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Rigidity of certain admissible pairs of rational homogeneous spaces of Picard number 1 which are not of the subdiagram type

Dedicated to Professor Lo Yang on the occasion of his 80th Birthday

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Abstract Recently Mok and Zhang introduced the notion of admissible pairs (X_0, X) of rational homogeneous spaces of Picard number 1 and prove rigidity of admissible pairs (X_0, X) of the subdiagram type whenever X_0 is nonlinear. It remains unsolved whether rigidity holds when (X_0, X) is an admissible pair **not** of the subdiagram type of nonlinear irreducible Hermitian symmetric spaces such that (X_0, X) is nondegenerate for substructures. In this article we provide sufficient conditions for confirming rigidity of such an admissible pair. In a nutshell our solution consists of an enhancement of the method of propagation of sub-VMRT structures along chains of minimal rational curves as is already implemented in the proof of the Thickening Lemma of Mok-Zhang. There it was proven that, for a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ on a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component and satisfying certain conditions so that in particular S is "uniruled" by open subsets of certain minimal rational curves on X, for a "good" minimal rational curve ℓ emanating from a general point $x \in S$, there exists an immersed neighborhood $\mathbf{N}_\ell$ of ℓ which is in some sense "uniruled" by minimal rational curves. By means of the Algeraicity Theorem of Mok-Zhang (2019), S can be completed to a projective subvariety $Z \subset X$. By the author's solution of the Recognition Problem for irreducible Hermitian symmetric spaces of rank $\geqslant 2$ (2008) and under Condition (F), which symbolizes the *fitting* of sub-VMRTs into VMRTs, we further prove that Z is the image under a holomorphic immersion of X_0 into X which induces an isomorphism on second homology groups. By studying $\mathbb{C}^*-$actions we prove that Z can be deformed via a one-parameter family of automorphisms to converge to $X_0 \subset X$. Under the additional hypothesis that all holomorphic sections in $\Gamma(X_0, T_X|_{X_0})$ lift to global holomorphic vector fields on X, we prove that the admissible pair (X_0, X) is rigid. As examples we check that (X_0, X) is rigid when X is the Grassmannian $G(n, n)$ of n-dimensional complex vector subspaces of $W \cong \mathbb{C}^{2n}$, $n \geqslant 3$, and when $X_0 \subset X$ is the Lagrangian Grassmannian consisting of Lagrangian vector subspaces of (W, σ) where σ is an arbitrary symplectic form on W.

Keywords rational homogeneous spaces, rigidity, uniruled projective manifolds, sub-VMRT structures

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1 Introduction

The prototype of geometric structures are G-structures over smooth manifolds. Riemannian manifolds on an m-dimensional smooth manifold gives a smooth reduction from $\mathrm{GL}(m;\mathbb{R})$ to $\mathrm{O}(m)$, a Kähler metric on a complex n-dimensional complex manifold X underlies a smooth $\mathrm{U}(n)$ structure on X, which is further reduced to an $\mathrm{SU}(n)$ structure in case the Kähler metric is Ricci flat.

In the case of holomorphic geometry, a holomorphic conformal structure on an n-dimensional complex manifold gives a holomorphic reduction of the holomorphic frame bundle from $\mathrm{GL}(n;\mathbb{C})$ to $\mathbb{C}^*\cdot\mathrm{O}(n;\mathbb{C})$, where $\mathrm{O}(n;\mathbb{C})$ is the complex orthogonal group of invertible n-by-n matrices A with coefficients in $\mathbb{C}$ satisfying $AA^t = I_n$. For $p, q \geqslant 2$, a (pq)-dimensional manifold is said to admit a holomorphic Grassmann structure modeled on the Grassmannian $G(p,q)$ if and only if the holomorphic frame bundle can be reduced from $\mathrm{GL}(pq;\mathbb{C})$ to the image $L \subset \mathrm{GL}(pq;\mathbb{C})$ of the group homomorphism $\Phi : \mathrm{GL}(p;\mathbb{C}) \times \mathrm{GL}(q;\mathbb{C})$ in $\mathrm{GL}(pq;\mathbb{C})$ defined by $\Phi(A,B)(Z) = BZA^t$. (Here in Harish-Chandra coordinates $G(p,q)$ is taken to be a compactification of the complex vector space $M(q,p;\mathbb{C})$ of q-by-p matrices with coefficients in $\mathbb{C}$.) Thus, a Grassmann structure on a complex manifold is an isomorphism $T_X \cong U \otimes V$ as holomorphic vector bundles in which U and V are holomorphic vector bundles on X of rank $\geqslant 2$. The hyperquadric is endowed with a flat holomorphic conformal structure, and a Grassmannian of rank $\geqslant 2$ with a flat Grassmann structure. Here and henceforth, without explicitly mentioning it, G-structures on complex manifolds are understood to be holomorphic.

In Riemannian geometry there is the Riemannian curvature tensor, and the flat Euclidean space is characterized locally as the underlying manifold of a Riemannian manifold with vanishing curvature tensor. In holomorphic geometry one can characterize the hyperquadric locally as the underlying complex manifold of a holomorphic conformal structure with vanishing Bochner-Weyl tensor. For each G-structure defined by a reductive linear subgroup $\mathrm{G} \subset \mathrm{GL}(m;\mathbb{R})$ Guillemin [Gu65] introduced a finite number of curvature-like tensors with the property that a flat model of the G-structure is characterized by the vanishing of the totality of such tensors, and the same applies to holomorphic geometry for reductive linear subgroups $\mathrm{G} \subset \mathrm{GL}(n;\mathbb{C})$.

Specializing to Fano manifolds, as one of the first results from our work on geometric structures modeled on VMRTs (varieties of minimal rational tangents), Hwang-Mok [HM97] characterized irreducible n-dimensional Hermitian symmetric spaces X of rank $\geqslant 2$ as the unique uniruled projective manifolds which admit irreducible G-structures for some reductive linear subgroups $\mathrm{G} \subsetneq \mathrm{GL}(V)$, $V \cong \mathbb{C}^n$. Here a G-structure is said to be irreducible if and only if G acts irreducibly on V.

When one considers X as an ambient manifold underlying a G-structure, in analogy to the study of (germs of) Riemannian submanifolds on Riemannian manifolds it is natural to consider (germs of) complex submanifolds on X which in some sense inherit geometric structures from X. This type of problems appeared in the first instance in the works of Bryant [Br01] and Walters [Wa97] on the question of Schur rigidity for smooth Schubert cycles X_0 on a Hermitian symmetric space X, where $X_0 \subset X$ is said to be Schur rigid if and only if for any positive integer r the only cycles $Z \subset X$ homologous to $r\cdot X_0$ must necessarily be of the form $\gamma_1 X_0 + \cdots + \gamma_r X_0$ for some $\gamma_1, \cdots, \gamma_r \in \mathrm{Aut}(X)$, In their works, in which among other things the problem was solved for certain sub-Grassmannians, a crucial step in the solution was a proof of Schubert rigidity for the pair (X_0, X). Here (X_0, X) is said to be Schubert rigid if, given any connected open subset $U \subset X$ and any complex submanifold $S \subset U$, $\dim(S) = \dim(X_0)$, such that for any $x \in S$, S is tangent at x to $\gamma_x(X_0)$ for some $\gamma_x \in \mathrm{Aut}(X)$, S must necessarily be an open subset of γX_0 for some $\gamma \in \mathrm{Aut}(X)$. The problem of Schur rigidity for smooth Schubert varieties on irreducible Hermitian symmetric spaces of the compact type was settled by Hong [Ho07], and the analogous problem for singular Schubert cycles on Grassmannians were in part solved by Hong [Ho05]. In Robles-The [RT12], a complete solution for Schur rigidity for Schubert cycles was obtained. We note that in all of the works cited in the above, a crucial part of the arguments relies on the study of cohomology groups associated to Lie algebras.

From a completely different perspective, as an outgrowth of the geometric theory of uniruled projective manifolds modeled on VMRTs of one of the authors with J.-M. Hwang, Hong-Mok [HoM10] and Hong-

Park [HoP11] studied via VMRT geometry the characterization of smooth Schubert cycles $X_0 \subset X$ on a rational homogeneous space X of Picard number 1. In [HoM10] we introduced the notion of VMRT-respecting germs of holomorphic maps and also the method of parallel transport of VMRTs along minimal rational curves. For instance, for a pair (X_0, X) of *nonlinear* rational homogeneous spaces of Picard number 1 of the subdiagram type, by [HoM10] and [HoP11] the image of any VMRT-respecting holomorphic immersion $i : (X_0; 0) \to (X; 0)$ must necessarily extend holomorphically to γX_0 for some $\gamma \in \mathrm{Aut}(X)$. In [HoM10] we introduced the notion of nondegeneracy for holomorphic mappings for pairs of VMRTs expressible in terms of projective second fundamental forms and established non-equidimensional Cartan-Fubini extension for VMRT-respecting germs of holomorphic immersions between uniruled projective manifolds of Picard number 1 under such an assumption of nondegeneracy. In Hong-Mok [HoM13] using VMRT structures we established the multiplicity-free case of Schur rigidity for nonlinear smooth Schubert cycles on rational homogeneous spaces of Picard number 1.

In Mok-Zhang [MZ19] we introduced the notion of admissible pairs (X_0, X) of rational homogeneous spaces of Picard number 1 and prove rigidity of admissible pairs (X_0, X) of the subdiagram type whenever X_0 is nonlinear. In the case of the subdiagram type this recovers Schubert rigidity in the symmetric case by a different method and extends the result to the general case of rational homogeneous spaces. Let $W \subset X$ be an open subset of a uniruled projective manifold X equipped with a minimal rational component and $S \subset W$ be a complex submanifold such that, defining $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$, the canonical map $\varpi : \mathscr{C}(S) \to S$ is dominant and it defines a sub-VMRT structure in some precise sense, we show that S is saturated with respect to open subsets of minimal rational curves of X lying on S under a condition of nondegeneracy of substructures expressible in terms of projective second fundamental forms and an additional condition, called Condition (T), concerning the intersection $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$, and prove that S can be extended to a projective subvariety $Z \subset X$. In the case where $S \subset W$ inherits a sub-VMRT structure modeled on (X_0, X) for some admissible pair (X_0, X) of rational homogeneous space of Picard number 1, the Condition (T), which is the condition $T_\alpha(\widetilde{\mathscr{C}_x}(S)) = T_x(\widetilde{\mathscr{C}_{[\alpha]}}(X)) \cap T_x(S)$, holds for any point $x \in S$ and a general member $\alpha \in \mathscr{C}_x(S)$ (where for a complex vector space V and for a subset $E \subset \mathbb{P}V$, $E \subset V - \{0\}$ denotes its affinization). We proved in [MZ19] that for an admissible pair (X_0, X) of the subdiagram type, $0 \in X_0 \subset X$, the pair $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ is nondegenerate for substructures if and only if it is nondegenerate for mappings, and conclude with the proofs of [HoM10] and [HoP11] that the pair (X_0, X) is rigid whenever (X_0, X) is of the subdiagram type and X_0 is nonlinear.

On the other hand admissible pairs (X_0, X) need not be of the subdiagram type. Specializing to the case where X_0 and X are of the Hermitian type, we have the prototypical examples of pairs of hyperquadrics (Q^m, Q^n), $3 \leqslant m \leqslant n$, where $Q^m \subset Q^n$ are embedded in the standard way, and the case of $(G^{\mathrm{III}}(n, n), G(n, n))$ of the standard embedding of the Lagrangian Grassmannian into the Grassmannian, $n \geqslant 3$. The pair $(Q^m, Q^n))$ is degenerate for substructures, but $(G^{\mathrm{III}}(n, n), G(n, n))$ is nondegenerate for substructures for $n \geqslant 3$. In [Zh14, Main Theorem 2] admissible pairs (X_0, X) not of the subdiagram type of nonlinear irreducible Hermitian symmetric spaces of the compact type were classified, and it was determined which of them are nondegenerate for substructures. It was shown by explicit examples that (X_0, X) is not rigid whenever (X_0, X) is degenerate for substructures.

It remains unsolved whether rigidity holds when (X_0, X) is an admissible pair **not** of the subdiagram type of nonlinear irreducible Hermitian symmetric spaces such that $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ is nondegenerate for substructures. In this article we provide a sufficient conditions for confirming rigidity of such an admissible pair. In a nutshell our solution consists of an enhancement of the method of propagation of sub-VMRT structures along chains of minimal rational curves as is already implemented in the proof of the Thickening Lemma of Mok-Zhang [MZ19, Proposition 6.1]. There it was proven that, for a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ on a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component and satisfying certain conditions so that in particular S is "uniruled" by open subsets of certain minimal rational curves of X, for a "good" minimal rational curve ℓ emanating from a general point $x \in S$, there exists an immersed neighborhood $\mathbf{N}_\ell$ of ℓ which is in some sense "uniruled" by minimal rational curves.

We return now to the case of sub-VMRT structures modeled on an admissible pair (X_0, X) of nonlinear

Hermitian symmetric spaces of the compact type. In this case X is uniruled by projective lines, the VMRT $\mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$ is linearly nondegenerate, the "collar" $\mathbf{N}_\ell \supset \ell$ may be taken to be embedded, and we deduce from the "Algebraicity Theorem" [MZ19, Main Theorem 2] that $S \subset Z$ for some projective subvariety $Z \subset X$ such that $\dim(Z) = \dim(S)$. On the one hand, there is a unique dominant component $\mathscr{C}(\mathbf{N}_\ell)$ of $\mathscr{C}^0(\mathbf{N}_\ell) := \mathscr{C}(X) \cap \mathbb{P}T(\mathbf{N}_\ell)$ obtained by tangential intersection, and the canonical projection $\varpi' : \mathscr{C}(\mathbf{N}_\ell) \to \mathbf{N}_\ell$ serves as a candidate for the analytic continuation of $\varpi : \mathscr{C}(S) \to S$ from S to the collar $\mathbf{N}_\ell$ of ℓ. On the other hand, making use of the deformation of minimal rational curves $\Lambda \subset \mathbf{N}_\ell$ and the explicit construction of such collars $\mathbf{N}_\ell$, where X_0 is endowed with a flat L_0-structure for a reductive linear subgroup $L_0 \subsetneq \mathrm{GL}(T_0(X_0))$, which is equivalently the identity component of the stabilizer subgroup of $\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X)$, we obtain by parallel transport of relative projective second fundamental forms of $(\mathscr{C}_x(S) \subset \mathscr{C}_x(X))$ along tautological liftings $\widehat{\Lambda}$ of minimal rational curves $\Lambda \subset \mathbf{N}_\ell$, and conclude from the method of parallel transport of Mok [Mo08b] the existence of an L_0-structure $\mathscr{C}(\mathbf{N}_\ell) \subset \mathbb{P}T(\mathbf{N}_\ell)$ on $\mathbf{N}_\ell$ extending the germ of L_0-structure on S at an initial point $x_0 \in S$, which is necessarily flat by analytic continuation in view of the characterization of flatness of G-structures for proper reductive linear subgroups G as given by Guillemin [Gu65].

