# HYBERBOLIC BELYI MAPS AND SHABAT-BLASCHKE PRODUCTS

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Abstract. We first introduce hyperbolic analogues of Belyi maps, Shabat polynomials and Grothendieck's dessins d'enfant. We then give arithmetic properties of the coefficients of the Chebyshev-Blaschke products and prove some Landen-type identities for theta functions.

## 1. Introduction

It is well known that there is a bijective correspondence between the connected compact Riemann surfaces and the nonsingular irreducible complex projective curves [12, p. 22-24]. In 1979, G. V. Belyi proved the following theorem:

**Theorem 1.1** (Belyi, [2]). *A connected compact Riemann surface $X$ is defined over the field $\overline{\mathbb{Q}}$ of algebraic numbers if and only if there exists a nonconstant holomorphic map $f : X \to \widehat{\mathbb{C}}$ with at most 3 critical values in the Riemann sphere $\widehat{\mathbb{C}}$. In such case, $f$ is isomorphic to a branched covering that is defined over $\overline{\mathbb{Q}}$.*

A branched covering $X$ of the Riemann sphere $\widehat{\mathbb{C}}$ ramified over at most three points $a, b, c$ has then been called a **Belyi map**. Inspired by Belyi's theorem, Grothendieck introduced in 1984 the theory of dessin d'enfant in his *Esquisse d'un programme* [9] in the hope of a better understanding of the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. The **dessin d'enfant** of a Belyi map has been defined to be the preimage under the Belyi map of the geodesic between $a$ and $b$.

Let $F_2 = \langle g_1, g_2 \rangle$ be the free group of rank 2 and $S_n$ be the symmetric group acting on $\{1, \ldots, n\}$. A **monodromy representation** is a group homomorphism $\rho : F_2 \to S_n$. We say a monodromy representation $\rho : F_2 \to S_n$ is **transitive** if $\rho(F_2)$ acts on $\{1, \ldots, n\}$ transitively. Two monodromy representations $\rho_1 : F_2 \to S_n$ and $\rho_2 : F_2 \to S_n$ are said to be **equivalent** if there exists a permutation $\iota$ on $\{1, \ldots, n\}$ such that $\rho_1(g_i) \circ \iota = \iota \circ \rho_2(g_i)$ for each $i = 1, 2$. It is easy to check that this indeed defines an equivalence relation on the collection of all monodromy representations, and that if two monodromy representations are equivalent and one of them is transitive, then so is the other. The category of Belyi maps, the category of dessins d'enfant and the category of transitive monodromies are all equivalent to each other. The detailed explanations can be found in [8] and [13]. In particular, one has

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**Theorem 1.2** ([8, p. 148-155])**.** *There is a bijective correspondence between the equivalence classes of Belyi maps onto the Riemann sphere and the equivalence classes of transitive monodromy representations.*

Roughly speaking, monodromies and dessins d'enfant, which are discrete objects, determine uniquely the Belyi maps which are arithmetic objects. Moreover, Grothendieck introduced the Galois action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the Belyi maps, and hence on the dessins d'enfant [9].

**Definition 1.3.** Let $F_2 := \langle g_1, g_2 \rangle$ be the rank 2 free group. Given a transitive monodromy representation $\rho : F_2 \to S_n$. Denote the numbers of cycles in $\rho(g_1)$ and $\rho(g_2)$ by $c_1$ and $c_2$ respectively. We say that $\rho$ is a **tree** if $c_1 + c_2 = n + 1$.

It is easy to check that if $\rho$ and $\rho'$ are equivalent transitive monodromy representations, then $\rho(g_i)$ and $\rho'(g_i)$ have the same number of cycles, for $i = 1, 2$. Hence if one of $\rho$ and $\rho'$ is a tree, then so is the other. A polynomial with at most two finite critical values is called a **Shabat polynomial**, which is clearly a Belyi map. The following subcorrespondence was proved by Shabat and Voevodsky [24]:

**Theorem 1.4** ([13, p. 84-85], [24])**.** *An equivalence class of transitive monodromy representations is a tree if and only if the corresponding equivalence class of Belyi maps consists of a Shabat polynomial.*

There is also a Galois action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the Shabat polynomials, and hence on the trees. It was proved by Lenstra and Schneps [23] that this action is faithful. Following Grothendieck, one hopes that the structures of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ can be revealed from the combinatorial properties of the trees or the dessins d'enfants.

We will establish a hyperbolic analogue (Theorem 3.4) of Grothendieck's theory of dessins d'enfant by replacing the Riemann sphere $\widehat{\mathbb{C}}$ with three marked points by the open unit disk $\mathbb{D}$ with two marked points. However, in this analogue the **hyperbolic Belyi maps** constructed from a given transitive monodromy are not rigid, i.e. they depend on the hyperbolic distance between the two marked points under the Poincaré metric. Moreover, we will establish a hyperbolic analogue (Theorem 5.2) of Shabat's correspondence. Indeed, the tree monodromies will correspond to finite Blaschke products with at most two critical values in $\mathbb{D}$, and such finite Blaschke products will be called **Shabat-Blaschke products**. Again, Shabat-Blaschke products constructed from a tree monodromy are not rigid and they depend on the hyperbolic distance between the two critical values in $\mathbb{D}$. We will also introduce and study the size of the hyperbolic dessin d'enfant of a Shabat-Blaschke product in Section 6.

It is natural to ask if there is a hyperbolic analogue of Belyi's theorem when one replaces the Riemann sphere $\widehat{\mathbb{C}}$ by the open unit disk $\mathbb{D}$ and $X$ is a noncompact topologically finite Riemann surface. To formulate such a result, there are two problems one needs to address first: i) What should be the algebraic object associated with the noncompact topologically finite Riemann surface $X$? ii) What should be used to replace $\overline{\mathbb{Q}}$? We do not know how to answer the first question, except some speculation given in Section 11. For the second question, it would be helpful to first study some concrete examples, in particular, the case $X = \mathbb{D}$ and the hyperbolic Belyi maps are Shabat-Blaschke products. It is known that the **Chebyshev polynomials** are examples of Shabat polynomials and their coefficients are integers. We will prove a hyperbolic analogue of this statement. **Chebyshev-Blaschke products**, which are hyperbolic analogues of Chebyshev polynomials, were studied by Ng, Tsang and Wang [18][19][20]. The Chebyshev-Blaschke products are examples of Shabat-Blaschke products. We will first recall the definition of Chebyshev-Blaschke products. Then we prove that the Chebyshev-Blaschke products are defined over

$$\mathbb{Z}\left[\sqrt{k}, \sqrt{k \circ s_n}, \frac{\omega_1 \circ s_n}{\omega_1}\right] \subseteq \overline{\mathbb{Q}(j)},$$

where $n$ is the degree of the Chebyshev-Blaschke product, $s_n$ is the scaling by $n$, $k$ and $\omega_1$ are defined in terms of Jacobi theta functions, and $j$ is the $j$-invariant. This leads us to eventually show that the Chebyshev-Blaschke products are defined over $\mathbb{Z}[[q^{1/4}]]$, the ring of power series in $q^{1/4}$ over $\mathbb{Z}$, where $q = e^{2\pi i \tau}$. Finally, we also obtain a family of Landen-type identities for theta functions as byproducts, which can degenerate to a family of trigonometric identities.

## 2. Preliminaries

Many properties of hyperbolic Belyi maps are topological, so we first recall some well-known results in topology that we are going to use.

**Lemma 2.1** (Homotopy lifting property, [10, p. 60])**.** *Suppose $X$, $Y$ and $Z$ are topological spaces, $p : X \to Y$ is a topological covering, and $f_t : Z \to Y$, $t \in [0,1]$, is a homotopy. Let $g : Z \to X$ be a continuous map such that $p \circ g = f_0$. Then there exists a unique homotopy $\widetilde{f_t} : Z \to X$ such that $\widetilde{f_0} = g$ and $p \circ \widetilde{f_t} = f_t$ for all $t \in [0,1]$.*

In particular, if $Z$ is a point, then we have the following:

**Lemma 2.2** (Path lifting property)**.** *Suppose $X$ and $Y$ are topological spaces and $p : X \to Y$ is a topological covering. Let $\gamma : [0,1] \to Y$ be a continuous path in $Y$. For any $x \in X$ with $p(x) = \gamma(0)$, there exists a unique continuous path $\widetilde{\gamma}_x : [0,1] \to X$ such that $\widetilde{\gamma}_x(0) = x$ and $p \circ \widetilde{\gamma}_x = \gamma$.*

**Lemma 2.3.** *Suppose $X$ and $Y$ are topological spaces and $p : X \to Y$ is a finite topological covering of degree $n$. Let $\gamma : [0,1] \to Y$ be a continuous path in $Y$. Let $E := p^{-1}(\gamma(0)) = \{x_1, \ldots, x_n\}$ and $F := p^{-1}(\gamma(1))$. Define $\sigma : E \to F$ by $\sigma(x_i) = \widetilde{\gamma}_{x_i}(1)$ for each $i$, where $\widetilde{\gamma}_{x_i}$ is defined in Lemma 2.2. Then $\sigma$ is bijective.*

*Proof.* Since $E$ and $F$ have the same finite cardinality, it suffices to show that $\sigma$ is injective. Let the reversed path of $\gamma$ be defined by $\gamma^{-1}(t) := \gamma(1-t)$ for all $t \in [0,1]$. For each $i$ and each $t \in [0,1]$, let $\widetilde{\gamma}_{x_i}^{-1}(t) := \widetilde{\gamma}_{x_i}(1-t)$, we have $p \circ \widetilde{\gamma}_{x_i}^{-1}(t) = p(\widetilde{\gamma}_{x_i}(1-t)) = \gamma(1-t) = \gamma^{-1}(t)$, so $p \circ \widetilde{\gamma}_{x_i}^{-1} = \gamma^{-1}$ for each $i$. Suppose $x_i, x_j \in E$ and $\sigma(x_i) = \sigma(x_j)$. Then $\widetilde{\gamma}_{x_i}^{-1}$ and $\widetilde{\gamma}_{x_j}^{-1}$ are liftings of $\gamma^{-1}$, and both start at the point $\widetilde{\gamma}_{x_i}(1) = \widetilde{\gamma}_{x_j}(1)$. By the uniqueness in Lemma 2.2, $\widetilde{\gamma}_{x_i}^{-1} = \widetilde{\gamma}_{x_j}^{-1}$. In particular, $x_i = \widetilde{\gamma}_{x_i}(0) = \widetilde{\gamma}_{x_i}^{-1}(1) = \widetilde{\gamma}_{x_j}^{-1}(1) = \widetilde{\gamma}_{x_j}(0) = x_j$. □

**Lemma 2.4.** *Suppose $X$ and $Y$ are topological spaces, $p : X \to Y$ is a topological covering, and $f_t : [0,1] \to Y$, $t \in [0,1]$, is a homotopy such that $f_t(0) = f_0(0)$ and $f_t(1) = f_0(1)$ for all $t \in [0,1]$. Suppose $g_0$ and $g_1$ are liftings of $f_0$ and $f_1$ respectively and $g_0(0) = g_1(0)$. Then $g_0$ and $g_1$ are homotopic and $g_0(1) = g_1(1)$.*

