ON PROPER HOLOMORPHIC MAPS BETWEEN BOUNDED SYMMETRIC DOMAINS

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Abstract. We study proper holomorphic maps between bounded symmetric domains D and Ω. In particular, when D and Ω are of the same rank ≥ 2 such that all irreducible factors of D are of rank ≥ 2, we prove that any proper holomorphic map from D to Ω is a totally geodesic holomorphic isometric embedding with respect to certain canonical Kähler metrics of D and Ω. We also obtain some results regarding holomorphic maps $F : D \to \Omega$ which map minimal disks of D properly into rank-1 characteristic symmetric subspaces of Ω. On the other hand, we obtain new rigidity results regarding semi-product proper holomorphic maps between D and Ω under a certain rank condition on D and Ω.

1. Introduction

In [Ts93], Tsai has proven that if F is a proper holomorphic map from an irreducible bounded symmetric domain D to a bounded symmetric domain Ω with the assumption that $\mathrm{rank}(D) \geq \mathrm{rank}(\Omega) \geq 2$, then $\mathrm{rank}(D) = \mathrm{rank}(\Omega)$ and F is a totally geodesic holomorphic isometric embedding with respect to the Bergman metrics up to a normalizing constant. In general, a proper holomorphic map f between reducible bounded symmetric domains D_1 and D_2 of equal rank ≥ 2 can be nonstandard (i.e., not totally geodesic) when the domain D_1 of f is reducible and has a rank-1 irreducible factor. We will give an example of such a proper holomorphic map. This example will also allow us to formulate an appropriate rigidity theorem (i.e., Theorem 1.2) for proper holomorphic maps between reducible bounded symmetric domains. Let D and Ω be irreducible bounded symmetric domains of rank ≥ 2. In [Ng15], Ng has proven that if a holomorphic map $f : D \to \Omega$ maps minimal disks of D properly into the rank-1 characteristic symmetric subspaces of Ω, then f is a totally geodesic holomorphic isometric embedding with respect to the Bergman metrics up to a normalizing constant.

In the first part of this article, we will study proper holomorphic maps between (reducible) bounded symmetric domains along the lines of Ng [Ng15]. For an irreducible bounded symmetric domain $U \Subset \mathbb{C}^n$, we let g_U be the canonical Kähler-Einstein metric on U normalized so that minimal disks of U are of constant Gaussian curvature -2, and we denote by ω_{g_U} the Kähler form of (U, g_U). Then, g_U agrees with the standard complex Euclidean metric of $\mathbb{C}^n$ at $\mathbf{0}$. For Kähler manifolds (M, g_M) and (N, g_N) with the corresponding Kähler forms ω_{g_M} and ω_{g_N} respectively, a holomorphic map $F : (M, g_M) \to (N, g_N)$ is said to be *isometric* if and only if $F^*\omega_{g_N} = \omega_{g_M}$. In addition, a holomorphic map $F : (M, g_M) \to (N, g_N)$ is said to be *an isometric map up to a normalizing constant* if and only if $F^*\omega_{g_N} = \lambda\omega_{g_M}$ for some positive real constant λ. Motivated by [Ng15, Proposition 1.2], we will

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also study holomorphic maps $F : D \to \Omega$ between (reducible) bounded symmetric domains D and Ω which map minimal disks of D properly into rank-1 characteristic symmetric subspaces of Ω. In this direction, we have the following generalization of [Ng15, Proposition 1.2].

Theorem 1.1. *Let $F : D \to \Omega$ be a holomorphic map between bounded symmetric domains D and Ω such that F maps the minimal disks of D properly into rank-1 characteristic symmetric subspaces of Ω. Suppose all irreducible factors of D are of rank at least two. Write $D = D_1 \times \cdots \times D_k$ and $\Omega = \Omega_1 \times \cdots \times \Omega_l$, where $D_j \Subset \mathbb{C}^{n_j}$, $1 \le j \le k$, and Ω_j, $1 \le j \le l$, are the irreducible factors of D and Ω respectively with $\mathrm{rank}(D_j) \ge 2$ for $1 \le j \le k$. Then, F is a totally geodesic isometric embedding from $(D_1, g_{D_1}) \times \cdots \times (D_k, g_{D_k})$ to $(\Omega_1, g_{\Omega_1}) \times \cdots \times (\Omega_l, g_{\Omega_l})$.*

In the consideration of proper holomorphic maps between bounded symmetric domains, we will deduce the following result from Theorem 1.1.

Theorem 1.2. *Let $F : D \to \Omega$ be a proper holomorphic map between bounded symmetric domains D and Ω such that $\mathrm{rank}(D) = \mathrm{rank}(\Omega)$. Suppose all irreducible factors of D are of rank at least two. Write $D = D_1 \times \cdots \times D_k$ and $\Omega = \Omega_1 \times \cdots \times \Omega_l$, where D_j, $1 \le j \le k$, and Ω_j, $1 \le j \le l$, are the irreducible factors of D and Ω respectively with $\mathrm{rank}(D_j) \ge 2$ for $1 \le j \le k$. Then, F is a totally geodesic isometric embedding from $(D_1, g_{D_1}) \times \cdots \times (D_k, g_{D_k})$ to $(\Omega_1, g_{\Omega_1}) \times \cdots \times (\Omega_l, g_{\Omega_l})$.*

In [Seo18], Seo has introduced semi-product proper holomorphic maps between (reducible) bounded symmetric domains. Then, Seo [Seo18] has proven that any proper rational map between (reducible) bounded symmetric domains is a semi-product proper holomorphic map. One of the main results in Seo [Seo18] is the classification of all proper holomorphic maps between (reducible) bounded symmetric domains of the same dimension (see [Seo18, Theorem 1.2]). Motivated by the work of Seo [Seo18], we will study semi-product proper holomorphic maps between non-equidimensional (reducible) bounded symmetric domains. Under certain rank conditions, we are able to get the complete description for such maps (see Theorem 4.5).

2. Preliminaries

For a (reducible) bounded symmetric domain $D = D_1 \times \cdots \times D_k$, where D_j, $1 \le j \le k$, are the irreducible factors of D, there is a Kähler metric g'_D on D such that $(D, g'_D) \cong (D_1, \lambda_1 g_{D_1}) \times \cdots \times (D_k, \lambda_k g_{D_k})$ for some positive real constants λ_j, $1 \le j \le k$, namely,

$$g'_D = \sum_{j=1}^{k} \lambda_j \pi_j^* g_{D_j},$$

where $\pi_j : D \to D_j$ is the canonical projection onto the j-th irreducible factor of D, $\pi_j(Z^1, \ldots, Z^k) = Z^j$ for $(Z^1, \ldots, Z^k) \in D_1 \times \cdots \times D_k = D$, $1 \le j \le k$. In what follows, for any bounded symmetric domain D we call such a metric g'_D on D a **canonical Kähler metric** and denote by $\mathrm{rank}(D)$ the rank of D. It is well-known that a bounded symmetric domain D is of rank 1 if and only if D is biholomorphic to a complex unit ball.

