SOME RECENT RESULTS ON HOLOMORPHIC ISOMETRIES OF THE COMPLEX UNIT BALL INTO BOUNDED SYMMETRIC DOMAINS AND RELATED PROBLEMS

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Abstract. In his seminal work Calabi established the foundation on the study of holomorphic isometries from a Kähler manifold with real analytic local potential functions into complex space forms, e.g., Fubini-Study spaces. This leads to interior extension results on germs of holomorphic isometries between bounded domains. General results on boundary extension were obtained by Mok under assumptions such as the rationality of Bergman kernels, which applies especially to holomorphic isometries between bounded symmetric domains in their Harish-Chandra realizations. Because of rigidity results in the cases where the holomorphic isometry is defined on an irreducible bounded symmetric domain of rank ≥ 2, we focus on holomorphic isometries defined on the complex unit ball $\mathbb{B}^n, n \geq 1$. We discuss results on the construction, characterization and classification of holomorphic isometries of the complex unit ball into bounded symmetric domains and more generally into bounded homogeneous domains. Furthermore, in relation to the study of the Hyperbolic Ax-Lindemann Conjecture for not necessarily arithmetic quotients of bounded symmetric domains, such holomorphic isometric embeddings play an important role. We also present some differential-geometric techniques arising from the study of the latter conjecture.

The subject of holomorphic isometries between Kähler manifolds is a classical topic in complex differential geometry going back to Bochner and Calabi. Especially, starting from the seminal work of Calabi [Ca53], in which questions of existence, uniqueness and analytic continuation of holomorphic isometries of Kähler manifolds into space forms such as the Euclidean and the Fubini-Study spaces were systematically studied, tools have been developed, notably using normalized potential functions called *diastases* defined in [Ca53], for the study of germs of holomorphic isometries between Kähler manifolds. The author was led to consider such questions especially for bounded symmetric domains equipped with the Bergman metric, in part to answer questions concerning commutants of modular correspondences on such domains raised by Clozel-Ullmo [CU03], who reduced rigidity problems in this context to questions in complex differential geometry including one on holomorphic isometries. These questions led the author to systematically study germs of holomorphic isometries between bounded domains in Euclidean spaces. Embedding a bounded domain by means of an orthonormal basis of the Hilbert space $H^2(U)$ of square integrable holomorphic functions into the

*Research partially supported by the GRF 17303814, of the HKRGC, Hong Kong.

Fubini-Study space $\mathbb{P}^\infty$ of countably infinite dimension, analytic continuation of germs of holomorphic isometries with respect to scalar multiples of the Bergman metric already follows from Calabi [Ca53], and the author has been focusing on boundary behavior of such analytic continuation as the domain of definition traverses the boundary of bounded domains under the assumption that Bergman kernels are rational.

In [Mo11] the author wrote a survey article "*Geometry of holomorphic isometries and related maps between bounded domains*", giving a historical account starting with Calabi [Ca53], explaining the motivation of the author's works in the subject area, and posing a number of open questions. There the related maps include holomorphic measure-preserving maps from a bounded symmetric domain into a Cartesian power of the domain in the terminology of Clozel-Ullmo [CU03]. In recent years the author has continued to work on various problems on the existence, uniqueness, characterization and classifications on holomorphic isometries especially between bounded symmetric domains (and sometimes more generally bounded homogeneous domains) with respect to scalar multiples of the Bergman metric. While the reader would benefit from reading the current article in conjunction with [Mo11], we note that the current article is *not a survey* on the subject, but rather an annotated account of the overall structure of works of the author and collaborators in the years since [Mo11] in the research area, focusing on the topic of holomorphic isometries between bounded symmetric domains and leaving aside other types of related holomorphic mappings. Statements of our principal results will be given together with brief discussions on the context, motivation and methodology, while we will mention recent developments on the topic from other researchers, and refer the reader to consult their original articles. As such the current article serves more as a *Leitfaden* on the author and his collaborators' recent works on the topic, together with an excursion on possible future links of the study of holomorphic isometries with other domains of research, notably with the theory of geometric structures and substructures, functional transcendence theory and the geometry of flag domains. For a discussion in those direction we refer the reader to §6 on "*Perspectives and concluding remarks"* and to the last two paragraphs of §5.

1. Introduction, first examples and background results

Given complex manifolds (X, ω_X) and (Y, ω_Y) equipped with respective Kähler forms, a holomorphic mapping $f : X \to Y$ is a holomorphic isometry if and only if $f^*\omega_Y = \omega_X$. If there exist global potential functions so that $\omega_X = \sqrt{-1}\partial\overline{\partial}\varphi_X$ and $\omega_Y = \sqrt{-1}\partial\overline{\partial}\varphi_Y$, then the holomorphic map $f : X \to Y$ is an isometry if and only if $\sqrt{-1}\partial\overline{\partial}(\varphi_X - f^*\varphi_Y) = 0$, i.e., $h(x) := \varphi(x) - \varphi_Y(f(x))$ is a pluriharmonic function, equivalently locally the real part of a holomorphic function. This simplification applies when we consider Bergman metrics on bounded domains U, since by definition there are global potential functions given by the $\log(k_U)$, where $k_U(x) = K_U(x, x)$ for the Bergman kernel $K_U(z, w)$ on U. To verify that f is a holomorphic isometry it is sufficient to check that $\varphi_X - f^*\varphi_Y$ is a constant, and this applies to give first examples of nonstandard holomorphic isometries from disks to polydisks. Equip the upper half-plane $\mathscr{H}$ with the Poincaré metric $ds^2_{\mathscr{H}} = \mathrm{Re}\frac{d\tau\otimes d\overline{\tau}}{(\mathrm{Im}\tau)^2}$ of constant Gaussian curvature -1 and $\mathscr{H}^2$ with the product metric. Then, the proper holomorphic map $f : \mathscr{H} \to \mathscr{H}^2$ given by $f(\tau) = \left(\sqrt{\tau}, i\sqrt{\tau}\right)$ is a holomorphic isometric embedding. More generally, we have the p-th root map given by

Proposition 1.1. (Mok [Mo12a]) *Let $p \geq 2$ be a positive integer and $\gamma = e^{\frac{\pi i}{p}}$. Then, the proper holomorphic mapping $f : (\mathscr{H}, ds^2_{\mathscr{H}}) \to (\mathscr{H}, ds^2_{\mathscr{H}})^p$ defined by*

$$f(\tau) = \left(\tau^{\frac{1}{p}}, \gamma\tau^{\frac{1}{p}}, \ldots, \gamma^{p-1}\tau^{\frac{1}{p}}\right)$$

is a holomorphic isometric embedding.

Proposition 1.1 results simply from the fact that we have for $p \geq 2$ the trigonometric identity $\sin\theta \sin\left(\frac{\pi}{p} + \theta\right) \cdots \sin\left(\frac{(p-1)\pi}{p} + \theta\right) = c_p \sin(p\theta)$ for some positive constant c_p.

For a positive integer g denote by $M_s(g, \mathbb{C})$ the complex vector space of symmetric g-by-g matrices, and write $\mathscr{H}_g \subset M_s(g, \mathbb{C})$ for the Siegel upper half-plane of genus g defined by $\mathscr{H}_g = \left\{\tau \in M_s(g, \mathbb{C}) : \mathrm{Im}(\tau) > 0\right\}$. Another early example of a nonstandard holomorphic isometric embedding is given by the following map of the upper half-plane $\mathscr{H}$ into $\mathscr{H}_3$, together with a verification that it does not arise from p-th root maps. From now on for a domain $U \subset \mathbb{C}^n$ biholomorphic to a bounded domain, we will denote by ds^2_U the Bergman metric on U. Note that when U is a homogeneous domain (U, ds^2_U) is Kähler-Einstein and its Ricci curvatures are equal to -1.

Proposition 1.2. (Mok [Mo12a]) *For $\zeta = \rho e^{i\varphi}$, $\rho > 0$, $0 < \varphi < \pi$, n a positive integer, we write $\zeta^{\frac{1}{n}} := \rho^{\frac{1}{n}} e^{\frac{i\varphi}{n}}$. Then, the holomorphic mapping $G : \mathscr{H} \to M_s(3, \mathbb{C})$ defined by*

$$G(\tau) = \begin{bmatrix} e^{\frac{\pi i}{6}}\tau^{\frac{2}{3}} & \sqrt{2}e^{-\frac{\pi i}{6}}\tau^{\frac{1}{3}} & 0 \\ \sqrt{2}e^{-\frac{\pi i}{6}}\tau^{\frac{1}{3}} & i & 0 \\ 0 & 0 & e^{\frac{\pi i}{3}}\tau^{\frac{1}{3}} \end{bmatrix}$$

maps $\mathscr{H}$ into $\mathscr{H}_3$, and $G : (\mathscr{H}, 2ds^2_{\mathscr{H}}) \to \left(\mathscr{H}_3, ds^2_{\mathscr{H}_3}\right)$ is a holomorphic isometry.

In what follows for an integer $p \geq 2$ we write $\rho_p : \mathscr{H} \to \mathscr{H}^p$ for the p-th root map as given by $f(\tau)$ in Proposition 1.1. We denote by $\iota : \mathscr{H}^p \to \mathscr{H}_p$ the standard inclusion of $\mathscr{H}^p$ into the Siegel upper half-plane $\mathscr{H}_p$ of genus p as a set of diagonal matrices given by $\iota(\tau_1, \cdots, \tau_p) = \mathrm{diag}(\tau_1, \cdots, \tau_p)$.