*Proof.* By Lemma 2.1, there exists a homotopy $\widetilde{f}_t : [0,1] \to X$ such that $\widetilde{f}_0 = g_0$ and $p \circ \widetilde{f}_t = f_t$ for all $t \in [0,1]$. Since the path $f_t(0)$ is a constant, by the uniqueness in Lemma 2.2, $\widetilde{f}_t(0) = \widetilde{f}_0(0)$ for all $t \in [0,1]$. Similarly, $\widetilde{f}_t(1) = \widetilde{f}_0(1)$ for all $t \in [0,1]$. Now both $\widetilde{f}_1$ and $g_1$ are liftings of $f_1$, and $\widetilde{f}_1(0) = \widetilde{f}_0(0) = g_0(0) = g_1(0)$. By Lemma 2.2, $g_1 = \widetilde{f}_1$, which is homotopic to $\widetilde{f}_0 = g_0$. Moreover, $g_0(1) = \widetilde{f}_0(1) = \widetilde{f}_1(1) = g_1(1)$. ☐

**Lemma 2.5** ([6, p. 22]). *Suppose $X$ is a connected Riemann surface, $Y$ is a Hausdorff topological space and $f : Y \to X$ is a local homeomorphism. Then there is a unique complex structure on $Y$ such that $f$ is holomorphic.*

**Lemma 2.6** ([6, p. 29]). *Suppose $X$ and $Y$ are connected Riemann surfaces, and $f : X \to Y$ is a nonconstant proper holomorphic mapping. Let $B$ be the set of all critical values of $f$, $X' := X \setminus f^{-1}(B)$ and $Y' = Y \setminus B$. Then $f : X' \to Y'$ is an unbranched holomorphic covering.*

**Lemma 2.7** ([6, p. 51]). *Suppose $X$ is a Riemann surface, $A \subset X$ is a closed discrete subset. Let $X' := X \setminus A$. If $Y'$ is a Riemann surface and $f' : Y' \to X'$ is an unbranched holomorphic covering, then $f'$ can be extended to a branched covering of $X$, i.e. there exists a Riemann surface $Y$, a nonconstant proper holomorphic mapping $f : Y \to X$, and a biholomorphism $\phi : Y \setminus f^{-1}(A) \to Y'$ such that $f|_{Y \setminus f^{-1}(A)} = f' \circ \phi$.*

**Lemma 2.8** ([6, p. 52]). *Suppose $X, Y$ and $Z$ are connected Riemann surfaces, $f : Y \to X$ and $g : Z \to X$ are nonconstant proper holomorphic mappings. Let $A \subset X$ be a closed discrete subset. Let $X' := X \setminus A, Y' := f^{-1}(X')$ and $Z' := g^{-1}(X')$. If $\phi' : Y' \to Z'$ is a biholomorphism such that $g \circ \phi' = f|_{Y'}$, then $\phi'$ can be extended to a biholomorphism $\phi : Y \to Z$ such that $g \circ \phi = f$.*

Let $n$ be a positive integer. The **Hecke congruence subgroup of level** $n$ is defined to be

$$\Gamma_0(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}) : c \equiv 0 \pmod{n} \right\}.$$

Let $\mathbb{H}$ be the open upper half plane, and $k$ be a positive even integer. A function $f : \mathbb{H} \to \mathbb{C}$ is said to be a **modular form of weight** $k$ **and of level** $n$, if all of the following conditions hold:

(1) $f$ is holomorphic;
(2) For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(n)$ and $\tau \in \mathbb{H}$,
$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau);$$
(3) $f$ is holomorphic at all the cusps [5, p. 16-17].

A function $f : \mathbb{H} \to \mathbb{C}$ satisfying condition (2) is said to be of **weight** $k$ **invariant under** $\Gamma_0(n)$.

**Lemma 2.9** ([5, p. 21, 24]). *If $f : \mathbb{H} \to \mathbb{C}$ is a modular form of weight $k$ and of level $n$, then $f(m\tau)$ is a modular form of weight $k$ and of level $mn$.*

A function $f : \mathbb{H} \to \mathbb{C} \cup \{\infty\}$ is said to be a **modular function of level** $n$ if all of the following conditions hold:

(1) $f$ is meromorphic,
(2) $f$ is invariant (i.e. weight 0 invariant) under $\Gamma_0(n)$,
(3) $f$ is meromorphic at all the cusps.

For any positive integer, let $j$ to be the $j$-invariant and $j_n(\tau) = j(n\tau)$. We have the following:

**Lemma 2.10** ([4, p. 229])**.** *Let $f : \mathbb{H} \to \mathbb{C}$ be a modular function of level $n$ whose Fourier expansion at $\infty$ has rational coefficients. Then $f \in \mathbb{Q}(j, j_n)$.*

**Lemma 2.11** ([4, p. 210])**.** *For any positive integer $n$, there exists a nonconstant polynomial $\Phi_n \in \mathbb{Q}[X, Y]$ such that $\Phi_n(j, j_n) = 0$.*

## 3. Hyperbolic Belyi maps

A domain $U \subset \mathbb{C}$ is said to be $n$**-connected** if $\widehat{\mathbb{C}} \setminus U$ has $n+1$ components. A simply connected domain is a 0-connected domain. A **doubly connected domain** is a 1-connected domain. A domain is **finitely connected** if it is $n$-connected for some $n \geq 0$. The open unit disk is denoted by $\mathbb{D}$.

Each doubly connected domain $U$ in $\mathbb{C}$ can be mapped conformally onto an annulus $A_r := \{z : r < |z| < 1\}$, for some $r \in [0, 1)$, or to $\widehat{\mathbb{C}} \setminus \{0, \infty\}$. Moreover, two annuli $A_{r_1}$ and $A_{r_2}$ are conformally equivalent if and only if $r_1 = r_2$, and none of the annuli is conformally equivalent to $\widehat{\mathbb{C}} \setminus \{0, \infty\}$, see for example [3, p. 96] and [22, p. 283]. The modulus of $U$, denoted by $M(U)$, is defined to be $\frac{1}{2\pi} \log(1/r)$ if $0 < r < 1$, and to be $+\infty$ if $r = 0$, when $U$ is conformally equivalent to $A_r$. Also, $M(U)$ is defined to be 0 if $U$ is conformally equivalent to $\widehat{\mathbb{C}} \setminus \{0, \infty\}$. For each $t \in (0, 1)$, the **Grötzsch's ring domain** $\mathfrak{G}_t$ is the doubly connected domain $\mathbb{D} - [0, t]$. The modulus of the Grötzsch's ring domain, $M(\mathfrak{G}_t)$, is a strictly decreasing continuous function in $t$ that maps onto $(0, +\infty)$ [15, p. 59-62]. Suppose $l$ is a geodesic between two distinct points $a, b \in \mathbb{D}$ equipped with the Poincaré metric [1]. Then $\mathbb{D} - l$ is conformal to a Grötzsch's ring domain. The modulus $M(\mathbb{D} - l)$ is uniquely determined by the hyperbolic length of $l$. Hence, if $l_1$ and $l_2$ are hyperbolic line segments of different lengths, then $M(\mathbb{D} - l_1) \neq M(\mathbb{D} - l_2)$.

A **hyperbolic Belyi map** is a tuple $(X, f, a, b)$, where $X$ is a connected Riemann surface, $a, b$ are two distinct points in the standard unit disk $\mathbb{D}$, and $f : X \to \mathbb{D}$ is a nonconstant proper holomorphic mapping whose critical values all lie in $\{a, b\}$. The **modulus** of a hyperbolic Belyi map $(X, f, a, b)$ is defined to be $M(\mathbb{D} - l)$, where $l$ is the geodesic between $a$ and $b$ under the Poincaré metric. Two hyperbolic Belyi maps $(X_1, f_1, a_1, b_1)$ and $(X_2, f_2, a_2, b_2)$ are said to be **equivalent** if there exists $\phi \in Aut(\mathbb{D})$ and biholomorphism $\varphi : X_1 \to X_2$ such that $\phi(a_1) = a_2$, $\phi(b_1) = b_2$, and $f_2 \circ \varphi = \phi \circ f_1$. It is easy to check that this indeed defines an equivalence relation on the collection of all hyperbolic Belyi maps.

Given a hyperbolic Belyi map $(X, f, a, b)$, we can associate a transitive monodromy representation to it in the following way:

Let $y \in \mathbb{D} \setminus \{a, b\}$. Let $\gamma_1, \gamma_2 : [0,1] \to \mathbb{D} \setminus \{a, b\}$ be some continuous paths both starting and ending at $y$ such that $\gamma_1$ is homotopic to an anticlockwise small circle around $a$ that separates the two points $a$ and $b$, and $\gamma_2$ is homotopic to an anticlockwise small circle around $b$ that separates the two points $a$ and $b$. Let $n := \deg f$ and $E := f^{-1}(y) = \{x_1, \ldots, x_n\}$. By Lemma 2.6 and the path lifting property (Lemma 2.2), $\gamma_1$ is lifted to $n$ paths $\widetilde{\gamma}_{1x_1}, \ldots, \widetilde{\gamma}_{1x_n}$. Similarly for the path $\gamma_2$. Now define $\sigma_1 : E \to E$ by $\sigma_1(x_i) = \widetilde{\gamma}_{1x_i}(1)$ for each $i = 1, \ldots, n$, and $\sigma_2 : E \to E$ by $\sigma_2(x_i) = \widetilde{\gamma}_{2x_i}(1)$ for each $i = 1, \ldots, n$. Then by Lemma 2.3, $\sigma_1$ and $\sigma_2$ are bijections on $E$. Let $\delta : E \to \{1, \ldots, n\}$ be a bijection. We define the group homomorphism $\rho : F_2 \to S_n$ by $g_i \mapsto \delta \circ \sigma_i \circ \delta^{-1}$, $i = 1, 2$. It is easy to show that this construction of $\rho$ is up to equivalence independent of the choice of the bijection $\delta$. By Lemma 2.4, $\sigma_1, \sigma_2$ and hence $\rho$ are independent of the choices of $\gamma_1$ and $\gamma_2$. Suppose $x$ is another point in $\mathbb{D} \setminus \{a, b\}$, and suppose $\beta_1, \beta_2$ are homotopic to $\gamma_1, \gamma_2$ respectively and they start and end at $x$. Let $D := f^{-1}(x)$. Define the bijections $\lambda_1, \lambda_2 : D \to D$ from $\beta_1, \beta_2$ in the same way as we define $\sigma_1, \sigma_2$ from $\gamma_1, \gamma_2$. There exists a continuous path $\alpha : [0,1] \to \mathbb{D} \setminus \{a, b\}$ such that $\alpha(0) = x$ and $\alpha(1) = y$. Define $\eta : D \to E$ by sending a point $p$ in $D$ to the endpoint of the lifting of $\alpha$ starting at $p$. By Lemma 2.3, $\eta$ is a bijection. By Lemma 2.4, we have $\lambda_i = \eta^{-1} \circ \sigma_i \circ \eta$ for $i = 1, 2$. Therefore, the construction of $\rho$ is independent of the choice of the base point $y$.