While at the initial point x_0, all minimal rational curves emanating from x_0 and tangent to $\mathbf{N}_\ell$ are "good", this is not obvious for $\mathscr{C}_z(\mathbf{N}_\ell) \subset \mathscr{C}_z(X)$ at a point $z \in \mathbf{N}_\ell$ obtained by propagation of sub-VMRTs. Modulo a property which we call Condition (F), which suggests the *fitting* of sub-VMRTs, $(\mathscr{C}_z(\mathbf{N}_\ell) \subset \mathscr{C}_z(X))$ is projectively equivalent to $(\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X))$, and every minimal rational curve emanating from z and tangent to $\mathbf{N}_\ell$ is a "good" minimal rational curves, and the argument of parallel transport can be iterated to allow us to show that $Z \subset X$ is an immersed projective manifold such that the normalization $\widetilde{Z}$ of Z, which is nonsingular, is endowed with a flat L_0-structure. Hence, the rationally connected and hence simply connected projective manifold $\widetilde{Z}$ is biholomorphically equivalent to the model manifold X_0, and $Z \subset X$ is the image of X_0 under a holomorphic immersion $h : X_0 \to X$ of degree 1 into X.

Cirtical to our proofs of the main results is the use of deformation theory on the cycle Z, $\dim(Z) =: s$. First of all, by means of a $\mathbb{C}^*$-action on X with an isolated fixed point x_0 which is a smooth point of Z, we can deform Z by flattening at x_0 to the sum of a model submanifold which is translate of X_0 together with an s-cycle A at infinity with respect to a choice of Harish-Chandra coordinates. Then we show that $A = \emptyset$ by a calculation of volumes based on the fact that aforementioned holomorphic immersion $h : X_0 \to X$ is of degree 1, which implies that, modulo a global automorphism, Z is a local deformation of X_0. From the local rigidity of $X_0 \subset X$ we conclude that Z is itself a translate of X_0 provided that small deformations of X_0 are induced by global automorphisms of X, which introduces for us a second condition, viz., the surjectivity of the restriction map $r : \Gamma(X, T_X) \to \Gamma(X_0, T_X|_{X_0})$ defined by $r(\eta) = \eta|_{X_0}$. In principle this property and Condition (F) can be checked case-by-case from the classification of admissible pairs of nonlinear irreducible Hermitian symmetric spaces of the compact type of Zhang [Zh14], but we will leave that aside and end the article for the purpose of illustration of applications of the Main Theorem with the examples of Lagrangian Grassmannians $G^{\mathrm{III}}(n,n)$ in Grassmannians $G(n,n)$, $n \geqslant 3$ (where the admissible pair $(G^{\mathrm{III}}(n,n), G(n,n))$ is nondegenerate for substructures), and those of hyperquadrics Q^m in hyperquadrics Q^n, $3 \leqslant m \leqslant n$ (where the admissible pair (Q^m, Q^n) is degenerate for substructures).

2 Background materials

We collect first of all basic definitions and results concerning VMRTs and sub-VMRTs relevant to the study of admissible pairs (X_0, X) of Hermitian symmetric spaces of the compact type. These are taken from Hwang-Mok [HM97], Mok [Mo08b] and Mok-Zhang [MZ19]. We also refer the reader to Hwang-Mok [HM99] and Mok [Mo08a] as general references on the geometric theory of VMRTs, and to the more recent survey article Mok [Mo16] for an exposition on VMRT theory incorporating the study of sub-VMRTs.

Theorem 2.1 (Hwang-Mok [HM97, Main Theorem])**.** *Let V be a finite dimensional complex vector space and* $\mathrm{G} \subset \mathrm{GL}(V)$ *be an irreducible faithful representation of a connected reductive complex Lie* *group* G*. Let X be a uniruled projective manifold endowed with a holomorphic* G*-structure. Then, the holomorphic* G*-structure is flat. Moreover, if* $\mathrm{G} \subsetneq \mathrm{GL}(V)$*, then X is biholomorphically equivalent to an irreducible Hermitian symmetric space of the compact type and of rank* $\geqslant 2$*.*

Theorem 2.2 (Special case of [Mo08b, Main Theorem])**.** *Let S be a Hermitian symmetric space of the compact type. For a reference point $0 \in S$, let $\mathscr{C}_0(S)$ denote the variety of minimal rational tangents on S. Let M be a Fano manifold of Picard number 1 and $\mathcal{K}$ be a minimal rational component on M. Suppose the variety of $\mathcal{K}$-rational tangents at a general point $x \in X$ is isomorphic to $\mathscr{C}_0(S)$ as a projective subvariety. Then, M is biholomorphic to S.*

Definition 2.1 ([MZ19, Definition 1.1])**.** Let X_0 and X be rational homogeneous spaces of Picard number 1, and $i : X_0 \hookrightarrow X$ be a holomorphic embedding equivariant with respect to a homomorphism of complex Lie groups $\Phi : \mathrm{Aut}_0(X_0) \to \mathrm{Aut}_0(X)$. We say that $(X_0, X; i)$ is an admissible pair (of rational homogeneous spaces of Picard number 1) if and only if (a) i induces an isomorphism $i_* : H_2(X_0, \mathbb{Z}) \xrightarrow{\cong} H_2(X, \mathbb{Z})$, and (b) denoting by $\mathcal{O}(1)$ the positive generator of $\mathrm{Pic}(X)$ and by $\rho : X \hookrightarrow \mathbb{P}(\Gamma(X, \mathcal{O}(1))^*) =: \mathbb{P}^N$ the first canonical projective embedding of X, $\rho \circ i : X_0 \hookrightarrow \mathbb{P}^N$ embeds X_0 as a (smooth) linear section of $\rho(X)$.

Definition 2.2 ([MZ19, Definition 1.2])**.** Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1, $W \subset X$ be a connected open subset in the complex topology, and $S \subset W$ be a complex submanifold. Consider the fibered space $\pi : \mathscr{C}(X) \to X$ of varieties of minimal rational tangents on X. For every point $x \in S$ define $\mathscr{C}_x(S) := \mathscr{C}_x(X) \cap \mathbb{P}T_x(S)$ and write $\varpi : \mathscr{C}(S) \to S$ for $\varpi = \pi|_{\mathscr{C}(S)}, \varpi^{-1}(x) := \mathscr{C}_x(S)$ for $x \in S$. We say that $S \subset W$ inherits a sub-VMRT structure modeled on (X_0, X) if and only if for every point $x \in S$ there exists a neighborhood U of x on S and a trivialization of the holomorphic projective bundle $\mathbb{P}T_X|_U$ given by $\Phi : \mathbb{P}T_X|_U \xrightarrow{\cong} U \times \mathbb{P}T_0(X)$ such that (1) $\Phi(\mathscr{C}(X)|_U) = U \times \mathscr{C}_0(X)$ and (2) $\Phi(\mathscr{C}(S)|_U) = U \times \mathscr{C}_0(X_0)$.

A central concept in our study of admissible pairs (X_0, X) of rational homogeneous spaces of Picard number 1 is the notion of rigidity on such pairs, as defined in [MZ18], as follows.

Definition 2.3. An admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1 is said to be *rigid* if and only if for any connected open subset $W \subset X$, any complex submanifold $S \subset W$ inheriting a sub-VMRT structure modeled on (X_0, X) must necessarily be an open subset of $\gamma(X_0) \subset X$ for some $\gamma \in \mathrm{Aut}(X)$. We also say equivalently that sub-VMRT structures modeled on (X_0, X) are rigid.

In [MZ19] the general notion of sub-VMRT structures was introduced for a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component. Denote by $\mathscr{Q}$ the irreducible component of the Chow space of X whose general member is a minimal rational curve on X belonging to $\mathcal{K}$. By the bad locus of $(X, \mathcal{K})$ we mean the smallest subvariety $B \subsetneq X$ such that all elements of $\mathscr{Q}$ not contained in B must necessarily be a minimal rational curve. By the enhanced bad locus $B' \supset B$ we mean the smallest subvariety of X outside which the (pointwise) tangent maps are birational morphisms.

Definition 2.4 ([MZ19, Definition 5.1])**.** Let X be a uniruled projective manifold, and $\mathcal{K}$ be a minimal rational component on X, $B' \subsetneq X$ be the enhanced bad locus of $(X, \mathcal{K}$. Let $\pi : \mathscr{C}(X) \to X$ be the underlying VMRT structure of $(X, \mathcal{K})$. Let $W \subset X - B'$ be a connected open subset in the complex topology, and $S \subset W$ be a complex submanifold. For every point $x \in S$ define $\mathscr{C}_x(S) := \mathscr{C}_x(X) \cap \mathbb{P}T_x(S)$ and write $\varpi : \mathscr{C}(S) \to S$ for $\varpi = \pi|_{\mathscr{C}(S)}, \varpi^{-1}(x) := \mathscr{C}_x(S)$ for $x \in S$. We say that $\varpi := \pi|_{\mathscr{C}(S)} : \mathscr{C}(S) \to S$ is a sub-VMRT structure on $(X, \mathcal{K})$ if and only if (a) the restriction of ϖ to each irreducible component of $\mathscr{C}(S)$ is surjective; (b) at a general point $x \in S$ and for any irreducible component Γ_x of $\mathscr{C}_x(S)$, we have $\Gamma_x \not\subset \mathrm{Sing}(\mathscr{C}_x(X))$; (c) for some positive integer m the fiber $\mathscr{C}_x(S)$ of $\varpi : \mathscr{C}(S) \to S$ has exactly m of irreducible components for every point x on S; and (d) for each irreducible component $\Gamma_{k,x}$ of $\mathscr{C}_x(S)$, $1 \leqslant k \leqslant m$, $\varpi : \mathscr{C}(S) \to S$ is a holomorphic submersion at a general point χ_k of $\Gamma_{k,x}$.

Definition 2.5 ([MZ19, Definition 5.4])**.** Let $\varpi : \mathscr{C}(S) \to S$, $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$, be a sub-VMRT structure on $S \subset W \subset X - B'$. For a point $x \in S$, and $[\alpha] \in \mathrm{Reg}(\mathscr{C}_x(S)) \cap \mathrm{Reg}(\mathscr{C}_x(X))$, we say that $(\mathscr{C}_x(S), [\alpha])$, or equivalently $(\widetilde{\mathscr{C}_x}(S), \alpha)$, satisfies Condition (T) if and only if $T_\alpha(\widetilde{\mathscr{C}_x}(S)) =$

$T_\alpha(\widetilde{\mathscr{C}_x}(X)) \cap T_x(S)$. We say that $\varpi : \mathscr{C}(S) \to S$ satisfies Condition (T) at x if and only if $(\widetilde{\mathscr{C}_x}(S), [\alpha])$ satisfies Condition (T) for a general point $[\alpha]$ of each irreducible component of $\mathrm{Reg}(\mathscr{C}_x(S)) \cap \mathrm{Reg}(\mathscr{C}_x(X))$. We say that $\varpi : \mathscr{C}(S) \to S$ satisfies Condition (T) if and only if it satisfies the condition at a general point $x \in S$.

In Mok-Zhang [MZ19] the notion of nondegeneracy for substructures for sub-VMRT structures was introduced. For our purpose we need the notion only for the special case where the ambient manifold is a rational homogeneous space X, and $\varpi : \mathscr{C}(S) \to S$ is a sub-VMRT structure modeled on an admissible pair of rational homogeneous spaces (X_0, X) of Picard number 1. Write $D \subset T_X$ for the distribution linearly spanned at a general point $x \in X$ by the affinized variety of minimal rational tangents $\widetilde{\mathscr{C}_x}(X) \subset T_x(X) - \{0\}$. We have

Definition 2.6 ([MZ19, Definition 3.2]). Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1. Let $0 \in X_0 \subset X$ be a reference point and $\alpha \in \widetilde{\mathscr{C}_0}(X_0)$ be arbitrary. Write $P_\alpha = T_\alpha(\widetilde{\mathscr{C}_0}(X))$, and denote by $\sigma_\alpha : S^2 P_\alpha \to T_0(X)/P_\alpha$ the second fundamental form of $\widetilde{\mathscr{C}_0}(X) \subset T_0(X) - \{0\}$ at α with respect to the flat connection on $T_0(X)$ as a Euclidean space. Denote by $\nu_\alpha : T_0(X)/P_\alpha \to T_0(X)/(P_\alpha + (D_0 \cap T_0(X_0)))$ the canonical projection. Writing $\tau_\alpha := \nu_\alpha \circ \sigma_\alpha$, so that $\tau_\alpha : S^2 P_\alpha \to T_0(X)/(P_\alpha + (D_0 \cap T_0(X_0)))$, define

$$\mathrm{Ker}\, \tau_\alpha(\,\cdot\,, T_\alpha(\widetilde{\mathscr{C}_0}(X_0))) := \Big\{ \eta \in T_\alpha(\widetilde{\mathscr{C}_0}(X)) : \tau_\alpha(\eta, \xi) = 0 \ \text{for every}\ \xi \in T_\alpha(\widetilde{\mathscr{C}_0}(X_0)) = P_\alpha \cap T_0(X_0). \Big\}.$$

We say that $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ is nondegenerate for substructures if and only if for a general point $\alpha \in \widetilde{\mathscr{C}_0}(X_0)$ we have $\mathrm{Ker}\, \tau_\alpha(\,\cdot\,, T_\alpha(\widetilde{\mathscr{C}_0}(X_0))) = T_\alpha(\widetilde{\mathscr{C}_0}(X_0))$, which is the same as $P_\alpha \cap T_0(X_0)$.

We also say that (X_0, X) is nondegenerate for substructures to mean that $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ is nondegenerate for substructures.

For the study of sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$, we have the following preparatory result relating the pointwise hypothesis concerning $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ and the family of sub-VMRTs $\mathscr{C}_x(S) = \mathscr{C}_x(X))\mathbb{P}T_x(S)$ as x varies over points x on S.

Lemma 2.1 ([MZ19, Lemma 1.1]). *Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1, $W \subset X$ be a connected open subset, and $S \subset W$ be a complex submanifold. Define $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$ and write $\varpi : \mathscr{C}(S) \to S$ for the canonical projection, $\varpi^{-1}(x) =: \mathscr{C}_x(S)$ for any point $x \in S$. Suppose $(\mathscr{C}_x(S) \subset \mathbb{P}T_x(X))$ is projectively equivalent to $(\mathscr{C}_0(X_0) \subset \mathbb{P}T_0(X))$ for any point $x \in S$. Then, $\varpi : \mathscr{C}(S) \to S$ is a holomorphic submersion.*

By the same type of arguments as in Lemma 2.1 we have the following well-known result for holomorphically fibered spaces with irreducible nonsingular compact fibers which we state for easy reference.

Lemma 2.2. *Let $\alpha : \mathscr{Z} \to \Delta$ be a proper holomorphic map over the unit disk Δ such that for any base point $t \in \Delta$ the fiber $Z_t := \alpha^{-1}(t)$ is connected, nonsingular and reduced. Then, $\alpha : \mathscr{Z} \to \Delta$ is a regular family of compact complex manifolds.*

For a uniruled projective manifold $(X, \mathcal{K})$ and a locally closed complex submanifold $S \subset X$ inheriting a sub-VMRT structure, we say that S is $\mathcal{K}$-saturated or just rationally saturated (when the choice of $\mathcal{K}$ is implicitly understood) if and only if for every $\mathcal{K}$-rational curve $\ell \subset X$ passing through some point $x_0 \in S$ and tangent to S at x_0, $(\ell \cap S; x_0)$ must necessarily agree with the germ of curve $(\ell; x_0)$.