Proposition 1.3. (Mok [Mo12a]) *The two holomorphic isometric embeddings $F, G : (\mathscr{H}, 2ds^2_{\mathscr{H}}) \hookrightarrow (\mathscr{H}_3, ds^2_{\mathscr{H}_3})$, $F := \iota \circ \rho_3$, are not congruent to each other. In fact, for any holomorphic isometric embedding $h : \mathscr{H} \hookrightarrow \mathscr{H} \times \mathscr{H} \times \mathscr{H}$, and for $H := \iota \circ h$, the two holomorphic embeddings $G, H : (\mathscr{H}, 2ds^2_{\mathscr{H}}) \hookrightarrow (\mathscr{H}_3, ds^2_{\mathscr{H}_3})$ are incongruent to each other.*

Here two holomorphic isometries $f, g : (D, \lambda ds_D^2) \to (\Omega, ds_\Omega^2)$ are said to be congruent to each other if and only if there exist $\varphi \in \mathrm{Aut}(D)$ and $\psi \in \mathrm{Aut}(\Omega)$ such that $g = \psi \circ f \circ \varphi$.

The main result of Mok [Mo12a] is the following theorem on the analytic continuation of germs of holomorphic isometries with respect to multiples of the Bergman metric under the assumption that Bergman kernels of both the domain and the target are rational.

Theorem 1.1. (Mok [Mo12a]) *Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded domains. Let $\lambda > 0$ and $f : (D, \lambda\, ds_D^2; x_0) \to (\Omega, ds_\Omega^2; y_0)$ be a germ of holomorphic isometry with respect to Bergman metrics up to a normalizing constant. Assume that the Bergman metrics on D and Ω are complete, that the Bergman kernel $K_D(z,w)$ on D extends to a rational function in $(z, \overline{w})$, and that analogously the Bergman kernel $K_\Omega(\xi,\eta)$ extends to a rational function in $(\xi, \overline{\eta})$. Then, f extends to a proper holomorphic isometric embedding $F : (D, \lambda\, ds_D^2) \hookrightarrow (\Omega, ds_\Omega^2)$. Moreover,* $\mathrm{Graph}(f) \subset D \times \Omega$ *extends to an affine-algebraic subvariety $V \subset \mathbb{C}^n \times \mathbb{C}^N$.*

Note that if in Theorem 1.1 we weaken the hypothesis to assuming that $K_D(z,w)$ extends to a meromorphic function in $(z, \overline{w})$ to a neighborhood of $\overline{D} \times D'$ and likewise $K_\Omega(\xi,\eta)$ extends to a meromorphic function in $(\xi, \overline{\eta})$ to a neighborhood of $\overline{\Omega} \times \Omega'$, where for a Euclidean domain $G \subset \mathbb{C}^n$ we write $G' = \{z \in \mathbb{C}^n : \overline{z} \in G\}$, Theorem 1.1 holds for the germ of holomorphic isometry f when the last sentence is replaced by the statement that $\mathrm{Graph}(f) \subset D \times \Omega$ extends to a complex-analytic subvariety on some neighborhood of $\overline{D} \times \overline{\Omega}$.

Consider the special case where D and Ω are complete circular domains, e.g., bounded symmetric domains in their Harish-Chandra realizations, and let $f : (D, \lambda\, ds_D^2; 0) \to (\Omega, ds_\Omega^2; 0)$ be a holomorphic isometry. From $f(0) = 0$ and the invariance of the Bergman kernels K_D and K_Ω under the circle group action, expanding in Taylor series at $0 \in D$ we have actually the equality $\log K_\Omega(f(z), f(z)) = \lambda \log K_\Omega(z,z) + a$ for some constant a, and hence by polarization the holomorphic functional identities $(\mathbf{I_{w_0}})$ $\log K_\Omega(f(z), f(w_0)) = \lambda \log K_\Omega(z, w_0) + a$ in z for any w_0 belonging to a sufficiently small neighborhood of $0 \in D$. If K_D and K_Ω are rational as assumed then differentiating the identities $(\mathbf{I_{w_0}})$ one can remove the logarithm and equivalently consider an infinite system $(\mathbf{J_{w_0}})$ of *algebraic* holomorphic identities. If now we replace $f(z)$ by the complex variables $\zeta = (\zeta_1, \cdots, \zeta_N)$, we obtain for each $w_0 \in U$ a subset $V_{w_0} \subset \mathbb{C}^n \times \mathbb{C}^N$ consisting of all (z, ζ) satisfying $(\mathbf{J_{w_0}})$, then the common zero set $V := \bigcap \{V_{w_0} : w_0 \in U\} \subset \mathbb{C}^n \times \mathbb{C}^N$ is an affine-algebraic subvariety containing $\mathrm{Graph}(f)$, and the difficulty was to prove that $\mathrm{Graph}(f) \subset V$ is an open subset, so that $V \supset \mathrm{Graph}(f)$ yields the desired extension, and that was precisely what was established in [Mo12a]. The same argumentation applies after modification to the general situation in Theorem 1.1 where D and Ω need not be circular domains and where $x_0 \in D$ and $y_0 \in \Omega$ are arbitrary base points to yield a proof of the theorem.

We note that in the special case concerning commutants of modular correspondences raised in Clozel-Ullmo [CU03], the unsolved case was when $D = \mathbb{B}^n$, $n \geq 2$, and $\Omega = \mathbb{B}^n \times \cdots \times \mathbb{B}^n$ (with p factors), and the germ of holomorphic isometry up to scaling factors is given $f = (f^1, \cdots f^p)$, $f^k : (\mathbb{B}^n; 0) \to (\mathbb{B}^n; 0)$, such that $\det(df^k) \not\equiv 0$. In that case the corresponding special case in Theorem 1.1 is much easier, and was already obtained by the author in Mok [Mo02] with the stronger conclusion that each f^k, $1 \le k \le p$ extends to an automorphism of $\mathbb{B}^n$, an assertion that follows after analytic continuation of each f^k across $\partial\mathbb{B}^n$ has been established by means of Alexander's theorem [Al70].

2. Existence and classification results

With reference to Theorem 1.1, the extension theorem for germs of holomorphic isometries, we will now specialize to the case where $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ are bounded symmetric domains in their Harish-Chandra realizations. As mentioned, in these cases the Bergman kernel $K_D(z, w)$ resp. $K_\Omega(\xi, \eta)$ is a rational function in $(z, \overline{w})$ resp. $(\xi, \overline{\eta})$. A bounded symmetric domain is always of nonpositive holomorphic bisectional curvature. If we restrict to the case where D is irreducible, as observed in Clozel-Ullmo [CU03], it follows from the proofs of Mok [Mo87] and [Mo89] on Hermitian metric rigidity that any germ of holomorphic isometry $f : (D, ds^2_D; x_0) \to (\Omega, ds^2_\Omega; y_0)$ must necessarily be totally geodesic whenever D is of rank ≥ 2. Thus, the interesting case is where $D \cong \mathbb{B}^n$ is the n-dimensional complex unit ball, $n \ge 2$, in which case $(\mathbb{B}^n, ds^2_{\mathbb{B}^n})$ is of strictly negative holomorphic bisectional curvature. The first examples were holomorphic isometric embeddings of the Poincaré disk into bounded symmetric domains, and for some time it was unknown whether nonstandard holomorphic isometries could exist when $n \ge 2$. This was raised as Problem 5.1.3 in [Mo11]. When Ω is itself a complex unit ball $\mathbb{B}^N$, it follows from Umehara [Um87] that any germ of holomorphic isometry $f : (\mathbb{B}^n, \lambda ds^2_{\mathbb{B}^n}; x_0) \to \left(\mathbb{B}^N, ds^2_{\mathbb{B}^N}; y_0\right)$ must necessarily be totally geodesic. In fact it was proven in [Um87] that any germ of Kähler-Einstein complex submanifold on $\left(\mathbb{B}^N, ds^2_{\mathbb{B}^N}\right)$ must necessarily be totally geodesic. For the case where Ω is an irreducible bounded symmetric domain of rank ≥ 2, *a priori* we have the following restriction on the maximal dimension of a holomorphically and isometrically embedded complex unit ball. Here for the formulation we normalize the scalar multiple of the Bergman metric, which is Kähler-Einstein, such that minimal disks are of constant Gaussian curvature -2. The canonical Kähler-Einstein metric on Ω chosen this way will be denoted by h, and those on a complex unit ball $\mathbb{B}^m$ will be denoted by g, or by g_n when the dimension n is important for the discussion.