Next, we want to prove that $\rho$ is transitive. Suppose $x_i, x_j \in E$. Since $X$ is a connected Riemann surface, $X$ is path connected. Since $X$ is a path connected Riemann surface and $f^{-1}(\{a, b\})$ is a finite set, $X \setminus f^{-1}(\{a, b\})$ is still path connected. So there exists a path $\beta : [0,1] \to X \setminus f^{-1}(\{a, b\})$ such that $\beta(0) = x_i$ and $\beta(1) = x_j$. Then $f \circ \beta$ is a closed path that starts and ends at $y$. Since the fundamental group $\pi_1(\mathbb{D} \setminus \{a, b\}, y)$ is generated by $\gamma_1$ and $\gamma_2$, we have that $f \circ \beta$ is homotopic (with endpoint $y$ fixed) to $\gamma_{m_1} \cdots \gamma_{m_k}$, where $m_1, \ldots, m_k = 1, 2$. Define

$$p_1 := \widetilde{\gamma_{m_1}}_{x_i}(1),\ p_2 := \widetilde{\gamma_{m_2}}_{p_1}(1),\ \ldots\ ,\ p_k := \widetilde{\gamma_{m_k}}_{p_{k-1}}(1).$$

Then

$$\begin{aligned} f \circ (\widetilde{\gamma_{m_1}}_{x_i} \widetilde{\gamma_{m_2}}_{p_1} \cdots \widetilde{\gamma_{m_k}}_{p_{k-1}}) &= (f \circ \widetilde{\gamma_{m_1}}_{x_i})(f \circ \widetilde{\gamma_{m_2}}_{p_1}) \cdots (f \circ \widetilde{\gamma_{m_k}}_{p_{k-1}}) \\ &= \gamma_{m_1} \gamma_{m_2} \cdots \gamma_{m_k}. \end{aligned}$$

By Lemma 2.4, $\widetilde{\gamma_{m_1}}_{x_i} \widetilde{\gamma_{m_2}}_{p_1} \cdots \widetilde{\gamma_{m_k}}_{p_{k-1}}$ ends at $x_j$. Define $\sigma := \sigma_{m_k} \circ \cdots \circ \sigma_{m_1}$. Then $\sigma(x_i) = p_k = x_j$, so $\rho$ is transitive.

**Lemma 3.1.** *Suppose $(X, f, a, b)$ and $(X', f', a', b')$ are two equivalent hyperbolic Belyi maps. Then the transitive monodromy representations $\rho$ and $\rho'$ associated respectively to these two hyperbolic Belyi maps are equivalent.*

*Proof.* Let $y, \gamma_1, \gamma_2, E, \sigma_1, \sigma_2$ be as above. Since $(X, f)$ and $(X', f')$ are equivalent, there exists a biholomorphism $\varphi : X \to X'$ and $\phi \in Aut(\mathbb{D})$ such that $\phi(a) = a'$, $\phi(b) = b'$ and $f' \circ \varphi = \phi \circ f$. Let $E' := f'^{-1}(\phi(y))$. Since winding numbers are invariant under biholomorphism, the closed paths $\phi \circ \gamma_1$ and $\phi \circ \gamma_2$ are representatives of generators of $\pi_1(\mathbb{D} \setminus \{a', b'\})$. Let $\sigma'_1, \sigma'_2$ be the bijections on $E'$ obtained respectively from $\phi \circ \gamma_1$ and $\phi \circ \gamma_2$. It is easy to check that $\varphi(E) = E'$, so $\varphi|_E : E \to E'$ is a bijection. Let $x_i \in E$. Since $f' \circ \varphi \circ \widetilde{\gamma_1}_{x_i} = \phi \circ \gamma_1$, we know that both $\widetilde{\phi \circ \gamma_1}_{\varphi(x_i)}$ and $\varphi \circ \widetilde{\gamma_1}_{x_i}$ are liftings of $\phi \circ \gamma_1$ by $f'$, and both paths start from $\varphi(x_i)$. By Lemma 2.2, $\widetilde{\phi \circ \gamma_1}_{\varphi(x_i)}(1) = \varphi \circ \widetilde{\gamma_1}_{x_i}(1)$, so $\sigma_1' \circ \varphi|_E(x_i) = \varphi|_E \circ \sigma_1(x_i)$. This is true for each $x_i \in E$, so $\sigma_1' \circ \varphi|_E = \varphi|_E \circ \sigma_1$. Similarly, $\sigma_2' \circ \varphi|_E = \varphi|_E \circ \sigma_2$. □

We also have the converse of the above lemma:

**Lemma 3.2.** *If $(X, f, a, b)$ and $(X', f', a', b')$ are hyperbolic Belyi maps of the same modulus whose associated transitive monodromy representations $\rho$ and $\rho'$ are equivalent, then $(X, f, a, b)$ and $(X', f', a', b')$ are equivalent.*

*Proof.* Let $y, E, \gamma_1, \gamma_2, \sigma_1, \sigma_2$ be the intermediate notations previously defined corresponding to $(X, f, a, b)$. Let $y', E', \gamma_1', \gamma_2', \sigma_1', \sigma_2'$ be that corresponding to $(X', f', a', b')$. Then there exists a bijection $\varphi : E \to E'$ such that $\sigma_i' \circ \varphi = \varphi \circ \sigma_i$, for $i = 1, 2$. Let $x_i \in E$ and $(f|_{X \setminus f^{-1}\{a,b\}})_*$ be the induced group homomorphism of $f|_{X \setminus f^{-1}\{a,b\}}$ pointed at $x_i$. We know that

$$(f|_{X \setminus f^{-1}\{a,b\}})_*(\pi_1(X \setminus f^{-1}\{a, b\}, x_i)) = \{\gamma \in \pi_1(\mathbb{D} \setminus \{a, b\}, y) | \widetilde{\gamma}_{x_i}(1) = x_i\},$$

which is in turn equal to the preimage under the group homomorphism $\pi_1(\mathbb{D} \setminus \{a, b\}, y) \to Bij(E)$ of the subgroup of bijections on $E$ fixing $x_i$. Since $\sigma_i' \circ \varphi = \varphi \circ \sigma_i$, for $i = 1, 2$, this group is then isomorphic (via $\gamma_i \mapsto \gamma_i'$) to the preimage under the group homomorphism $\pi_1(\mathbb{D} \setminus \{a', b'\}, y') \to Bij(E')$ of the subgroup of bijections on $E'$ fixing $\varphi(x_i)$, which is equal to

$$\begin{aligned} & \{\gamma \in \pi_1(\mathbb{D} \setminus \{a', b'\}, y') | \widetilde{\gamma}_{\varphi(x_i)}(1) = \varphi(x_i)\} \\ = \;& (f'|_{X' \setminus f'^{-1}\{a',b'\}})_*(\pi_1(X' \setminus f'^{-1}\{a', b'\}, \varphi(x_i))). \end{aligned}$$

Since the two hyperbolic Belyi maps are of the same modulus, there exists $\phi \in Aut(\mathbb{D})$ such that $\phi(a) = a'$ and $\phi(b) = b'$. By Proposition 1.37 in [10, p. 67], the two coverings

$$X \setminus f^{-1}\{a, b\} \xrightarrow{f} \mathbb{D} \setminus \{a, b\}$$

and

$$X' \setminus f'^{-1}\{a', b'\} \xrightarrow{f'} \mathbb{D} \setminus \{a', b'\} \xrightarrow{\phi^{-1}} \mathbb{D} \setminus \{a, b\}$$

are isomorphic as topological coverings. By Lemma 2.5, these two holomorphic coverings are isomorphic. By Lemma 2.8, the two hyperbolic Belyi maps are equivalent. □

We also have the following variant of the Riemann existence theorem:

**Lemma 3.3.** *Given a $\lambda \in (0, +\infty)$ and a transitive monodromy representation $\rho$, there exists a hyperbolic Belyi map $(X, f, a, b)$ of modulus $\lambda$ whose associated transitive monodromy representation is equivalent to $\rho$.*

*Proof.* Suppose the codomain of $\rho$ is $S_n$ for some positive integer $n$. Let $\Delta_1, \ldots, \Delta_n$ be copies of the open unit disk slitted along $[0, 1)$. Let $t \in (0, 1)$ such that the modulus of $\mathbb{D} \setminus [0, t]$ is $\lambda$. Assume along the slit, each slitted disk $\Delta_i$ has two disjoint copies of $(0, t)$ attached, named $\alpha_i^u$ and $\alpha_i^l$, and has two disjoint copies of $(t, 1)$ attached, named $\beta_i^u$ and $\beta_i^l$. We construct a connected topological space $Z$ from $\sqcup_i \Delta_i$ by gluing the edges $\alpha_i^l$ and $\alpha_j^u$ together whenever $\rho(g_1)(i) = j$, and gluing the edges $\beta_i^l$ and $\beta_j^u$ together whenever $\rho(g_2)(\rho(g_1)(i)) = j$. We define $g : Z \to \mathbb{D} \setminus \{0, t\}$ to be the canonical projection onto $\mathbb{D} \setminus \{0, t\}$, which is a topological covering. By Lemma 2.5, there exists a complex structure on $Z$ such that $g : Z \to \mathbb{D} \setminus \{0, t\}$ is a holomorphic covering. By Lemma 2.7, $g$ extends to a branched covering $f : X \to \mathbb{D}$ whose critical values all lie in the set $\{0, t\}$. Now $f$ is the desired hyperbolic Belyi map. $\square$

By Lemma 3.1, Lemma 3.2 and Lemma 3.3, we have the following:

**Theorem 3.4.** *For each $\lambda \in (0, +\infty)$, there is a bijective correspondence between the equivalence classes of hyperbolic Belyi maps of modulus $\lambda$ and the equivalence classes of transitive monodromy representations.*

This theorem is the hyperbolic analogue of Theorem 1.2. The main difference between the two versions is that in the hyperbolic analogue, we have the extra parameter $\lambda$ that depends on the hyperbolic distance between the two critical values in $\mathbb{D}$.

## 4. Difference in Euler characteristics

A Riemann surface is said to be **topologically finite** if it is homeomorphic to a closed surface with at most finitely many closed disks and points removed.

**Lemma 4.1** (Riemann-Hurwitz: topologically finite version, [20][25])**.** *Suppose $f : X \to Y$ is a nonconstant proper holomorphic mapping between Riemann surfaces. If $Y$ is topologically finite and $f$ has finitely many critical values, then $X$ is also topologically finite and the following Riemann-Hurwitz formula holds,*

$$\deg R_f = \deg f \cdot \chi(Y) - \chi(X),$$

*where $R_f$ is the ramification divisor of $f$, hence $\deg R_f$ is the sum of the order of the critical points of $f$, and $\chi(X)$ and $\chi(Y)$ are Euler characteristic of $X$ and $Y$ respectively.*

By Lemma 4.1, given any hyperbolic Belyi map $(X, f, a, b)$, we know that $X$ is topologically finite. Moreover, since $\mathbb{D}$ is noncompact, and $f$ is continuous and surjective, we know $X$ is also noncompact, so $X$ is homeomorphic to a closed surface with at least a disk or a point removed.