Denote by $\Delta^k = \{(z_1, \ldots, z_k) \in \mathbb{C}^k : |z_j| < 1,\ 1 \le j \le k\}$ the k-disk in $\mathbb{C}^k$ for any integer $k \ge 1$. We let $\mathbb{B}^n$ be the complex unit ball in the complex n-dimensional

Euclidean space $\mathbb{C}^n$ with respect to the standard complex Euclidean metric, i.e.,

$$\mathbb{B}^n := \left\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 1 \right\}.$$

For any complex manifold M, we denote by $T_x^{1,0}(M) = T_x(M)$ the holomorphic tangent space to M at $x \in M$.

Let $D \cong G/K$ be a bounded symmetric domain in $\mathbb{C}^n$ and let $r := \mathrm{rank}(D)$, where G is the identity component of the automorphism group of D and $K \subset G$ is the isotropy subgroup of G at $\mathbf{0} \in \mathbb{C}^n$. By the Polydisk Theorem (cf. [Mok89, p. 88], [Wo72]), there is a totally geodesic complex submanifold $\Pi \cong \Delta^r$ of D such that

$$D = \bigcup_{k \in K} k \cdot \Pi.$$

A vector $v \in T_x(D)$, $x \in D$, is said to be a **characteristic vector** of D at x if v is tangent to any direct factor of a totally geodesic r-disk of D (cf. [Ng15, Section 2]). Write $D = D_1 \times \cdots \times D_k$, where D_j, $1 \le j \le k$, are the irreducible factors of D. Then, it follows from Wolf [Wo72] that any rank-1 characteristic symmetric subspace of D is of the form $\{x_1\} \times \cdots \times \{x_{j-1}\} \times B_j \times \{x_{j+1}\} \times \cdots \times \{x_k\}$ for some j, $1 \le j \le k$, where $B_j \cong \mathbb{B}^{m_j}$ is a rank-1 characteristic symmetric subspace of D_j, $x_\mu \in D_\mu$ is a point for each $\mu \neq j$, and m_j is a positive integer depending on D_j. Here, we also know that $(B_j, g_{D_j}|_{B_j})$ is holomorphically isometric to $(\mathbb{B}^{m_j}, g_{\mathbb{B}^{m_j}})$, which is of constant holomorphic sectional curvature -2. For the notion of characteristic symmetric subspaces of bounded symmetric domains, we refer the readers to Mok-Tsai [MT92].

3. Proper holomorphic maps between bounded symmetric domains of equal rank ≥ 2

Motivated by the study in Tsai [Ts93] and Ng [Ng15], we are concerning proper holomorphic maps between (reducible) bounded symmetric domains of the same rank ≥ 2. In [Ts93], Tsai has proven that if $F : D \to \Omega$ is a proper holomorphic map between bounded symmetric domains D and Ω, then $\mathrm{rank}(D) \le \mathrm{rank}(\Omega)$. Thus, it is natural to ask the following question.

Question 3.1. Let $F : D \to \Omega$ be a proper holomorphic map between bounded symmetric domains D and Ω. If $\mathrm{rank}(D) = \mathrm{rank}(\Omega) \ge 2$, then is F a totally geodesic holomorphic isometric embedding with respect to some canonical Kähler metrics on D and Ω?

Remark 3.2. Tsai [Ts93, Main Theorem] has an affirmative answer to Question 3.1 under the assumption that D is irreducible.

However, we have a negative answer to Question 3.1 if D is reducible and some irreducible factor of the domain D is a complex unit ball, namely, we have

Example 3.3. We also denote by $M(p, q; \mathbb{C})$ the space of p-by-q complex matrices. A type-I irreducible bounded symmetric domain is given by

$$D^{\mathrm{I}}_{p,q} := \left\{ Z \in M(p, q; \mathbb{C}) : \mathbf{I}_q - \overline{Z}^t Z > 0 \right\},$$

where p and q are positive integers. We refer the readers to Mok [Mok89] for details about bounded symmetric domains.

For any integer $n \geq 2$, it is well-known that there is a positive integer q and a proper holomorphic map $f : \mathbb{B}^n \to \mathbb{B}^{q-2}$ which is not a holomorphic isometry from $(\mathbb{B}^n, \lambda g_{\mathbb{B}^n})$ to $(\mathbb{B}^{q-2}, g_{\mathbb{B}^{q-2}})$ for any real constant $\lambda > 0$ (cf. D'Angelo [D88]). More precisely, from D'Angelo [D88, p. 84] we may let $q = 2n + 1$ and

$$f(z_1, \ldots, z_n) := (z_1, \ldots, z_{n-1}, z_1 z_n, z_2 z_n, \ldots, z_{n-1} z_n, z_n^2).$$

Writing $f = (f_1, \ldots, f_{q-2})$, we define a map $F : \mathbb{B}^n \times D^{\mathrm{I}}_{2,2} \to D^{\mathrm{I}}_{3,q}$ by

$$F(z_1, \ldots, z_n; \mathbf{W}) = \begin{pmatrix} f_1(z_1, \ldots, z_n) & \cdots & f_{q-2}(z_1, \ldots, z_n) & \mathbf{0} \\ 0 & \cdots & 0 & \mathbf{W} \end{pmatrix}$$

for $(z_1, \ldots, z_n) \in \mathbb{B}^n$ and $\mathbf{W} \in D^{\mathrm{I}}_{2,2}$. Then, F is a proper holomorphic map between the bounded symmetric domains $\mathbb{B}^n \times D^{\mathrm{I}}_{2,2}$ and $D^{\mathrm{I}}_{3,q}$ of rank three such that F is not a holomorphic isometry with respect to any canonical Kähler metrics of $\mathbb{B}^n \times D^{\mathrm{I}}_{2,2}$ and $D^{\mathrm{I}}_{3,q}$.

Remark 3.4.

(1) It is known from Chan-Xiao-Yuan [CXY17] and Mok [Mok12] that any holomorphic isometry between bounded symmetric domains with respect to the canonical Kähler metrics is a proper holomorphic map. In [Ch19], we have shown that any holomorphic isometry between bounded symmetric domains of the same rank with respect to the canonical Kähler metrics is totally geodesic. From Example 3.3, we know that this result from [Ch19] cannot be generalized to the case of proper holomorphic maps unless we impose additional assumptions on the bounded symmetric domains.
(2) Example 3.3 shows that Theorems 1.1 and 1.2 cannot be generalized to the case where some irreducible factor of the domain D is of rank 1.