Theorem 2.1. (Mok [Mo16a]) *Let $\Omega \subset \Sigma$ be the Borel embedding of an irreducible bounded symmetric domain Ω into its dual Hermitian symmetric manifold Σ of the compact type, where* $\mathrm{Pic}(\Sigma) \cong \mathbb{Z}$, *generated by the positive line bundle $\mathcal{O}(1)$. Let g resp. h be the canonical Kähler-Einstein metric on $\mathbb{B}^n$ resp. Ω normalized so that minimal disks on $\mathbb{B}^n$ resp. Ω are of constant Gaussian curvature -2. Let $p = p(\Omega)$ be the nonnegative integer such that $K_\Sigma^{-1} \cong \mathcal{O}(p+2)$. Suppose $F : (B^n, g) \to (\Omega, h)$ is a holomorphic isometry (which is necessarily a proper holomorphic isometric embedding). Then $n \le p + 1$.*

For a uniruled projective manifold X equipped with a minimal rational component $\mathcal{K}$ and for a standard minimal rational curve ℓ (assumed smooth for convenience) belonging to $\mathcal{K}$ we have the Grothendieck decomposition $T(X)|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some $p \geq 0$ independent of the choice of ℓ. From the deformation theory of rational curves, for a general point $x \in X$ and denoting by $\mathscr{C}_x(X)$ the variety of minimal rational tangents (VMRT) of $(X, \mathcal{K})$ at x, i.e., the variety of projectivizations of tangents to minimal rational curves passing through x (cf. Mok [Mo16b]), p is exactly the dimension of $\mathscr{C}_x(X)$. When X is of Picard number 1 and $\mathcal{K}$ pertains to a uniruling by rational curves of degree 1, as is the case of irreducible Hermitian symmetric spaces Σ of the compact type, we have $K_X^{-1} \cong \mathcal{O}(p+2)$. Since Σ is in particular homogeneous, the positive integer p in Theorem 2.1 is exactly the dimension of the VMRT at any point of Σ. We have also

Theorem 2.2. (Mok [Mo16a]) *Let $n \geq 1$, $\lambda > 0$, and $F : (\mathbb{B}^n, \lambda\, g) \to (\Omega, h)$ be a holomorphic isometry such that $\overline{F(\mathbb{B}^n)} \cap \mathrm{Reg}(\partial\Omega) \neq \emptyset$. Then, $\lambda = 1$ and $n \leq p + 1$.*

The basis of both Theorem 2.1 and Theorem 2.2 lies in the structure of $\partial\Omega$ (cf. Wolf [Wo72]). Writing $G_0 = \mathrm{Aut}_0(\Omega)$ and $r := \mathrm{rank}(\Omega)$, $\partial\Omega$ decomposes into r disjoint union of G_0-orbits E_i, $1 \leq i \leq r$, such that $E_{k+1} \subset \overline{E_k}$. E_1 is the same as the smooth part $\mathrm{Reg}(\partial\Omega)$, which is foliated by maximal complex submanifolds of dimension $N - p - 1$. Consider the strictly plurisubharmonic function $\varphi_\Omega(z) := \frac{1}{p+2} \log K_\Omega(z, z)$, where $K_\Omega(z, w)$ stands for the Bergman kernel on Ω. Then, $\varphi_\Omega = -\log(-\rho_\Omega)$ where ρ_Ω is a real-analytic defining function of $\partial\Omega$ at any point on $\mathrm{Reg}(\partial\Omega)$.

By Theorem 1.1, for any holomorphic isometry F from the complex unit ball $\mathbb{B}^n$ to Ω with respect to scalar multiples of the Bergman metric, $\mathrm{Graph}(F)$ must necessarily extend to an affine-algebraic variety $V \subset \mathbb{C}^n \times \mathbb{C}^N$. From the assumptions in either Theorem 2.1 or Theorem 2.2 one can deduce that for a general point $a \in \partial\mathbb{B}^n$ there exists an open neighborhood G of a in $\mathbb{C}^n$ such that $F|_{G\cap\mathbb{B}^n}$ extends holomorphically to a holomorphic embedding $F^\sharp$ of G onto an n-dimensional complex submanifold $Z \subset U$ of some open neighborhood U of $b = f(a) \in \mathrm{Reg}(\partial\Omega)$. (For b sufficiently general the embedding can be chosen such that $(F^\sharp)^*\rho_\Omega$ is a defining function of $\mathbb{B}^n$ at a (i.e., it has nonzero gradient at a) although this fact need not be used in what follows.) From the strict pseudoconvexity of $\mathbb{B}^n$ it follows that the nonnegative Levi form $\sqrt{-1}\partial\overline{\partial}\rho_\Omega|_{T_b^{1,0}(Z\cap\partial\Omega)}$ has $n-1$ positive eigenvalues, and the dimension estimate $n \leq p + 1$ follows from the foliated structure of $\mathrm{Reg}(\partial\Omega)$ described in the last paragraph, proving both Theorem 2.1 and Theorem 2.2.

Note that for a local strictly pseudoconvex domain U with smooth boundary defined by $\rho < 0$, where $d\rho$ is nowhere zero and ρ is strictly plurisubharmonic, letting θ be the Kähler metric with Kähler form $\sqrt{-1}\partial\overline{\partial}(-\log(-\rho))$, by a computation of Klembeck [Kl78] (U, θ) is asymptotically of constant holomorphic sectional curvature -2 along the real hypersurface $\rho = 0$.

Going in the opposite direction we proved in [Mo16a] the existence of nonstandard holomorphic isometric embeddings of the $(p+1)$-dimensional complex unit ball into Ω.

Theorem 2.3. (Mok [Mo16a]) *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2 and denote by Σ the irreducible Hermitian symmetric manifold of the compact type dual to Ω. Denoting by $\delta \in H^2(\Sigma,\mathbb{Z}) \cong \mathbb{Z}$ the positive generator of the second integral cohomology group of Σ, we write $c_1(\Sigma) = (p+2)\delta$. Then, there exists a nonstandard proper holomorphic isometric embedding $F : \big(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\big) \hookrightarrow (\Omega, ds^2_\Omega)$. More precisely, letting $\Omega \subset \Sigma$ be the Borel embedding of Ω into its dual Hermitian symmetric space of the compact type Σ, and denoting by $\mathscr{V}_x$ the union of all minimal rational curves on Σ passing through a point $x \in \Sigma$, for any smooth boundary point $q \in \mathrm{Reg}(\partial\Omega)$, the intersection $V_q := \mathscr{V}_q \cap \Omega$ is the image of a holomorphic isometric embedding $F_q : \big(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\big) \hookrightarrow (\Omega, ds^2_\Omega)$.*

The proof of Theorem 2.3 is geometric, and it follows from the aforementioned computation of Klembeck [Kl78] on asymptotic curvature behavior along strictly pseudoconvex boundary points. The bounded symmetric domain $\Omega \Subset \mathbb{C}^N$ in its Harish-Chandra realization is a complete circular domain such that, given any complex line ℓ passing through $0 \in \Omega$, $\ell \cap \Omega$ is a disk centered at $0 \in \ell$ of radius between 1 and $\sqrt{r}$, $r := \mathrm{rank}(\Omega)$, and it is of radius 1 if and only if $[T_0\ell] \in \mathscr{C}_0(\Sigma)$. Consider $V_0 = \mathscr{V}_0 \cap \Omega$. Then $V_0 = \mathscr{V}_0 \cap \mathbb{B}^N$, and thus $\partial V_0 \subset \mathscr{V}_0$ is a strictly pseudoconvex real hypersurface. When $r \geq 2$, $\mathscr{V}_0$ is smooth except for the isolated singularity at 0. By Klembeck [Kl78], the normalized Bergman metric $h := \frac{1}{p+2} ds^2_\Omega$ is asymptotically of constant holomorphic sectional curvature -2 along ∂V_0. Fix any line ℓ_0 passing through 0 such that $[T_0(\ell_0)] \in \mathscr{C}_0(\Sigma)$, pick any boundary point q of the minimal disk $\Delta_0 := \ell_0 \cap \Omega$, and consider a real one-parameter subgroup $\big\{\Phi_t : t \in \mathbb{R}\big\}$ of transvections in $\mathrm{Aut}_0(\Omega)$ fixing Δ_0 as a set such that $\Phi_t(0), t \geq 0$, traverses a geodesic ray and converges to q in the Euclidean topology as $t \to \infty$. Then, $\Phi_t(V_0) = V_{\Phi_t(0)}$ and $V_{\Phi_t(0)}$ converges to V_q as subvarieties as $t \to \infty$. As a Kähler submanifold of (Ω, h), the local differential geometry of $(V_q, h|_{V_q})$ is identical to the asymptotic geometry of $\big(V_0, h|_{\mathrm{Reg(V_0)}}\big)$, hence $(V_q, h|_{V_q})$ is of constant holomorphic sectional curvature -2, and it must be the image of a holomorphic isometry of $(\mathbb{B}^{p+1}, g)$ into (Ω, h) by Theorem 1.1. (In this case it already follows from Calabi [Ca53]). Since $ds_{\mathbb{B}^{p+1}} = (p+2)g$ and $ds_\Omega = (p+2)h$, we have equivalently that $F : \big(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\big) \hookrightarrow (\Omega, ds^2_\Omega)$ is a holomorphic isometric embedding.

After constructing examples of holomorphic isometric embeddings from complex unit balls into irreducible bounded symmetric domains Ω as in Theorem 2.3, and given the dimension estimates on the complex unit balls on which such isometries may be defined, it is natural to study the set of all holomorphic isometries from $(\mathbb{B}^{p+1}, g)$ into (Ω, h). For this reason the author proposes an approach which reduces the problems to questions in the theory of geometric substructures (VMRT-substructures) on projective manifolds uniruled by projective lines. We note that this approach, which will be discussed in §3, applies in principle to all irreducible bounded symmetric domains excepting those of type IV, i.e., the n-dimensional Lie spheres D_n^{IV}, $n \geq 3$, and for that reason for some time it was not clear whether one should expect nonstandard holomorphic isometries of $F : \left(\mathbb{B}^{n-1}, ds^2_{\mathbb{B}^{n-1}}\right) \hookrightarrow (D_n^{IV}, ds^2_{D_n^{IV}})$ other than those defined by cones of minimal rational curves as given in Theorem 2.3. It turns out that such examples do exist. In fact, by a manipulation of polarized forms of functional equations arising from equating potential functions for Kähler metrics as in [Mo12a], Chan-Mok [CM17a] was able to completely classify and describe all holomorphic isometries of complex unit balls into D_n^{IV} as given in the following theorem and its corollary.