**Lemma 4.2.** *Let $(X, f, a, b)$ be a hyperbolic Belyi map and $\sigma_1, \sigma_2$ be defined as in Section 3. Then the number of cycles of $\sigma_1$ equals the cardinality of $f^{-1}(a)$ and the number of cycles of $\sigma_2$ equals the cardinality of $f^{-1}(b)$.*

*Proof.* By a theorem on the local behavior of a holomorphic mapping [6, p. 10], we can choose a small circle $\beta$ around $a$ such that it is lifted by $f$ to $|f^{-1}(a)|$ cycles of paths, each cycle goes around a preimage of $a$ and the number of paths the cycle contains is equal to the multiplicity of that preimage. Next, we join the base point $y$ to the circle $\beta$ by a path $\alpha$ in $\mathbb{D} \setminus \{a, b\}$. Then the closed path $\alpha\beta\alpha^{-1}$ is lifted to cycles of paths having the same property. Since $\alpha\beta\alpha^{-1}$ is homotopic to $\gamma_1$, by Lemma 2.4, the lifting of $\gamma_1$ is again cycles of paths having the same property. Hence, the number of cycles of $\sigma_1$ equals $|f^{-1}(a)|$. Similarly, the number of cycles of $\sigma_2$ equals $|f^{-1}(b)|$. $\square$

Fix a $\lambda \in (0, +\infty)$. Given a transitive monodromy representation $\rho : F_2 := \langle g_1, g_2 \rangle \to S_n$. Let $(X, f, a, b)$ be the hyperbolic Belyi map of modulus $\lambda$ associated to $\rho$. Let $(\overline{X}, \overline{f}, c, d, e)$ be the Belyi map onto the Riemann sphere associated to $\rho$. We have that $\deg f = n = \deg \overline{f}$. By Lemma 4.2 and its analogue for Belyi maps onto the Riemann sphere, $|\overline{f}^{-1}(c)| = c_1 = |f^{-1}(a)|$, similarly $|\overline{f}^{-1}(d)| = c_2 = |f^{-1}(b)|$. By the Riemann-Hurwitz formula in Lemma 4.1,

$$\begin{aligned}
\chi(\overline{X}) - \chi(X) &= \deg \overline{f} \cdot \chi(\overline{\mathbb{C}}) - \deg R_{\overline{f}} - \deg f \cdot \chi(\mathbb{D}) + \deg R_f \\
&= 2 \deg \overline{f} - \deg R_{\overline{f}} - \deg \overline{f} + \deg R_f \\
&= \deg \overline{f} - (3 \deg \overline{f} - |\overline{f}^{-1}(c)| - |\overline{f}^{-1}(d)| - |\overline{f}^{-1}(e)|) \\
&\quad + (2 \deg \overline{f} - |f^{-1}(a)| - |f^{-1}(b)|) \\
&= |\overline{f}^{-1}(e)|.
\end{aligned}$$

Therefore, $\chi(X)$ is $\chi(\overline{X})$ minus the number of cycles in $\rho(g_1)^{-1} \circ \rho(g_2)^{-1}$, as $\gamma_c\gamma_d\gamma_e = 1$, where $\gamma_c, \gamma_d, \gamma_e$ are closed continuous paths in $\widehat{\mathbb{C}}$ with the same base point in $\widehat{\mathbb{C}} \setminus \{c, d, e\}$ that goes around $c$, $d$ and $e$ respectively, just like how $\gamma_1, \gamma_2$ goes around $a$ and $b$ in Section 3.

**Remark 4.3.** Alternatively, by comparing $X$ and $\overline{X}$ constructed using the cutting and pasting surgery in the Riemann Existence Theorem, one can see that $X$ differs from $\overline{X}$ by missing $c_3$ number of closed disks, where $c_3$ is the number of cycles in $\rho(g_1)^{-1} \circ \rho(g_2)^{-1}$. Since taking away a disk from a surface will decrease its Euler characteristic by 1, we obtain the same formula as before.

When $\overline{X} = \widehat{\mathbb{C}}$, $\overline{f}$ is a nonconstant Shabat polynomial, and $e = \infty$, we have $|\overline{f}^{-1}(e)| = 1$. Then $\chi(X) = \chi(\overline{X}) - |\overline{f}^{-1}(e)| = 2 - 1 = 1$. By Liouville's theorem, $X$ cannot be biholomorphic to $\mathbb{C}$, so $X$ is biholomorphic to $\mathbb{D}$. Therefore, in the next section we will study the hyperbolic Belyi maps $(X, f, a, b)$ when $X = \mathbb{D}$.

## 5. Shabat-Blaschke products

**Definition 5.1.** A **Shabat-Blaschke product** is a triple $(g, a, b)$, where $g : \mathbb{D} \to \mathbb{D}$ is a finite Blaschke product whose critical values all lie in $\{a, b\}$. The **modulus** of a Shabat-Blaschke product is defined to be $M(\mathbb{D} - l)$, where $l$ is the geodesic between $a$ and $b$. We say that two Shabat-Blaschke products $(g_1, a_1, b_1)$ and $(g_2, a_2, b_2)$ are **equivalent** if there exists $\phi, \varphi \in Aut(\mathbb{D})$ such that $\phi(a_1) = b_1$, $\phi(b_1) = b_2$, and $g_2 \circ \varphi = \phi \circ g_1$.

The following result gives a characterization of Shabat-Blaschke products:

**Theorem 5.2.** *An equivalence class of hyperbolic Belyi maps consists of a hyperbolic Belyi map of the form $(\mathbb{D}, g, a, b)$, where $(g, a, b)$ is a Shabat-Blaschke product, if and only if the corresponding equivalence class of transitive monodromy representations is a tree.*

*Proof.* Let $(X, f, a, b)$ be a hyperbolic Belyi map and let $\rho : F_2 \to S_n$ be its corresponding transitive monodromy representation. By the Riemann-Hurwitz formula in Lemma 4.1 and by Lemma 4.2,

$$\begin{aligned}
\chi(X) &= \deg f \cdot \chi(\mathbb{D}) - \deg R_f \\
&= n - (2n - |f^{-1}(a)| - |f^{-1}(b)|) \\
&= -n + c_1 + c_2.
\end{aligned}$$

If $\rho$ is a tree, then $\chi(X) = 1$, so $X$ is homeomorphic to the open unit disk $\mathbb{D}$. By Liouville's theorem, $X$ cannot be biholomorphic to $\mathbb{C}$. By the uniformization theorem,

$X$ is thus biholomorphic to $\mathbb{D}$. Hence $(X, f, a, b)$ is equivalent to a hyperbolic Belyi map $(\mathbb{D}, g, a, b)$. Since $g : \mathbb{D} \to \mathbb{D}$ is a nonconstant proper holomorphic mapping, $g$ is a finite Blaschke product by a theorem of Fatou [21, p. 212]. Hence $(g, a, b)$ is a Shabat-Blaschke product. Conversely, if $(X, f, a, b)$ is equivalent to $(\mathbb{D}, g, a, b)$, where $(g, a, b)$ is a Shabat-Blaschke product, then $\chi(X) = 1$, so $c_1 + c_2 = n + 1$, i.e. its corresponding transitive monodromy representation is a tree. $\square$

Via the transitive monodromy representations, we obtain for each fixed $\lambda \in (0, +\infty)$, a bijective correspondence between the equivalence classes of Belyi maps onto the Riemann sphere and the equivalence classes of hyperbolic Belyi maps of modulus $\lambda$. By Belyi's theorem, in each equivalent class of Belyi map onto the Riemann sphere, there is a representative element $(X, f, a, b, c)$ such that $X$ and $f$ are defined over $\overline{\mathbb{Q}}$. There is a group action by the absolute Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the set of equivalence classes of Belyi maps onto the Riemann sphere, which is defined by acting on the algebraic coefficients of $X$ and $f$ [13, p. 115-117][8, p. 250][9]. Thus for each fixed $\lambda$, there is an induced Galois action on the set of equivalence classes of hyperbolic Belyi maps of modulus $\lambda$. Similarly, there is a Galois action on the set of equivalence classes of Shabat polynomials and we have the following:

**Theorem 5.3.** *For each fixed $\lambda \in (0, +\infty)$, there is a bijective correspondence between the eqivalence classes of Shabat polynomials and the equivalence classes of Shabat-Blaschke products of modulus $\lambda$, we also have an induced Galois action on the set of equivalence classes of Shabat-Blaschke products of modulus $\lambda$.*

*Proof.* Theorem 1.4 says that there is a bijective correspondence between the equivalence classes of tree monodromies and the equivalence classes of Shabat polynomials. Theorem 5.2 implies that for each fixed $\lambda \in (0, +\infty)$, there is a bijective correspondence between the equivalence classes of tree monodromies and the equivalence classes of Shabat-Blaschke products of modulus $\lambda$. The first claim follows by compositing the two correspondences. The Galois action on the set of equivalence classes of Shabat-Blaschke products of modulus $\lambda$ is induced from that on the set of equivalence classes of Shabat polynomials. $\square$

## 6. Size of hyperbolic dessins d'enfant in $\mathbb{D}$

The **dessin d'enfant** of a Belyi map $(X, f, a, b, c)$ onto the Riemann sphere is defined to be the preimage of a geodesic in $\widehat{\mathbb{C}}$ joining $a$ and $b$ [13, p. 80][9]. We can define the **hyperbolic dessin d'enfant** of a hyperbolic Belyi map $(X, f, a, b)$ similarly to be the preimage of the geodesic $l$ joining $a$ and $b$ under the Poincaré metric. We regard the preimages of $a$ and $b$ as the white and black vertices respectively of the dessin, while the $n$ liftings of the geodesic as the edges of the dessin, where $n$ is the degree of $f$. If the associated transitive monodromy representation of a (hyperbolic) Belyi map is a tree, then by Lemma 4.2, $|f^{-1}(a)|+|f^{-1}(b)| = c_1+c_2 = n+1$, so its associated dessin has $n+1$ vertices and $n$ edges, and such dessin is a connected bipartite tree embedded in $X$, where $X = \widehat{\mathbb{C}}$ if the Belyi map is onto $\widehat{\mathbb{C}}$, and $X = \mathbb{D}$ if the Belyi map is hyperbolic.

Let $(B, a, b)$ be a Shabat-Blaschke product and $l$ be the geodesic joining $a$ and $b$ under the Poincaré metric of $\mathbb{D}$. Then $\mathbb{D} \setminus B^{-1}(l)$ is doubly connected and the modulus of it will be called the **size** of the hyperbolic dessin d'enfant $B^{-1}(l)$.