We first recall the following lemma obtained from Mok-Tsai [MT92] and Tsai [Ts93], which is known by Ng [Ng15, p. 224].

Lemma 3.5 (cf. Mok-Tsai [MT92], Tsai [Ts93] and Ng [Ng15])**.** *Let $F : D \to \Omega$ be a proper holomorphic map between bounded symmetric domains D and Ω. Suppose* $\mathrm{rank}(D) = \mathrm{rank}(\Omega) \geq 2$. *Then, F maps rank-*1 *characteristic symmetric subspaces of D properly into rank-*1 *characteristic symmetric subspaces of Ω. In particular, F maps minimal disks of D properly into rank-*1 *characteristic symmetric subspaces of Ω.*

This yields the following obvious corollary.

Corollary 3.6. Let $F : D \to \Omega$ be a proper holomorphic map between bounded symmetric domains D and Ω such that $\mathrm{rank}(D) = \mathrm{rank}(\Omega)$. If Ω is of tube type, then so is D.

Proof. Suppose $\mathrm{rank}(D) = \mathrm{rank}(\Omega) = 1$. Then, D (resp. Ω) is biholomorphic to a complex unit ball. Since Ω is of tube type, Ω is the complex unit disk and thus D can only be the complex unit disk as well. In particular, D is of tube type.

Now, we suppose $\mathrm{rank}(D) = \mathrm{rank}(\Omega) \geq 2$. Note that rank-1 characteristic symmetric subspaces of Ω are precisely the minimal disks of Ω because Ω is of tube type (cf. Mok-Tsai [MT92] and Wolf [Wo72]). By Lemma 3.5, F maps rank-1 characteristic symmetric subspaces of D properly into rank-1 characteristic symmetric subspaces of Ω. Therefore, rank-1 characteristic symmetric subspaces of D could only be unit disks. Hence, all irreducible factors of D are of tube type and so is D by Wolf [Wo72]. □

We observe that [Ng15, Proposition 1.2] actually holds by [Ng15, Proof of Proposition 1.2] even when the target bounded symmetric domain is reducible, namely, we have

Proposition 3.7 (cf. Proposition 1.2 in Ng [Ng15])**.** Let D and Ω be bounded symmetric domains of rank ≥ 2 and let $F : D \to \Omega$ be a holomorphic map. Suppose D is irreducible and F maps the minimal disks of D properly into the rank-1 characteristic symmetric subspaces of Ω. Write $\Omega = \Omega_1 \times \cdots \times \Omega_l$, where Ω_j, $1 \leq j \leq l$, are the irreducible factors of Ω. Then, F is a totally geodesic isometric embedding from (D, g_D) to $(\Omega_1, g_{\Omega_1}) \times \cdots \times (\Omega_l, g_{\Omega_l})$.

Remark 3.8. From the proof of Proposition 1.2 in Ng [Ng15], we know that F is a totally geodesic isometric embedding from $(D, \lambda g_D)$ to $(\Omega_1, g_{\Omega_1}) \times \cdots \times (\Omega_l, g_{\Omega_l})$ for some positive real constant λ. But then by the fact that F maps minimal disks of D properly into the rank-1 characteristic symmetric subspaces of Ω and F is totally geodesic, we can deduce that $\lambda = 1$.

By making use of Proposition 3.7 and results in Ng [Ng15], we are ready to prove Theorem 1.1, as follows.

Proof of Theorem 1.1. We write $Z^j = (Z^j_1, \ldots, Z^j_{n_j}) \in D_j \Subset \mathbb{C}^{n_j}$ for the Harish-Chandra coordinates of D_j, $1 \leq j \leq k$. For $W = (W^1, \ldots, W^k) \in D$, we let $\iota_{j,W} : D_j \hookrightarrow D$ be the natural embedding given by

$$\iota_{j,W}(Z^j) := (W^1, \ldots, W^{j-1}, Z^j, W^{j+1}, \ldots, W^k)$$

for $Z^j \in D_j$, $1 \leq j \leq k$. Then, each $F \circ \iota_{j,W} : D_j \to \Omega$ is a holomorphic map which maps the minimal disks of D_j properly into rank-1 characteristic symmetric subspaces of Ω, $1 \leq j \leq k$. Since D_j is an irreducible bounded symmetric domain of rank ≥ 2, it follows from Proposition 3.7 that $F \circ \iota_{j,W}$ is a totally geodesic holomorphic isometric embedding from (D_j, g_{D_j}) to $(\Omega_1, g_{\Omega_1}) \times \cdots \times (\Omega_l, g_{\Omega_l})$, $1 \leq j \leq k$. Let g'_Ω be the canonical Kähler metric on Ω such that $(\Omega, g'_\Omega) \cong (\Omega_1, g_{\Omega_1}) \times \cdots \times (\Omega_l, g_{\Omega_l})$. Therefore, for any $W \in D$ we have $\iota^*_{j,W}(F^*\omega_{g'_\Omega}) = \omega_{g_{D_j}}$ for $1 \leq j \leq k$.

Write $h := F^* g'_\Omega$ and $\omega_h := F^*\omega_{g'_\Omega}$. For $1 \leq j \leq k$, let $\pi_j : D \to D_j$ be the canonical projection onto the j-th factor, i.e., $\pi_j(W^1, \ldots, W^k) = W^j$ for $(W^1, \ldots, W^k) \in D$. Let $Z = (Z^1, \ldots, Z^k) \in D$. Note that $T_Z(D) = T_{Z^1}(D_1) \oplus \cdots \oplus T_{Z^k}(D_k)$. For any $v \in T_Z(D)$, we write $v = v_1 + \ldots + v_k$, where $v_j \in T_{Z^j}(D_j)$ for $1 \leq j \leq k$. Furthermore, for $1 \leq j \leq k$ and for any tangent vector $v_j \in T_{Z^j}(D_j)$ we may write $v_j = \sum_{\mu=1}^{r_j} v_{j,\mu} e^{(j)}_\mu$ in normal form (cf. Mok [Mok89, p. 252]), where $r_j := \mathrm{rank}(D_j) \geq 2$ and $\{e^{(j)}_\mu\}_{\mu=1}^{r_j}$ is the standard basis for the holomorphic tangent space of a totally geodesic r_j-disk of D_j through the point $Z^j \in D_j$. In this situation, $e^{(j)}_\mu$, $1 \leq \mu \leq r_j$, are characteristic vectors of $T_{Z^j}(D_j)$, for $1 \leq j \leq k$. From [Ng15, Proof of Lemma 3.1] we have