The irreducible bounded symmetric domain in $\mathbb{C}^n$, $n \geq 3$, of type IV is given by

$$D_n^{IV} = \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 2,\ \sum_{j=1}^n |z_j|^2 < 1 + \left| \tfrac{1}{2} \sum_{j=1}^n z_j^2 \right|^2 \right\}$$

and the Kähler form corresponding to the Bergman metric $ds^2_{D_n^{IV}}$ on D_n^{IV} is given by $\omega_{ds^2_{D_n^{IV}}} = -n\sqrt{-1}\partial\overline{\partial} \log \left(1 - \sum_{j=1}^n |z_j|^2 + \left| \frac{1}{2} \sum_{j=1}^n z_j^2 \right|^2 \right)$. We have $h = \frac{1}{n} ds^2_{D_n^{IV}}$. For $\mathbf{v} \in M(1, n; \mathbb{C})$ we write $V_{\mathbf{v}} \subseteq \mathbb{C}^n$ for the affine-algebraic subvariety defined by $\sum_{j=1}^n v_j z_j - \frac{1}{2} \sum_{j=1}^n z_j^2 = 0$, and we write $\Sigma_{\mathbf{v}} := V_{\mathbf{v}} \cap D_n^{IV}$. Manipulating the functional equations as in Mok [Mo12a] relating Bergman kernels $K_{\mathbb{B}^m}(z, w)$ and $K_{D_n^{IV}}(\xi, \eta)$ via holomorphic isometries $F : (\mathbb{B}^m, \lambda ds^2_{\mathbb{B}^m}) \hookrightarrow (D_n^{IV}, ds^2_{D_n^{IV}})$, $\xi := F(z), \eta = F(w)$, we were able to completely classify holomorphic isometries from complex unit balls into type-IV domains with respect to scalar multiples of the Bergman metric, as follows.

Theorem 2.4. (Chan-Mok [CM17a]) *Let $F : (\mathbb{B}^m, \lambda ds^2_{\mathbb{B}^m}) \hookrightarrow (D_n^{IV}, ds^2_{D_n^{IV}})$ be a holomorphic isometric embedding, where $n \geq 3$ and $m \geq 1$ are integers. Then, either $\lambda = \frac{n}{m+1}$ or $\lambda = \frac{2n}{m+1}$ and we have the following.*

1) *If $\lambda = \frac{n}{m+1}$, then $1 \leq m \leq n-1$ and $F = \widetilde{f} \circ \rho$ for some holomorphic isometric embedding $\widetilde{f} : (\mathbb{B}^{n-1}, g_{n-1}) \hookrightarrow (D_n^{IV}, h)$ and some totally geodesic holomorphic isometric embedding $\rho : (\mathbb{B}^m, g_m) \hookrightarrow (\mathbb{B}^{n-1}, g_{n-1})$.*
2) *If $\lambda = \frac{n}{m+1}$ and $m = n-1$, then F is congruent to a nonstandard holomorphic isometric embedding $\widehat{F}_{\mathbf{c}} : (\mathbb{B}^{n-1}, g_{n-1}) \hookrightarrow (D_n^{IV}, h)$ such that $\widehat{F}_c(\mathbb{B}^{n-1})$ is the irreducible component of $\Sigma_{\mathbf{c}}$ containing $\mathbf{0}$ for some $\mathbf{c} \in M(1, n; \mathbb{C})$ satisfying $\mathbf{c}\overline{\mathbf{c}}^t = 1$. In addition, F is congruent to $F_q : \left(\mathbb{B}^{n-1}, ds^2_{\mathbb{B}^{n-1}}\right) \hookrightarrow \left(D_n^{IV}, h\right)$ for $q \in \mathrm{Reg}(\partial\Omega)$ (as given in Theorem 2.3) if and only if F is congruent to $\widehat{F}_{\mathbf{c}}$ for some $\mathbf{c}$ satisfying $\mathbf{c}\mathbf{c}^t = 0$.*
3) *If $\lambda = \frac{2n}{m+1}$, then $m = 1$ and $F : (\Delta, n ds^2_{\Delta}) \hookrightarrow \left(D_n^{IV}, ds^2_{D_n^{IV}}\right)$ is totally geodesic.*

Corollary 2.1. (Chan-Mok [CM17a]) *Let $F : (\mathbb{B}^m, g_m) \hookrightarrow (D_n^{IV}, h)$ be a holomorphic isometric embedding, where $1 \leq m \leq n-2$ and $n \geq 3$. Then F is induced by some holomorphic isometric embedding $\widetilde{f} : (\mathbb{B}^{n-1}, g_{n-1}) \hookrightarrow (D_n^{IV}, h)$ via slicing of $\mathbb{B}^{n-1}$. More precisely $F = \widetilde{f} \circ \rho$ for some totally geodesic holomorphic isometric embedding $\rho : (\mathbb{B}^m, g_m) \hookrightarrow (\mathbb{B}^{n-1}, g_{n-1})$.*

Upmeier-Wang-Zhang [UWZ17] and Xiao-Yuan [XY16] have independently obtained the classification result in Theorem 2.4 (on type-IV domains) for the case of $m = n-1$ and they give explicit parametrizations of the maps. Functional equations were made use of in [XY16] while in [UWZ17] the authors studied operators on Hilbert spaces induced by holomorphic isometries and made use of Jordan algebras. Moreover, for an arbitrary bounded symmetric domain Ω of rank ≥ 2 they gave an interesting characterization in terms of Jordan algebras of the holomorphic isometric embedding $F_q : \left(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\right) \hookrightarrow (\Omega, ds^2_\Omega)$, as defined in Theorem 2.3, among all holomorphic isometric embeddings of $\mathbb{B}^{p+1}$ into Ω, where $p = p(\Omega)$.

Finally, we state a result with a sketch of the proof concerning bounded homogeneous domains which can be established along the lines of argument of Theorem 2.3. The proof of the latter theorem involves an argument ascertaining the convergence of a sequence of subvarieties which are the images of $V_0 = \mathscr{V}_0 \cap \Omega$ by a sequence of automorphisms, thereby obtaining in the limit a complex submanifold V_q which reflects the asymptotic geometry of V_0 near strictly pseudoconvex boundary points. For this type of argument to work on a bounded symmetric domain Ω, it is sufficient to have from the very beginning a local complex submanifold $Z \subset U$ on a neighborhood of a smooth boundary point $b \in \partial U$ such that Z intersects $\partial\Omega$ transversally along $Z \cap \partial\Omega$, $Z \cap \Omega \subset Z$ is strictly pseudoconvex along $Z \cap \partial\Omega$, a sequence of points $x_n \in Z$, $n \geq 1$, converging to b, and a sequence of automorphisms $\Phi_n \in \mathrm{Aut}(\Omega)$ such that $\Phi_n(x_n) = x_0$, where $x_0 \in \Omega$ is some fixed base point. It then follows from the fact that $(Z \cap \Omega, g|_{Z\cap\Omega})$ is asymptotically of holomorphic sectional curvature -2 that $Z_n := \Phi_n(Z \cap \Omega) \subset \Phi_n(U \cap \Omega)$ converges as a subvariety to some Kähler submanifold $Z_\infty \subset \Omega$ of constant holomorphic sectional curvature -2. Now the same set-up can be applied to the class of bounded homogeneous domains. These are bounded domains biholomorphic to homogeneous Siegel domains of the first or second kind constructed by Pyatetskii-Shapiro [Py69], which are biholomorphic to bounded domains via canonical isomorphisms (cf. Xu [Xu05]) and the description in Mok [Mo14b, §5]). A bounded homogeneous domain $\mathscr{D}$ is weakly pseudoconvex and there is in a canonical realization $\mathscr{D} \subset \mathbb{C}^N$ a dense subset of smooth boundary points in the semi-algebraic boundary $\partial\mathscr{D}$. Moreover, the Bergman kernel $K_\mathscr{D}(\xi, \eta)$ is a rational function in $(\xi, \overline{\eta})$, from which one deduces from the aforementioned "*rescaling*" argument the existence of proper holomorphic isometric embeddings of the complex unit ball whose dimension is equal to the number of positive eigenvalues of the Levi form of a smooth defining function on the complex tangent spaces at such points. We have

Theorem 2.5. *Let $\mathscr{D} \Subset \mathbb{C}^N$ be a canonical realization of a bounded homogeneous domain. Let $b_0 \in \mathrm{Reg}(\partial\mathscr{D})$ and let ρ be a smooth local defining function of $\mathscr{D}$ on a neighborhood of $b \in \mathbb{C}^N$. Suppose for b lying on some neighborhood of b_0 on $\partial\mathscr{D}$ the Levi form $\sqrt{-1}\partial\overline{\partial}\rho$ restricted to the complex tangent space $T^{1,0}_b(\partial\mathscr{D})$ has exactly s positive eigenvalues. Then, there exists a proper holomorphic isometric embedding $F : (\mathbb{B}^{s+1}, g) \hookrightarrow (\mathscr{D}, h)$ with respect to the normalized canonical Kähler-Einstein metric g resp. h on $\mathbb{B}^{s+1}$ resp. $\mathscr{D}$ such that* $\mathrm{Graph}(F) \subset \mathbb{C}^{s+1} \times \mathbb{C}^N$ *extends to an affine-algebraic subvariety.*

Here as before $(\mathbb{B}^{s+1}, g)$ is normalized so that minimal disks are of constant Gaussian curvature -2. On the other hand, $(\mathscr{D}, h)$ is normalized so that the Kähler form ω_h is given by $\omega_h = \sqrt{-1}\partial\overline{\partial}(-\log(-\rho'))$ where ρ' is a smooth defining function of D at a general point of $\partial\mathscr{D}$.