Let $A_r := \{z : r < |z| < 1\}$. We have the following lemma:

**Lemma 6.1.** *If $f : U \to A_r$ is a proper unbranched covering of degree $n$, then $U$ is biholomorphic to the doubly-connected domain $A_{r^{1/n}}$.*

*Proof.* The fundamental group $\pi_1(A_r)$ is $\mathbb{Z}$. There is a bijective correspondence between the subgroups of $\pi_1(A_r)$ and the power maps $g_m : A_{r^{1/m}} \to A_r$, $m \geq 1$, defined by $g_m(z) = z^m$. This implies that the power maps are all the proper unbranched coverings of $A_r$ up to isomorphism. Since $f$ is of degree $n$, we know that $f$ is isomorphic to the power map $g_n$, so the domain $U$ is biholomorphic to $A_{r^{1/n}}$. □

**Theorem 6.2.** *Suppose $(B, a, b)$ is a Shabat-Blaschke product of degree $n$ and of modulus $\lambda$. Let $l$ be the geodesic between $a$ and $b$ under the Poincaré metric of $\mathbb{D}$. Then the modulus of $\mathbb{D} - B^{-1}(l)$ is $\lambda/n$.*

*Proof.* There exists a biholomorphism $\phi$ that maps $\mathbb{D} - l$ onto an annulus $A_r$ for some $r \in (0, 1)$. The map $\phi \circ B : \mathbb{D} - B^{-1}(l) \to A_r$ is a proper unbranched covering. By Lemma 6.1, we have

$$M(\mathbb{D} - B^{-1}(l)) = M(A_{r^{1/n}}) = \frac{1}{2\pi} \log(1/r^{1/n}) = \frac{1}{n} M(A_r) = \frac{1}{n} M(\mathbb{D} - l) = \lambda/n.$$

□

Loosely speaking, the above theorem showed quantitatively how the size of a hyperbolic dessin d'enfant of a Shabat-Blaschke product depends on the degree and the modulus of the Shabat-Blaschke product. Note however that there is no such concept of the size of a dessin d'enfant of a Shabat polynomial, since the Riemann sphere with a connected tree taken away is a simply-connected domain.

## 7. Jacobi elliptic functions

For any $\tau$ in the open upper-half plane $\mathbb{H}$, and $q = e^{2\pi i \tau}$, the **Jacobi theta functions** are defined as follows:

$$\vartheta_1(v, \tau) = \sum_{n=-\infty}^{\infty} i^{2n-1} q^{(n+1/2)^2} e^{(2n+1)iv},$$

$$\vartheta_2(v, \tau) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{(2n+1)iv},$$

$$\vartheta_3(v, \tau) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2niv},$$

$$\vartheta_0(v, \tau) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2niv}.$$

We also define

$$\omega_1(\tau) = \vartheta_3^2(0,\tau), \quad k(\tau) = \frac{\vartheta_2^2(0,\tau)}{\vartheta_3^2(0,\tau)}, \quad \sqrt{k(\tau)} = \frac{\vartheta_2(0,\tau)}{\vartheta_3(0,\tau)}.$$

The theta functions can be expressed in terms of each other, for example:

$$\vartheta_0(v,\tau) = \vartheta_3(v+1/2,\tau). \tag{1}$$

The modular transformations [26, p. 475][14, p. 17] for the theta functions $\vartheta_3$ and $\vartheta_2$ are:

$$\vartheta_3(v,\tau+1) = \vartheta_3(v,\tau), \tag{2}$$

$$\vartheta_2(v,\tau+1) = \vartheta_2(v,\tau), \tag{3}$$

$$\vartheta_3(v,-1/\tau) = (-i\tau/2)^{1/2} e^{\frac{i\tau v^2}{2\pi}} \vartheta_3(\tau v/2, \tau/4), \tag{4}$$

and

$$\vartheta_2(v,-1/\tau) = (-i\tau/2)^{1/2} e^{\frac{i\tau v^2}{2\pi}} \vartheta_0(\tau v/2, \tau/4). \tag{5}$$

We also have

$$\vartheta_3(0,\tau-1/2) = \vartheta_0(0,\tau) \tag{6}$$

and

$$\vartheta_3^4(0,\tau) = \vartheta_2^4(0,\tau) + \vartheta_0^4(0,\tau). \tag{7}$$

The **Jacobi elliptic functions** are defined as follows:

$$sn(u,\tau) = \frac{\vartheta_3(0,\tau)}{\vartheta_2(0,\tau)} \cdot \frac{\vartheta_1(u/\omega_1(\tau),\tau)}{\vartheta_0(u/\omega_1(\tau),\tau)},$$

$$cn(u,\tau) = \frac{\vartheta_0(0,\tau)}{\vartheta_2(0,\tau)} \cdot \frac{\vartheta_2(u/\omega_1(\tau),\tau)}{\vartheta_0(u/\omega_1(\tau),\tau)},$$

$$dn(u,\tau) = \frac{\vartheta_0(0,\tau)}{\vartheta_3(0,\tau)} \cdot \frac{\vartheta_3(u/\omega_1(\tau),\tau)}{\vartheta_0(u/\omega_1(\tau),\tau)},$$

$$cd(u,\tau) = \frac{cn(u,\tau)}{dn(u,\tau)}.$$

It is known that [14, p. 26]

$$\lim_{\tau\to+i\infty} cd(u,\tau) = \cos(u). \tag{8}$$

## 8. Chebyshev-Blaschke products

Let $n$ be a positive integer, $\tau \in \mathbb{R}_{>0}i$, and $\mathbb{D}$ be the open unit disk. The **Chebyshev-Blaschke product** $f_{n,\tau} : \mathbb{D} \to \mathbb{D}$ introduced in [20][18][19] is defined by

$$f_{n,\tau}(z) = \sqrt{k(n\tau)}cd(n\omega_1(n\tau)u, n\tau), \tag{9}$$

where

$$z = \sqrt{k(\tau)}cd(\omega_1(\tau)u,\tau).$$

The Chebyshev-Blaschke products $f_{n,\tau}, \tau \in \mathbb{R}_{>0}i$, are hyperbolic analogues of the **Chebyshev polynomial** $T_n : \mathbb{C} \to \mathbb{C}$ defined by

$$T_n(z) = cos(nu), \quad z = cos(u).$$

Chebyshev polynomials are examples of Shabat polynomials since they have exactly two critical values in $\mathbb{C}$.

If we regard $f_{n,\tau}$ as a rational function on $\widehat{\mathbb{C}}$ and let $\mathcal{T}_{n,\tau} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be defined by

$$\mathcal{T}_{n,\tau}(z) = \frac{f_{n,\tau}(\sqrt{k(\tau)}z)}{\sqrt{k(n\tau)}},$$

then $\mathcal{T}_{n,\tau}$ is referred to as an **elliptic rational function**. The elliptic rational functions have applications in filter design in engineering [16, Chapter 12]. It was shown in [19] that

$$\lim_{\tau\to+i\infty} \mathcal{T}_{n,\tau}(z) = T_n(z). \tag{10}$$

The zeros of this Chebyshev-Blaschke product is computed in [18][19], so it can also be written as a rational function,

$$f_{n,\tau}(z) = z^{(1-(-1)^n)/2} \prod_{i=1}^{\lfloor n/2 \rfloor} \frac{z^2 - b_i}{1 - b_i z^2}, \tag{11}$$

where

$$b_i = \frac{\vartheta_2^2((2i-1)\pi/2n, \tau)}{\vartheta_3^2((2i-1)\pi/2n, \tau)}, \quad 1 \le i \le \lfloor n/2 \rfloor. \tag{12}$$

The Chebyshev-Blaschke product $f_{n,\tau}$ has exactly two critical values in $\mathbb{D}$, $\sqrt{k(n\tau)}$ and $-\sqrt{k(n\tau)}$ [19]. Therefore, the Chebyshev-Blaschke products $f_{n,\tau}$, $n \ge 1$, $\tau \in \mathbb{R}_{>0}i$, are examples of Shabat-Blaschke products, whose modulus is $\lambda_{n,\tau} = n\pi\tau/(4i)$ [20]. For all $\tau$, the monodromy of the Chebyshev-Blaschke product $f_{n,\tau}$ is the same as that of the Chebyshev polynomial $T_n$ [20][19]. The hyperbolic dessins d'enfant of the Chebyshev-Blaschke products are chains in $\mathbb{D}$.

## 9. Rings of definition of Chebyshev-Blaschke products

For each $j = 1, \dots, \lfloor n/2 \rfloor$, let

$$S_{n,j} = \sum_{1 \le i_1 < \cdots < i_j \le \lfloor n/2 \rfloor} b_{i_1} \dots b_{i_j}, \tag{13}$$

where $b_i$ is given by (12). The $S_{n,j}$ are $\pm$ of the coefficients of $f_{n,\tau}$ when expanded, as

$$f_{n,\tau}(z) = z^{(1-(-1)^n)/2} \frac{z^{2\lfloor n/2 \rfloor} + \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j S_{n,j}(\tau) z^{2\lfloor n/2 \rfloor - 2j}}{1 + \sum_{j=1}^{\lfloor n/2 \rfloor} (-1)^j S_{n,j}(\tau) z^{2j}}. \tag{14}$$

Since the $b_i$'s, as functions in $\tau$, are actually defined and meromorphic on the open upper half plane $\mathbb{H}$, so are $S_{n,j}$ and $f_{n,\tau}^{(k)}(0)$.

**Lemma 9.1.** *For each positive integer $n \ge 2$, and $j = 1, \dots, \lfloor n/2 \rfloor$,*

$$S_{n,j} \in \mathbb{Q}(f_{n,\tau}(0), f'_{n,\tau}(0), f''_{n,\tau}(0), \dots),$$

*a subfield of the field $\mathcal{M}(\mathbb{H})$ of meromorphic functions on the upper half plane.*

*Proof.* From (14) we have for each $\tau \in \mathbb{R}_{>0}i$,

$$f_{n,\tau}(z)\left(1+\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j S_{n,j} z^{2j}\right) = z^{(1-(-1)^n)/2}\left(z^{2\lfloor n/2\rfloor}+\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j S_{n,j} z^{2\lfloor n/2\rfloor-2j}\right).$$

