SCHUR RIGIDITY OF SCHUBERT VARIETIES IN RATIONAL HOMOGENEOUS MANIFOLDS OF PICARD NUMBER ONE

JAEHYUN HONG AND NGAIMING MOK

Abstract. Given a rational homogeneous manifold $S = G/P$ of Picard number one and a Schubert variety S_0 of S, the pair (S, S_0) is said to be homologically rigid if any subvariety of S having the same homology class as S_0 must be a translate of S_0 by the automorphism group of S. The pair (S, S_0) is said to be Schur rigid if any subvariety of S with homology class equal to a multiple of the homology class of S_0 must be a sum of translates of S_0. Earlier we completely determined homologically rigid pairs (S, S_0) in case S_0 is homogeneous and answered the same question in smooth non-homogeneous cases. In this article we consider Schur rigidity, proving that (S, S_0) exhibits Schur rigidity whenever S_0 is a non-linear smooth Schubert variety.

Modulo a classification result of the first author's, our proof proceeds by a reduction to homological rigidity by deforming a subvariety Z of S with homology class equal to a multiple of the homology class of S_0 into a sum of distinct translates of S_0, and by observing that the arguments for the homological rigidity apply since any two translates of S_0 intersect in codimension at least two. Such a degeneration is achieved by means of the $\mathbb{C}^*$-action associated with the stabilizer of the Schubert variety T_0 opposite to S_0. By transversality of general translates, a general translate of Z intersects T_0 transversely and the $\mathbb{C}^*$-action associated with the stabilizer of T_0 induces a degeneration of Z into a sum of translates of S_0, not necessarily distinct. After investigating the Bialynicki-Birular decomposition associated with the $\mathbb{C}^*$-action we prove a refined form of transversality to get a degeneration of Z into a sum of distinct translates of S_0.

1. Introduction

A rational homogeneous manifold S is a complex projective algebraic variety on which a complex linear algebraic group G acts transitively. In the current article we focus on rational homogeneous manifolds $S = G/P$ of Picard number 1. It may happen that the same projective manifold S can be presented as a rational homogeneous manifold in two different ways. In other words, taking G to be identity component of the automorphism group of S, and $P \subset G$ to be a maximal parabolic subgroup, there may exist a simple complex Lie group $G' \subsetneq G$ such that writing $P' := P \cap G'$ the canonical map $G'/P' \hookrightarrow G/P = S$ is a biholomorphism. Specifically, when S is a rational homogeneous manifold of type (B_ℓ, α_ℓ) (respectively, of type $(C_\ell; \alpha_1)$ or of type (G_2, α_1)), the automorphism group of S is of type $D_{\ell+1}$ (respectively, of type $A_{2\ell}$ or of type B_3) and thus we will regard S as a rational homogeneous manifold of type $(D_{\ell+1}, \alpha_{\ell+1})$ (respectively, of type $(A_{2\ell}, \alpha_1)$ or of type $(B_3; \alpha_1)$).

We consider a rigidity problem on a pair (S, S_0) consisting of a rational homogeneous manifold S of Picard number 1 and a smooth Schubert variety $S_0 \subset S$. In the event that $S = G/P = G'/P'$ as in the above where $G' \subsetneq G$, $P' \subsetneq P$, taking a Borel subgroup $B' \subset P'$ and a Borel subgroup $B \subset P$ containing B', homology groups of S are generated

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by the finitely many Schubert varieties which are topological closures of the B-orbits (which are affine cells) on S. *A priori* each B-orbit decomposes into the union of a finite number of B'-orbits. However, since the sum of the ranks of homology groups of S equals the number of B-orbits resp. the number of B'-orbits, each B-orbit must already be a B'-orbit, so that the set of Schubert varieties on S does not change when $S = G'/P'$ is rewritten as $S = G/P$, which we will do throughout the article.

The ray generated by the homology class of a Schubert variety S_0 is extremal in the sense that if the sum of the homology classes of two effective cycles is contained in the ray, then both classes are contained in the ray. It is an interesting problem in algebraic geometry to describe the space of all effective cycles representing homology classes in this ray. Trivially, any sum of translates of S_0 by G belongs to this space, but it contains more, for example, when S is the projective linear space $\mathbb{P}^n$ and S_0 is the projective line $\mathbb{P}^1$ in $\mathbb{P}^n$. On the other hand, when S is the Grassmannian of d-dimensional subspaces of $\mathbb{C}^n$ and S_0 is a sub-Grassmannian, if both are non-linear, then it consists only of sums of translates of S_0 ([5], [18]).