Regarding the very first examples of holomorphic isometric embeddings from the unit disk into polydisks, viz., the p-th root maps and maps obtained from them by means of composition, those maps remain the only known holomorphic isometric embeddings. It is a tempting yet challenging problem to ask whether the p-th root maps are in a certain sense the generators of the set $\mathbf{HI}(\Delta, \Delta^p)$ of all holomorphic isometric embeddings of the unit disk into polydisks. This was Problem 5.1.2 of Mok [Mo11] which remains unsolved. For the more accessible problem of characterization the p-th root map, we have recently the work of Chan [Ch16] which completed a partial result of Ng [Ng10] solving the problem for $p = 2$ and for p odd, settling in the affirmative a characterization problem for the p-th root map in terms of sheeting numbers.

Theorem 2.6. (Chan [Ch16], Ng [Ng10] for $p = 2$ and for p odd) *Let $p \geq 2$ be an integer. If $f : (\Delta, ds^2_\Delta) \hookrightarrow (\Delta^p, ds^2_{\Delta^p})$ is a holomorphic isometric embedding with sheeting number $n = p$, then f is the p-th root embedding up to reparametrization.*

Here by Theorem 1.1 Graph(f) extends to an irreducible subvariety $V \subset \mathbb{P}^1 \times (\mathbb{P}^1)^p$, and the sheeting number n is by definition the sheeting number of the canonical projection $\pi : V \to \mathbb{P}^1$ onto the first factor. By a reparametrization of f we mean the composition $\psi \circ f \circ \varphi$, where $\varphi \in \mathrm{Aut}(\Delta)$ and $\psi \in \mathrm{Aut}(\Delta^p)$ (which includes permutations of the components). On top of Theorem 2.6, $\mathbf{HI}(\Delta, \Delta^p)$ is now completely determined for $p \leq 4$ (Ng [Ng10] for $p = 2, 3$ and Chan [Ch17a] for $p = 4$).

Concerning holomorphic isometries of the complex unit ball into irreducible bounded symmetric domains Ω, other than the rank-1 case $\Omega \cong \mathbb{B}^n$, in which case all such maps are totally geodesic, and the case of Lie spheres $\Omega \cong D^{IV}_n, n \geq 3$, where there is a complete classification given by Theorem 2.4, other than some examples (as given in §1) not much is known about the set $\mathbf{HI}(\mathbb{B}^m, \Omega)$ of holomorphic isometries up to normalizing constants from the complex unit ball to Ω. On the existential side by considering holomorphic isometries of $\Delta \times \mathbb{B}^m$ into Ω for Ω of rank ≥ 2 not biholomorphic to a Lie sphere Chan and Yuan [CY17] have now obtained new examples of maps from complex unit balls into Ω incongruent to those obtained by restriction from $F_q : \mathbb{B}^{p+1} \hookrightarrow \Omega$ as given in Theorem 2.3 constructed from cones of minimal rational curves.

3. Structural and uniqueness results

Following up on the discussion on holomorphic isometries of the complex unit ball into irreducible bounded symmetric domain Ω in §2, Chan and Mok [CM17a] have obtained the following general results on *bona fide* holomorphic isometric embeddings of the complex unit ball into Ω, i.e., for holomorphic isometries with respect to the Bergman metric or with respect to the normalized canonical Kähler-Einstein metrics (subject to the requirement that minimal disks are of constant Gaussian curvature -2) without normalizing constants, in which we consider $\Omega \subset \Sigma$ canonically as an open subset of its dual Hermitian symmetric space Σ of the compact type by the Borel embedding.

Theorem 3.1. (Chan-Mok [CM17a]) *Let $f : (\mathbb{B}^n, g_n) \hookrightarrow (\Omega, h)$ be a holomorphic isometric embedding, where $n \geq 1$ and $\Omega \Subset \mathbb{C}^N$ is an irreducible bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization. Let $\Omega \subset \Sigma$ be the Borel embedding of Ω into its dual Hermitian symmetric space of the compact type Σ. Denote by $\iota : \Sigma \hookrightarrow \mathbb{P}\big(\Gamma(\Sigma, \mathcal{O}(1))^*\big)$ the minimal canonical projective embedding of Σ. Then, $f(\mathbb{B}^n)$ is an irreducible component of a complex-analytic subvariety $V \subseteq \Omega$ satisfying $\iota(V) = P \cap \iota(\Omega)$ for some projective linear subspace $P \subset \mathbb{P}\big(\Gamma(\Sigma, \mathcal{O}(1))^*\big)$.*

For $\mathbf{HI}_1(\mathbb{B}^m, \Omega)$, i.e., *bona fide* holomorphic isometric embeddings from $\mathbb{B}^{p+1}$ into Ω, $p = p(\Omega)$, our belief is that $F_q : \mathbb{B}^{p+1} \hookrightarrow \Omega$ of Theorem 2.3 are the only holomorphic isometries whenever Ω is not biholomorphic to a Lie sphere, which we confirm in the rank-2 cases, as follows.

Theorem 3.2. (Mok-Yang [MY18]) *Let $\Omega \subset \Sigma$ be the Borel embedding of an irreducible bounded symmetric domain Ω of rank 2 not biholomorphic to any type-IV domain $D_n^{IV}, n \geq 3$. Let $F : \big(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\big) \hookrightarrow (\Omega, ds^2_\Omega)$ be a holomorphic isometric embedding, $p := p(\Omega)$, and write $Z := f(\mathbb{B}^{p+1})$. Then, there exists $q \in$ Reg$(\partial\Omega)$ such that $Z = \mathscr{V}_q \cap \Omega$, where $\mathscr{V}_q \subset \Sigma$ is the union of minimal rational curves on Σ passing through q, and $F : \big(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\big) \hookrightarrow (\Omega, ds^2_\Omega)$ is congruent to the holomorphic isometric embedding $F_q : \big(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\big) \hookrightarrow (\Omega, ds^2_\Omega)$ given in Theorem 2.3.*

Recall here that p is given by $c_1(\Sigma) = (p + 2)\delta$ and equivalently by $p = \dim(\mathscr{C}_x(\Sigma))$ for the VMRT $\mathscr{C}_x(\Sigma)$ at any point $x \in \Sigma$.

Theorem 3.2 covers the case where Ω is a type-I domain $D^I(2, q)$, $q \geq 3$ (which is dual to the Grassmannian $G(2, q)$), the type-II domain $D^{II}(5, 5)$ (which is dual to the 10-dimensional orthogonal Grassmannian $G^{II}(5, 5)$) and the 16-dimensional exceptional domain D^V (which is of type E_6). The analogous uniqueness results when Ω is a type-I domain $D^I(3, q)$, $q \geq 3$, or when Ω is the 27-dimensional exceptional domain D^{VI} (which is of type E_7) have also been established, cf. Yang [Ya17]. The first open case is that of the type-III domain $D^{III}(3, 3)$ (which is biholomorphic to the Siegel upper half-plane $\mathscr{H}_3$ and dual to the 6-dimensional Lagrangian Grassmannian $G^{III}(3, 3)$), the only remaining irreducible bounded symmetric domain of rank 3.

Here is a sketch of an approach that the author has proposed for proving uniqueness up to reparametrization of nonstandard holomorphic isometries $F : (\mathbb{B}^{p+1}; g) \hookrightarrow (\Omega, h)$, $p = p(\Omega)$, for irreducible bounded symmetric domains Ω of rank ≥ 2 not biholomorphic to a type-IV domain D_n^{IV}, $n \geq 3$. Recall that $\Omega \subset \Sigma$ is the Borel embedding, and that for $x \in \Sigma$, $\mathscr{V}_x$ is a union of minimal rational curves (i.e., projective lines on Σ with respect to the first canonical embedding $\nu : \Sigma \hookrightarrow \mathbb{P}(\Gamma(\Sigma, \mathcal{O}(1))^*)$). $\mathscr{V}_x$ has an isolated singularity x, and $\mathscr{V}_x$ is homogeneous under the stabilizer $H \subset \mathrm{Aut}(\Sigma)$ of $\mathscr{V}_x$. Thus, for $y_1, y_2 \in \mathscr{V}_x - \{x\}$, there exists $\varphi \in \mathrm{Aut}(\Sigma)$ such that $\varphi(y_1) = \varphi(y_2)$ and such that $d\varphi(T_{y_1}(\mathscr{V}_x)) = T_{y_2}(\mathscr{V}_x)$. Given $Z = F(\mathbb{B}^{p+1})$, the author proposed to show that at a general point $z \in Z$, the inclusion $(T_z(Z) \subset T_z(\Sigma))$ is transformed to $(T_y(\mathscr{V}_x) \subset T_y(\Sigma))$ for a smooth point y on $\mathscr{V}_x$ by $d\psi$ for some $\psi \in \mathrm{Aut}(\Sigma)$ such that $\psi(z) = y$. Our strategy consists more precisely of (a) identifying the isomorphism class of the inclusion $(T_z(Z) \subset T_z(\Sigma))$ under the action of $\mathrm{Aut}(\Sigma)$ as described; (b) reconstructing Z as an open subset of some $\mathscr{V}_x$, and (c) proving that $x = q \in \partial\Omega$.