By expanding $f_{n,\tau}$ in power series at 0 and comparing coefficients on both sides, and using the identity theorem, the tuple $(S_{n,1},\dots,S_{n,\lfloor n/2\rfloor})$ satisfies a system $(*)$ of countably many linear equations in $\lfloor n/2\rfloor$ $\mathcal{M}(\mathbb{H})$-variables whose coefficients are in

$$\mathbb{Q}(f_{n,\tau}(0), f'_{n,\tau}(0), f''_{n,\tau}(0),\dots).$$

On the other hand, suppose $(T_1(\tau),\dots,T_{\lfloor n/2\rfloor}(\tau)) \in \mathcal{M}(\mathbb{H})^{\lfloor n/2\rfloor}$ is a solution to $(*)$. Since the poles of $T_j$ are discrete, there exists $\tau \in \mathbb{R}_{>0}i$ and a small open ball $\mathcal{N}$ around $\tau$ such that $T_j$ is analytic on $\mathcal{N}$ for all $j$. Then for each $\tau \in \mathbb{R}_{>0}i \cap \mathcal{N}$, we have

$$f_{n,\tau}(z)\left(1+\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j T_j z^{2j}\right) = z^{(1-(-1)^n)/2}\left(z^{2\lfloor n/2\rfloor}+\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j T_j z^{2\lfloor n/2\rfloor-2j}\right)$$

on $\mathbb{D}$, and hence on $\mathbb{P}^1$ by the identity theorem. Therefore,

$$\begin{aligned} & z^{(1-(-1)^n)/2}\frac{z^{2\lfloor n/2\rfloor}+\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j S_{n,j} z^{2\lfloor n/2\rfloor-2j}}{1+\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j S_{n,j} z^{2j}} \\ = \quad & z^{(1-(-1)^n)/2}\frac{z^{2\lfloor n/2\rfloor}+\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j T_j z^{2\lfloor n/2\rfloor-2j}}{1+\sum_{j=1}^{\lfloor n/2\rfloor}(-1)^j T_j z^{2j}} \end{aligned}$$

on $\mathbb{P}^1$. Since the numerators of both sides are monic, we have $S_{n,j}(\tau) = T_j(\tau)$ for all $j = 1,\dots,\lfloor n/2\rfloor$. Since this holds for all $\tau \in \mathbb{R}_{>0}i\cap\mathcal{N}$, by the identity theorem, $S_{n,j} = T_j$ on the upper half plane, for all $j$. This shows that $(S_{n,1},\dots,S_{n,\lfloor n/2\rfloor})$ is the unique solution to $(*)$. By Gaussian elimination, the infinite system $(*)$ is equivalent to a system $(**)$ of at most $\lfloor n/2\rfloor$ linear equations in $\lfloor n/2\rfloor$ $\mathcal{M}(\mathbb{H})$-variables, whose coefficients are again in

$$\mathbb{Q}(f_{n,\tau}(0), f'_{n,\tau}(0), f''_{n,\tau}(0),\dots).$$

Now $(S_{n,1},\dots,S_{n,\lfloor n/2\rfloor})$ is the unique solution to $(**)$, so the system $(**)$ should have exactly $\lfloor n/2\rfloor$ linear equations, and by Cramer's rule,

$$S_{n,j} \in \mathbb{Q}(f_{n,\tau}(0), f'_{n,\tau}(0), f''_{n,\tau}(0),\dots)$$

for all $j = 1,\dots,\lfloor n/2\rfloor$. □

From (11), we know that the Chebyshev-Blaschke product $f_{n,\tau}$ is even if $n$ is even and it is odd if $n$ is odd. By (9) and the properties on the Jacobi elliptic functions [26, p. 499-500], we have

$$f_{n,\tau}(0) = \begin{cases} (-1)^{n/2}\sqrt{k(n\tau)} & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

By differentiating (9) and putting $z=0$, we have

$$f'_{n,\tau}(0) = \begin{cases} 0 & \text{if } n \text{ is even;} \\ (-1)^{(n-1)/2} \cdot \dfrac{n\omega_1(n\tau)\sqrt{k(n\tau)}}{\omega_1(\tau)\sqrt{k(\tau)}} & \text{if } n \text{ is odd.} \end{cases}$$

It was proved in [18] that for each $n \geq 1$ and $\tau \in \mathbb{R}_{>0}i$, the Chebyshev-Blaschke product $f_{n,\tau}$ satisfies the following nonlinear differential equation:

$$\begin{aligned} \omega_1^2(\tau)(k(\tau)-z^2)(1-k(\tau)z^2)\tfrac{d^2w}{dz^2} + \omega_1^2(\tau)[2k(\tau)z^3 - (1+k^2(\tau))z]\tfrac{dw}{dz} \\ +n^2\omega_1^2(n\tau)[(1+k^2(n\tau))w - 2k(n\tau)w^3] = 0. \end{aligned} \tag{15}$$

Hence we have

$$f''_{n,\tau}(0) = \begin{cases} (-1)^{n/2} \cdot \dfrac{n^2\omega_1^2(n\tau)\sqrt{k(n\tau)}}{\omega_1^2(\tau)k(\tau)}(k^2(n\tau)-1) & \text{if } n \text{ is even;} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Let

$$A(z) := \omega_1^2(\tau)(k(\tau)-z^2)(1-k(\tau)z^2)$$

and

$$B(z) := \omega_1^2(\tau)[2k(\tau)z^3 - (1+k^2(\tau))z].$$

Denote the binomial coefficients by $C^\alpha_\beta$. By applying Leibniz rule twice, for $i \geq 2$,

$$\begin{aligned} (w^3)^{(i)} &= 3(w^2w^{(1)})^{(i-1)} \\ &= 3\sum_{j=0}^{i-1} C^{i-1}_j (w^2)^{(j)} w^{(i-j)} \\ &= 3w^2w^{(i)} + 3\sum_{j=1}^{i-1} C^{i-1}_j (2ww^{(1)})^{(j-1)} w^{(i-j)} \\ &= 3w^2w^{(i)} + 6\sum_{j=1}^{i-1}\sum_{k=0}^{j-1} C^{i-1}_j C^{j-1}_k w^{(k)}w^{(j-k)}w^{(i-j)}, \end{aligned}$$

so by taking the $i$-th derivative of (15), we have for $i \geq 2$,

$$\begin{aligned} Aw^{(i+2)} + \sum_{j=1}^{i}(C^i_jA^{(j)} + C^i_{j-1}B^{(j-1)})w^{(i-j+2)} + B^{(i)}w' + n^2\omega_1^2(n\tau)(1+k^2(n\tau))w^{(i)} \\ -6n^2\omega_1^2(n\tau)k(n\tau)\left[w^2w^{(i)} + 2\sum_{j=1}^{i-1}\sum_{k=0}^{j-1} C^{i-1}_j C^{j-1}_k w^{(k)}w^{(j-k)}w^{(i-j)}\right] = 0. \end{aligned} \tag{16}$$

For $j \geq 5$, the $j$-th derivatives of the polynomials $A$ and $B$ vanish, so for $i \geq 4$,

$$\begin{aligned} Aw^{(i+2)} + \sum_{j=1}^{4}(C^i_jA^{(j)} + C^i_{j-1}B^{(j-1)})w^{(i-j+2)} + n^2\omega_1^2(n\tau)(1+k^2(n\tau))w^{(i)} \\ -6n^2\omega_1^2(n\tau)k(n\tau)\left[w^2w^{(i)} + 2\sum_{j=1}^{i-1}\sum_{k=0}^{j-1} C^{i-1}_j C^{j-1}_k w^{(k)}w^{(j-k)}w^{(i-j)}\right] = 0. \end{aligned}$$

Putting $w = f_{n,\tau}$, $z = 0$, we get for $i \geq 4$,

$$\begin{aligned}&\omega_1^2(\tau)k(\tau)f_{n,\tau}^{(i+2)}(0) - i^2\omega_1^2(\tau)(1+k^2(\tau))f_{n,\tau}^{(i)}(0)\\&+i(i-1)^2(i-2)\omega_1^2(\tau)k(\tau)f_{n,\tau}^{(i-2)}(0) + n^2\omega_1^2(n\tau)(1+k^2(n\tau))f_{n,\tau}^{(i)}(0)\\&-6n^2\omega_1^2(n\tau)k(n\tau)\left[f_{n,\tau}(0)^2 f_{n,\tau}^{(i)}(0) + 2\sum_{j=1}^{i-1}\sum_{k=0}^{j-1} C_j^{i-1}C_k^{j-1} f_{n,\tau}^{(k)}(0)f_{n,\tau}^{(j-k)}(0)f_{n,\tau}^{(i-j)}(0)\right] = 0.\end{aligned} \tag{17}$$

If $n$ and $i$ are of the same parity and $i \geq 4$, then by (17),

$$\begin{aligned}&f_{n,\tau}^{(i+2)}(0) + \left[\frac{n^2\omega_1^2(n\tau)[1+(3(-1)^{n-1}-2)k^2(n\tau)]}{\omega_1^2(\tau)k(\tau)} - i^2\left(\frac{1}{k(\tau)} + k(\tau)\right)\right] f_{n,\tau}^{(i)}(0)\\&+i(i-1)^2(i-2)f_{n,\tau}^{(i-2)}(0)\\&-\frac{12n^2\omega_1^2(n\tau)k(n\tau)}{\omega_1^2(\tau)k(\tau)}\sum_{j=1}^{i-1}\sum_{k=0}^{j-1} C_j^{i-1}C_k^{j-1} f_{n,\tau}^{(k)}(0)f_{n,\tau}^{(j-k)}(0)f_{n,\tau}^{(i-j)}(0) = 0.\end{aligned} \tag{18}$$

Differentiating (15) once, and putting $w = f_{n,\tau}$ and $z = 0$, we have

$$\begin{aligned}&\omega_1^2(\tau)k(\tau)f_{n,\tau}^{(3)}(0) + [n^2\omega_1^2(n\tau)(1+k^2(n\tau)) - \omega_1^2(\tau)(1+k^2(\tau))]f'_{n,\tau}(0)\\&-6n^2\omega_1^2(n\tau)k(n\tau)f_{n,\tau}^2(0)f'_{n,\tau}(0) = 0.\end{aligned}$$

Then if $n$ is odd,

$$f_{n,\tau}^{(3)}(0) = (-1)^{(n+1)/2}\cdot\frac{n\omega_1(n\tau)\sqrt{k(n\tau)}}{\omega_1(\tau)\sqrt{k(\tau)}^3}\left[\frac{n^2\omega_1^2(n\tau)}{\omega_1^2(\tau)}(1+k^2(n\tau)) - (1+k^2(\tau))\right].$$

If $n$ is even, $f_{n,\tau}^{(3)}(0) = 0$.
By (16), if $n$ is even,

$$f_{n,\tau}^{(4)}(0) = (-1)^{n/2}\cdot\frac{n^2\omega_1^2(n\tau)\sqrt{k(n\tau)}}{\omega_1^2(\tau)k^2(\tau)}(1-k^2(n\tau))\left[\frac{n^2\omega_1^2(n\tau)}{\omega_1^2(\tau)}(1-5k^2(n\tau)) - 4(1+k^2(\tau))\right].$$

If $n$ is odd, then $f_{n,\tau}^{(4)}(0) = 0$.
By (16) again, if $n$ is odd,

$$\begin{aligned}f_{n,\tau}^{(5)}(0) \quad = \quad & (-1)^{(n-1)/2}\cdot\frac{n\omega_1(n\tau)\sqrt{k(n\tau)}}{\omega_1(\tau)\sqrt{k(\tau)}^5}\left[\frac{n^4\omega_1^4(n\tau)}{\omega_1^4(\tau)}(k^4(n\tau)+14k^2(n\tau)+1)\right.\\&\left.-10\frac{n^2\omega_1^2(n\tau)}{\omega_1^2(\tau)}(1+k^2(\tau))(1+k^2(n\tau)) + 3(3k^4(\tau)+2k^2(\tau)+3)\right].\end{aligned}$$

If $n$ is even, then $f_{n,\tau}^{(5)}(0) = 0$. By the expressions for $f_{n,\tau}(0), f'_{n,\tau}(0), \cdots, f_{n,\tau}^{(5)}(0)$, together with (18) and induction, we know that for each $n$ and $i$, as a function in $\tau$ on the upper half plane,

$$f_{n,\tau}^{(i)}(0) \in \mathbb{Q}\left(\sqrt{k(\tau)}, \sqrt{k(n\tau)}, \frac{\omega_1(n\tau)}{\omega_1(\tau)}\right).$$

Hence by Lemma 9.1, we have the following:

**Theorem 9.2.** *For each $n \geq 1$, the coefficients of the Chebyshev-Blaschke product $f_{n,\tau}$ are in the field*

$$\mathbb{Q}\left(\sqrt{k}, \sqrt{k \circ s_n}, \frac{\omega_1 \circ s_n}{\omega_1}\right),$$

*where $s_n(\tau) = n\tau$. By multiplying both the numerator and denominator of $f_{n,\tau}$ by the product of the denominators of the $S_{n,j}$, $j = 1, \ldots, \lfloor n/2 \rfloor$, we know that $f_{n,\tau}$ is defined over the ring*

$$\mathbb{Z}\left[\sqrt{k}, \sqrt{k \circ s_n}, \frac{\omega_1 \circ s_n}{\omega_1}\right].$$

*Moreover,*

$$\frac{\vartheta_2((2i-1)\pi/2n, \tau)}{\vartheta_3((2i-1)\pi/2n, \tau)}, \qquad i = 1, \ldots, \lfloor n/2 \rfloor,$$

*are algebraic over this field since they are zeros of $f_{n,\tau}$.*

This theorem is analogous to the fact that the coefficients of the Chebyshev polynomials are in $\mathbb{Z}$, and also the fact that for each $n \geq 2$ and $i = 1, \ldots, \lfloor n/2 \rfloor$,

$$\cos\left(\frac{(2i-1)\pi}{2n}\right)$$

is an algebraic number. Note also that when $\tau \to +i\infty$,

$$\sqrt{k(\tau)} \to 0 \quad \text{and} \quad \frac{\omega_1(n\tau)}{\omega_1(\tau)} \to 1,$$

so the ring in the theorem degenerates to $\mathbb{Z}$ when $\tau \to +i\infty$.