Theorem 2.3 ([MZ19, Theorem 1.4]). *Let $(X, \mathcal{K})$ be a uniruled projective manifold X equipped with a minimal rational component $\mathcal{K}$ with associated VMRT structure given by $\pi : \mathscr{C}(X) \to X$. Assume that at a general point $x \in X$, the VMRT $\mathscr{C}_x(X)$ is irreducible. Write $B' \subset X$ for the enhanced bad locus of $(X, \mathcal{K})$. Let $W \subset X - B'$ be a connected open subset, and $S \subset W$ be a complex submanifold such that, writing $\mathscr{C}(S) := \mathscr{C}(X)|_S \cap \mathbb{P}T(S)$ and $\varpi := \pi|_{\mathscr{C}(S)}$, $\varpi : \mathscr{C}(S) \to S$ is a sub-VMRT structure satisfying* Condition (T). *Suppose furthermore that for a general point x on S and for each of the irreducible components $\Gamma_{k,x}$ of $\mathscr{C}_x(S)$, $1 \leqslant k \leqslant m$, the pair $(\Gamma_{k,x}, \mathscr{C}_x(X))$ is nondegenerate for substructures. Then, S is rationally saturated with respect to $(X, \mathcal{K})$.*

By a distribution $\mathscr{D}$ on a complex manifold M we mean a coherent subsheaf of the tangent sheaf $\mathscr{T}(M)$.

$\mathscr{D}$ is said to be bracket generating if and only if, defining inductively $\mathscr{D}_1 = \mathscr{D}$, $\mathscr{D}_{k+1} = \mathscr{D}_k + [\mathscr{D}, \mathscr{D}_k]$, for a general point $x \in M$, we have $\mathscr{D}_m|_U = \mathscr{T}(U)$ on some neighborhood U of x for m sufficiently large. Here and in what follows we say that a Fano manifold X of Picard number 1 is uniruled by (projective) lines if it is equipped with a minimal rational component $\mathcal{K}$ such that the homology class of each member of $\mathcal{K}$ is a generator of the second Betti group $H_x(X;\mathbb{Z}) \cong \mathbb{Z}$.

Theorem 2.4 ([MZ19, Main Theorem 2]). *In Theorem 2.3 suppose furthermore that $(X, \mathcal{K})$ is a projective manifold of Picard number 1 uniruled by lines and that the distribution $\mathscr{D}$ on S defined by $\mathscr{D}_x := \mathrm{Span}(\widetilde{\mathscr{C}_x}(S))$ is bracket generating. Then, there exists an irreducible subvariety $Z \subset X$ such that $S \subset Z$ and such that $\dim(Z) = \dim(S)$.*

The hypothesis that $\mathscr{D} := \mathrm{Span}(\widetilde{\mathscr{C}_x}(S))$ is bracket generating is trivially satisfied when $\mathscr{D}_x \subset T_0(S)$ is linearly nondegenerate for a general point $x \in S$. This is in particular the case when we deal with sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$ modeled on an admissible pair (X_0, X) of nonlinear Hermitian symmetric spaces of the compact type, which are our main objects of study in the current article. Of crucial importance for our argumentation is the following proposition also called the *Thickening Lemma* which allows us to construct *collars* around certain minimal rational curves which are compactifications of holomorphic curves lying on the support S of a sub-VMRT structure.

Proposition 2.1 ([MZ19, Proposition 6.1]). *Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component, $\dim(X) =: n$, and $\varpi : \mathscr{C}(S) \to S$ be a sub-VMRT structure as in Theorem 1.1, $\dim(S) =: s$. Let $[\alpha] \in \mathscr{C}(S)$ be a smooth point of both $\mathscr{C}(S)$ and $\mathscr{C}(X)$ such that $\varpi : \mathscr{C}(S) \to S$ is a submersion at $[\alpha]$, $\varpi([\alpha]) =: x$, and $[\ell] \in \mathcal{K}$ be the minimal rational curve assumed smooth at x such that $T_x(\ell) = \mathbb{C}\alpha$, and $f : \mathbf{P}_\ell \to \ell$ be the normalization of ℓ, $\mathbf{P}_\ell \cong \mathbb{P}^{\mathbf{1}}$. Suppose $(\mathscr{C}_x(S), [\alpha])$ satisfies* Condition (T)*. Then, there exists an s-dimensional complex manifold $\mathbf{E}_\ell$, $\mathbf{P}_\ell \subset \mathbf{E}_\ell$, and a holomorphic immersion $F : \mathbf{E}_\ell \to X$ such that $F|_{\mathbf{P}_\ell} \equiv f$ and $F(\mathbf{E}_\ell)$ contains a neighborhood of x on S.*

Let now X be an irreducible Hermitian symmetric space of the compact type and of rank $\geqslant 2$. Write $X = G/P$, where $G = \mathrm{Aut}_0(X)$, $P \subset G$ denotes some maximal parabolic subgroup defining X, i.e., $P \subset G$ is the isotropy subgroup at some point $0 \in X$, $P = K^{\mathbb{C}} \cdot M^-$ for a Levi decomposition of P in the standard notation of the Harish-Chandra decomposition, where $K^{\mathbb{C}} \subset P$ is a Levi factor, and $M^- \subset P$ is the unipotent radical (which is a normal subgroup). Then, defining $\Phi : P \to \mathrm{GL}(T_0(X))$ by $\Phi(\gamma) = d\gamma(0)$ we have $M^- = \mathrm{Ker}(\Phi)$, and Φ induces a group isomorphism of $K^{\mathbb{C}} \equiv P/M^-$ onto a reductive linear subgroup $L \subset \mathrm{GL}(T_0(X))$. Thus X admits a holomorphic G-structure with a reduction of the holomorphic frame bundle from $\mathrm{GL}(T_0(X))$ to the reductive linear group $\mathrm{G} = L \subsetneq \mathrm{GL}(T_0(X))$. L is precisely the identity component of the stablilizer subgroup of the VMRT $\mathscr{C}_0(X) \subsetneq \mathbb{P}T_0(X)$. (If we replace G by $\mathrm{Aut}(X)$, which has possibly more than one but at most finitely many connected components, define Φ on $\mathrm{Aut}(X;0)$ analogously and denote by $\widetilde{L} \subsetneq \mathrm{GL}(T_0(X))$ the image of $\mathrm{Aut}(X;0)$ under Φ, then $\widetilde{L} \subset \mathrm{GL}(T_0(X))$ is precisely the stabilizer subgroup of $\mathscr{C}_0(X)$). We will be considering admissible pairs $(X_0;X)$ of rational homogeneous manifolds of Picard number 1 and of the Hermitian symmetric type. When a locally closed complex submanifold $S \subset X$ inherits a sub-VMRT structure modeled on the pair (X_0, X), then S inherits a G-structure where $\mathrm{G} = L_0 \subset \mathrm{GL}(T_0(X_0))$, where the definition $L_0 \subset \mathrm{GL}(T_0(X_0))$ is analogous to that of $L \subset \mathrm{GL}(T_0(X))$, replacing $X = G/P$ by $X_0 = G_0/P_0$ in an obvious way, etc.

3 Statement of Results

For the formulation of the results we will introduce a property, called Condition (F) for the pair $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ of VMRTs associated to an admissible pair (X_0, X) of rational homogeneous spaces of Picard number 1, as follows. The name Condition (F) refers to the way that the pair $(\mathscr{C}_y(S), \mathscr{C}_y(X))$ is *fitted* to the model pair $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$, after $\mathscr{C}_x(S)$ is propagated by parallel transport from $x \in S$ to a point $y \notin S$ lying on a minimal rational curve ℓ emanating from x whose germ $(\ell;x)$ at x lies on $(S;x)$. It is thus a condition on the fitting of sub-VMRTs to the model.

Definition 3.1. Let (X_0, X) be an admissible pair of rational homogeneous spaces of Picard number 1, and denote by $\mathscr{C}(X_0) \subset \mathbb{P}T_{X_0}$ resp. $\mathscr{C}(X) \subset \mathbb{P}T_X$ the VMRT structure on X_0 resp. X defined by the minimal rational component of projective lines with respect to the minimal canonical projective embedding $i_0 : X_0 \hookrightarrow \mathbb{P}\Gamma(X_0, \mathcal{O}(1))^*$ resp. $i : X \hookrightarrow \mathbb{P}\Gamma(X, \mathcal{O}(1))^*)$. Let $0 \in X_0 \subset X$ be a reference point. We say that $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ satisfies Condition (F) if and only if the following holds. Let $\lambda : \mathscr{C}_0(X_0) \hookrightarrow \mathscr{C}(X)$ be an arbitrary holomorphic embedding such that, denoting by $\mathbb{P}V_\lambda \subset \mathbb{P}T_0(X)$ the projective linear span of $\lambda(\mathscr{C}_0(X_0))$, we have $\lambda(\mathscr{C}_0(X_0)) = \mathbb{P}V_\lambda \cap \mathscr{C}_0(X)$ and the inclusion $(\lambda(\mathscr{C}_0(X_0)) \subset \mathbb{P}V_\lambda)$ is projectively equivalent to the inclusion $(\mathscr{C}_0(X_0) \subset \mathbb{P}T_0(X_0))$. Then, there exists a projective linear automorphism $\Lambda : \mathbb{P}T_0(X) \xrightarrow{\cong} \mathbb{P}T_0(X)$ such that $\Lambda|_{\mathscr{C}_0(X)} : \mathscr{C}_0(X) \xrightarrow{\cong} \mathscr{C}_0(X)$ and such that $\Lambda|_{\mathscr{C}_0(X_0)} \equiv \lambda$.

The following intermediate result is a key step towards proving the main result of this paper concerning rigidity of certain admissible pairs (X_0, X) of rational homogeneous manifolds of the Hermitian symmetric type. Here and in what follows on a Fano manifold Y of Picard number 1 we will denote by $\mathcal{O}(1)$ the positive generator of $\mathrm{Pic}(Y) \cong \mathbb{Z}$.

Proposition 3.2. *Let (X_0, X) be an admissible pair of irreducible Hermitian symmetric spaces of the compact type of rank $\geqslant 2$, $0 \in X_0 \subset X$. Suppose the pair of VMRTs $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ is nondegenerate for substructures and it satisfies* Condition (F). *Let $W \subset X$ be an open subset in the complex topology, and $S \subset W$ be a complex submanifold such that, defining $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$ by tangential intersection, the canonical projection $\varpi : \mathscr{C}(S) \to S$ defines a sub-VMRT structure modeled on (X_0, X). Then, there exists a projective subvariety $Z \subset X$ such that $S \subset Z$ and $\dim(Z) = \dim(S)$. Moreover, writing $\nu : \widetilde{Z} \to Z$ for the normalization, there is a biholomorphism $\Phi : X_0 \xrightarrow{\cong} \widetilde{Z}$, and for any choice of such a biholomorphism Φ the holomorphic map $h := \nu \circ \Phi : X_0 \to X$ is of degree 1, i.e., $h^*(\mathcal{O}(1)) \cong \mathcal{O}(1)$.*

From Proposition 3.1 we will deduce the following main result on the rigidity of certain admissible pairs of rational homogeneous manifolds.

Main Theorem *Let (X_0, X) be an admissible pair of irreducible Hermitian symmetric spaces of the compact type of rank $\geqslant 2$, $0 \in X_0 \subset X$. Suppose the pair of VMRTs $(\mathscr{C}_0(X_0), \mathscr{C}_0(X))$ is nondegenerate for substructures and it satisfies* Condition (F). *Assume furthermore that the restriction map $r : \Gamma(X, T_X) \to \Gamma(X_0, T_X|_{X_0})$ defined by $r(\eta) = \eta|_{X_0}$ is surjective. Then, (X_0, X) is rigid.*

The same conditions as in Main Theorem apply in the case where (X_0, X) is degenerate for substructures to show that a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ modeled on (X_0, X) is a translate γX_0 of X_0 by some $\gamma \in \mathrm{Aut}(X)$ provided that S is *by assumption* saturated by open subsets of minimal rational curves of X. In this direction a first evidence of such a result was obtained by Zhang [Zh14, Theorem 5.1] in which it was shown that this holds for the pair (Q^m, Q^n), $3 \leqslant m < n$, by means of explicit computation. From the proof of Main Theorem we have immediately the following generalization characterizing $X_0 \subset X$ as the unique linearly saturated sub-VMRT structure modeled on (X_0, X) modulo translation by global automorphisms of X, under certain assumptions on the admissible pair of rational homogeneous spaces (X_0, X) of the Hermitian symmetric type. As we will check, the conditions are satisfied in the case of the pair (Q^m, Q^n), $3 \leqslant m < n$, by which we obtain an alternative and more conceptual proof of the aforementioned rigidity result of Zhang [Zh14].

Theorem 3.1. *Let (X_0, X) be an admissible pair of irreducible Hermitian symmetric spaces of the compact type of rank $\geqslant 2$, $0 \in X_0 \subset X$. Suppose the pair of VMRTs $(\mathscr{C}_0(X_0), \mathscr{C}(X_0))$ satisfies* Condition (F), *and assume furthermore that the restriction map $r : \Gamma(X, T_X) \to \Gamma(X_0, T_X|_{X_0})$ defined by $r(\eta) = \eta|_{X_0}$ is surjective. Let now $W \subset X$ be an open subset in the complex topology and assume that $S \subset W$ inherits by tangential intersection a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ modeled on (X_0, X), where $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$. Assume that $S \subset W$ is saturated with respect to open subsets of minimal rational curves tangent to S. Then, there exists $\gamma \in \mathrm{Aut}(X)$ such that S is an open subset of γX_0.*

4 Proof of Proposition 3.1

Proof of Proposition 3.1. Recall that in the statement of Proposition 3.1 (X_0, X) is an admissible pair of rational homogeneous spaces of the Hermitian symmetric type. Let $W \subset X$ be a connected open subset in the complex topology, and $S \subset W$ be a complex submanifold such that, defining $\mathscr{C}_x(S) := \mathscr{C}_x(X) \cap \mathbb{P}T_x(S)$ for $x \in S$, the inclusion $(\mathscr{C}_x(S) \subset \mathscr{C}_x(X))$ is transformed to the inclusion $(\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X))$ by $[\Lambda_x]$ for some linear isomorphism $\Lambda_x : T_x(X) \xrightarrow{\cong} T_0(X)$. For later use note that since $\mathrm{Aut}_0(X)$ acts transitively on X and since $\widetilde{L} \subset \mathrm{GL}(T_0(X))$ is precisely the stabilizer subgroup of $\mathscr{C}_0(X)$, we may take $\Lambda_x = d\varphi_x$ for some $\varphi_x \in \mathrm{Aut}_0(X)$ that $\varphi_x(x) = 0$.