$$h\big(e^{(i)}_\mu, \overline{e^{(j)}_\nu}\big) = 0$$

for distinct i, j, $1 \leq i, j \leq k$, and for any μ, ν, $1 \leq \mu \leq r_i$, $1 \leq \nu \leq r_j$. In general, letting $\alpha_\mu \in T_{Z^\mu}(D_\mu) \subset T_Z(D)$ be characteristic vectors, $1 \leq \mu \leq k$, we have $h(\alpha_i, \overline{\alpha_j}) = 0$ for distinct i, j, $1 \le i, j \le k$. This implies that for tangent vectors $v_\mu \in T_{Z^\mu}(D_\mu) \subset T_Z(D)$, $1 \le \mu \le k$, we have $h(v_i, \overline{v_j}) = 0$ for distinct i, j, $1 \le i, j \le k$. In particular, we have

$$\omega_h = \sqrt{-1}\sum_{j=1}^{k} \sum_{1\le\mu,\nu\le n_j} h^{(j)}_{\mu\overline{\nu}}(Z) dZ^j_\mu \wedge d\overline{Z^j_\nu}$$

on D. Recall that for $W \in D$ we have $\iota^*_{i,W}\omega_h = \omega_{g_{D_i}}$ for $1 \le i \le k$. Thus, for $1 \le j \le k$, each $h^{(j)}_{\mu\overline{\nu}}(Z)$, $1 \le \mu, \nu \le n_j$, only depends on Z^j, i.e., $h^{(j)}_{\mu\overline{\nu}}(Z) \equiv h^{(j)}_{\mu\overline{\nu}}(Z^j)$. In addition, for $1 \le j \le k$ we have

$$\sqrt{-1} \sum_{1\le\mu,\nu\le n_j} h^{(j)}_{\mu\overline{\nu}}(Z^j) dZ^j_\mu \wedge d\overline{Z^j_\nu} = \pi_j^*\omega_{g_{D_j}}$$

by $\pi_j^*\omega_{g_{D_j}} = \pi_j^*(\iota^*_{j,W}\omega_h) = (\iota_{j,W} \circ \pi_j)^*\omega_h$. Then, we have $\omega_h = \sum_{j=1}^k \pi_j^*\omega_{g_{D_j}}$ and thus $F^*g'_\Omega = h = \sum_{j=1}^k \pi_j^* g_{D_j}$. Hence, F is a (proper) holomorphic isometric embedding from $(D, \sum_{j=1}^k \pi_j^* g_{D_j})$ to (Ω, g'_Ω). Since the irreducible factors of D are of rank ≥ 2, it follows from the arguments of [Mok12, Proof of Theorem 1.3.2] that the second fundamental form of $(F(D), g'_\Omega|_{F(D)})$ in (Ω, g'_Ω) vanishes identically and thus F is totally geodesic, as desired. □

As a consequence, we have a simple proof of Theorem 1.2 in the following. (Noting that Theorem 1.2 actually provides an affirmative answer to Question 3.1 under the assumption that all irreducible factors of the domain D are of rank ≥ 2.)

Proof of Theorem 1.2. By Lemma 3.5, F maps minimal disks of D properly into rank-1 characteristic symmetric subspaces of Ω. Then, the result follows from Theorem 1.1. □

Now, we study holomorphic maps $f : D \to \Omega$ which map minimal disks of D properly into rank-1 characteristic symmetric subspaces of Ω, where D and Ω are bounded symmetric domains such that Ω is reducible. The case where the reducible bounded symmetric domain Ω has an irreducible factor of rank ≥ 2 can be quite complicated in general if some irreducible factors of the domain D are complex unit balls (See Example 3.3). Therefore, we will focus on the simplest case where the target Ω is a product of complex unit balls. We first recall a result of Ng [Ng15].

Lemma 3.9 (cf. Proposition 2.3 in [Ng15])**.** *Let $F : \Delta \times U \to \mathbb{B}^n$ be a holomorphic map such that $F|_{\Delta\times\{\mathbf{0}\}} : \Delta \cong \Delta \times \{\mathbf{0}\} \to \mathbb{B}^k$ is a proper map, where $U \Subset \mathbb{C}^m$ is a bounded domain containing $\mathbf{0}$. Then, for any $(z, w) \in \Delta \times U$ we have $F(z, w) = F(z, \mathbf{0})$.*

On the other hand, by Mok [Mok16] and Yuan-Zhang [YZ12], we observe the non-existence of holomorphic isometries between certain bounded symmetric domains with respect to the canonical Kähler metrics, as follows.

Proposition 3.10. Let $\Omega \Subset \mathbb{C}^N$ be a bounded symmetric domain such that Ω has an irreducible factor of rank ≥ 2, i.e., $\Omega = \Omega_1 \times \cdots \times \Omega_n$ and there exists j, $1 \le j \le n$, such that $\text{rank}(\Omega_j) \ge 2$, where Ω_i, $1 \le i \le n$, are the irreducible factors of Ω. Equip a Kähler metric g'_Ω on Ω so that $(\Omega, g'_\Omega) \cong (\Omega_1, \lambda_1 g_{\Omega_1}) \times \cdots \times (\Omega_n, \lambda_n g_{\Omega_n})$ for some positive real constants λ_j, $1 \le j \le n$. Then, there does not exist a holomorphic isometry from (Ω, g'_Ω) to $(\mathbb{B}^{N_1}, \mu_1 g_{\mathbb{B}^{N_1}}) \times \cdots \times (\mathbb{B}^{N_m}, \mu_m g_{\mathbb{B}^{N_m}})$, where μ_l, $1 \le l \le m$, are positive real constants.