Given a Schubert variety S_0 of a rational homogeneous manifold S, the pair (S, S_0) is said to be *Schur rigid* if any subvariety $Z \subset S$ having homology class equal to a multiple of the homology class of S_0, must be a sum of translates of S_0 by G. In this paper we consider the question of Schur rigidity. Our main result is

Theorem 1.1. *Let $S = G/P$ be a rational homogeneous manifold of Picard number one and let S_0 be a non-linear smooth Schubert variety of S. Then, the pair (S, S_0) is Schur rigid.*

Here, after embedding S into a projective space $\mathbb{P}^N$ by the ample generator of the Picard group of S, we say S_0 is linear if S_0 is a linear subspace of $\mathbb{P}^N$. Clearly, (S, S_0) is not Schur rigid if S_0 is linear and is not maximal, i.e., there is a linear Schubert variety S_0' of S containing S_0 properly. For the case when S_0 is a maximal linear space, see Proposition 4.12.

One of the methods to determine Schur rigidity is to use differential systems. Given a Borel subgroup of G, Schubert varieties of $S = G/P$ are indexed by a certain subset W^P of the Weyl group W of G. For $w \in W^P$, let $S(w)$ denote the Schubert variety of type w. Kostant constructed a representative of the cohomology class Poincaré dual to the homology class of $S(w)$ by a closed positive (k, k)-form $\phi(w)$ ([28]). The Schur differential system is defined by the space of tangent subspaces of S on which $\phi(v)$ vanish for all v with $\dim S(w) = \dim S(v)$ and $S(w) \neq S(v)$, and the Schubert differential system is defined by the space of tangent spaces of G-translates of $S(w)$. Then the Schur rigidity problem is reduced to two problems: (1) the equality of the Schur differential system and the Schubert differential system, and (2) the uniqueness of integral varieties of the Schubert differential system up to the action of G.

Bryant introduced these differential systems and investigated their integral varieties for various Schubert varieties in compact irreducible Hermitian symmetric spaces ([37], [5]). Hong proved the Schur rigidity of smooth Schubert varieties in compact irreducible Hermitian symmetric spaces and some singular Schubert varieties in Grassmannians by reducing the problems (1) and (2) to the vanishing of certain Lie algebra cohomology spaces ([18], [17]). Robles-The developed this method further to characterize Schur rigid Schubert varieties in compact irreducible Hermitian symmetric spaces ([34], for the flexibility see [6] and [35]). It is not easy to apply this differential geometric method to Schubert varieties of rational homogeneous manifolds other than compact Hermitian symmetric spaces: neither of the two steps (1) and (2) can be reduced to Lie algebra cohomology computations.

There is another form of rigidity weaker than Schur rigidity. We say that the pair (S, S_0) is *homologically rigid* if any subvariety $Z \subset S$ having homology class equal to $[S_0]$ must be a G-translate of S_0. Coskun determined homologically rigid Schubert varieties of Grassmannians and orthogonal Grassmannians by using algebro-geometric methods ([7], [8], for a survey see [9]).

In the previous work ([15]) we used a different method, the geometric theory of uniruled projective manifolds based on varieties of minimal rational tangents, which was developed by Hwang-Mok and was applied to characterize uniruled projective manifolds of Picard number one and to prove their deformation rigidity ([19], [20], [21], [22], [23], [24]). We generalized the theory to the pair consisting of a uniruled projective manifold and one of its projective submanifold ([14]) and proved that smooth Schubert varieties in a rational homogeneous manifold of Picard number one are homologically rigid with certain obvious exceptions (Theorem 1.1 of [15], Theorem 1.4 of [13]).

Recently, Mok-Zhang proved that Schubert varieties of rational homogeneous manifolds S of Picard number one, which are associated to subdiagrams of the Dynkin diagram of S, are Schubert rigid, i.e., integral varieties of their Schubert differential systems are unique up to the action of G ([31]). The equality of the Schur differential system and the Schubert differential system will imply Schur rigidity of these Schubert varieties. However, at the moment we don't have any tool to prove the equality of the two differential systems.

These rigidity problems go back to the smoothability problem ([5]). A homology class $\mathbf{s}$ in $H(S, \mathbb{Z})$ is said to be *smoothly representable* if there is a nonsingular subvariety of S whose homology class is $\mathbf{s}$. If the pair (S, S_0) is Schur rigid and S_0 is singular, then any multiple $r[S_0]$ of the homology class of S_0 is not smoothly representable. For other kinds of smoothability and related results, see [12], [5], [7], [8], [9].

The pair (S, S_0) is Schur rigid if (and only if) it is homologically rigid and there is no irreducible reduced subvariety of S whose homology class is r times the homology class of S_0 for any $r \geq 2$. Our strategy is to deform an irreducible reduced subvariety Z of S representing $r[S_0]$ to a sum Z_∞ of G-translates of S_0 such that at least one of the irreducible components has multiplicity one. We impose the condition on the multiplicity to have a neighborhood of a minimal rational curve in the smooth locus of Z_∞.