Let $x \in S$. Let $H_x \subset \mathbb{P}(\mathrm{GL}(T_x(X))$ be the identity component of the stabilizer subgroup of $\mathscr{C}_x(X) \cong \mathscr{C}_0(X)$. Since X is of Hermitian symmetric type, hence of long-root type, H_x acts transitively on $\mathscr{C}_x(X)$. Let $J_x \subset \mathbb{P}(\mathrm{GL}(T_x(S))$ be the stabilizer subgroup of $\mathscr{C}_x(S) \cong \mathscr{C}_0(X_0)$. Since again X_0 is of Hermitian symmetric type, J_x acts transitively on $\mathscr{C}_x(S)$. In particular, $\mathscr{C}_x(X)$ is nonsingular, and $\mathscr{C}_x(S) \subset \mathscr{C}_x(X)$ is a projective submanifold. From the transitivity of J_x on $\mathscr{C}_x(S)$ it follows that the Thickening Lemma (Proposition 2.1) applies to any $[\alpha] \in \mathscr{C}_x(S)$ and hence to any minimal rational curve $\ell \subset X$ emanating from x such that $[T_x(\ell)] := [\alpha] \in \mathscr{C}_x(S)$. Note that the germ $(\ell; x)$ of ℓ at $x \in S$ lies on the germ $(S; x)$ of S at x, i.e., the connected component of $\ell \cap W$ containing x must lie on S. Since the minimal embedding $i : X \hookrightarrow \mathbb{P}\Gamma(X, \mathcal{O}(1))^*$ embeds and hence identifies X as a projective submanifold uniruled by projective lines, $\ell \subset X$ is nonsingular, and the normalization $f : \mathbf{P}_\ell \to \ell$ is an isomorphism. By Proposition 2.1, writing $s := \dim(S)$, there exists an s-dimensional complex manifold $\mathbf{E}_\ell$, $\mathbf{P}_\ell \subset \mathbf{E}_\ell$, and a holomorphic immersion $F : \mathbf{E}_\ell \to X$ such that $F|_{\mathbf{P}_\ell} \equiv f$ and $F(\mathbf{E}_\ell)$ contains a neighborhood of x on S. From the facts that $F : \mathbf{E}_\ell \to X$ is an immersion and that $f : \mathbf{P}_\ell \cong \ell$, by shrinking $\mathbf{E}_\ell$ we may actually take $F : \mathbf{E}_\ell \to X$ to be an embedding and hence identify $\mathbf{E}_\ell$ with $F(\mathbf{E}_\ell) =: \mathbf{N}_\ell$.

By the process of adjunction of minimal rational curves and the fact that $\mathscr{C}_x(S) \subset \mathbb{P}T_x(S)$ is linearly nondegenerate it follows by Theorem 2.4 that there exists an irreducible projective subvariety $Z \subset X$, $\dim(Z) = \dim(S) = s$ such that $S \subset Z$. We claim that $Z \subset X$ is an immersed submanifold, i.e., denoting by $\nu : \widetilde{Z} \to Z$ the normalization of Z, $\widetilde{Z}$ is a projective manifold and $\nu : \widetilde{Z} \to Z \subset X$ is a generically injective holomorphic immersion. The key idea is the propagation of VMRT structures along chains of rational curves, as already implicit in the solution of the Recognition Problem for irreducible Hermitian symmetric spaces of rank $\geqslant 2$ given in Theorem 2.2. There we have an ambient uniruled projective manifold $(Z, \mathcal{K})$ equipped with a minimal rational component. In contrast, here we start with a germ of complex submanifold $(S; x_0) \subset (X; x_0)$ and we need to construct the immersed complex submanifold $Z \subset X$. For this purpose we will retrace the construction of Z in our special situation. For the proof of Proposition 3.1 we will only make use of $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$, which equips S with a holomorphic L_0-structure modeled on the irreducible Hermitian symmetric space X_0 of rank $\geqslant 2$ for a reductive linear subgroup $L_0 \subsetneq \mathrm{GL}(T_0(X_0))$ as defined in §2.

Here the propagation of the VMRT structure along a minimal rational curve has to be coupled with the proof of the Thickening Lemma. Starting with any point $x \in S$, take any minimal rational curve ℓ on X emanating from x such that $[T_x(\ell)] = [\alpha] \in \mathscr{C}_x(S) = \mathscr{C}_x(X) \cap \mathbb{P}T_x(S)$, so that $(\ell; x) \subset (S, x)$, and write $\mathbf{N}_\ell \supset \ell$ for a collar around ℓ as defined in the last paragraph. In what follows we will describe the construction of $\mathbf{N}_\ell$ as given in the proof of the Thickening Lemma. The construction will allow us to propagate the L_0-structure on S to $\mathbf{N}_\ell$.

Let $T(\mathbf{N}_\ell|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^a \oplus \mathcal{O}^b$ be the Grothendieck decomposition of the tangent bundle $T(\mathbf{N}_\ell)$ over ℓ. Then, $\dim(\mathscr{C}_x(S)) = a$. The positive part $P_\ell := \mathcal{O}(2) \oplus \mathcal{O}(1)^a \subset T(\mathbf{N}_\ell)|_\ell$ is well defined independent of the choice of the Grothendieck summands. Shrinking S if necessary let $\mathscr{X} \subset S$ be a b-dimensional complex submanifold, $\mathscr{X} \cong \Delta^b$, passing through x such that $T_x(\mathscr{X})$ is transverse to $P_{\ell,x}$. Parametrize $\mathscr{X}$ by holomorphic coordinates $t = (t_1, \cdots, t_b) \in \Delta^b$, and write $x(t) \in \mathscr{X}$ for the point corresponding to $t \in \Delta^b$, $x(0) = x$. Without loss of generality we may assume that there exists a holomorphic family of minimal rational curves $\{\ell(t) : t \in \Delta^b\}$ on X such that (a) $\ell(0) = \ell$ and such that for every $t \in \Delta^b$ we have (b) $x(t) \in \ell(t)$, (c) $[T_{x(t)}(\ell(t))] \in \mathscr{C}_{x(t)}(S)$ and (d) $\mathscr{X}$ is transverse to $P_{\ell(t),x(t)}$ (where the latter is defined in analogy to $P_{\ell,x} = P_{\ell(0),x(0)}$). From (c) and Theorem 2.3 we have for all $t \in \Delta^b$ the inclusion

$(\ell(t); x(t)) \subset (S; x(t))$ between germs of subvarieties. We may also assume that $\ell(t) \cap \ell(t') = \emptyset$ whenever $t \neq t'$, so that $\mathscr{L} := \bigcup \{\ell(t) : t \in \Delta^b\} \subset X$ is an embedded $(b+1)$-dimensional locally closed complex submanifold. There is a canonical projection $\lambda : \mathscr{L} \to \Delta^b$ defined by the property $z \in \ell(\lambda(z))$. $\mathscr{Z} \subset \mathscr{L}$ is the image of a holomorphic section of $\lambda : \mathscr{L} \to \Delta^b$. Without loss of generality we may assume that there exists a holomorphic section of λ with image $\mathscr{Z}' = \{x'(t) : t \in \Delta^b\}$ such that $\mathscr{Z} \cap \mathscr{Z}' = \emptyset$. For $0 < \epsilon < 1$ we write $\mathscr{L}(\epsilon) := \lambda^{-1}(\Delta^b(\epsilon))$. For $t \in \Delta^b$ define now $\mathcal{V}_t := \bigcup \{\ell_{[\alpha_t]} : [\alpha_t] \in \mathscr{C}_{x(t)}(S)\}$ to be the cone of minimal rational curves on X passing through $x(t)$ whose germs at $x(t)$ lie on $(S; x(t))$, and $\mathcal{V}'_t := \bigcup \{\ell_{[\alpha'_t]} : [\alpha'_t] \in \mathscr{C}_{x'(t)}\}$ to be the cone of minimal rational curves on X passing through $x'(t)$ whose germs at $x'(t)$ lie on $(S; x'(t))$.

For each $x(t) \in \mathscr{Z}$ we have $[\alpha(t)] := [T_{x(t)}(\ell(t))] \in \mathscr{C}_{(x(t)}(S)$. Write $\mathcal{A} := \{[\alpha(t)] : t \in \Delta^b\}$. Similarly for each $x'(t) \in \mathscr{Z}'$ we have $[\alpha'(t)] := [T_{x'(t)}(\ell(t))] \in \mathscr{C}_{x'(t)}(S)$, and we write $\mathcal{A}' := \{[\alpha'(t)] : t \in \Delta^b\}$. Let $\{\mathscr{O}_m : m \in \mathbb{N}\}$ be a decreasing sequence of neighborhoods of the complex submanifold $\mathcal{A} \subset \mathscr{C}(S)|_{\mathscr{Z}}$ such that $\bigcap \{\mathscr{O}_m : m \in \mathbb{N}\} = \mathcal{A}$. Similarly let $\{\mathscr{O}'_m : m \in \mathbb{N}\}$ be a decreasing sequence of neighborhoods of the complex submanifold $\mathcal{A}' \subset \mathscr{C}(S)|_{\mathscr{Z}'}$ such that $\bigcap \{\mathscr{O}'_m : m \in \mathbb{N}\} = \mathcal{A}'$. For $0 < \epsilon < 1$ and $m \in \mathbb{N}$, write $\mathcal{V}(\epsilon) := \bigcup \{\mathcal{V}_t : t \in \Delta^b(\epsilon)\}$ and $\mathcal{V}(\epsilon, m) := \bigcup \{\ell_{[\beta_t]} : t \in \Delta^b(\epsilon), [\beta_t] \in \mathscr{O}_m\}$. We have $\mathcal{V}(\epsilon) \supset \mathcal{V}(\epsilon, m)$ and $\bigcap \{\mathcal{V}(\epsilon, m) : m \in \mathbb{N}\} = \mathscr{L}(\epsilon)$. Similarly, $\mathcal{V}'(\epsilon)$ and $\mathcal{V}'(\epsilon, m)$ are analogously defined with $\mathcal{V}_t$ being replaced by $\mathcal{V}'_t$ and $\mathscr{O}_m$ being replaced by $\mathscr{O}'_m$. We have $\mathcal{V}'(\epsilon) \subset \mathcal{V}'(\epsilon, m)$ and $\bigcap \{\mathcal{V}'(\epsilon, m) : m \in \mathbb{N}\} = \mathscr{L}(\epsilon)$.

For $0 < \epsilon < 1$ and $m \in \mathbb{N}$ write $\mathbf{N}_\ell(\epsilon, m) := \mathcal{V}(\epsilon, m) \cup \mathcal{V}'(\epsilon, m)$. Note that $\mathbf{N}_\ell(\epsilon, m)$ is connected since both $\mathcal{V}(\epsilon, m)$ and $\mathcal{V}'(\epsilon, m)$ contain the rational curve ℓ. From the proof of the Thickening Lemma (Proposition 2.1 here), for $\epsilon_0 > 0$ sufficiently small and $m_0 \in \mathbb{N}$ sufficiently large, $\mathbf{N}_\ell(\epsilon_0, m_0)$ is a collar around ℓ extending S. Since $\ell \subset X$ is embedded, for simplicity and without loss of generality we assume that for such pairs (ϵ_0, m_0) the collar $N(\epsilon_0, m_0)$ is embedded in X. (Note that irrespective of how two (embedded) collars $\mathbf{N}^1_\ell$ and $\mathbf{N}^2_\ell$ of ℓ are constructed, their germs along ℓ are the same in view of the Identity Theorem for holomorphic functions.) In other words $\dim \mathbf{N}_\ell(\epsilon_0, m_0) = \dim(S) = s$ and $\mathbf{N}_\ell(\epsilon_0, m_0)$ contains a neighborhood of x on S. Fix such a positive real number ϵ_0 and such a natural number m_0, and write $\mathbf{N}_\ell$ for $\mathbf{N}_\ell(\epsilon_0, m_0)$. Since $\ell \subset \mathbf{N}_\ell$ is a standard minimal rational curve, and the latter property is an open condition in the complex topology, without loss of generality we will further assume that all minimal rational curves (projective lines) of X lying on $\mathbf{N}_\ell$ are standard.

We have $\mathbf{N}_\ell = \mathcal{V}(\epsilon_0, m_0) \cup \mathcal{V}'(\epsilon_0, m_0)$. Removing $\mathscr{Z}(\epsilon_0) := \{x_t : t \in \Delta^b(\epsilon_0)\}$ from $\mathcal{V}(\epsilon_0, m_0)$ and $\mathscr{Z}'(\epsilon_0) := \{x'_t : t \in \Delta^b(\epsilon_0)\}$ from $\mathcal{V}'(\epsilon_0, m_0)$, and defining $\mathcal{W} := \mathcal{V}(\epsilon_0, m_0) - \mathscr{Z}(\epsilon_0)$, $\mathcal{W}' := \mathcal{V}'(\epsilon_0, m_0) - \mathscr{Z}'(\epsilon_0)$, we may write $\mathbf{N}_\ell = \mathcal{W} \cup \mathcal{W}'$ and regard $\mathbf{N}_\ell$ as being covered by two coordinate charts $\mathcal{W}$ and $\mathcal{W}'$. More precisely, noting that $\mathcal{V}$ (resp. $\mathcal{V}'$) is the image of a holomorphic $\mathbb{P}^1$-bundle under a tautological map which collapses a distinguished section into $\mathscr{Z}$ (resp. $\mathscr{Z}'$), $\mathcal{W}$ (resp. $\mathcal{W}'$) is the total space of a holomorphic $\mathbb{C}$-bundle, and, shrinking $\mathscr{Z}$, $\mathscr{Z}'$ and $\epsilon_0 > 0$ if necessary we may assume $\mathcal{W} \cong \Omega \times \mathbb{C}$, $\mathcal{W}' \cong \Omega' \times \mathbb{C}$, where Ω and Ω' are Euclidean domains and where the fibers $\cong \mathbb{C}$ are mapped under the implicit isomorphisms onto affine lines.

Define now $\mathscr{C}^0(\mathbf{N}_\ell) = \mathscr{C}(X) \cap \mathbb{P}T(\mathbf{N}_\ell)$. Thus, for $x \in S$ we have $\mathscr{C}^0_x(\mathbf{N}_\ell) = \mathscr{C}_x(S) \cap \mathbb{P}T(\mathbf{N}_\ell)$. Let now $\mathscr{C}(\mathbf{N}_\ell) \subset \mathscr{C}^0(\mathbf{N}_\ell)$ be the unique irreducible component containing $\mathscr{C}(S)$. To prove Proposition 3.1 we will show first of all that the canonical projection $\varpi : \mathscr{C}(\mathbf{N}_\ell) \to \mathbf{N}_\ell$ defines on $\mathbf{N}_\ell$ an L_0-structure which is moreover flat. (Shrinking $\mathbf{N}_\ell$ if necessary we may assume the latter to be simply connected, and the flatness of the L_0-structure $\varpi : \mathscr{C}(\mathbf{N}_\ell) \to \mathbf{N}_\ell$ implies that $\mathbf{N}_\ell$ can be identified with an open subset of the model Hermitian symmetric space X_0 of the compact type and of rank $\geqslant 2$.)