The approach turns out to work for the rank-2 cases in Theorem 3.2 and the rank-3 cases in Yang [Ya17]. Step (a) was established using methods of local differential geometry based on the Gauss equations for the holomorphic isometry $F : \big(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}};\big) \hookrightarrow (\Omega, ds^2_\Omega)$. Step (b) is implemented by means of techniques of reconstructing germs of complex submanifolds $(S; x_0)$ equipped with sub-VMRT structures (cf. Mok-Zhang [MZ$_1$18]) modeled on certain uniruled projective subvarieties of classical Fano manifolds of Picard number 1. In general such reconstruction consists of proving linear saturation (i.e., the property that the germ of a projective line $(\ell; x)$ tangent to S at $x \in S$ must necessarily lie on $(S; x_0)$) by verifying certain nondegeneracy conditions expressed in terms of second fundamental forms, followed by a process of adjunction of minimal rational curves, as introduced in Mok-Zhang [MZ$_1$18] and discussed in the expository article Mok [Mo16b]. In the case at hand the models are the Schubert subvarieties $\mathscr{V}_x \subset \Sigma$ which are uniruled by projective lines outside the isolated singularity $x \in \mathscr{V}_x$ (with the exception of Lagrangian Grassmannians for which the method does not apply). Once we have identified Z as an open subset of some $\mathscr{V}_x$, it follows from Theorem 1.1 that $x \notin \Omega$ since $Z \subset \Omega$ must be nonsingular. From the identity theorem for real-analytic functions it follows easily that $x \in \mathrm{Reg}(\partial\Omega)$.

4. Boundary behavior of holomorphic isometries

Regarding the boundary behavior of holomorphic isometric embeddings of the Poincaré disk into a bounded symmetric domain, we have proved the following general result on the boundary behavior of locally closed holomorphic curves on a bounded symmetric domain Ω when the holomorphic curves exit $\partial\Omega$.

Theorem 4.1. (Chan-Mok [CM17b]) *Let $b_0 \in \partial\Delta$, U be an open neighborhood of a point b_0 in $\mathbb{C}$, $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain in its Harish-Chandra realization, and let $\mu : U \hookrightarrow \mathbb{C}^N$ be a holomorphic embedding such that $\mu(U \cap \Delta) \subset \Omega$ and $\mu(U \cap \partial\Delta) \subset \partial\Omega$. Then, μ is asymptotically totally geodesic at a general point $b \in U \cap \partial\Delta$. More precisely, denoting by $\sigma(z)$ the second fundamental form of $\mu(U \cap \Delta)$ in (Ω, ds^2_Ω) at $z = \mu(w)$, for a general point $b \in U \cap \partial\Delta$ we have $\lim_{w \in U \cap \Delta,\, w \to b} \|\sigma(\mu(w))\| = 0$.*

Since by Theorem 1.1 the graph of any holomorphic isometry (with respect to scalar multiples of the Bergman metric) between bounded symmetric domains in their Harish-Chandra realizations extends to an affine-algebraic variety, it follows from Theorem 4.1 that we have the following result on holomorphic isometries from the unit disk to bounded symmetric domains.

Theorem 4.2. *Let* $f : (\Delta, \lambda ds^2_\Delta) \hookrightarrow (\Omega, ds^2_\Omega)$ *be a holomorphic isometric embedding, where* λ *is a positive constant and* $\Omega \Subset \mathbb{C}^N$ *is a bounded symmetric domain in its Harish-Chandra realization. Then,* f *is asymptotically totally geodesic at a general point* $b \in \partial\Delta$.

As a consequence of Theorem 4.2, we have

Theorem 4.3. (Chan-Mok [CM17b], Clozel [Cl07] for the classical cases) *Let* D *and* Ω *be bounded symmetric domains,* $\Phi : \mathrm{Aut}_0(D) \to \mathrm{Aut}_0(\Omega)$ *be a group homomorphism, and* $F : D \to \Omega$ *be a* Φ*-equivariant holomorphic map. Then,* F *is totally geodesic.*

Theorem 4.1 was first proved by Mok [Mo14a] under the stronger assumption that $\mu(U \cap \partial\Delta) \subset \mathrm{Reg}(\partial\Omega)$. In that case we obtained at the same time the estimate that for a general point $b \in U \cap \partial\Delta$, there exists a relatively compact open neighborhood U_0 of b in U and a constant $C \geq 0$ such that the estimate $\|\sigma(\mu(w))\| \leq C(1 - \|w\|)$ holds for $w \in U_0 \cap \Delta$. The proof in Mok [Mo14a] is direct and elementary. Although it is tempting to generalize the arguments of [Mo14a] to the general situation where $\mu(U \cap \Delta)$ exits an arbitrary stratum of the boundary (in its decomposition into $\mathrm{Aut}_0(\Omega)$-orbits), the problem is more delicate than it appears, and in Chan-Mok [CM17b]) we presented instead an indirect proof involving rescaling and the use of the Poincaré-Lelong equation adopting a methodology which is of independent interest in its own right.

The proof in [CM17b], which is sketched below, is by argument by contradiction, and that is the reason why an asymptotic estimate of $\|\sigma\|$ is lacking in general. Write $Z = \mu(U \cap \Delta)$ and $Z^\sharp = \mu(U)$. For a general point $b \in U$, $Z^\sharp$ is smooth at $\mu(b)$ and the restriction of the Bergman metric of Ω on Z is of asymptotically constant Gaussian curvature at $\mu(b)$. Suppose for the sake of argument by contradiction that μ is not asymptotically totally geodesic at b. By rescaling one extracts a holomorphic mapping F of the Poincaré disk which reflects the asymptotic behavior of μ at b. In particular F is a holomorphic isometric embedding since μ is asymptotically of constant Gaussian curvature. We may further rescale F if necessary and assume that the holomorphic isometry F is as "uniform" as one desires (e.g., we may require that the norm of the second fundamental form to be constant), and we obtain a contradiction to the existence of a certain "rescaled" hypothetical and nonstandard holomorphic isometric embedding of the Poincaré disk by reducing it to the case where $Z' := F(\Delta)$ lies on a tube domain of rank $s \leq r := \mathrm{rank}(\Omega)$, and where nonzero vectors tangents to Z' are of rank s, and by applying the Poincaré-Lelong equation to the logarithm of the (constant) norm of some "tautological" section of an $\mathrm{Aut}(\Omega)$-homogeneous holomorphic line bundle over Z' (cf. [CM17b]). In the case of type-III domains (which are biholomorphic to Siegel upper half-planes) the tautological section is a "twisted determinant" on tangents to the curve when tangent vectors are identified with symmetric matrices.

Theorem 4.2 should be contrasted with the existence result Theorem 2.3. There, for Ω an irreducible bounded symmetric domain of rank ≥ 2 and for $q \in \mathrm{Reg}(\partial\Omega)$ the holomorphic isometric embedding $F_q : \left(\mathbb{B}^{p+1}, ds^2_{\mathbb{B}^{p+1}}\right) \hookrightarrow (\Omega, ds^2_\Omega)$ such that $F_q(\mathbb{B}^{p+1}) = V_q = \mathscr{V}_q \cap \Omega$ is never asymptotically totally geodesic.

5. Zariski closures of images of algebraic sets under uniformization

For a bounded symmetric domain Ω denote by $\Omega \Subset \mathbb{C}^N \subset \Sigma$ the standard inclusions incorporating both the Harish-Chandra embedding $\Omega \Subset \mathbb{C}^m$ into a Euclidean space and the Borel embedding $\Omega \subset \Sigma$ into its dual Hermitian symmetric space of the compact type. A subvariety $S \subset \Omega$ is said to be an irreducible algebraic (sub)set if and only if it is an irreducible component of the intersection $\mathscr{V} \cap \Omega$ for some projective subvariety $\mathscr{V} \subset \Sigma$. An algebraic subset $S \subset \Omega$ is by definition the union of a finite number of irreducible algebraic subsets of Ω. In the case where $\Omega = \mathbb{B}^n$, $n \geq 2$, note that a totally geodesic complex submanifold of $\mathbb{B}^n$ is precisely a non-empty intersection of the form $\Pi \cap \mathbb{B}^n$, where $\Pi \subset \mathbb{P}^n$ is a projective linear subspace of $\mathbb{P}^n$.

Any totally geodesic complex submanifold $\Xi \subset \Omega$ is an open subset of its dual Hermitian symmetric space of the compact type Θ, $\Theta \subset \Sigma$, so that $\Xi \subset \Omega$ is an example of an algebraic subset. In connection with problems on a “dual” projective geometry on quotients $X_\Gamma := \mathbb{B}^n/\Gamma$ of the complex unit ball by a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$, the author was led first of all to study Zariski closures of images of totally geodesic complex submanifolds $S \subset \Omega$ under the uniformization map $\pi : \mathbb{B}^n \to X_\Gamma := \mathbb{B}^n/\Gamma$. From a geometric perspective the same problem can be raised when $\mathbb{B}^n$ is replaced by a bounded symmetric domain Ω. It transpires that similar questions were raised in number theory and functional transcendence theory. In fact, it was conjectured that the Zariski closure of the image of an algebraic subset $S \subset \Omega$ under the uniformization map $\pi : \Omega \to X_\Gamma := \Omega/\Gamma$ must necessarily be a totally geodesic subset when the lattice Γ is *arithmetic*. The latter is known as the Hyperbolic Ax-Lindemann Conjecture, and it is one of the two components for giving an unconditional proof of the André-Oort Conjecture following the scheme of proof of Pila-Zannier [PZ08]. (See last two paragraphs of §5 for more details.) Here X_Γ is equipped with a canonical quasi-projective structure as given by Baily-Borel [BB66]. From a purely geometric perspective there is no reason why one needs to restrict to arithmetic lattices, although assuming Ω to be irreducible by Margulis [Ma84] nonarithmetic lattices $\Gamma \subset \mathrm{Aut}(\Omega)$ only occur in the rank-1 cases, i.e., in the cases of $\Omega = \mathbb{B}^n$, $n \geq 1$. Focusing on the rank-1 cases and using methods of complex differential geometry we have proven the following theorems.