**Remark 9.3.** Another result similar to Theorem 9.2 is Theorem 4.1 in the paper [11] of Ismail and Zhang. It was proved that the coefficients of Ramanujan entire functions are lying in a polynomial ring over $\mathbb{C}(q)$ generated by expressions in terms of $q^{1/4}, \vartheta_2(0,\tau)$ and $\vartheta_3(0,\tau)$.

Next, we will prove that the coefficients of the Chebyshev-Blaschke products are in the algebraic closure $\overline{\mathbb{Q}(j)}$, where $j$ is the $j$-invariant.

**Lemma 9.4.** *For any positive integer $n$, $\vartheta_2^4(0, n\tau)$ and $\vartheta_3^4(0, n\tau)$ are modular forms of weight 2 with respect to the Hecke congruence subgroup $\Gamma_0(4n)$.*

*Proof.* By (1), (5) and [7, p. 338], both $\vartheta_3(0,\tau)$ and $\vartheta_2(0,\tau)$ are holomorphic on the open upper half plane $\mathbb{H}$. By (2) and (4), we have

$$\vartheta_3(0, \tau+1) = \vartheta_3(0,\tau) \quad \text{and} \quad \vartheta_3\left(0, \frac{\tau}{4\tau+1}\right) = (4\tau+1)^{1/2}\vartheta_3(0,\tau).$$

Since $\Gamma_0(4)$ is generated by

$$\pm\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \pm\begin{pmatrix} 1 & 0 \\ 4 & 1 \end{pmatrix},$$

see [5, p. 21], we have that $\vartheta_3^4(0,\tau)$ is weight 2 invariant under $\Gamma_0(4)$. Similarly, by (3) and (5), $\vartheta_2^4(0,\tau)$ is weight 2 invariant under $\Gamma_0(4)$. It is known that the $n$th-coefficient $d_n$ of the Fourier series of $\vartheta_3^4(0,\tau)$ is the number of ways to express $n$ as an ordered sum of squares of four integers, so $|d_n| \leq (2n+1)^4 \leq 3^4 n^4$ for $n > 0$, so the $n$-th coefficient $e_n$ of the Fourier series in $q^{1/4}$ satisfies $|e_n| \leq (3/4)^4 n^4$ for

$n>0$. By Proposition 1.2.4 in [5, p. 17], $\vartheta_3^4(0,\tau)$ is a weight 2 modular form with respect to $\Gamma_0(4)$. Let $\widetilde{q}=q^{1/4}$. Then

$$\vartheta_2^4(0,\tau)=\left(\sum_{n=-\infty}^{\infty}\widetilde{q}^{(2n+1)^2}\right)^4$$

whose $n$-th coefficient $c_n$ count the number of ways to express $n$ as an ordered sum of squares of four odd integers, so $c_n$ satisfies a similar bound for $n>0$, and hence $\vartheta_2^4(0,\tau)$ is a weight 2 modular form with respect to $\Gamma_0(4)$. By Lemma 2.9, $\vartheta_2^4(0,n\tau)$ and $\vartheta_3^4(0,n\tau)$ are weight 2 modular forms with respect to $\Gamma_0(4n)$. □

**Corollary 9.5.** *The functions $k^2(\tau),k^2(n\tau)$, and $\frac{\omega_1^2(n\tau)}{\omega_1^2(\tau)}$ are modular functions with respect to $\Gamma_0(4n)$.*

**Theorem 9.6.** *For each positive integer $n$, the coefficients of the Chebyshev-Blaschke product $f_{n,\tau}$ are in the algebraic closure $\overline{\mathbb{Q}(j)}$, where $j$ is the $j$-invariant.*

*Proof.* Since the coefficients of the Fourier expansions at $\infty$ of $\vartheta_3^4(0,n\tau)$ and $\vartheta_2^4(0,n\tau)$ are rational, those of $k^2(\tau)$, $k^2(n\tau)$ and $\frac{\omega_1^2(n\tau)}{\omega_1^2(\tau)}$ are also rational. Hence by Corollary 9.5 and Lemma 2.10,

$$k^2(\tau),k^2(n\tau),\frac{\omega_1^2(n\tau)}{\omega_1^2(\tau)}\in\mathbb{Q}(j,j_{4n}),$$

where $j_{4n}(\tau)=j(4n\tau)$. By Lemma 2.11, there exists nonconstant polynomial $\Phi_{4n}\in\mathbb{Q}[X,Y]$ such that

$$\Phi_{4n}(j,j_{4n})=0.$$

This shows that $j_{4n}\in\overline{\mathbb{Q}(j)}$, so $\mathbb{Q}(j,j_{4n})\subseteq\overline{\mathbb{Q}(j)}$. Then

$$\sqrt{k(\tau)},\sqrt{k(n\tau)},\frac{\omega_1(n\tau)}{\omega_1(\tau)}\in\overline{\mathbb{Q}(j)}.$$

Hence by Theorem 9.2, the coefficients of $f_{n,\tau}$ are in $\overline{\mathbb{Q}(j)}$. □

**Theorem 9.7.** *For each positive integer $n$, the coefficients of the Chebyshev-Blaschke product $f_{n,\tau}$ are in $Frac(\mathbb{Z}[[q^{1/4}]])$, the field of fraction of the ring of power series in $q^{1/4}$ over $\mathbb{Z}$. By multiplying both the numerator and the denominator of $f_{n,\tau}$ by a suitable element in $\mathbb{Z}[[q^{1/4}]]$, the denominators of the coefficients are cleared out and hence $f_{n,\tau}$ is defined over $\mathbb{Z}[[q^{1/4}]]$.*

*Proof.* By abuse of notations, we use $\sqrt{k(q)}$ and $\omega_1(q)$ to denote the $q$-expansions of $\sqrt{k(\tau)}$ and $\omega_1(\tau)$ respectively. Since

$$\begin{aligned}\mathbb{Q}\left(\sqrt{k(q)},\sqrt{k(q^n)},\tfrac{\omega_1(q^n)}{\omega_1(q)}\right)&\subseteq\mathbb{Q}(\vartheta_2(0,q),\vartheta_3(0,q),\vartheta_2(0,q^n),\vartheta_3(0,q^n))\\&\subseteq Frac(\mathbb{Z}[[q^{1/4}]]),\end{aligned}$$

by Theorem 9.2, the coefficients of $f_{n,\tau}$ are in $Frac(\mathbb{Z}[[q^{1/4}]])$. □

**Remark 9.8.** It would be interesting to find more examples of other family of Shabat-Blaschke products that are defined over a finite extension of $Frac(\mathbb{Z}[[q^{1/4}]])$, or defined over a finite extension of $\mathbb{Q}\left(\sqrt{k},\sqrt{k\circ s_n},\frac{\omega_1\circ s_n}{\omega_1}\right)$, or defined over $\overline{\mathbb{Q}(j)}$. One would also like to see if there is a deformation of the Belyi's theorem formulated using the above fields.

## 10. Landen-type identities for theta functions

In this section, we will obtain some Landen-type identities for theta functions, which will degenerate to some trigonometric identities.

From (13), we know that for each positive integer $n \geq 2$, and $j = 1, \dots, \lfloor n/2 \rfloor$, $S_{n,j}$ can be expressed in terms of

$$\vartheta_2((2i-1)\pi/2n, \tau) \quad \text{and} \quad \vartheta_3((2i-1)\pi/2n, \tau),$$

where $1 \leq i \leq \lfloor n/2 \rfloor$. On the other hand, we know from Theorem 9.2 that $S_{n,j}$ can be expressed in terms of

$$\vartheta_2(0,\tau), \vartheta_3(0,\tau), \vartheta_2(0,n\tau), \vartheta_3(0,n\tau).$$

Therefore, for each positive integer $n \geq 2$, and each $j = 1, \dots, \lfloor n/2 \rfloor$, we have a theta identity relating those theta functions. For example, when $n$ is even and $j = \lfloor n/2 \rfloor$, we have

$$\prod_{1 \leq i \leq \lfloor n/2 \rfloor} \frac{\vartheta_2^2((2i-1)\pi/2n, \tau)}{\vartheta_3^2((2i-1)\pi/2n, \tau)} = \frac{\vartheta_2(0, n\tau)}{\vartheta_3(0, n\tau)}$$

which coincides with the Landen transformation of even order $n$ evaluated at $z = 0$ [14, p. 23, 253-254, 259]. When $n$ is odd and $j = \lfloor n/2 \rfloor$, we have

$$\prod_{1 \leq i \leq \lfloor n/2 \rfloor} \frac{\vartheta_2^2((2i-1)\pi/2n, \tau)}{\vartheta_3^2((2i-1)\pi/2n, \tau)} = \frac{n\vartheta_2(0, n\tau)\vartheta_3(0, n\tau)}{\vartheta_2(0,\tau)\vartheta_3(0,\tau)}$$

which coincides with the Landen transformation of odd order $n$ evaluated at $z = 0$ [14, p. 253-256]. However, we also get other theta identities in which the left hand sides are other symmetric polynomials in

$$\frac{\vartheta_2^2((2i-1)\pi/2n, \tau)}{\vartheta_3^2((2i-1)\pi/2n, \tau)}, \quad i = 1, \dots, \lfloor n/2 \rfloor.$$

We display some examples of theta identities when $n$ is small below. When $n = 2$, we only have one identity

$$\frac{\vartheta_2^2(\pi/4, \tau)}{\vartheta_3^2(\pi/4, \tau)} = \frac{\vartheta_2(0, 2\tau)}{\vartheta_3(0, 2\tau)}.$$