From now on Λ will denote a minimal rational curve of X such that either $\Lambda = \ell_{[\beta_t]}$ for some $[\beta_t] \in \mathscr{O}_{m_0}$ lying over $t \in \Delta^b(\epsilon_0)$, or $\Lambda = \ell_{[\beta'_t]}$ for some $[\beta'_t] \in \mathscr{O}'_{m_0}$ lying over $t \in \Delta^b(\epsilon_0)$, and denote the set of all such minimal rational curves by $\mathbf{A}(\epsilon_0, m_0)$. Thus, by assumption $\Lambda \subset \mathbf{N}_\ell$ (when $\Lambda \in \mathbf{A}(\epsilon_0, m_0)$). For convenience we may assume that S is a complex submanifold on an open subset $W \subset X$ where W is a complex ball in a Harish-Chandra coordinate chart. By the convexity of the complex ball any nonempty intersection of a minimal rational curve on X with W must be connected. Since the subvariety $S \subset W$ contains a nonempty open subset of the connected set $\Lambda \cap W$, we have $\Lambda \cap S \subset \Lambda \cap W$ is a subvariety of the same dimension, hence $\Lambda \cap S = \Lambda \cap W$ is connected. Consider the restriction $\mathscr{C}(X)|_\Lambda$. There is a tautological lifting $\widehat{\Lambda}$ of Λ to $\mathscr{C}(X)|_\Lambda$ where $z \in \Lambda$ is lifted to $[T_z(\Lambda)] \in \mathscr{C}_z(X)$ and moreover any point

$x \in \Lambda \cap S$ is lifted to $[T_x(\Lambda)] \in \mathscr{C}_x(X) \cap \mathbb{P}T_x(S) = \mathscr{C}_x(S)$.

We are going to argue by the method of parallel transport along minimal rational curves on $\mathbf{N}_\ell$ that $\varpi : \mathscr{C}(\mathbf{N}_\ell) \to \mathbf{N}_\ell$ defines an L_0-structure. From the hypothesis defining sub-VMRT structures modeled on (X_0, X) it follows readily that for a *general* point $z \in \mathbf{N}_\ell$, $(\mathscr{C}_z(\mathbf{N}_\ell) \subset \mathbb{P}T_z(\mathbf{N}_\ell))$ is projectively equivalent to $(\mathscr{C}_x(\mathbf{N}_\ell) \subset \mathbb{P}T_x(\mathbf{N}_\ell))$ for any point $x \in S$. For $\Lambda \in \mathbf{A}(\epsilon_0, m_0)$ we will consider parallel transport from $\mathscr{C}_x(\mathbf{N}_\ell)$ for $x \in \Lambda \cap S$ to an arbitrary point $z \in \Lambda$, and see that the key argument for parallel transport in the proof of Theorem 2.2 still applies to show that the same remains true for an *arbitrary* point $z \in \Lambda$. Recall that $\mathbf{N}_\ell = \mathcal{W} \cup \mathcal{W}'$. The parallel transport argument will allow us to show that $\mathscr{C}(\mathbf{N}_\ell)|_{\mathcal{W}}$ and $\mathscr{C}(\mathbf{N}_\ell)|_{\mathcal{W}'}$ define holomorphic L_0-structures, which implies that $\varpi : \mathscr{C}(\mathbf{N}_\ell) \to \mathbf{N}_\ell$ underlies a holomorphic L_0-structure. (The two L_0-structures on the overlap $\mathcal{W} \cap \mathcal{W}'$ agree by definition.)

The argument of parallel transport involves VMRTs, while $\mathscr{C}(\mathbf{N}_\ell)$ is defined by tangential intersection (and throwing away irreducible components which do not cover $\mathbf{N}_\ell$). Thus for $z \in \mathbf{N}_\ell$, $\mathscr{C}_z(\mathbf{N}_\ell)$ may be interpreted as potential VMRTs of a geometric substructure, but we have to show that they are actually VMRTs of an immersed complex submanifold uniruled by lines. For $\Lambda \in \mathbf{A}(\epsilon_0, m_0)$ and $\Lambda' \in \mathbf{A}'(\epsilon_0, m_0)$ to relate $\mathscr{C}_z(\mathbf{N}_\ell)$ with minimal rational curves we consider $\Lambda \subset \mathbf{N}_\ell$ as minimal rational curves inside the noncompact "uniruled" complex manifold $\mathbf{N}_\ell$. Although $\mathbf{N}_\ell$ is noncompact, we can still study the infinitesimal deformation of the minimal rational ℓ (of X) inside $\mathbf{N}_\ell$. $\ell \subset \mathbf{N}_\ell$ is a free rational curve. The deformation of ℓ inside $\mathbf{N}_\ell$ is unobstructed, and it defines a germ of complex manifold $\mathcal{K}_{\mathbf{N}_\ell}$ at $[\ell]$ whose points correspond to deformations of $[\ell]$, and a germ of universal family $\rho : \mathcal{U}_{\mathbf{N}_\ell} \to \mathcal{K}_{\mathbf{N}_\ell}$ which is a germ at $[\ell]$ of a holomorphic $\mathbb{P}^1$-bundle over $\mathcal{K}_{\mathbf{N}_\ell}$. Shrinking $\mathcal{K}_{\mathbf{N}_\ell}$ if necessary we may assume without loss of generality that all points of $\mathcal{K}_{\mathbf{N}_\ell}$ represent standard rational curves ℓ' on $\mathbf{N}_\ell$, i.e., free rational curves such that $T_{\mathbf{N}_\ell}|_{\ell'} \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^a \oplus \mathcal{O}^b$. We may represent the germ $\mathcal{K}_{\mathbf{N}_\ell}$ at $[\ell]$ by an actual complex manifold bearing the same name so that $\mathbf{A}(\epsilon_0, m_0) \subset \mathcal{K}_{\mathbf{N}_\ell}$ (but $K_{\mathbf{N}_\ell}$ also contains small deformations of each $\Lambda \in \mathbf{A}(\epsilon_0, m_0)$ fixing an arbitrary given point $z \in \Lambda$). Denote by $\mu : \mathcal{U}_{\mathbf{N}_\ell} \to X$ the evaluation, which is a holomorphic submersion since $\mathcal{K}_{\mathbf{N}_\ell}$ consists of free rational curves (cf. Hwang-Mok [HM98, Proposition 4]). The tangent map $\tau : \mathcal{U}_{\mathbf{N}_\ell} \to \mathbb{P}T_{\mathbf{N}_\ell}$ is well-defined on $\mathcal{U}_{\mathbf{N}_\ell}$. Since minimal rational curves on $\mathbf{N}_\ell$ are standard rational curves, the tangent map τ is a holomorphic immersion (cf. Mok [08a, Lemma 2]). Since these standard rational curves are furthermore projective lines on X, the tangent map τ is a holomorphic embedding (cf. Mok [Mo08a, Lemma 3]), hence its image in $\mathbb{P}T_{\mathbf{N}_\ell}$ defines a germ of complex submanifold $\mathscr{C}^\flat(\mathbf{N}_\ell)$ along the lifting $\widehat{\ell}$ of ℓ to $\mathbb{P}T_{\mathbf{N}_\ell}$.

Since $\tau : \mathcal{U}_{\mathbf{N}_\ell} \to \mathbb{P}T_{\mathbf{N}_\ell}$ is a holomorphic embedding, by abuse of notation we will also denote by $\mu : \mathscr{C}^\flat(\mathbf{N}_\ell) \to \mathbf{N}_\ell$ the canonical projection, which is a holomorphic submersion. Note that $\mathscr{C}^\flat(\mathbf{N}_\ell) \subset \mathbb{P}T_{\mathbf{N}_\ell}$ is only defined as a germ of complex submanifold along $\widehat{\ell}$, but we may represent the germ $\mathscr{C}^\flat(\mathbf{N}_\ell)$ along $\widehat{\ell}$ by an actual locally closed complex submanifold bearing the same name which contains the tautological lifting $\widehat{\Lambda}$ for any $\Lambda \in \mathbf{A}(\epsilon_0, m_0)$. Restricting to such minimal rational curves Λ on $\mathbf{N}_\ell$, we have $\mu|_\Lambda : \mathscr{C}^\flat(\mathbf{N}_\ell)|_\Lambda \to \Lambda$. Denote by $\sigma := \sigma|_{\mathscr{C}^\flat(\mathbf{N}_\ell)|\mathbb{P}T_{\mathbf{N}_\ell}}$ the fiberwise projective second fundamental form of $\mathscr{C}^\flat(\mathbf{N}_\ell) \subset \mathbb{P}T_{\mathbf{N}_\ell}$ with respect to the canonical projection $\pi|_\Lambda : \mathbb{P}T_{\mathbf{N}_\ell} \to \Lambda$. Thus, for $z \in \Lambda$ and $[\gamma] \in \mathscr{C}_z(\mathbf{N}_\ell)$, $\sigma([\gamma])$ is the projective second fundamental at $[\gamma]$ of $\mathscr{C}_z(\mathbf{N}_\ell) \subset \mathbb{P}T_z(\mathbf{N}_\ell)$, and it takes value in the normal space $N_{[\gamma]} := T_{\mathbb{P}T_z(\mathbf{N}_\ell),[\gamma]}/T_{\mathscr{C}_z(\mathbf{N}_\ell),[\gamma]}$. Consider now $[\gamma] = [T_z(\Lambda)]$. Varying z over Λ the normal spaces $N_{[T_z(\Lambda)]}$ put together constitute the relative normal bundle $N_{\widehat{\Lambda}}$ of $\mathscr{C}^\flat(\mathbf{N}_\ell) \subset \mathbb{P}T(\mathbf{N}_\ell)$ along $\widehat{\Lambda}$ with respect to $\pi|_\Lambda : \mathbb{P}T(\mathbf{N}_\ell)|_\Lambda \to \Lambda$, and we have now the relative second fundamental form $\sigma_{\widehat{\Lambda}} : S^2T_\pi|_{\widehat{\Lambda}} \to N_{\widehat{\Lambda}}$.

By the same proof of Theorem 2.2 (from Mok [Mo08b]), $\mathrm{Hom}(S^2T_\pi|_{\widehat{\Lambda}}, N_{\widehat{\Lambda}}) \cong \mathcal{O}^{\frac{ba(a+1)}{2}}$ is a trivial holomorphic vector bundle, and hence $\sigma_{\widehat{\Lambda}}$ is a parallel section with respect to the flat connection on $\mathcal{O}^{\frac{ba(a+1)}{2}}$. (As a consequence for instance $\dim(\mathrm{Ker}(\sigma_{\widehat{\Lambda}}([T_z(\Lambda)]))$ is independent of $z \in \Lambda$.) Since now $\mathscr{C}^\flat(\mathbf{N}_\ell)$ contains $\widehat{\Lambda}$ we may also take $\mathscr{C}^\flat(\mathbf{N}_\ell)$ as being well-defined as a germ of complex submanifold along $\widehat{\Lambda}$. For $x \in S$, we have $\mathscr{C}_x(S) = \mathscr{C}_x(\mathbf{N}_\ell)$. Note that the germ of $\mathscr{C}_x(\mathbf{N}_\ell)$ at $[T_x(\Lambda)]$ agrees with $\mathscr{C}^\flat_x(\mathbf{N}_\ell)$ as a germ at $[T_x(\Lambda)]$, hence by analytic continuation the same holds true when $x \in \Lambda \cap S$ is replaced by an arbitrary point $z \in \Lambda$. (Here recall that in general $\mathscr{C}_z(S)$ is defined by tangential intersection $\mathscr{C}^0_z(S) = \mathscr{C}_z(X) \cap \mathbb{P}T_z(\mathbf{N}_\ell)$ and by removing irrelevant irreducible components.) We have thus proven by parallel transport along $\Lambda \in \mathbf{A}(\epsilon_0, m_0)$ that for any $z \in \mathbf{N}_\ell$, $(\mathscr{C}_z(\mathbf{N}_\ell) \subset \mathbb{P}T_z(\mathbf{N}_\ell))$ is projectively equivalent to $(\mathscr{C}_x(\mathbf{N}_\ell) \subset \mathbb{P}T_x(\mathbf{N}_\ell))$ for $x \in S$ and hence to $(\mathscr{C}_0(X_0) \subset \mathbb{P}T_0(X_0)$ for the model Hermitian symmetric space X_0 of the compact type and of rank $\geqslant 2$, and we can conclude by Lemma 2.1 that $\varpi : \mathscr{C}(\mathbf{N}_\ell) \to \mathbf{N}_\ell$ underlies a holomorphic L_0-structure.

We claim that the holomorphic L_0-structure $\varpi : \mathscr{C}(\mathbf{N}_\ell) \to \mathbf{N}_\ell$ is flat. At each point $z \in \mathbf{N}_\ell$, there is a nonempty open subset $U_z \subset \mathscr{C}_z(\mathbf{N}_\ell)$ such that for each $[\gamma] \in U_z$, denoting by $\Lambda_{[\gamma]}$ the unique projective line on X passing through z such that $T_z(\Lambda_{[\gamma]}) = \mathbb{C}\gamma$, we have $\Lambda_{[\gamma]} \subset \mathbf{N}_\ell$. In the proof of Theorem 2.1 (from Hwang-Mok [HM97]) the flatness of the G-structure modeled on an irreducible Hermitian symmetric space of the compact type of rank $\geqslant 2$ is obtained by proving the vanishing of curvature-like tensors (cf. Guillemin [Gu65]) along minimal rational curves, and deducing from linear algebra that this is sufficient for proving the vanishing of the curvature-like tensors. For the argument to work, by the identity theorem for analytic functions, in place of proving the vanishing of curvature-like tensors along all minimal rational curves it is sufficient to prove, for any $z \in \mathbf{N}_\ell$, the same for a nonempty open set of minimal rational curves passing through z. (Actually it is sufficient to prove the latter for a nonempty open set of points z on $\mathbf{N}_\ell$.) Since the proof in Hwang-Mok [HM97] of the vanishing of curvature-like tensors on a Fano manifold X admitting L_0-structure relied only on the fact that the splitting of the holomorphic tangent bundle T_X over a minimal rational curve ℓ is of the standard type, i.e., $T_X|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$ for some nonnegative integers p and q, the same argument for a nonempty open set of minimal rational curves passing through $z \in \mathbf{N}_\ell$, for any $z \in \mathbf{N}_\ell$ implies by the aforementioned use of the identity theorem for analytic functions that the relevant curvature-like tensors of Guillemin [Gu65] vanish identically on $\mathbf{N}_\ell$. From this observation we have proven that $\mathscr{C}(\mathbf{N}_\ell)$ defines a flat L_0-structure on $\mathbf{N}_\ell$.

Recall that $(X, \mathcal{K})$ is the ambient Hermitian symmetric space of the compact type equipped with the minimal rational component $\mathcal{K}$ of projective lines on X with respect to the minimal canonical projective embedding, i.e., the embedding defined by $\Gamma(X, \mathcal{O}(1))$. For $z \in X$ denote by $\mathcal{K}(z) \subset \mathcal{K}$ the smooth cycle consisting of projective lines passing through z. Let now $x_0 \in S$ be any base point. Write $\mathscr{V}_0 := \{x_0\}$. Define $\mathscr{V}_1 := \bigcup \{\ell \in \mathcal{K}(x_0) : [T_{x_0}(\ell)] \in \mathscr{C}_{x_0}(S)\}$.

