}\|m'}}\sum_{\boldsymbol{k}}\sum_{\boldsymbol{k}'}\mu_{\boldsymbol{k}}^{z,\ell}\mu_{\boldsymbol{k}'}^{z,\ell'} S_{\boldsymbol{k}}(\alpha_\phi(p))S_{\boldsymbol{k}'}(\overline{\alpha_\phi(p)}).\end{aligned}$$

With Remark 7.3, we see that

$$\Lambda(T)^{-1}\sum_{\phi\in\mathcal{H}_T}\omega_\phi^z(x) = \Sigma_\omega + \mathcal{E}_\omega \tag{9.6}$$

where

$$\begin{aligned}\Sigma_\omega &= \sum_{m,m'} \frac{e^{-mm'/x}}{mm'} \prod_{\substack{p^\ell \| m \\ p^{\ell'} \| m'}} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{k}'} \mu_{\boldsymbol{k}}^{z,\ell} \mu_{\boldsymbol{k}'}^{z,\ell'} \int_{S^{1^n}/\mathfrak{S}_n} S_{\boldsymbol{k}} \overline{S_{\boldsymbol{k}'}}\, d\mu_p \\ \mathcal{E}_\omega &\ll T^{-1/2} \sum_{m,m'} (mm')^{L-1} e^{-mm'/x} \prod_{\substack{p^\ell \| m \\ p^{\ell'} \| m'}} \sum_{\boldsymbol{k}} \sum_{\boldsymbol{k}'} |\mu_{\boldsymbol{k}}^{z,\ell} \mu_{\boldsymbol{k}'}^{z,\ell'}|.\end{aligned}$$

As

$$\begin{aligned}\sum_{\boldsymbol{k}} \sum_{\boldsymbol{k}'} |\mu_{\boldsymbol{k}}^{z,\ell} \mu_{\boldsymbol{k}'}^{z,\ell'}| &\ll C(n|z|,\ell) C(n|z|,\ell')(\ell+1)^n (\ell'+1)^n \\ &\leq \tau(p^\ell)^{n-1} \tau(p^{\ell'})^{n-1} r^{|z|}(p^\ell) r^{|z|}(p^{\ell'}),\end{aligned}$$

we deduce that

$$\begin{aligned}\mathcal{E}_\omega &\ll T^{-1/2} \Big(\sum_m m^L e^{-m/(2x)} r^{|z|}(m) \Big)^2 \\ &\ll T^{-1/2} x^{L+1} \exp(\mathsf{c} z' (\log_2 x + \log_2 z'))\end{aligned} \tag{9.7}$$

by (9.3).

By Remark 8.1, (8.6), the Mellin transform of $\Gamma(s)$ and Lemma 8.2 (3) with the choice of $\sigma = 1 - 1/\log z'$, we obtain that

$$\begin{aligned}\Sigma_\omega &= \sum_{m,m'} \frac{e^{-mm'/x}}{mm'} \mathbb{E}[\lambda^z(m) \check{\lambda}^z(m')], \\ &= \frac{1}{2\pi \mathtt{i}} \int_{(1)} \mathbb{E}[L(1+s,\rho_{\mathrm{St}})^z L(1+s,\check{\rho}_{\mathrm{St}})^z] \Gamma(s) x^s\, ds \\ &= \mathbb{E}[|L(1,\rho_{\mathrm{St}})|^{2z}] + O\big(x^{-1/\log z'} \exp\big(\mathsf{c} z' \log_2 z'\big)\big).\end{aligned} \tag{9.8}$$

In view of the O-terms in (9.5), we set $x = \exp(\varpi \rho(\log T)/(\log_2 T))$ where $\varpi > 0$ is a small enough constant. Then the O-terms in (9.5), (9.7) and (9.8) are

$$\ll \exp\left(\mathsf{c}\left(z' \log_2 T - \rho \frac{\log T}{\log_2 T}\right)\right) + \exp\left(\mathsf{c}'\left(z' \log_2 z' - \frac{\rho \log T}{(\log_2 T)(\log z')}\right)\right),$$

which is absorbed in $O(\exp(-\mathsf{c}\rho(\log T)/(\log_2 T)^2))$ when

$$z' \leq \delta Z(T) = \delta \rho (\log T)/((\log_2 T)^2 \log_3 T)$$

for a small constant $\delta > 0$. Taking a smaller δ if necessary, we see from (9.2) that this O-term suppresses $E^z(x,T)$ as well. Thus, from (9.4)-(9.8),

$$\begin{aligned}&\Lambda(T)^{-1} \sum_{\phi \in \mathcal{K}_T} |L(1,\phi)|^{2z} \\ &= \mathbb{E}[|L(1,\rho_{\mathrm{St}})|^{2z}] + O\bigg(\exp\bigg(-\mathsf{c} \frac{\log T}{(\log_2 T)^2 (\log_3 T)^{1+\eta}} \bigg)\bigg)\end{aligned}$$

for $|z| \leq \delta Z(T)$. Our result follows by replacing $\Lambda(T)$ with $\#(\mathcal{K}_T)$, which is legitimate in light of Theorem 2.1 (1) and $\#(\mathcal{H}_T) = \Lambda(T)(1 + O(T^{-1/2}))$; the constants c and δ are suppressed as the exponents of $\log_3 T$'s are relaxed (to bigger values).

9.3. **Proof of Theorems 2.2 and 2.3.** Theorem 2.2 follows from Theorem 2.3 with the choice of $\tau = \log_2 T - (2 + o(1)) \log_3 T$, so $\tau = \{1 + o(1)\} \log_2 T$ and $\tau^{-1} e^{\tau} = (\log T)/(\log_2 T)^{3+o(1)}$. With Proposition 8.3 and Theorem 2.4, one may follow [36, Section 7] to obtain Theorem 2.3. The shorter range in our case is due to the smaller region of z in Theorem 2.4. This ends our proof.

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