Theorem 5.1. (Mok [Mo17a]) *Let $n \geq 2$ be an integer and let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a torsion-free lattice. Denote by $X_\Gamma := \mathbb{B}^n/\Gamma$ the quotient manifold, of finite volume with respect to the canonical Kähler-Einstein metric $ds^2_{X_\Gamma}$ induced from the Bergman metric $ds^2_{\mathbb{B}^n}$. Let $\pi : \mathbb{B}^n \to X_\Gamma$ be the universal covering map and denote by $S \subset \mathbb{B}^n$ an irreducible algebraic subset. Then, the Zariski closure $Z \subset X_\Gamma$ of $\pi(S)$ in X_Γ is a totally geodesic subset.*

Theorem 5.2. (Mok [Mo17a]) *Let A be any set of indices and $\Sigma_\alpha \subset X_\Gamma$, $\alpha \in A$, be a family of closed totally geodesic subsets of X_Γ of positive dimension. Write $E := \bigcup\{\Sigma_\alpha : \alpha \in A\}$. Then, the Zariski closure of E in X_Γ is a union of finitely many totally geodesic subsets.*

For a possibly *nonarithmetic* lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$, X_Γ in the above is endowed a canonical quasi-projective structure defined by a compactification result of Mok [Mo12b]. The latter result is deduced from L^2-estimates of $\overline{\partial}$ and from the compactification theorem of Siu-Yau [SY82] yielding a Moishezon compactification of complete Kähler manifolds of finite volume and of pinched strictly negative sectional curvature.

Proposition 5.1. (Mok [Mo12b]) *Writing X for X_Γ in the notation of the preceding theorems, there exists a projective variety $\overline{X}_{\min}$ such that $X = \overline{X}_{\min} - \{p_1, \cdots, p_m\}$, where each p_i, $1 \leq i \leq m$, is a normal isolated singularity of $\overline{X}_{\min}$.*

In [Mo17a] we introduce a new framework into problems for functional transcendence on not necessarily arithmetic finite-volume quotients of bounded symmetric domains, although the methods were only applied there to the rank-1 case. In what follows let $\Omega \Subset \mathbb{C}^N$ be any possibly reducible bounded symmetric domain, G_0 be the identity component of $\mathrm{Aut}(\Omega)$, $\Gamma \subset G_0$ be a torsion-free lattice, and $\Omega \subset \Sigma$ be the Borel embedding. Write G for the identity component of $\mathrm{Aut}(\Sigma)$. Let now $\mathscr{V} \subset \Sigma$ be an irreducible subvariety and $S \subset \Omega$ be an irreducible algebraic subset, $\dim(S) =: s$, which is an irreducible component of $\mathscr{V} \cap \Omega$, and for the ensuing discussion assume for convenience that the reduced subvariety $\mathscr{V} \subset \Sigma$ corresponds to a smooth point of some irreducible component $\mathcal{K}$ of the Chow space $\mathrm{Chow}(\Sigma)$ of Σ. Let $\rho : \mathscr{U} \to \mathcal{K}$, $\mu : \mathscr{U} \to \Sigma$ be the universal family of $\mathcal{K}$, and write $\mu_0 : \mathscr{U}_0 := \mathscr{U}|_{\mu^{-1}(\Omega)} \to \Omega$ be the restriction of $\mathscr{U}$ over Ω. Then G acts on $\mathscr{U}_0$ and by restriction G_0 acts on $\mathscr{U}_0$.

Write $X_\Gamma := \Omega/\Gamma$, which is equipped with a canonical quasi-projective structure, $\pi_\Gamma : \Omega \to X_\Gamma$ for the uniformization map and define $Z \subset \overline{\pi(S)}^{\mathscr{Z}ar}$ for the Zariski closure of $\pi(S)$ in X_Γ, which admits a canonical quasi-projective structure. $\mu_0 : \mathscr{U}_0 \to \Omega$ descends to X_Γ to give a locally homogeneous holomorphic fiber bundle $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$. In case $\Gamma \subset G_0$ is cocompact, then $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ is projective. When $\Gamma \subset G_0$ is a nonuniform lattice, by the differential-geometric method of compactification of Mok-Zhong [MZ$_2$89] we have a quasi-projective compactification $\overline{\mu_\Gamma} : \overline{\mathscr{U}_\Gamma} \to \overline{X_\Gamma}$. In this context the meromorphic foliation $\mathscr{F}$ on $\mathscr{U}$ defined by tautological liftings of $\mathscr{W} \subset \Sigma$ (of members $\mathscr{W}$ belonging to $\mathcal{K}$) extends meromorphically to $\overline{\mathscr{U}_\Gamma}$. (The proof of the extension was only written for the rank-1 case, but can be strengthened using [MZ$_2$89] to the general case.) The simplifying assumption that $\mathscr{V}$ corresponds to a smooth point of $\mathcal{K}$ implies that $\mathscr{F}$ is holomorphic at a general point of $\mathscr{V}$. We take the Zariski closure of the tautological lifting $\mathscr{S} \subset \mathscr{U}_\Gamma$ to obtain $\mathscr{Z} = \overline{\mathscr{S}}^{\mathscr{Z}ar} \subset \mathscr{U}_\Gamma$ and we have $Z = \mu_\Gamma(\mathscr{Z})$. We proved in [Mo17a] that $\mathscr{Z}$ is saturated with respect to the foliation $\mathscr{F}$. Let now $\widetilde{Z} \subset \Omega$ be an irreducible component of $\pi_\Gamma^{-1}(Z)$. From the saturation of $\mathscr{Z}$ under $\mathscr{F}$ it follows that $\widetilde{Z}$ is in a neighborhood of a general point $x \in S$ the union of an analytic family

Φ of (connected open subsets of) members of $\mathcal{K}$, and the union of the members of Φ then contains the germ of an s-dimensional complex submanifold Ξ of Σ at $b_0 \in \partial\widetilde{Z}$.

Now we restrict to the rank-1 situation covered by Theorem 5.1 and Theorem 5.2. At a general point $b \in \xi \cap \partial\mathbb{B}^n$, $\Xi \cap \mathbb{B}^n \subset \Xi$ is *strictly pseudoconvex* at b, and from a computation of Klembeck [Kl78] $\widetilde{Z}$ is asymptotically of constant negative holomorphic sectional curvature. By a comparison with curvatures of $(\mathbb{B}^n, g)$ we conclude that $\widetilde{Z} \subset \mathbb{B}^n$ is asymptotically totally geodesic. Now $Z = \widetilde{Z}/\Gamma'$ for some infinite subgroup $\Gamma' \subset \Gamma$. In case $\Gamma \subset G_0$ is cocompact let $U \Subset \widetilde{Z}$ be an open relatively compact subset such that $\pi(U) = Z$. From strict pseudoconvexity of $\widetilde{Z}$ at b there exists a sequence of elements $\gamma_n \in \Gamma'$ such that $\gamma_n(x)$ converges to b for any $x \in U$, and we conclude that $\widetilde{Z} \subset \mathbb{B}^n$ is totally geodesic, which gives Theorem 5.1 and Theorem 5.2 in the cocompact case. For the modification in this last step to the case of a nonuniform lattice $\Gamma \subset G_0$ we refer the reader to [Mo17a].

Here we get the total geodesy of $\widetilde{Z} \Subset \mathbb{B}^n$ directly. A slight reformulation of this last step makes it applicable to the higher rank situation, in the event that $\widetilde{Z}$ happens to be strictly pseudoconvex at b, as follows. From the asymptotic curvature property of $\widetilde{Z}$ at b we conclude that $\widetilde{Z}$ is the image of a holomorphic isometry $F : \mathbb{B}^s \to \Omega$ with respect to multiples of the Bergman metric. By Mok [Mo12b], $\widetilde{Z} \subset \Omega$ is algebraic. In other words, both the domain and the target of the covering map $\pi_\Gamma|_{\widetilde{Z}} : \widetilde{Z} \to Z$ are algebraic. If the lattice $\Gamma \subset G_0$ is arithmetic, then we can conclude that $\widetilde{Z} \subset \Omega$ and $Z \subset X_\Gamma$ are totally geodesic, by Ullmo-Yafaev [UY11]. Without the arithmeticity assumption but using the special property that $\widetilde{Z}$ is strictly pseudoconvex at a general point of $\partial\widetilde{Z}$, one can easily show that $\widetilde{Z}$ is homogeneous under an algebraic subgroup of G_0. Since $\widetilde{Z}$ admits a quotient of finite volume, this implies that $\widetilde{Z}$ is a holomorphic isometric copy of a bounded symmetric domain, and hence totally geodesic by Chan-Mok [CM17b, Theorem 5.19].