For any $x_1 \in \mathscr{V}_1$ other than x_0, there is a unique minimal rational curve ℓ_1 joining x_0 to x_1, and there exists an embedded collar $\mathbf{N}_{\ell_1} \supset \ell_1$ such that $\mathbf{N}_{\ell_1}$ also contains a nonempty open neighborhood of x_0 on S. At $x_1 \in \ell_1$ by the method of parallel transport we have defined $\mathscr{C}_{x_1}(\mathbf{N}_{\ell_1})$ such that $(\mathscr{C}_{x_1}(\mathbf{N}_{\ell_1}) \subset \mathbb{P}T_{x_1}(\mathbf{N}_{\ell_1}))$ is projectively equivalent to $(\mathscr{C}_0(X_0) \subset \mathbb{P}T_0(X_0))$. By Condition (F), the underlying projective linear isomorphism extends to a projective linear isomorphism $\gamma_x : \mathbb{P}T_{x_1}(X) \xrightarrow{\cong} \mathbb{P}T_0(X)$ such that $\varphi_x|_{\mathscr{C}_{x_1}(X)} : \mathscr{C}_{x_1}(X) \xrightarrow{\cong} \mathscr{C}_0(X)$. (Actually there exists an automorphism $\varphi_x \in \mathrm{Aut}(X)$ such that $\varphi_x(x) = 0$ and such that $d\varphi_x(x) = \gamma_x$.) It follows that $\varpi : \mathscr{C}(\mathbf{N}_{\ell_1}) \to \mathbf{N}_{\ell_1}$ is a sub-VMRT structure modeled on (X_0, X). In particular, the pair $(\mathscr{C}_{x_1}(\mathbf{N}_{\ell_1}), \mathscr{C}_{x_1}(X))$ is nondegenerate for substructures and it satisfies Condition (T). Define now $\mathscr{V}_2 := \bigcup \{\ell \in \mathcal{K}(x_1) : x_1 \in \mathscr{V}_1 - \{x_0\}, [T_{x_1}(\ell)] \in \mathscr{C}_{x_1}(S)\}$. Any point $x_2 \in \mathscr{V}_2 - \mathscr{V}_1$ is the end-point of an ordered triple (x_0, x_1, x_2) of distinct points linked by an ordered pair (ℓ_1, ℓ_2) of minimal rational curves, so that $x_0, x_1 \in \ell_1$ and $T_{[x_0]}(\ell_1) \in \mathscr{C}_{x_0}(\mathbf{N}_{\ell_1}) = \mathscr{C}_{x_0}(S)$, while $x_1, x_2 \in \ell_2$ and $T_{[x_1]}(\ell_2) \in \mathscr{C}_{x_1}(\mathbf{N}_{\ell_2})$.

The construction of $\mathscr{V}_2$ yields by iteration an inductive construction of projective varieties $\mathscr{V}_k$, $k \in \mathbb{N}$ such that for some natural number $m < \dim(S)$ we have $\mathscr{V}_0 \subset \mathscr{V}_1 \subsetneq \cdots \subsetneq \mathscr{V}_k \subsetneq \cdots \subsetneq \mathscr{V}_m$ while $\mathscr{V}_m = \mathscr{V}_{m+1} = \cdots\cdots$. For $1 \leqslant k \leqslant m$ each point $x_k \in \mathscr{V}_k - \mathscr{V}_{k-1}$ is the end-point of an ordered $(k+1)$-tuple $(x_0, \cdots, x_k)$ of distinct points linked by an ordered k-tuple of minimal rational curves $(\ell_1, \cdots, \ell_k)$. We will call ℓ_k the last leg of a chain of minimal rational curves linking $(x_0, \cdots, x_k)$, and denote by $\mathscr{L}(x_k)$ the set of all possible such last legs of chains of minimal rational curves for various choices of $x_1, \cdots, x_{k-1}$ and $(\ell_0, \cdots, \ell_k)$ linking $(x_0, \cdots, x_k)$. From the proof of Theorem 2.4 we have $S \subset Z$ where $Z \subset X$ is a projective subvariety and $\dim(Z) = \dim(S)$, and where $Z = \mathscr{V}_m$. For $0 \leqslant k \leqslant m-1$ let now $\mathbf{L}_k$ denote the set of all lines belonging to $\bigcup \big\{\mathcal{K}(x_k) : x_k \in \mathscr{V}_k - \mathscr{V}_{k-1} : T_{x_k}(\ell_k) \in \mathscr{C}_{[x_k]}(\mathbf{N}_{\ell_k}) \text{ for some } \ell_k \in \mathscr{L}(x_k)\big\}$, and define $\mathbf{L} := \mathbf{L}_1 \cup \cdots \cup \mathbf{L}_m$. Then, $Z = \mathscr{V}_m = \bigcup \{\ell : \ell \in \mathbf{L}\}$. We have also $Z = \bigcup \{\mathbf{N}_\ell : \ell \in \mathbf{L}\}$. By compactness of Z and Heine-Borel there exist a finite number of minimal rational curves $\Lambda_1, \cdots, \Lambda_s$ and their corresponding embedded collars $\mathbf{N}_{\Lambda_i}$, $1 \leqslant i \leqslant s$ such that $Z = \mathbf{N}_{\Lambda_1} \cup \cdots \cup \mathbf{N}_{\Lambda_s}$. Thus $Z \subset X$ is an immersed submanifold, and its normalization $\nu : \widetilde{Z} \to Z$ is nonsingular and projective. By construction Z is rationally connected by projective lines, thus $\widetilde{Z}$ is a rationally connected projective manifold, hence simply connected (cf. Debarre [De01]). It follows that $\widetilde{Z}$ is biholomorphically equivalent to the model Hermitian symmetric space X_0 of the compact type and of rank $\geqslant 2$. The proof of Proposition 3.1 is complete. □

5 Proof of Main Theorem

Proof of Main Theorem. From the classification of admissible pairs (X_0, X) of Hermitian symmetric spaces of the compact type and of rank $\geqslant 2$ there exist a Harish-Chandra coordinate chart on $\mathcal{W} \subset X$ such that the intersection $\mathcal{W} \cap X_0$ is given by a linear subspace in the chart. At the origin 0 of the coordinate chart, $0 \in X_0 \subset X$, in terms of the chosen Harish-Chandra coordinates the scalar multiplications $T_t(z) = tz, t \in \mathbb{C}^*$, defines a $\mathbb{C}^*$-action such that 0 is an isolated fixed point. Let now $W \subset X$ be an open neighborhood of 0 in the complex topology, and $S \subset W$ be a complex submanifold which underlies a sub-VMRT structure modeled on (X_0, X). Let now $h : X_0 \to X$ be a holomorphic immersion such that $Z := h(X_0)$ is the immersed complex submanifold of X giving an analytic continuation of S, and we may assume without loss of generality that $0 \in S$ and that 0 lies on the smooth locus of Z. By assumption $(\mathscr{C}_0(S) \subset \mathscr{C}_0(X))$ is projectively equivalent to $(\mathscr{C}_0(X_0) \subset \mathscr{C}_0(X))$. $\mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$ underlies a reductive L-structure in the notation of the last paragraph of §2. In this case any projective linear transformation on $\mathbb{P}T_0(X)$ preserving $\mathscr{C}_0(X)$ must necessarily belong to L. Hence, applying a linear transformation belonging to $L \subset \mathrm{GL}(T_0(X))$ (cf.§2), without loss of generality we may assume that $\mathscr{C}_0(S) = \mathscr{C}_0(X_0)$. Now for $t \neq 0$ define $Z_t := T_{\frac{1}{t}}(Z) = \{w : tw \in Z\}$. As $t \to 0$, $Z_t \cap \mathcal{W} \subset \mathcal{W}$ converges as a subvariety to $X_0 \cap \mathcal{W}$ inside the Harish-Chandra coordinate chart $\mathcal{W} \cong \mathbb{C}^s$. We claim that actually Z_t converges as a subvariety to X_0 as $t \to 0$.

Let ω be a Kähler form on X such that minimal rational curves are of unit volume. Recall that $h : X_0 \to X$, $h(X_0) = Z$, is a holomorphic immersion such that minimal rational curves are mapped biholomorphically onto minimal rational curves on X. It follows that minimal rational curves on X_0 are also of unit volume with respect to $h^*\omega$. Since $\mathrm{Pic}(X_0) \cong \mathbb{Z}$, the two Kähler forms $\omega|_{X_0}$ and $h^*\omega$ are cohomologous, hence

$$\mathrm{Volume}(X_0, h^*\omega) = \int_{X_0} \frac{(h^*\omega)^s}{s!} = \int_{X_0} \frac{\omega^s}{s!} = \mathrm{Volume}\left(X_0, \omega|_{X_0}\right).$$

We have $\mathrm{Volume}(X_0, h^*\omega) = \mathrm{Volume}(\mathrm{Reg}(Z), \omega|_{\mathrm{Reg}(Z)})$. On the other hand, the subvarieties Z_t, $t \in \mathbb{C}^*$, are deformations of each other, in particular of $Z_1 = Z$, so that

$$\mathrm{Volume}(\mathrm{Reg}(Z_t), \omega|_{\mathrm{Reg}(Z_t)}) = \mathrm{Volume}(\mathrm{Reg}(Z), \omega|_{\mathrm{Reg}(Z)})$$

$$= \mathrm{Volume}(X_0, h^*\omega) = \mathrm{Volume}\left(X_0, \omega|_{X_0}\right).$$

The subvarieties $\{Z_t \subset X : |t| < 1\}$ all have the same volume with respect to ω, and, for any sequence $\{t_n\}_{n \geqslant 0}$ such that $t_n \to 0$, there exists a subsequence $\{t_{\sigma(n)}\}_{n \geqslant 0}$ for some strictly monotonically increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $Z_{t_{\sigma(n)}}$ converges as subvarieties to some subvariety $Z^\sharp \subset X$. From the above, for any $n \in \mathbb{N}$,

$$\mathrm{Volume}(Z^\sharp, \omega) = \mathrm{Volume}(Z_{t_{\sigma(n)}}, \omega).$$

On the other hand, $Z_t \cap \mathcal{W}$ converges as subvarieties to $X_0 \cap \mathcal{W}$, hence as a cycle $Z^\sharp = X_0 + V$, where V is an s-dimensional cycle with support lying inside the divisor $D = X - \mathcal{W}$. From $\mathrm{Volume}(X_0, \omega) = \mathrm{Volume}(Z_{t(\sigma_n)}, \omega)$ it follows that $\mathrm{Volume}(V, \omega) = 0$, i.e., $V = \emptyset$. Since this works for any sequence $\{t_n\}$ on the unit disk Δ converging to 0, we conclude that $Z_t \to X_0$ as $t \to 0$.

We claim that $Z \subset X$ is actually a complex submanifold. In other words, in the notation of Proposition 3.1, the holomorphic immersion $h = \nu \circ \Phi : X_0 \to X$ is actually a holomorphic embedding. Recall that $Z = h(X_0)$ and that $Z_t := T_{\frac{1}{t}}(Z) = \{w : tw \in Z\}$, where in terms of the chosen Harish-Chandra coordinates chart $\mathcal{U} \subset Z$ with the origin at some smooth point $a \in Z$, with Harish-Chandra coordinates z, for $\lambda \in \mathbb{C}^*$ we have $T_\lambda(z) = \lambda z$ on $\mathcal{U}$. For any $t \in \mathbb{C}^*$, $\mathrm{Sing}(Z_t) = T_{\frac{1}{t}}(\mathrm{Sing}(Z))$. We have proven that $Z_t \to X_0$ as subvarieties of X as $t \to 0$. Suppose now $\mathrm{Sing}(Z) \neq \emptyset$, and let p be any point on $\mathrm{Sing}(Z)$. Writing $p_n := T_n(z)$ for any positive integer n and letting q be any limit point of $\{p_n\}_{n \geqslant 1}$, by the lower semicontinuity property of Lelong numbers for a convergent family of closed positive currents, we have

$$\nu([X_0]; q) \geqslant \limsup_{n\to\infty} \nu\big(\big[Z_{\frac{1}{n}}\big]; p_n\big) = \limsup_{n\to\infty} \nu\big((T_n)_*[Z]; T_n(p)\big) = \nu([Z]; p) \geqslant 2,$$

where $\nu(S; x)$ denotes the Lelong number of a closed positive (k,k)-current S on X at a point x and where $[V]$ denotes the integral current of a pure $(n-k)$-dimensional complex-analytic subvariety $V \subset X$. The inequality $\nu([X_0]; q) \geqslant 2$ gives a plain contradiction to the smoothness of $X_0 \subset X$, proving our claim that $Z \subset X$ is a complex submanifold. In other words, $h : X_0 \to X$, $Z = h(X_0)$, is a holomorphic embedding.

Write $Z_0 := X_0$. Define now $\mathscr{Z}' := \big\{(t,z) : t \in \Delta, z \in Z_t\big\} \subset \Delta \times X$. $\mathscr{Z}' \subset \Delta \times X$ is a complex-analytic subvariety. Let $\alpha : \mathscr{Z}' \to \Delta$ be the canonical projection onto the first factor. Then, all fibers $\{t\} \times Z_t, t \in \Delta$, are equidimensional smooth and reduced subvarieties of $\Delta \times X$. By Lemma 2.2, $\alpha : \mathscr{Z}' \to \Delta$ is a regular family of projective submanifolds. To prove Main Theorem it remains to establish.

Lemma 5.1. *Let $\alpha : \mathscr{Z}' \to \Delta$, $\mathscr{Z}' \subset \Delta \times X$, $Z'_t := \alpha^{-1}(t)$, be regular family of projective submanifolds of $\Delta \times X$ such that $Z'_t = \{t\} \times Z_t$ for $t \in \Delta$, where $Z_0 := X_0$. Suppose the restriction map $r : \Gamma(X, T_X) \to \Gamma(X_0, T_X|_{X_0})$ given by $r(\eta) = \eta|_{X_0}$ is surjective. Then, for $\epsilon > 0$ sufficiently small, there exists a holomorphic map $\gamma : \Delta(\epsilon) \to \mathrm{Aut}_0(X)$ such that $\gamma(0) = \mathrm{id}_X$ and such that $Z_t = \{t\} \times \gamma_t(X_0)$ for every $t \in \Delta(\epsilon)$.*