To get such a deformation, we use the Schubert variety T_0 opposite to S_0. By transversality of general translates, there is a translate of Z which intersects T_0 transversely. The stabilizer of T_0 is a parabolic subgroup of G. Applying the $\mathbb{C}^*$-action λ associated with this parabolic subgroup and taking its limit, we get a degeneration of Z to a sum of translates of S_0. But this is not enough, because irreducible components of the limit may have multiplicities > 1. To overcome this difficulty we introduce a transversality stronger than the usual transversality.

We say that Z intersects T_0 transversely with respect to the $\mathbb{C}^*$-action λ if Z intersects T_0 transversely and the λ-limits of points in the intersection $Z \cap T_0$ are all distinct. Then the $\mathbb{C}^*$-action λ gives a degeneration of Z into a sum Z_∞ of r distinct translates of S_0. When S_0 is a smooth Schubert variety of a rational homogeneous manifold S of Picard number one, any two translates of S_0 intersects in codimension ≥ 2. From this it follows the existence of a line lying on the smooth part of Z_∞.

The problem of perturbing Z to a subvariety Z' intersecting T_0 transversely with respect to the $\mathbb{C}^*$-action λ, is investigated in Section 3. The stabilizer P_I^- of T_0 has an open orbit $\mathcal{O}$ in T_0, which is an L_I-homogeneous bundle with fiber a U_I^--orbit F, where L_I is the reductive part of P_I^- and U_I^- is the unipotent part of P_I^-. Then the closure $T_0 = \overline{\mathcal{O}}$ is the union of L_I-translates of the closure $\overline{F}$. The problem is reduced to the question whether, for two points p_1, p_2 in $Z \cap T_0$ lying in F, the isotropy of G at p_1 can move p_2 in a direction outside the sum of the tangent space $T_{p_2}Z$ of Z and the tangent space $T_{p_2}F$ of F. Here, the $\mathbb{C}^*$-action λ plays a role again. Especially, the Bialynicki-Birular ($\pm$)-decomposition $\coprod_\sigma \mathcal{O}_\sigma^\pm$ associated with λ, which is nothing but the $P_I^\pm$-orbit decomposition of S, and the relation among $\mathcal{O}_\sigma^+$ and $\mathcal{O}_\tau^-$ are used. For a detailed argument see Proposition 3.3.

As we explained in the above, the relation among Schubert varieties, parabolic subgroups and $\mathbb{C}^*$-actions is one of the main ingredients in the proof. We collect notations, definitions, basic properties in Section 2. In Section 4 we complete the proof by generalizing the arguments for $r = 1$ when the special fiber is a smooth Schubert variety to the case for $r \geq 2$ when the special fiber is a sum of its translates (Proposition 4.3) and by showing that any two translates of a smooth Schubert variety in a rational homogeneous manifold of Picard number one intersect in codimension ≥ 2 (Proposition 4.8). At the end of the paper, we prove that a maximal linear Schubert variety of S is Schur rigid with the exception of some obvious cases (Proposition 4.12).

This paper deals with smooth Schubert varieties in rational homogeneous manifolds of Picard number one. While most of the arguments work in Section 3 for Schubert varieties in rational homogeneous manifolds without the conditions on smoothness and on the Picard number, these two conditions are essential in Section 4 (Proposition 4.3). While it is perceivable that generalizations to the case of smooth Schubert varieties on a rational homogeneous manifold of higher Picard number (where we have to deal with several minimal rational components) could be obtained by modifying the methods of Hong-Mok [15] and the current article, there are intrinsic difficulties in the case of singular Schubert varieties even when the ambient rational homogeneous manifold S is of Picard number 1. In fact, it is crucial to use Kodaira Stability Theorem, which is an application of deformation theory of rational curves. To do the same for a singular Schubert variety S_0 on S of Picard number 1 we need at least to have an ample supply of minimal rational curves of S lying on the smooth locus of S_0. Unfortunately, the latter fails to be the case for certain singular Schubert varieties (cf. Remark 4.1).

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2. Preliminaries

We fix notations and explain properties which will be used later in this paper. A basic reference is [36].

2.1. **Parabolic subgroups.** Let G be a connected simple algebraic group over $\mathbb{C}$. Fix a Borel subgroup B of G and a maximal torus T in B. Denote by Δ^+ the system of positive roots of G and by $\Phi = \{\alpha_1, \dots, \alpha_\ell\}$ the system of simple roots of G. For any subgroup H of G invariant under the conjugation by T, let $\Delta(H)$ denote the set of all roots α whose root space $\mathfrak{g}_\alpha$ is contained in the Lie algebra of H. For example, $\Delta(B) = \Delta^+$.