Proof. Assume the contrary that there exists a holomorphic isometry f from (Ω, g'_Ω) to $(\mathbb{B}^{N_1}, \mu_1 g_{\mathbb{B}^{N_1}}) \times \cdots \times (\mathbb{B}^{N_m}, \mu_m g_{\mathbb{B}^{N_m}})$, where μ_l, $1 \le l \le m$, are positive real constants. Then, by restricting to the irreducible factor Ω_j of Ω, we have a holomorphic isometry $\hat{f}$ from (Ω_j, g_{Ω_j}) to $(\mathbb{B}^{N_1}, \mu'_1 g_{\mathbb{B}^{N_1}}) \times \cdots \times (\mathbb{B}^{N_m}, \mu'_m g_{\mathbb{B}^{N_m}})$, where $\mu'_l := \frac{\mu_l}{\lambda_j}$ for $1 \le l \le m$. Write Ω_j for an irreducible factor of Ω such that $\mathrm{rank}(\Omega_j) \ge 2$. Then, it follows from [Mok16] that there exists a nonstandard (i.e., not totally geodesic) holomorphic isometry F from $(\mathbb{B}^k, g_{\mathbb{B}^k})$ to (Ω_j, g_{Ω_j}) for some integer $k \ge 2$. This gives a holomorphic isometry $\hat{f} \circ F$ from $(\mathbb{B}^k, g_{\mathbb{B}^k})$ to $(\mathbb{B}^{N_1}, \mu'_1 g_{\mathbb{B}^{N_1}}) \times \cdots \times (\mathbb{B}^{N_m}, \mu'_m g_{\mathbb{B}^{N_m}})$. By the rigidity theorem of Yuan-Zhang [YZ12], $\hat{f} \circ F$ is totally geodesic. This contradicts with the fact that F is not totally geodesic. Hence, there does not exist such a holomorphic isometry f, as desired. □

Now, by making use of the technique in Ng [Ng15], we have the following structure theorem for holomorphic maps from a bounded symmetric domain D to a product Ω of complex unit balls which map minimal disks of D properly into rank-1 characteristic symmetric subspaces of Ω.

Theorem 3.11. *Let $D = D_1 \times \cdots \times D_k$ be a bounded symmetric domain of rank ≥ 2 and $\Omega := \mathbb{B}^{m_1} \times \cdots \times \mathbb{B}^{m_l}$ be a product of complex unit balls, where D_i, $1 \le i \le k$, are irreducible bounded symmetric domains. Let $f : D \to \Omega$ be a holomorphic map which maps minimal disks of D properly into rank-1 characteristic symmetric subspaces of Ω. Write $f = (f_1, \ldots, f_l)$, where $f_j : D \to \mathbb{B}^{m_j}$, $1 \le j \le l$, are holomorphic maps. Then, we have $k \le l$ and up to a permutation of the irreducible factors of Ω, we have*

$$f(Z^1, \ldots, Z^k) = \begin{cases} (f_1(Z^1), \ldots, f_k(Z^k), f_{k+1}(Z), \ldots, f_l(Z)) & \text{if } k < l, \\ (f_1(Z^1), \ldots, f_k(Z^k)) & \text{if } k = l, \end{cases}$$

for $Z = (Z^1, \ldots, Z^k) \in D_1 \times \cdots \times D_k = D$. Moreover, $D_i \cong \mathbb{B}^{n_i}$ is a complex unit ball for some $n_i \ge 1$, $1 \le i \le k$, i.e., D is also a product of complex unit balls.

Proof. We may assume without loss of generality that $f(\mathbf{0}) = \mathbf{0}$. For each i, $1 \le i \le k$, we choose a minimal disk $\Delta^{(i)} := \{(z, 0, \ldots, 0) \in D_i : |z| < 1\} \subseteq D_i$. Write $M_i := D_1 \times \cdots \times D_{i-1} \times \Delta^{(i)} \times D_{i+1} \times \cdots \times D_k \subseteq D$ for $1 \le i \le k$. Restricting f to the minimal disk $\hat{\Delta}^{(i)} := \{(0, \ldots, 0)\} \times \Delta^{(i)} \times \{(0, \ldots, 0)\} \subseteq M_i \subseteq D$, we have $f(\hat{\Delta}^{(i)}) \subseteq B_i \subset \Omega$ for some rank-1 characteristic symmetric subspace B_i of Ω which contains $\mathbf{0}$. Note that such a rank-1 characteristic symmetric subspace B_i is exactly $\{(0, \ldots, 0)\} \times \mathbb{B}^{m_{j_i}} \times \{(0, \ldots, 0)\}$ for some j_i, $1 \le j_i \le l$. Thus, $f_j(\hat{\Delta}^{(i)}) = \{\mathbf{0}\}$ for $j \ne j_i$, and $f_{j_i}|_{\hat{\Delta}^{(i)}} : \hat{\Delta}^{(i)} \to B_i \cong \mathbb{B}^{m_{j_i}}$ is a proper holomorphic map. We write $Z = (Z^1, \ldots, Z^k) \in D_1 \times \cdots \times D_k = D$ and $Z^i \in D_i$ is the Harish-Chandra coordinates of D_i, $1 \le i \le k$. By [Ng15, Proposition 2.3] (i.e., Lemma 3.9), we have

$$f_{j_i}|_{M_i}(Z^1, \ldots, Z^{i-1}; z, 0, \ldots, 0; Z^{i+1}, \ldots, Z^k) \equiv f_{j_i}|_{M_i}(0, \ldots, 0; z, 0, \ldots, 0; 0, \ldots, 0)$$

and thus $f_{j_i}(Z^1, \ldots, Z^k) \equiv f_{j_i}(\mathbf{0}; Z^i; \mathbf{0})$ by the definition of M_i. In other words, f_{j_i} is independent of the variables Z^μ for all $\mu \ne i$, i.e., $f_{j_i}(Z) \equiv f_{j_i}(Z^i)$. It then follows that for distinct i_1, i_2, $1 \le i_1, i_2 \le k$, we have $j_{i_1} \ne j_{i_2}$ and thus $k \le l$. We may assume that $j_i = i$ for $1 \le i \le k$ after a permutation of the irreducible factors of Ω. Then, from the above results we have

$$f(Z^1, \dots, Z^k) = \begin{cases} (f_1(Z^1), \dots, f_k(Z^k), f_{k+1}(Z), \dots, f_l(Z)) & \text{if } k < l, \\ (f_1(Z^1), \dots, f_k(Z^k)) & \text{if } k = l. \end{cases}$$

Assume the contrary that D has an irreducible D_j which is of rank ≥ 2, i.e., $\mathrm{rank}(D_j) \ge 2$. Then, by restricting to $\{\mathbf{0}\} \times D_j \times \{\mathbf{0}\} \subset D$, we have a holomorphic map F from D_j to $\Omega = \mathbb{B}^{m_1} \times \cdots \times \mathbb{B}^{m_l}$ which maps minimal disks of D_j properly into rank-1 characteristic symmetric subspaces of Ω. By Proposition 3.7, $F : D_j \to \Omega$ is a totally geodesic holomorphic isometric embedding with respect to certain canonical Kähler metrics on D_j and Ω, which contradicts with the result of Proposition 3.10. Hence, all irreducible factors of D_j are of rank 1, i.e., $D_i \cong \mathbb{B}^{n_i}$ for some positive integer n_i, $1 \le i \le k$, as desired. □

In general, for a holomorphic map $f : D \to \Omega$ between bounded symmetric domains D and Ω of the same rank ≥ 2 which maps minimal disks of D properly into rank-1 characteristic symmetric subspaces of Ω, we do not have the analogous structure theorem as in Theorem 3.11 if Ω is not a product of complex unit balls. Actually, we have the following trivial example.