Proof. Let $\mathscr{Q}$ be an irreducible component of the Chow scheme $\mathrm{Chow}(X)$ which contains the point $[X_0]$ representing the multiplicity-free s-dimensional cycle with support X_0. Let $\mathscr{O} \subset \mathrm{Chow}(X)$ be the subscheme consisting of the orbit of $[X_0]$ under the action of $G = \mathrm{Aut}_0(X)$, i.e., the set of all translates $[\gamma X_0]$ as γ runs over G. We claim that $\mathscr{O} \subset \mathscr{Q}$. (This needs a justification since there may *a priori* exist two components $\mathscr{Q}$ and $\mathscr{Q}'$ such that $[X_0] \in \mathscr{Q}$ and $[X_0] \in \mathscr{Q}'$.) To see this note that $[X_0] \in \mathscr{O}$ is the limit of a sequence of points $[W_m] \in \mathscr{Q}$, $m \geqslant 1$, where each $[W_m]$ is a smooth point of $\mathrm{Chow}(X)$. For each $m \geqslant 1$ the orbit $G \cdot [W_m]$ must lie on the unique irreducible component $\mathscr{Q}$ of $\mathrm{Chow}(X)$ containing $[W_m]$. Hence, for every $\gamma \in G$, $\gamma \cdot [X_0] = \lim_{m\to\infty} \gamma \cdot [W_m] \in \mathscr{Q}$ since $\mathscr{Q} \subset \mathrm{Chow}(X)$ is closed, and we have proven the claim that $\mathscr{O} \subset \mathscr{Q}$. Hence, $\dim \mathscr{O} \leqslant \dim \mathscr{Q}$. Here and in what follows dimensions are complex dimensions. The dimension of $\mathscr{Q}$ at $[X_0]$ satisfies

$$\dim \mathscr{Q} \leqslant \dim \Gamma(X_0, N_{X_0|X}),$$

where $N_{X_0|X} = T_X|_{X_0}/T_{X_0}$ denotes the holomorphic normal bundle on X_0 for the inclusion $X_0 \subset X$. Denote by $H \subset G$ the subgroup which fixes $X_0 \subset X$ as a subset, and by $Q \subset H$ the normal subgroup which fixes every point on X_0. The identity component of H/Q agrees with G_0, so that $\dim H = \dim \Gamma(X_0, T_{X_0}) + \dim Q$. Note that the kernel of the restriction map $r : \Gamma(X, T_X) \to \Gamma(X_0, T_X|_{X_0})$ consists of holomorphic vector fields on X which vanish identically on X_0. Thus, $\mathrm{Ker}(r)$ agrees with the Lie algebra $\mathfrak{q}$ of Q, so that $\dim \Gamma(X, T_X) = \dim \Gamma(X_0, T_X|_{X_0}) + \dim Q$ from the hypothesis that r is surjective. Noting this, for the orbit $\mathscr{O} = G \cdot [X_0]$, we have

$$\begin{aligned} \dim \mathscr{O} &= \dim G/H = \dim \Gamma(X, T_X) - \dim H \\ &= (\dim \Gamma(X_0, T_X|_{X_0}) + \dim Q) - (\dim \Gamma(X_0, T_{X_0})) + \dim Q) \\ &= \dim \Gamma(X_0, T_X|_{X_0}) - \dim \Gamma(X_0, T_{X_0}). \end{aligned}$$

From the tangent sequence

$$0 \longrightarrow T_{X_0} \longrightarrow T_X|_{X_0} \longrightarrow N_{X_0|X} \longrightarrow 0,$$

its associated long exact sequence and from $H^1(X_0, T_{X_0}) = 0$ (cf. Calabi-Vesentini [CV60]) we obtain a short exact sequence of vector spaces

$$0 \longrightarrow \Gamma(X_0, T_{X_0}) \longrightarrow \Gamma(X_0, T_X|_{X_0}) \longrightarrow \Gamma(X_0, N_{X_0|X}) \longrightarrow 0,$$

so that $\dim\Gamma(X_0, T_X|_{X_0}) - \dim\Gamma(X_0, T_{X_0})) = \dim\Gamma(X_0, N_{X_0|X})$, hence

$$\dim\mathscr{O} = \dim\Gamma(X_0, T_X|_{X_0}) - \dim\Gamma(X_0, T_{X_0})$$
$$= \dim\Gamma(X_0, N_{X_0|X}) \geqslant \dim\mathscr{Q} \geqslant \dim\mathscr{O}\,.$$

As a consequence, we have

$$\dim\mathscr{O} = \dim\mathscr{Q} = \dim\Gamma(X_0, N_{X_0|X})\,,$$

implying that $\mathscr{Q} = \mathscr{O} = G\cdot[X_0]$. Lemma 5.1 follows immediately. □ □

Proof of Main Theorem cont. By Lemma 5.1, for $t \in \Delta$ sufficiently small, we have $T_{\frac{1}{t}}Z =: Z_t = \gamma_t X_0$ for some $\gamma_t \in G = \mathrm{Aut}_0(X)$. It follows that $Z = T_t(\gamma_t X_0) = (T_t \circ \gamma_t)(X_0)$, for t sufficiently small, is a translate of X_0 by an element of G, as desired. □

Proof of Theorem 3.1. The same proof as that of Main Theorem yields Theorem 3.1, when the linear saturation of the given sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ is taken as a hypothesis in case (X_0, X) is degenerate for substructures. □

6 Examples

(6.1) In this final section we examine examples of admissible pairs of rational homogeneous manifolds not of the sub-diagram type for which Main Theorem and Theorem 3.1 apply. We consider first of all the admissible pairs $(G^{\mathrm{III}}(n,n), G(n,n))$, $n \geqslant 3$, in the standard embedding of the Lagrangian Grassmannian $G^{\mathrm{III}}(n,n)$ into the Grassmannian $G(n,n)$. $(G^{\mathrm{III}}(n,n), G(n,n))$ is an admissible pair of rational homogeneous spaces not of the sub-diagram type. By Zhang [Zh14, Theorem 4.14], for $n \geqslant 3$ the pair $(G^{\mathrm{III}}(n,n), G(n,n))$ is nondegenerate for substructures. When $n = 2$, the admissible pair $(G^{\mathrm{III}}(2,2), G(2,2))$ is equivalently the pair (Q^3, Q^4), which is degenerate for substructures. We defer to (4.2) the discussion on the admissible pairs of hyperquadrics $(Q^m, Q^n), 3 \leqslant m < n$.

Let (W, s) be a complex $(2n)$-dimensional vector space W equipped with a symplectic form s. Choosing an appropriate basis $\{e_1, \cdots, e_{2n}\}$ of W we may assume that s is represented by the matrix $\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$. Denote by $\mathcal{M}(n)$ the complex vector space of n-by-n matrices Z with complex coefficients. Identifying $Z \in \mathcal{M}(n)$ with the n-dimensional vector subspace $\Pi_Z \subset W$ spanned by the n column vectors of $\begin{bmatrix} I_n \\ Z \end{bmatrix}$, we realize $\mathcal{M}(n)$ as an open subset of $G(n,n)$. The Lagrangian Grassmannian $G^{\mathrm{III}}(n,n) \subset G(n,n)$ is the complex submanifold consisting of n-dimensional isotropic subspaces with respect to s. In the chosen basis, $\{e_1, \cdots, e_n\}$ is an isotropic subspace of (W, s). Consider now $Z \in \mathcal{M}(n)$. Then, the n column vectors of $\begin{bmatrix} I_n \\ Z \end{bmatrix}$ span an n-dimensional isotropic subspace of (W, s) if and only if

$$[I_n, Z^t] \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} I_n \\ Z \end{bmatrix} = 0, \quad \text{i.e.,}$$

$$[-Z^t, I_n] \begin{bmatrix} I_n \\ Z \end{bmatrix} = 0 \quad \text{i.e., } Z^t = Z.$$

$G^{\mathrm{III}}(n,n) \cap \mathcal{M}(n)$ is precisely the vector subspace $\mathcal{M}_s(n) \subset \mathcal{M}(n)$ consisting of symmetric n-by-n matrices with complex coefficients. The standard $\mathbb{C}^*$-action $\{T_t : t \in \mathbb{C}^*\}$ on $G(n,n)$, when restricted to the big Schubert cell $\mathcal{M}(n) \subset G(n,n)$, is given by $T_t(Z) = tZ$. Clearly $\mathcal{M}_s(n)$ and hence $G^{\mathrm{III}}(n,n) = \overline{\mathcal{M}_s(n)}$ are preserved under the $\mathbb{C}^*$-action $\{T_t\}_{t\in\mathbb{C}^*}$.

We claim that the admissible pair $(G^{\mathrm{III}}(n,n), G(n,n))$ satisfies Condition (F). Let $\lambda : \mathscr{C}_0(G^{\mathrm{III}}(n,n)) \hookrightarrow \mathscr{C}_0(G(n,n))$ be an arbitrary holomorphic embedding such that, denoting by $\mathbb{P}V_\lambda \subset \mathbb{P}T_0(G(n,n))$ the projective linear span of $\lambda(\mathscr{C}_0(G^{\mathrm{III}}(n,n)))$, the inclusion $(\lambda(\mathscr{C}_0(G^{\mathrm{III}}(n,n)) \subset \mathbb{P}V_\lambda)$ is projectively equivalent to the inclusion $(\mathscr{C}_0(G^{\mathrm{III}}(n,n)) \subset \mathbb{P}T_0(G^{\mathrm{III}}(n,n)))$.

Let $\pi_E : E \to G(n,n))$ be the tautological vector bundle on the Grassmannian $G(n,n)$. Thus, given an n-dimensional vector subspace $\Pi \subset W$, we have $E_{[\Pi]} = \Pi \subset W$. Write $U := E^*$. Denote by $\mathbf{W} = W \times G(n,n)$ the trivial complex vector bundle over $G(n,n)$ of rank $2n$ with fibers identified with W, and write $U' := \mathbf{W}/E$. For the tangent bundle on the Grassmannian $G(n,n)$ we have $T_{G(n,n)} \cong \mathrm{Hom}(E,U') = E^* \otimes U' = U \otimes U'$. We have $\mathscr{C}_0(G(n,n)) = \mathbb{P}U_0 \times \mathbb{P}U_0' \subset \mathbb{P}(U_0 \otimes U_0')$, where the inclusion is given by the Segre embedding.

Over $G(n,n)$ we have the short exact sequence $0 \longrightarrow E \longrightarrow \mathbf{W} \longrightarrow U' \longrightarrow 0$ of holomorphic vector bundles. Restricted to the Lagrangian Grassmannian $G^{\mathrm{III}}(n,n) \subset G(n,n)$, for an n-dimensional vector subspace $\Pi \subset W$ isotropic with respect to the symplectic space (W,s) the symplectic form s induces a linear isomorphism $W/\Pi \cong \Pi^*$, i.e., $U'_{[\Pi]} \cong E^*_{[\Pi]} = U_{[\Pi]}$. Thus over the Lagrangian Grassmannian $G^{\mathrm{III}}(n,n)$ we have the canonical isomorphism $U'|_{G^{\mathrm{III}}(n,n)} \cong U|_{G^{\mathrm{III}}(n,n)}$ induced by the symplectic form s on W. In what follows we will restrict our attention to the Lagrangian Grassmannian $G^{\mathrm{III}}(n,n)$ and make the identification of U' with U over $G^{\mathrm{III}}(n,n)$ by means of the aforementioned canonical isomorphism. With this identification we have $\mathscr{C}_0(G^{\mathrm{III}}(n,n)) = \nu(\mathbb{P}U_0) \subset \mathbb{P}(S^2U_0)$, where $\nu : \mathbb{P}U_0 \to \mathbb{P}(S^2U_0)$ is the Veronese embedding.

Denoting by $\pi_1 : \mathbb{P}U_0 \times \mathbb{P}U_0' \to \mathbb{P}U_0$ and $\pi_2 : \mathbb{P}U_0 \times \mathbb{P}U_0' \to \mathbb{P}U_0'$ the canonical projections, we have $\mathcal{O}_{\mathbb{P}(U_0 \otimes U_0')}(1)|_{\mathbb{P}(U_0)\times\mathbb{P}(U_0')} \cong \pi_1^*\mathcal{O}_{\mathbb{P}U_0}(1) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}U_0'}(1)$, and $\nu^*\left(\mathcal{O}_{\mathbb{P}(U_0\otimes U_0')}(1)|_{\mathbb{P}(U_0)\times\mathbb{P}(U_0)}\right) = \mathcal{O}_{\mathbb{P}U_0}(2)$. By hypothesis $(\lambda(\mathscr{C}_0(G^{\mathrm{III}}(n,n)) \subset \mathbb{P}V_\lambda)$ is projectively equivalent to the inclusion $(\mathscr{C}_0(G^{\mathrm{III}}(n,n)) \subset \mathbb{P}T_0(G^{\mathrm{III}}(n,n)))$. Hence, $\dim(V_\lambda) = \dim(S^2U_0) = \frac{n(n+1)}{2} > n$, and $\lambda^*\left(\mathcal{O}_{\mathbb{P}(U_0\otimes U_0)}(1)|_{\mathbb{P}(U_0)\times\mathbb{P}(U_0')}\right) = \mathcal{O}_{\mathbb{P}U_0}(2)$, i.e., $\mathcal{O}_{\mathbb{P}U_0}(2) \cong \lambda^*\pi_1^*\mathcal{O}_{\mathbb{P}U_0}(1) \otimes \lambda^*\pi_2^*\mathcal{O}_{\mathbb{P}U_0'}(1)$. It follows that either (a) $\lambda^*\pi_1^*\mathcal{O}_{\mathbb{P}U_0}(1) \cong \lambda^*\pi_2^*\mathcal{O}_{\mathbb{P}U_0}(1) \cong \mathcal{O}_{\mathbb{P}U_0}(1)$, or (b) we have a degenerate situation where one of the two scenarios below occurs: (i) $\lambda^*\pi_1^*\mathcal{O}_{\mathbb{P}U_0}(1) \cong \mathcal{O}_{\mathbb{P}U_0}(2)$ and $\lambda^*\pi_2^*\mathcal{O}_{\mathbb{P}U_0}(1) \cong \mathcal{O}_{\mathbb{P}U_0}$; (ii) $\lambda^*\pi_2^*\mathcal{O}_{\mathbb{P}U_0}(1) \cong \mathcal{O}_{\mathbb{P}U_0}(2)$ and $\lambda^*\pi_1^*\mathcal{O}_{\mathbb{P}U_0}(1) \cong \mathcal{O}_{\mathbb{P}U_0}$. In the degenerate cases (b) the projective linear span $\mathbb{P}V_\lambda$ of $\lambda(\mathscr{C}_0(G^{\mathrm{III}}(n,n)))$ is of dimension $\leqslant \dim(\mathbb{P}U_0) = n-1$, hence $\dim(V_\lambda) \leqslant n$, a plain contradiction. Thus only (a) occurs, in which case $\pi_i \circ \lambda : \mathscr{C}_0(G^{\mathrm{III}}(n,n)) \to \mathbb{P}U_0$ must be an isomorphism for $i = 1,2$, which implies the existence of projective linear isomorphisms $\Lambda_1 : \mathbb{P}U_0 \xrightarrow{\cong} \mathbb{P}U_0$ and $\Lambda_2 : \mathbb{P}U_0 \xrightarrow{\cong} \mathbb{P}U_0'$ such that the embedding $\lambda : \mathscr{C}_0(G^{\mathrm{III}}(n,n) \to \mathscr{C}_0(G(n,n))$ is induced by a linear embedding from S^2U_0 into $U \otimes U_0'$, also denoted by λ, which is given by $\lambda(u \otimes u) = \Lambda_1(u) \otimes \Lambda_2(u)$ and extended to S^2U_0 by polarization, verifying Condition (F) for the admissible pair $(G^{\mathrm{III}}(n,n), G(n,n))$.