For the reader who may like to see how complex differential geometry could interact with questions in diophantine geometry and functional transcendence, here is a digression around the Hyperbolic Ax-Lindemann Conjecture. We start with a special case of Ax's Theorem (Ax [Ax71]) on $\mathbb{C}^n$, with Euclidean coordinates $(z_1, \cdots z_n)$. Let $V \subset \mathbb{C}^n$ be an m-dimensional irreducible affine algebraic subvariety. Assume that 0 is a smooth point on V and that $(z_1, \cdots, z_m)$ serve as holomorphic local coordinates on V at 0. For $1 \le k \le n$ define $f_k = 2\pi i z_k|_V$. Define $\pi : \mathbb{C}^n \to (\mathbb{C}^*)^n$ by $\pi(z_1, \cdots, z_n) = (e^{2\pi i z_1}, \cdots, e^{2\pi i z_n})$. Then, by [Ax71] the transcendence degree of the field of functions on V generated by $\{f_1, \cdots, f_n; e^{f_1}, \cdots, e^{f_n}\}$ is equal to $n+m$ unless V lies on a $\mathbb{Q}$-hyperplane of $\mathbb{C}^n$, i.e., unless $\pi(V)$ is contained in an algebraic torus, equivalently a totally geodesic subvariety of $(\mathbb{C}^*)^n$ with respect to the Kähler metric on $(\mathbb{C}^*)^n$ induced from the (translation-invariant) Euclidean metric $ds^2_{\rm euc}$ on $\mathbb{C}^n$ by the uniformization map π. In particular, when the n component functions of $\pi|_V$ are algebraically dependent, it follows readily that the Zariski closure of $\pi(V) \subset (\mathbb{C}^*)^n$ is an algebraic torus $T \subsetneq (\mathbb{C}^*)^n$. This gives the Ax-Lindemann Theorem for the exponential map.

If for Ax-Lindemann we replace $\mathbb{C}^n$ by a bounded symmetric domain and denote now by $\pi : \Omega \to \Omega/\Gamma =: X_\Gamma$ the uniformization map for a torsion-free *arithmetic* lattice $\Gamma \subset \mathrm{Aut}(\Omega)$, and replace $V \subset \mathbb{C}^n$ by an algebraic subset $S \subset \Omega$, the analogous conjectural statement was that the Zariski closure of $\pi(S)$ in X_Γ is necessarily a totally geodesic subset, commonly called the Hyperbolic Ax-Lindemann Conjecture. Pila-Zannier [PZ08] adopted a strategy for conjectures regarding special points. For the André-Oort Conjecture regarding Zariski closures of sets of special points on Shimura varieties it led to a reduction of the conjecture into two components, a number-theoretic component concerning lower bounds for the sizes of Galois orbits and a geometric component which is precisely the Hyperbolic Ax-Lindemann Conjecture, a conjecture by now confirmed by Ullmo-Yafaev [UY14] in the cocompact case, by Pila-Tsimerman [PT14] for $\mathscr{A}_g$, and by Klingler-Ullmo-Yafaev [KUY16] in the general case. A crucial ingredient is o-minimal geometry, especially counting arguments on rational points of Pila-Wilkie [PW06] in a model-theoretic context (cf. also Bombieri-Pila [BP89]), and methods of Peterzil-Starchenko [PS09] on tame complex analysis. Complex differential geometry entered into play in [KUY16], where volume estimates of Hwang-To [HT02] on subvarieties on bounded symmetric domains were used in an essential way. Applications of hyperbolic Ax-Lindemann to number theory are given in Ullmo [Ull14], and uses of o-minimality in functional transcendence are expounded in Pila [Pi15]. For the broader context of problems in arithmetic and geometry related to unlikely intersection, we refer the reader to Zannier [Za12].

6. Perspectives and concluding remarks

In this article the author has been discussing various aspects of his recent works in part with collaborators on the topic of holomorphic isometries in Kähler geometry, together with some references to related recent results by other researchers. Our discussion was on research problems intrinsic to complex differential geometry and also on the use of holomorphic isometries in other contexts. To gauge how research on the topic could develop in the future the author would venture to examine it (a) from the point of view of a complex differential geometer, (b) in connection with applications to other subject areas in mathematics and (c) with an eye on identifying new directions of research. As is the flavor of this article, these are only reflections from the author prompted by his own research involvement and does not represent a comprehensive overview on the subject.

The study of holomorphic isometries on Kähler geometry is by its very definition intrinsic to complex differential geometry, and the subject took shape from works of Bochner and Calabi, notably the seminal work of Calabi [Ca53]. In the tradition of classical differential geometry and focusing on bounded symmetric domains, recent progress on the subject include existence and uniqueness results, structural and characterization theorems and results on the asymptotic geometry of holomorphic isometric embeddings into bounded symmetric domains. The structure of the full set of holomorphic isometries between two given bounded symmetric domains is a natural object of study, but so far only in very special cases are we in a position to classify such maps. For instance, in the case of maps from the Poincaré disk into polydisks, while classical techniques involving functional identities arising from diastases and the study of branched coverings of compact Riemann surfaces have led to some neat classification results in low dimensions, the complexity of such objects grows very fast with dimensions, and it appears that in this and other characterization problems new tools are necessary for general results.

As an example, in the study of holomorphic isometries from complex unit balls of maximal dimension into an irreducible bounded symmetric domain, the geometric theory of varieties of minimal rational tangents has been a source of methods both for existence results and for structural results notably in the reconstruction of images of holomorphic isometric embeddings via the method of geometric substructures. Another new source of methods is the use of Jordan algebras and operator theory on Hilbert spaces in Upmeier-Wang-Zhang [UWZ17]. It is for instance tempting to study holomorphic isometric embeddings from the Poincaré disk into polydisks via linear isometries between Hilbert spaces of functions, and the methods developed could shed light on factorization problems especially the question whether the p-th root maps are the "generators" of holomorphic isometries between polydisks. The space of holomorphic isometries from the complex unit ball into a bounded symmetric domain forms a real algebraic variety, and the result that images of *bona fide* holomorphic isometries arise from linear sections with respect to the minimal canonical embedding gives an effective bound on the number of parameters for its description. From Chan [Ch17b]) one would expect that the same is valid for holomorphic isometries with other normalizing constants, and a confirmation of that belief in the general case would be a unifying result for bounded symmetric domains.

The author was led to consider holomorphic isometries of Kähler manifolds and related topics (such as holomorphic measure-preserving maps) in order to answer questions concerning modular correspondences on finite-volume quotients of bounded symmetric domains, notably questions in arithmetic dynamics raised by Clozel-Ullmo [CU03] concerning commutants of such correspondences. Thus, it was around bounded symmetric domains that the author started his investigation, and it is gratifying to see that the study of holomorphic isometries from complex unit balls into bounded symmetric domains finds its way into problems in functional transcendence theory. A primary issue in functional transcendence theory is the question of generating algebraic subsets from processes which are *a priori* complex-analytic, and the use of holomorphic isometries provides such a means, viz., from the algebraicity of such maps due to the rationality of Bergman kernels. The link between holomorphic isometries with the uniformization map arises when some *strictly pseudoconvex* complex tangential directions are picked up as complex-analytic families of algebraic subsets traverse the boundary of bounded symmetric domains, and the asymptotic curvature behavior is then recaptured through rescaling arguments. Further input from complex differential geometry into problems in functional transcendence theory could involve the study of asymptotic geometric behavior as complex-analytic families of algebraic subsets exit the boundary of bounded symmetric domains (or more generally flag domains, cf. second last paragraph). In this regard, the work of Chan-Mok [CM17b] on the asymptotic curvature behavior when holomorphic curves exit boundaries of bounded symmetric domains is a step in this direction, and for the general question the fine structure of boundaries of bounded symmetric domains in their Harish-Chandra realizations (cf. Wolf [Wo72]) will enter into play.

In other directions holomorphic isometries in Kähler geometry are a source of examples for further study in other areas of mathematics. In several complex variables they may give new examples of proper holomorphic maps exhibiting new qualitative behavior (e.g., Chan-Xiao-Yuan [CXY17]). They may give algebraic subsets of bounded symmetric domains admitting interesting geometric substructures (e.g., holomorphic isometries of complex unit balls into Lie spheres give rise to possibly degenerate holomorphic conformal structures), and they are also related to the study of some special Schubert varieties (cf. [Mo16a], [Mo17b]). The latter examples could motivate differential-geometric characterizations of (open subsets of) wider classes of Schubert varieties.

In the seminal paper of Calabi [Ca53] complex space forms endowed with pseudo-Kähler metrics were already studied. Beyond bounded symmetric domains it would be natural, both from the point of view of developing the theory of holomorphic isometries in a pseudo-Kählerian context, and in connection with the study of functional transcendence, to generalize to the study of quotients of large classes of flag domains (cf. Fels-Huckleberry-Wolf [FHW06]), especially those admitting invariant pseudo-Kähler metrics and compatible filtrations of the holomorphic tangent bundles. These include the period domains for complex variations of Hodge structures, but ought to be broader in scope, and it will be interesting to construct horizontal algebraic subvarieties of such flag domains and to study *horizontal* holomorphic isometries into such flag domains and their connection to various questions in functional transcendence.

Finally, research on holomorphic isometries in Kähler geometry may have acted as a *catalyst* to highlight the relevance of complex differential geometry in the study of Shimura varieties which are by their very definition arithmetic quotients of bounded symmetric domains. Years ago the author had harbored the hope that Bergman metrics could be made use of in treating number-theoretic problems on Shimura varieties. With the advance of methods from diophantine geometry and model theory and the schematic reduction of outstanding problems on special points to issues involving unlikely intersection, the hope is already reality. From the point of view of complex differential geometry the study of holomorphic isometries serves perhaps as a path through the fascinating territory of Shimura varieties and their generalizations, and its proper role in tackling problems in functional transcendence will depend on future interaction between analytic, algebraic and model-theoretic perspectives in the study of such varieties.

Acknowledgment The author would like to thank the organizers of KSCV for their invitation, and it is a pleasure for him to dedicate this article to Professor

Kang-Tae Kim, initiator of this long series of international symposia in several complex variables, on the occasion of his sixtieth birthday.

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