Example 3.12. Let p_j, q_j, $1 \le j \le 4$, be positive integers such that $p_j \le q_j$ for $1 \le j \le 4$. Let $f : D^{\mathrm{I}}_{p_1,q_1} \times D^{\mathrm{I}}_{p_2,q_2} \times D^{\mathrm{I}}_{p_3,q_3} \times D^{\mathrm{I}}_{p_4,q_4} \to D^{\mathrm{I}}_{p_1+p_2,q_1+q_2} \times D^{\mathrm{I}}_{p_3+p_4,q_3+q_4}$ be the holomorphic map defined by

$$f(Z) := (f_1(Z), f_2(Z))$$

for $Z = (Z^1, Z^2, Z^3, Z^4) \in D^{\mathrm{I}}_{p_1,q_1} \times D^{\mathrm{I}}_{p_2,q_2} \times D^{\mathrm{I}}_{p_3,q_3} \times D^{\mathrm{I}}_{p_4,q_4}$ with

$$f_1(Z) := \begin{bmatrix} Z^1 & \mathbf{0} \\ \mathbf{0} & Z^2 \end{bmatrix}, \quad f_2(Z) := \begin{bmatrix} Z^3 & \mathbf{0} \\ \mathbf{0} & Z^4 \end{bmatrix}.$$

It is clear that f is a proper holomorphic map between bounded symmetric domains of the same rank $\sum_{j=1}^4 p_j \ge 4$ but none of the f_1, f_2 depends only on one of the $Z^1, \dots, Z^4$.

4. Semi-product proper holomorphic maps between bounded symmetric domains

Motivated by the recent work of Seo [Seo18], we will study semi-product proper holomorphic maps between (reducible) bounded symmetric domains in this section. Let $f : D_1 \times \cdots \times D_k \to \Omega_1 \times \cdots \times \Omega_l$ be a proper holomorphic map, where D_i, $1 \le i \le k$, and Ω_j, $1 \le j \le l$, are irreducible bounded symmetric domains. Write Z^j (or W^j) for the Harish-Chandra coordinates of D_j for $1 \le j \le k$. In [Seo18], Seo introduced the notion of semi-product proper holomorphic maps between (reducible) bounded symmetric domains, as follows.

Definition 4.1 (cf. Seo [Seo18])**.** The map f is said to be a **semi-product proper holomorphic map** if for any $i \in \{1, \dots, k\}$, there exists $j \in \{1, \dots, l\}$ such that the map $f_{i,j,W} : D_i \to \Omega_j$ defined by

$$f_{i,j,W}(Z^i) = f_j(W^1, \dots, W^{i-1}, Z^i, W^{i+1}, \dots, W^k)$$

is a proper holomorphic map for $W = (W^1, \ldots, W^{i-1}, W^{i+1}, \ldots, W^k)$ in some dense open subset of $D_1 \times \cdots \times \widehat{D_i} \times \cdots \times D_k$. Here, $\widehat{D_i}$ means that D_i is omitted. On the other hand, we say that the map f is a **product map** if $k = l$ and

$$f(Z^1, \ldots, Z^k) = (f_1(Z^{\sigma(1)}), \ldots, f_k(Z^{\sigma(k)}))$$

for some permutation $\sigma \in \Sigma_k$ so that each holomorphic map f_j only depends on the holomorphic coordinates of $D_{\sigma(j)}$ for $1 \le j \le k$.

A map F from a bounded domain $D \Subset \mathbb{C}^n$ to a bounded domain $\Omega \Subset \mathbb{C}^N$ is said to be **rational** if all component functions of F are rational functions in $z = (z_1, \ldots, z_n) \in D \Subset \mathbb{C}^n$, i.e., $F = (F_1, \ldots, F_N)$ and $F_j(z) = \frac{P_j(z)}{Q_j(z)}$, $1 \le j \le N$, for some complex polynomials $P_j, Q_j \in \mathbb{C}[z]$. Then, Seo [Seo18] has shown that any rational proper holomorphic map between (reducible) bounded symmetric domains is a semi-product proper holomorphic map, namely, we have

Proposition 4.2 (cf. Proposition 3.5 in Seo [Seo18])**.** Let $f : D_1 \times \cdots \times D_k \to \Omega_1 \times \cdots \times \Omega_l$ be a proper holomorphic map, where D_i, $1 \le i \le k$, and Ω_j, $1 \le j \le l$, are irreducible bounded symmetric domains. If f is rational, then f is a semi-product proper holomorphic map.

Motivated by the example of a proper holomorphic map from $D^{\mathrm{I}}_{2,2}$ to $D^{\mathrm{I}}_{3,3}$ constructed by Tsai [Ts93, p. 124], we give an example of a semi-product proper holomorphic map between certain reducible bounded symmetric domains which is neither a product map nor totally geodesic.

Example 4.3. Let $f : D^{\mathrm{I}}_{2,2} \times D^{\mathrm{I}}_{2,2} \to D^{\mathrm{I}}_{3,3} \times D^{\mathrm{I}}_{3,3}$ be a holomorphic map given by

$$f(Z^1, Z^2) = \left(\begin{bmatrix} Z^1 & 0 \\ 0 & h_1(Z^2) g_1(Z^1) \end{bmatrix}, \begin{bmatrix} Z^2 & 0 \\ 0 & h_2(Z^1) g_2(Z^2) \end{bmatrix} \right)$$

for $(Z^1, Z^2) \in D^{\mathrm{I}}_{2,2} \times D^{\mathrm{I}}_{2,2}$, where h_j and g_j are holomorphic functions on $D^{\mathrm{I}}_{2,2}$ such that for any $W \in D^{\mathrm{I}}_{2,2}$ we have $|h_j(W)| < 1$ and $|g_j(W)| < 1$, $j = 1, 2$. Then, it is clear that f is a semi-product proper holomorphic map but not a product map. In addition, we can choose the holomorphic functions h_j and g_j, $j = 1, 2$, such that f is not totally geodesic. This also shows the existence of a semi-product proper holomorphic map between bounded symmetric domains which is not a rational map.