By (8), we know that

$$\lim_{\tau \to +i\infty} \frac{1}{k(\tau)} \frac{\vartheta_2^2(\pi/4, \tau)}{\vartheta_3^2(\pi/4, \tau)} = \cos^2\left(\frac{\pi}{4}\right).$$

By considering the Fourier expansions, we have

$$\lim_{\tau \to +i\infty} \frac{1}{k(\tau)} \frac{\vartheta_2(0, 2\tau)}{\vartheta_3(0, 2\tau)} = \frac{1}{2}.$$

By the above identity and limits, we get

$$\cos^2\left(\frac{\pi}{4}\right) = \frac{1}{2}.$$

When $n = 3$, we again only have one identity

$$\frac{\vartheta_2^2(\pi/6, \tau)}{\vartheta_3^2(\pi/6, \tau)} = \frac{3\vartheta_2(0, 3\tau)\vartheta_3(0, 3\tau)}{\vartheta_2(0,\tau)\vartheta_3(0,\tau)}.$$

By multiplying $1/k(\tau)$ on both sides and taking limits, we get the trigonometric identity

$$\cos^2\left(\frac{\pi}{6}\right)=\frac{3}{4}.$$

When $n=4$, we have two theta identities,

$$\frac{\vartheta_2^2(\pi/8,\tau)}{\vartheta_3^2(\pi/8,\tau)}+\frac{\vartheta_2^2(3\pi/8,\tau)}{\vartheta_3^2(3\pi/8,\tau)}=\frac{8\vartheta_2(0,4\tau)}{\vartheta_2^2(0,\tau)\vartheta_3^2(0,\tau)}\cdot\frac{\vartheta_3^4(0,4\tau)-\vartheta_2^4(0,4\tau)}{\vartheta_3(0,4\tau)-\vartheta_2(0,4\tau)}$$

and

$$\frac{\vartheta_2^2(\pi/8,\tau)}{\vartheta_3^2(\pi/8,\tau)}\frac{\vartheta_2^2(3\pi/8,\tau)}{\vartheta_3^2(3\pi/8,\tau)}=\frac{\vartheta_2(0,4\tau)}{\vartheta_3(0,4\tau)}.$$

By multiplying $1/k(\tau)$ on both sides of the first identity and taking limits, and by multiplying $1/k^2(\tau)$ on both sides of the second identity and taking limits, we get the trigonometric identities

$$\cos^2\left(\frac{\pi}{8}\right)+\cos^2\left(\frac{3\pi}{8}\right)=1$$

and

$$\cos^2\left(\frac{\pi}{8}\right)\cos^2\left(\frac{3\pi}{8}\right)=\frac{1}{8}.$$

When $n=5$, we have two theta identities,

$$\begin{aligned}&\frac{\vartheta_2^2(\pi/10,\tau)}{\vartheta_3^2(\pi/10,\tau)}+\frac{\vartheta_2^2(3\pi/10,\tau)}{\vartheta_3^2(3\pi/10,\tau)}\\ =\;&\frac{5\vartheta_2(0,5\tau)\vartheta_3(0,5\tau)}{6\vartheta_2^2(0,\tau)\vartheta_3^2(0,\tau)}\cdot\frac{\vartheta_3^4(0,\tau)+\vartheta_2^4(0,\tau)-25(\vartheta_3^4(0,5\tau)+\vartheta_2^4(0,5\tau))}{5\vartheta_2(0,5\tau)\vartheta_3(0,5\tau)-\vartheta_2(0,\tau)\vartheta_3(0,\tau)}\end{aligned}$$

and

$$\frac{\vartheta_2^2(\pi/10,\tau)}{\vartheta_3^2(\pi/10,\tau)}\frac{\vartheta_2^2(3\pi/10,\tau)}{\vartheta_3^2(3\pi/10,\tau)}=\frac{5\vartheta_2(0,5\tau)\vartheta_3(0,5\tau)}{\vartheta_2(0,\tau)\vartheta_3(0,\tau)}.$$

By multiplying $1/k(\tau)$ on both sides of the first identity and taking limits, and by multiplying $1/k^2(\tau)$ on both sides of the second identity and taking limits, we get the trigonometric identities

$$\cos^2\left(\frac{\pi}{10}\right)+\cos^2\left(\frac{3\pi}{10}\right)=\frac{5}{4}$$

and

$$\cos^2\left(\frac{\pi}{10}\right)\cos^2\left(\frac{3\pi}{10}\right)=\frac{5}{16}.$$

When $n=6$, we have three theta identities,

$$\begin{aligned}&\frac{\vartheta_2^2(\pi/12,\tau)}{\vartheta_3^2(\pi/12,\tau)}+\frac{\vartheta_2^2(3\pi/12,\tau)}{\vartheta_3^2(3\pi/12,\tau)}+\frac{\vartheta_2^2(5\pi/12,\tau)}{\vartheta_3^2(5\pi/12,\tau)}\\ =\;&\frac{6\vartheta_2(0,6\tau)(\vartheta_2^2(0,6\tau)+\vartheta_3^2(0,6\tau))}{\vartheta_2^2(0,\tau)\vartheta_3^2(0,\tau)}\\ &\cdot\frac{3\vartheta_2^2(0,\tau)\vartheta_3^2(0,\tau)\vartheta_2(0,6\tau)-\vartheta_3(0,6\tau)[\vartheta_2^4(0,\tau)+\vartheta_3^4(0,\tau)+45\vartheta_2^4(0,6\tau)-9\vartheta_3^4(0,6\tau)]}{\vartheta_2^2(0,\tau)\vartheta_3^2(0,\tau)-18\vartheta_2(0,6\tau)\vartheta_3(0,6\tau)(\vartheta_2^2(0,6\tau)+\vartheta_3^2(0,6\tau))},\end{aligned}$$

$$\begin{aligned}
&\frac{\vartheta_2^2(\pi/12,\tau)}{\vartheta_3^2(\pi/12,\tau)}\frac{\vartheta_2^2(3\pi/12,\tau)}{\vartheta_3^2(3\pi/12,\tau)}+\frac{\vartheta_2^2(\pi/12,\tau)}{\vartheta_3^2(\pi/12,\tau)}\frac{\vartheta_2^2(5\pi/12,\tau)}{\vartheta_3^2(5\pi/12,\tau)}+\frac{\vartheta_2^2(3\pi/12,\tau)}{\vartheta_3^2(3\pi/12,\tau)}\frac{\vartheta_2^2(5\pi/12,\tau)}{\vartheta_3^2(5\pi/12,\tau)}\\
= \quad &\frac{6\vartheta_2(0,6\tau)(\vartheta_2^2(0,6\tau)+\vartheta_3^2(0,6\tau))}{\vartheta_2^2(0,\tau)\vartheta_3^2(0,\tau)}\\
&\cdot\frac{3\vartheta_2^2(0,\tau)\vartheta_3^2(0,\tau)\vartheta_3(0,6\tau)-\vartheta_2(0,6\tau)[\vartheta_2^4(0,\tau)+\vartheta_3^4(0,\tau)-9\vartheta_2^4(0,6\tau)+45\vartheta_3^4(0,6\tau)]}{\vartheta_2^2(0,\tau)\vartheta_3^2(0,\tau)-18\vartheta_2(0,6\tau)\vartheta_3(0,6\tau)(\vartheta_2^2(0,6\tau)+\vartheta_3^2(0,6\tau))},
\end{aligned}$$

and

$$\frac{\vartheta_2^2(\pi/12,\tau)}{\vartheta_3^2(\pi/12,\tau)}\frac{\vartheta_2^2(3\pi/12,\tau)}{\vartheta_3^2(3\pi/12,\tau)}\frac{\vartheta_2^2(5\pi/12,\tau)}{\vartheta_3^2(5\pi/12,\tau)}=\frac{\vartheta_2(0,6\tau)}{\vartheta_3(0,6\tau)}.$$

By multiplying $1/k(\tau)$ on both sides of the first identity and taking limits, by multiplying $1/k^2(\tau)$ on both sides of the second identity and taking limits, by multiplying $1/k^3(\tau)$ on both sides of the third identity and taking limits, we get the trigonometric identities

$$\cos^2\left(\frac{\pi}{12}\right)+\cos^2\left(\frac{3\pi}{12}\right)+\cos^2\left(\frac{5\pi}{12}\right)=\frac{3}{2},$$

$$\cos^2\left(\frac{\pi}{12}\right)\cos^2\left(\frac{3\pi}{12}\right)+\cos^2\left(\frac{\pi}{12}\right)\cos^2\left(\frac{5\pi}{12}\right)+\cos^2\left(\frac{3\pi}{12}\right)\cos^2\left(\frac{5\pi}{12}\right)=\frac{9}{16},$$

and

$$\cos^2\left(\frac{\pi}{12}\right)\cos^2\left(\frac{3\pi}{12}\right)\cos^2\left(\frac{5\pi}{12}\right)=\frac{1}{32}.$$

## 11. Conjecture

Finally, we list some further problems that one may try to study. Suppose $\rho$ is a tree monodromy representation that gives rise to Shabat-Blaschke products of degree $n$ with exactly two critical values in $\mathbb{D}$. For each fixed $\tau\in\mathbb{R}_{>0}i$, motivated by the critical values of the Chebyshev-Blaschke products, let $B_\tau:\mathbb{D}\to\mathbb{D}$ be the Shabat-Blaschke product associated to $\rho$ whose critical values are $-\sqrt{k(n\tau)}$ and $\sqrt{k(n\tau)}$. Let $s_n(\tau)=n\tau$ for all $\tau\in\mathbb{H}$. We have the following conjecture:

**Conjecture 11.1.** *There exists $a_1,\dots,a_n,b_1,\dots,b_m$ in*

$$\overline{\mathbb{Q}\left(\sqrt{k},\sqrt{k\circ s_n},\frac{\omega_1\circ s_n}{\omega_1}\right)}\subseteq\mathcal{M}(\mathbb{H})$$

*(where $\mathcal{M}(\mathbb{H})$ is the field of meromorphic functions on the upper half-plane $\mathbb{H}$) such that*

- *for any $\tau\in\mathbb{R}_{>0}i$, $a_0(\tau),\dots,a_n(\tau),b_0(\tau),\dots,b_m(\tau)\in\mathbb{C}$.*
- *for any $\tau\in\mathbb{R}_{>0}i$ and $z\in\mathbb{D}$,*

$$B_\tau(z)=\frac{a_n(\tau)z^n+a_{n-1}(\tau)z^{n-1}+\cdots+a_0(\tau)}{b_m(\tau)z^m+b_{m-1}(\tau)z^{m-1}+\cdots+b_0(\tau)}.$$

One can also have weaker conjectures by replacing the algebraic closure in the above conjecture by $\overline{\mathbb{Q}(j)}$ or $\overline{Frac(\mathbb{Z}[[q^{1/4}]])}$.

In a paper [17] by Maskit, there is a way to embed a topologically finite Riemann surface into a compact Riemann surface. One may try to use such embedding to formulate a deformation of Belyi's theorem.

Another version of the Belyi's theorem says that a compact Riemann surface $X$ admits a Belyi map if and only if $X$ can be uniformized by a finite index subgroup of a Fuchsian triangle group [12, p. 71]. One may try to formulate a deformation of this version of Belyi's theorem.

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