Finally, to check that Main Theorem applies to $(G^{\mathrm{III}}(n,n), G(n,n))$ it remains to show that the restriction map $r : \Gamma(G(n,n), T_{G(n,n)}) \to \Gamma(G^{\mathrm{III}}(n,n), T_{G(n,n)})$ defined by $r(\eta) = \eta|_{G^{\mathrm{III}}(n,n)}$ is surjective. Now in the notation above we have $T_{G(n,n)} = U \otimes U'$ where U and U' are universal semipositive bundles over the Grassmannian $G(n,n)$. Note that $G(n,n) = U(2n)/S(U(n) \times U(n))$, so that the isotropy subgroup is isomorphic to $U(1) \times (SU(n) \times SU(n))$ and the universal bundle U over $G(n,n)$ arises from the standard representation ρ of $U(n)$ (being $U(1) \times SU(n)$ where $SU(n)$ is the first Cartesian factor) on $\mathbb{C}^n$ (being the first tensor factor of $\mathbb{C}^n \otimes \mathbb{C}^n$), up to a character on $U(1)$. In what follows we restrict our attention to $G^{\mathrm{III}}(n,n)$ and denote the restriction $U|_{G^{\mathrm{III}}(n,n)}$ by U too, recalling also that $U|_{G^{\mathrm{III}}(n,n)} \cong U'|_{G^{\mathrm{III}}(n,n)}$ canonically. We have the holomorphic direct sum decomposition $U \otimes U = S^2U \oplus \Lambda^2U$ of homogeneous holomorphic vector bundles on $G^{\mathrm{III}}(n,n)$, which up to a character on $U(1)$, corresponds to the decomposition of $\rho \otimes \rho$ on $\mathbb{C}^n \otimes \mathbb{C}^n$ into irreducible representations. Thus, $U \otimes U = S^2U \oplus \Lambda^2U$ is a direct sum decomposition into irreducible homogeneous holomorphic vector bundles on $G^{\mathrm{III}}(n,n)$. Write $G_0 = \mathrm{Aut}(G^{\mathrm{III}}(n,n)) = \mathbb{P}\mathrm{Sp}(n;\mathbb{C})$ and $G = \mathrm{Aut}_0(G(n,n)) = \mathbb{P}\mathrm{GL}(n;\mathbb{C})$. Note that the injective holomorphic group homomorphism $G_0 \hookrightarrow G$ induces an injective homomorphism $\mathfrak{g}_0 = \Gamma(G^{\mathrm{III}}(n,n), T_{G^{\mathrm{III}}(n,n)}) \hookrightarrow \Gamma(G(n,n), T_{G(n,n)}) = \mathfrak{g}$, so that $\mathfrak{g}_0$ is contained in the image of the restriction map r. To prove the surjectivity of the restriction map $r : \Gamma(G(n,n), T_{G(n,n)}) \cong \Gamma(G(n,n), U \otimes U') \to \Gamma(G^{\mathrm{III}}(n,n), T_{G(n,n)}) = \Gamma(G^{\mathrm{III}}(n,n), S^2U) \oplus \Gamma(G^{\mathrm{III}}(n,n), \Lambda^2U)$ it suffices to check that (a) the projection of the image of r to $\Gamma(G^{\mathrm{III}}(n,n), \Lambda^2U)$ is nonempty and that (b) $\Gamma(G^{\mathrm{III}}(n,n), \Lambda^2U)$ is irreducible under the action of G_0.

For the proof of (a) it suffices to observe that from the transitivity of G on $G(n,n)$ that there exists a holomorphic one-parameter family $\{\gamma_t : t \in \Delta\} \subset G$ such that $\gamma_0 = \mathrm{id}_{G_0}$ and $0 \neq \frac{\partial \gamma_t}{\partial t}\Big|_{t=0} \bmod T_{G^{\mathrm{III}}(n,n)} \in$

$\Gamma(G^{\rm III}(n,n), N_{G^{\rm III}(n,n)|G(n,n)})$; $N_{G^{\rm III}(n,n)|G(n,n)} \cong \Lambda^2 U$.

The statement (b) follows from the Bott-Borel-Weil Theorem. More precisely, if $\mathscr{E}_0$ is an irreducible P_0-representation space, equivalently, if $\mathscr{E} = G_0 \times_{P_0} \mathscr{E}_0$ is irreducible homogeneous vector bundle on $X_0 = G_0/P_0$, then the complex vector space $\Gamma(X_0, \mathscr{E})$ of global holomorphic sections is irreducible as a G-representation space by the Bott-Borel-Weil Theorem.

We are now ready to show

Theorem 6.1. *For $n \geqslant 3$, the admissible pair $(G^{\rm III}(n,n), G(n,n))$ is rigid. In other words, if $S \subset G(n,n)$ is a complex submanifold of some open subset of $G(n,n)$ inheriting a sub-VMRT structure modeled on $(\mathscr{C}_0(G^{\rm III}(n,n)), \mathscr{C}_0(G(n,n))$, i.e., $(\nu(\mathbb{P}^{n-1}), \zeta(\mathbb{P}^{n-1})$, where ν denotes the Veronese embedding and ζ denotes the Segre embedding, then S is an open subset of a projective submanifold $Z \subset G(n,n)$ and there exists $\gamma \in {\rm Aut}(G(n,n))$ such that $Z = \gamma(G^{\rm III}(n,n))$.*

Remark. In term of the universal rank-n semipositive vector bundles U and U' on $G(n,n)$ such that $T_{G(n,n)} = U \otimes U'$ the hypothesis on the locally closed submanifold $S \subset G(n,n)$ is equivalently the hypothesis that, shrinking S if necessary, there exists an isomorphism $\varphi : U'|_S \xrightarrow{\cong} U|_S$ such that, identifying $U'|_S$ with $U|_S$ by means of φ, T_S can be naturally identified with $S^2U|_S$, hence inheriting a holomorphic G-structure modeled on the Lagrangian Grassmannian $G^{\rm III}(n,n)$.

Proof of Theorem 6.1 Maintaining the notation $X_0 = G^{\rm III}(n,n)$, $X = G(n,n)$ for $n \geqslant 3$ we have proven that the admissible pair (X_0, X) satisfies Condition (F). To conclude that (X_0, X) is a rigid pair it remains to justify that the restriction map $r : \Gamma(X, T_X) \to \Gamma(X_0, T_X|_{X_0})$ defined by $r(\eta) = \eta|_{X_0}$ is surjective, and we have reduced the problem to proving that $\Gamma(X_0, \Lambda^2 U)$, $U \cong E^*$, is irreducible. By Proposition 6.1 and Proposition 6.2 we know that all sections in $\Gamma(X_0, \Lambda^2 U)$ descend from $\Lambda^2 W$. The symplectic group ${\rm Sp}(n;\mathbb{C})$ acts irreducibly on W, and $\Lambda^2 W$ splits into the direct sum of two irreducible components. More precisely, the symplectic form s gives a linear map $\psi : \Lambda^2 W \to \mathbb{C}$, and we have $\Lambda^2 W = A \oplus B$ where $A = {\rm Ker}\psi \subset \Lambda^2 W$ is of codimension 1, and $B = \mathbb{C}\zeta$, where $\zeta \in \Lambda^2 W$, is the element corresponding to $s \in \Lambda^2 W^*$ under the isomorphism $W^* \cong W$ induced by s. Denote by $\varpi : \Lambda^2 W \to \Gamma(X_0, \Lambda^2 U)$ the natural projection. At every point $x \in X_0$, $\varpi(\zeta(x)) = 0$ as can be checked using the normal form of s. It follows that $\Gamma(X_0, \Lambda^2 U) \cong A$ is irreducible, and hence $r : \Gamma(X, T_X) \to \Gamma(X_0, T_X|_{X_0})$ is surjective. By Main Theorem (as stated in §3) the pair (X_0, X) is rigid, as desired. □

(6.2) We consider now the case $(X_0, X) = (Q^m, Q^n)$ where $3 \leqslant m < n$. In this case (X_0, X) is degenerate for substructures, but under the assumption that a germ of sub-VMRT structure on $(S, x_0) \subset (X.x_0)$ modeled on (Q^m, Q^n) is linearly saturated, Theorem 3.1 still applies. We will check that (a) Condition (F) holds and that the restriction map $r : \Gamma(Q^n, T_{Q^n}) \to \Gamma(Q^m, T_{Q^n}|_{Q^m})$ is surjective. This gives a new proof of the main result of Zhang [Zh14, §5]. We have

Theorem 6.2 (from Zhang [Zh14])**.** *Let $3 \leqslant m \leqslant n$, and $S \subset Q^n \subset \mathbb{P}^{n+1}$ be locally closed complex submanifold such that the standard holomorphic conformal structure on Q^n restricts to a holomorphic conformal structure on S. Suppose S is linearly saturated. In other words, suppose for any $x \in S$ and for any projective line $\ell \subset Q^n$ which passes through x and which is tangent to S at x we must have the inclusion $(\ell; x) \subset (S, x)$ of germs of complex submanifolds. Then, there exists $\gamma \in {\rm Aut}(Q^n)$ such that $Z := \gamma(Q^m)$ contains S is an open subset.*

Proof. By Theorem 3.1 it suffices to verify that (a) the admissible pair (Q^m, Q^n) satisfies Condition (F) on the fitting of sub-VMRTs; and that (b) the restriction map $r : \Gamma(Q^n, T_{Q^n}) \to \Gamma(Q^m, T_{Q^n}|_{Q^m})$ is surjective. □

We proceed with verifying (a) on the fitting of sub-VMRTs. We have $\mathscr{C}_0(Q^d) \cong Q^{d-2} \hookrightarrow \mathbb{P}T_0(Q^d) \cong \mathbb{P}^{d-1}$ for any integer $d \geqslant 3$. Suppose $\lambda : \mathscr{C}_0(Q^m) \hookrightarrow \mathscr{C}_0(Q^n)$ is given such that $(\mathscr{C}_0(Q^m) \subset \mathbb{P}T_0(Q^m))$ is projectively equivalent to $(\lambda(\mathscr{C}_0(Q^m) \subset \mathbb{P}V_\lambda)$, where $\mathbb{P}V_\lambda$ stands for the projective linear span of the image of λ, and $\lambda(\mathscr{C}_0(Q^m)) = \lambda(\mathscr{C}_0(Q^n)) \cap \mathbb{P}V_\lambda$. Let $\mathfrak{q}$ be any choice of a quadratic bilinear form on $T_0(Q^n)$ which is a representative (unique up to a nonzero multiplicative constant) of the conformal class at 0 of the standard holomorphic conformal structure of Q^n. By the hypothesis the restriction of $\mathfrak{q}$ to V_λ is nondegenerate, and λ is induced by a linear map $\Lambda : T_0(Q^m) \xrightarrow{\cong} V_\lambda$. For a linear subspace $E \subset T_0(Q^n)$

denote by $E^\perp$ the annihilator of E with respect to $\mathfrak{q}$. When $\mathfrak{q}|_E$ is nondegenerate, $E^\perp \cap E = 0$ and we have $T_0(Q^n) = E \oplus E^\perp$, and $\mathfrak{q}|_{E^\perp}$ is also nondegenerate. Applied to the current situation with $E = T_0(Q^m)$ or V_λ, there is a linear isomorphism $\Lambda' : T_0(Q^m)^\perp \to V_\lambda^\perp$ which preserves restrictions of the quadratic form $\mathfrak{q}$. Λ and Λ' together determine a linear map $\widetilde{\Lambda} : T_0(Q^n) \xrightarrow{\cong} T_0(Q^n)$ which preserves $\mathfrak{q}$, thus $\widetilde{\Lambda}$ determines a projective linear transformation $\widetilde{\Lambda}$ extending Λ such that $\widetilde{\Lambda} : \mathscr{C}_0(Q^n) \xrightarrow{\cong} \mathscr{C}_0(Q^n)$, proving (a).

For the proof of (b) note first of all that the tangent sequence $0 \longrightarrow T_{Q^m} \longrightarrow T_{Q^n}|_{Q^m} \longrightarrow N_{Q^m|Q^n} \longrightarrow 0$ splits holomorphically. To see this, note that the projective second fundamental form $\sigma : S^2T_{Q^n} \to N_{Q^n|\mathbb{P}^{n+1}}$ is everywhere nondegenerate. For each point $x \in Q^m$, the restriction of σ_x to $S^2T_x(Q^m)$ is also nondegenerate, and $\mathcal{N}_x := T_x(Q^m)^\perp$ is complementary to $T_x(Q^m)$. Varying x on Q^m, by the holomorphicity of σ we obtain a holomorphic vector subbundle $\mathcal{N} \subset T_{Q^m}$ such that $T_{Q^n}|_{Q^m} = T_{Q^m} \oplus \mathcal{N}$, proving that the tangent sequence on $Q^m \subset Q^n$ splits holomorphically. We have $\Gamma(Q^m, T_{Q^n}|_{Q^m}) = \Gamma(Q^m, T_{Q^m}) \oplus \Gamma(Q^m, \mathcal{N})$. From the injective group homomorphism $\mathrm{Aut}_0(Q^m) \hookrightarrow \mathrm{Aut}_0(Q^n)$ it follows by passing to Lie algebras that all holomorphic tangent vector fields on Q^m lift to holomorphic vector fields on Q^n. It remains therefore to prove that all sections in $\Gamma(Q^m, \mathcal{N})$ lift to holomorphic vector fields on Q^n.

By the homogeneity of Q^n under $\mathrm{Aut}_0(Q^n)$, for any $x \in Q^m$ there exists a global holomorphic vector field η on Q^n such that $\eta(x) \notin T_x(Q^m)$. Thus, the vector subspace $A \subset \Gamma(Q^m, N_{Q^m|Q^n})$ induced by global holomorphic vector fields is nonzero. Since $Q^m \subset Q^n$ is a complete linear section by $(n-m)$ hyperplanes, we have $N_{Q^m|Q^n} \cong \mathcal{O}(1)^{n-m}$. To prove the surjectivity of $r : \Gamma(Q^n, T_{Q^n}) \to \Gamma(Q^m, T_{Q^n}|_{Q^m})$ consider first of all the case where $n = m+1$. Then, $\Gamma(Q^m, N_{Q^m|Q^{m+1}}) \cong \Gamma(Q^m, \mathcal{O}(1))$ is an $(m+2)$ dimensional vector space on which $\mathrm{Aut}_0(Q^m)$ acts irreducibly. Hence, from $A \neq 0$ it follows that $A = \Gamma(Q^m, N_{Q^m|Q^{m+1}})$.

In general, identifying $\mathcal{N}$ with $\mathcal{O}(1)^{n-m}$, it suffices that for any direct summand $\mathcal{L} \cong \mathcal{O}(1)$ in $\mathcal{N} \cong \mathcal{O}(1)^{n-m}$ we can prove that each section ξ in $\Gamma(Q^m, \mathcal{L}) \subset \Gamma(Q^m, \mathcal{N})$ lifts to a global holomorphic vector field on Q^n. We may insert a hyperquadric Q^{m+1}, $Q^m \subset Q^{m+1} \subset Q^n$. We have $T_{Q^{m+1}}|_{Q^m} = T_{Q^m} \oplus \mathcal{L}$ where $\mathcal{L} \cong \mathcal{O}(1)$ and the arguments in the last paragraph shows that any $\xi \in \Gamma(Q^m, \mathcal{L})$ lifts to some holomorphic vector field η' on Q^{m+1}. Then, η' lifts to $\eta \in \Gamma(Q^n, T_{Q^n})$ from the equivariant embedding $Q^{m+1} \hookrightarrow Q^n$. Arguing for each of the direct summand of $\mathcal{N} = \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)$ it follows that each section $\xi \in \Gamma(Q^m, \mathcal{N}) = \Gamma(Q^m, \mathcal{O}(1)) \oplus \cdots \oplus \Gamma(Q^m, \mathcal{O}(1))$ lifts to a global holomorphic vector field $\eta \in \Gamma(Q^n, T_{Q^n})$, verifying (b) and hence completing the proof of the theorem. $\square$

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