We can actually obtain lots of holomorphic maps from $D^{\mathrm{I}}_{2,2}$ to $\Delta := \{w \in \mathbb{C} : |w| < 1\}$. Write $W = \left(w_{ij}\right)_{1 \le i,j \le 2}$ and let $p(W)$ be a polynomial in $(w_{11}, w_{12}, w_{21}, w_{22})$. Let $M := \sup_{W \in \overline{D^{\mathrm{I}}_{2,2}}} |p(W)|$. Then, we have $M < +\infty$ by the boundedness of $D^{\mathrm{I}}_{2,2}$. Moreover, by the maximum modulus principle we actually have $|p(W)| < M$ for any $W \in D^{I}_{2,2}$ because p is a non-constant holomorphic function on the bounded domain $D^{\mathrm{I}}_{2,2}$. We define $h(W) := \frac{1}{M} p(W)$. Then, for any $W \in D^{\mathrm{I}}_{2,2}$ we have $|h(W)| = \frac{1}{M}|p(W)| < 1$. Thus, $h : D^{\mathrm{I}}_{2,2} \to \mathbb{C}$ is a holomorphic function such that for any $W \in D^{\mathrm{I}}_{2,2}$ we have $|h(W)| < 1$. In general, we may replace the polynomial $p(W)$ by any non-constant bounded holomorphic function on $D^{\mathrm{I}}_{2,2}$ in the above.

In analogy to Lemma 3.9, Seo [Seo18] obtained the following result.

Lemma 4.4 (cf. Corollary 2.3 in [Seo18])**.** *Let D and Ω be irreducible bounded symmetric domains such that* $\mathrm{rank}(D) \ge \mathrm{rank}(\Omega)$*. We also let $F : D \times U \to \Omega$ be a* *holomorphic map such that* $F|_{D\times\{w\}} : D \cong D \times \{w\} \to \Omega$ *is a proper holomorphic map for each* $w \in U$*, where* $U \Subset \mathbb{C}^m$ *is a connected bounded domain. Then,* f *does not depend on* $w \in U$*.*

For any (reducible) bounded symmetric domain U, we write

$$R_U := \{\mathrm{rank}(U') : U' \text{ is an irreducible factor of } U\}$$

and we define $r_{\min}(U) := \min R_U$ and $r_{\max}(U) := \max R_U$. We remark here that there are reducible bounded symmetric domains D and Ω such that $r_{\min}(D) \geq r_{\max}(\Omega)$ and $\mathrm{rank}(D) < \mathrm{rank}(\Omega)$. For example, for $D = D^{\mathrm{I}}_{3,p_1} \times D^{\mathrm{I}}_{3,p_2}$ and $\Omega = D^{\mathrm{I}}_{3,q_1} \times D^{\mathrm{I}}_{3,q_2} \times D^{\mathrm{I}}_{3,q_3}$, where $p_1, p_2, q_1, q_2, q_3 \geq 3$ are integers, we have $r_{\min}(D) = 3 = r_{\max}(\Omega)$ but $\mathrm{rank}(D) = 6 < 9 = \mathrm{rank}(\Omega)$.

From Example 4.3, there is a semi-product proper holomorphic map $f : D \to \Omega$ which is nonstandard and not a product map even if $r_{\min}(D) = r_{\max}(\Omega)-1$ and that D and Ω have the same number of irreducible factors. Therefore, for a semi-product proper holomorphic map $f : D \to \Omega$ between bounded symmetric domains D and Ω, by imposing a certain rank condition on D and Ω, namely, $r_{\min}(D) \geq r_{\max}(\Omega)$, we have

Theorem 4.5. *Let* $D = D_1 \times \cdots \times D_k$ *and* $\Omega = \Omega_1 \times \cdots \times \Omega_l$ *be bounded symmetric domains, where* D_i*,* $1 \leq i \leq k$*, and* Ω_j*,* $1 \leq j \leq l$*, are irreducible bounded symmetric domains. Let* $f = (f_1, \ldots, f_l) : D \to \Omega$ *be a semi-product proper holomorphic map. If* $r_{\min}(D) = \min\{\mathrm{rank}(D_i) : 1 \leq i \leq k\} \geq \max\{\mathrm{rank}(\Omega_j) : 1 \leq j \leq l\} = r_{\max}(\Omega)$*, then* $k \leq l$ *and we have the following.*

(1) *Suppose* $k = l$*. Then, we have*
  (a) $\mathrm{rank}(D) = \mathrm{rank}(\Omega)$, $r_{\min}(D) = r_{\max}(\Omega) =: r$ *and* $\mathrm{rank}(D_i) = \mathrm{rank}(\Omega_j)$ $= r$ *for all* i*,* j*,* $1 \leq i \leq k$*,* $1 \leq j \leq l$*.*
  (b) f *is a product map, i.e.,*

$$f(Z^1, \ldots, Z^k) = (f_1(Z^{\sigma(1)}), \ldots, f_k(Z^{\sigma(k)}))$$

  for some permutation $\sigma \in \Sigma_k$*, where* $Z^j \in D_j$ *for* $j = 1, \ldots, k$*.*

  If in addition that $r_{\max}(\Omega) \geq 2$*, then* $f : D \to \Omega$ *is a totally geodesic holomorphic isometric embedding with respect to certain canonical Kähler metrics on* D *and* Ω*.*

(2) *Suppose* $k < l$*. Then, up to a permutation of the irreducible factors* Ω_j*,* $1 \leq j \leq l$*, of* Ω*, we have*

$$f(Z^1, \ldots, Z^k) = (f_1(Z^1), \ldots, f_k(Z^k), f_{k+1}(Z), \ldots, f_l(Z))$$

  for $Z = (Z^1, \ldots, Z^k) \in D_1 \times \cdots \times D_k = D$*, and for each* i*,* $1 \leq i \leq k$*, we have* $\mathrm{rank}(D_i) = r_{\min}(D) = r_{\max}(\Omega) = \mathrm{rank}(\Omega_i)$ *and* $f_i : D_i \to \Omega_i$ *is a proper holomorphic map. If in addition that* $r_{\max}(\Omega) \geq 2$*, then for* $1 \leq i \leq k$*,* $f_i : D_i \to \Omega_i$ *is a totally geodesic holomorphic isometric embedding with respect to the Bergman metrics up to a normalizing constant.*

Proof. Our method here is inspired by the proof of Proposition 3.4 in Seo [Seo18]. Since f is a semi-product map, for $1 \leq i_1 < i_2 \leq k$, there are $j_1, j_2 \in \{1, \ldots, l\}$ such that $f_{i_\mu, j_\mu, w^{(\mu)}} : D_{i_\mu} \to \Omega_{j_\mu}$ defined by

$$f_{i_\mu, j_\mu, w^{(\mu)}}(Z^{i_\mu}) = f_{j_\mu}(w^{(\mu)}_1, \ldots, w^{(\mu)}_{i_\mu - 1}, Z^{i_\mu}, w^{(\mu)}_{i_\mu + 1}, \ldots, w^{(\mu)}_k)$$

is a proper holomorphic map for $w^{(\mu)} = (w_1^{(\mu)}, \dots, w_{i_\mu-1}^{(\mu)}, w_{i_\mu+1}^{(\mu)}, \dots, w_k^{(\mu)}) \in D_1 \times \cdots \times \widehat{D_{i_\mu}} \times \cdots \times D_k$, $\mu = 1, 2$. Here, $\widehat{D_{i_\mu}}$ means that the factor D_{i_μ} is omitted. If $j_1 = j_2$, then

$$f_{j_1}(w_1, \dots, w_{i_1-1}, \cdot, w_{i_1+1}, \dots, w_{i_2-1}, \cdot, w_{i_2+1}, \dots, w_k) : D_{i_1} \times D_{i_2} \to \Omega_{j_1}$$

is a proper holomorphic map, a plain contradiction because $\mathrm{rank}(D_{i_1} \times D_{i_2}) > \mathrm{rank}(D_{i_1}) \geq \mathrm{rank}(\Omega_{j_1})$ by the assumption (cf. [Ts93]). Thus, we have $j_1 \neq j_2$. In particular, for any $i \in \{1, \dots, k\}$, there exists $n_i \in \{1, \dots, k\}$ such that $f_{i,n_i,w} : D_i \to \Omega_{n_i}$ is a proper holomorphic map for $w \in D_1 \times \cdots \times \widehat{D_i} \times \cdots \times D_k$ and $n_i \neq n_\mu$ whenever $i \neq \mu$. Then, we have $k \leq l$.

Now, we may assume that $n_i = i$ for $i = 1, \dots, k$ after permuting the irreducible factors of Ω. Note that $\mathrm{rank}(D_i) \geq r_{\min}(D) \geq r_{\max}(\Omega) \geq \mathrm{rank}(\Omega_i)$ for $1 \leq i \leq k$. Applying Corollary 2.3 in Seo [Seo18] (i.e., Lemma 4.4) to $f_{i,i,w}$ for each i, we obtain that f_i depends only on $Z^i \in D_i$ and $f_i : D_i \to \Omega_i$ is a proper holomorphic map for $1 \leq i \leq k$.

Case (1) Suppose $k = l$. Then, f is a product map. Moreover, we have

$$\mathrm{rank}(D) \geq k r_{\min}(D) \geq k r_{\max}(\Omega) = l r_{\max}(\Omega) \geq \mathrm{rank}(\Omega), \tag{4.1}$$

i.e., $\mathrm{rank}(D) \geq \mathrm{rank}(\Omega)$. From Tsai [Ts93, p. 129], we have $\mathrm{rank}(D) = \mathrm{rank}(\Omega)$. Thus, each inequality in Eq. (4.1) is actually an equality. In particular, we have

$$\mathrm{rank}(D_i) = r_{\min}(D) = r_{\max}(\Omega) = \mathrm{rank}(\Omega_j)$$

for all i, j, $1 \leq i \leq k$, $1 \leq j \leq l$.

If in addition that $r_{\max}(\Omega) \geq 2$, then we have $\mathrm{rank}(D_i) \geq r_{\min}(D) \geq r_{\max}(\Omega) \geq \mathrm{rank}(\Omega_i) \geq 2$ for $1 \leq i \leq k$. By Tsai [Ts93, Main Theorem], $f_i : D_i \to \Omega_i$ is a totally geodesic holomorphic isometric embedding with respect to the Bergman metrics up to a normalizing constant for $1 \leq i \leq k$. Hence, $f : D \to \Omega$ is a totally geodesic holomorphic isometric embedding with respect to certain canonical Kähler metrics on D and Ω. (Noting that the result also follows directly from Theorem 1.2 in this situation.)

Case (2) Suppose $k < l$. From the above, we have

$$f(Z^1, \dots, Z^k) = (f_1(Z^1), \dots, f_k(Z^k), f_{k+1}(Z), \dots, f_l(Z))$$

after permuting the irreducible factors Ω_j, $1 \leq j \leq k$, of Ω. The rest follows from the arguments in Case (1).

□

Remark 4.6.

(1) By Proposition 3.5 in Seo [Seo18] and Theorem 4.5, we know that any rational proper holomorphic map f from $D_1 \times \cdots \times D_k$ to $\Omega_1 \times \cdots \times \Omega_k$ is a product map whenever $\mathrm{rank}(D_i) = \mathrm{rank}(\Omega_j) = r$ for all i, j, $1 \leq i, j \leq k$ and r is independent of i and j.

(2) Define a holomorphic map $f : D_3^{\mathrm{III}} \times D_3^{\mathrm{III}} \to D_{3,q_1}^{\mathrm{I}} \times D_{3,q_2}^{\mathrm{I}} \times \Delta$ by

$$f(Z^1, Z^2) := \left(\begin{bmatrix} Z^1 & \mathbf{0} \end{bmatrix}, \begin{bmatrix} Z^2 & \mathbf{0} \end{bmatrix}, h(Z^1, Z^2) \right),$$

where $q_1, q_2 \geq 3$ are integers, and $h : D_3^{\mathrm{III}} \times D_3^{\mathrm{III}} \to \mathbb{C}$ is a holomorphic function such that $|h(Z^1, Z^2)| < 1$ on $D_3^{\mathrm{III}} \times D_3^{\mathrm{III}}$. (Noting that $r_{\min}(D_3^{\mathrm{III}} \times D_3^{\mathrm{III}}) = 3 = r_{\max}(D_{3,q_1}^{\mathrm{I}} \times D_{3,q_2}^{\mathrm{I}} \times \Delta)$ because $q_1, q_2 \geq 3$.) Then, we may choose a function h so that $f : D_3^{\mathrm{III}} \times D_3^{\mathrm{III}} \to D_{3,q_1}^{\mathrm{I}} \times D_{3,q_2}^{\mathrm{I}} \times \Delta$ is a semi-product proper holomorphic map which is not totally geodesic. That means in Case (2) of Theorem 4.5, it is possible that such a semi-product proper holomorphic map is not totally geodesic.

(3) By Proposition 3.5 in [Seo18] (i.e., Proposition 4.2), the statement of Theorem 4.5 still holds true if we assume that f is rational instead of f is semi-product. In other words, Theorem 4.5 gives a complete description of all rational proper holomorphic maps $f : D \to \Omega$ between (reducible) bounded symmetric domains when $r_{\min}(D) \geq r_{\max}(\Omega)$.

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