For a subset I of Φ denote by $P_I^\pm$ the parabolic subgroup of G whose Lie algebra is

$$(\mathfrak{t} + \sum_{\alpha \in \mathbb{Z}I \cap \Delta} \mathfrak{g}_\alpha) + \sum_{\alpha \in \Delta^+ \backslash \mathbb{Z}I} \mathfrak{g}_{\pm\alpha}.$$

Let L_I be its reductive part containing T and $U_I^\pm$ be its unipotent part. Then $\Delta(L_I) = \Delta \cap \mathbb{Z}I$ and $\Delta(U_I^\pm) = \Delta^\pm \backslash \mathbb{Z}I$. For $I = \emptyset \subset \Phi$, $P_\emptyset^\pm$ is a Borel subgroup $B^\pm$ of G (B^+ is the given Borel subgroup B and B^- is the Borel subgroup opposite to B) and $U_\emptyset^\pm$ is the unipotent part $U^\pm$ of $B^\pm$. As in the case of B, we will sometimes use notations P_I, U_I, U instead of P_I^+, U_I^+, $U_\emptyset^+$.

To each simple root α_k we associate a parabolic subgroup $P_{\Phi - \{\alpha_k\}}$ of G. The homogeneous manifold $S = G/P_{\Phi - \{\alpha_k\}}$ is called the rational homogeneous manifold associated to the simple root α_k or of type (G, α_k). From now on we fix a simple root α_k and set $P = P_{\Phi - \{\alpha_k\}}$, $L_P = L_{\Phi - \{\alpha_k\}}$, $U_P = U_{\Phi - \{\alpha_k\}}$ and $S = G/P$.

2.2. **Schubert varieties.** Let W be the Weyl group of G and let W_P be the Weyl group of the reductive part of P. Then the set of T-fixed points in $S = G/P$ is indexed by the set of right cosets W/W_P: letting x_0 be the base point of S with the isotropy group P, the map $[w] \in W/W_P \mapsto x_w := wx_0$ is a bijective map from W/W_P to the T-fixed point set in S. The B-orbit decomposition of S is given by $S = \coprod_{[w] \in W/W_P} B.x_w$.

For an element $w \in W$ define a subset $\Delta(w)$ of Δ^+ by $\Delta(w) = \{\beta \in \Delta^+ : w(\beta) \in -\Delta^+\}$. Let W^P be the subset of W consisting of $w \in W$ such that $\Delta(w) \subset \Delta(U_P)$. Then W^P is a set of representatives of W/W_P so that we have

$$S = \coprod_{w \in W^P} B.x_w.$$

For each $w \in W^P$, the closure $S(w)$ of $B.x_w$ is called the *Schubert variety of type* w. For the Borel subgroup B^- opposite to B we also have a decomposition $S = \coprod_{w \in W^P} B^-.x_w$, and we call the closure $T(w)$ of $B^-.x_w$ the *opposite Schubert variety of type* w, or, the *Schubert variety opposite to* $S(w)$.

Each orbit $B^\pm.x_w$ is biholomorphic to an affine cell $\mathbb{C}^{k^\pm}$ for some $k^\pm$ depending on w, so that we get a coordinate system on it. We describe this coordinate system more precisely. Consider a root α as a character of T, i.e., a homomorphism from T to the multiplicative group $\mathbb{C}^*$, so that, for a root vector $e_\alpha \in \mathfrak{g}_\alpha$, we have $Ad(h)e_\alpha = \alpha(h)e_\alpha$ for $h \in T$. Then there exist a unique closed subgroup $U_\alpha \subset G$ and an isomorphism $u_\alpha : \mathbb{C} \to U_\alpha$ such that $\operatorname{Im} du_\alpha = \mathfrak{g}_\alpha$ and for $h \in T$

$$hu_\alpha(x)h^{-1} = u_\alpha(\alpha(h)x)$$

where $x \in \mathbb{C}$ (8.1.1 of [36]).

Let U_P^- be the unipotent part of P^-, the parabolic subgroup of G opposite to P. Then $U_P^-.x_0$ is an open neighborhood of x_0 biholomorphic to $U_P^- = \prod_{\alpha\in\Delta(U_P)} U_{-\alpha}$. Thus, for $w \in W^P$, $w(U_P^-).x_w = wU_P^-.x_0$ is an open neighborhood of x_w in $S = G/P$ biholomorphic to $w(U_P^-) = \prod_{\alpha\in\Delta(U_P)} U_{-w(\alpha)}$. Similarly, $B.x_w = (U \cap w(U_P^-)).x_w$ is an open neighborhood of x_w in $S(w)$ biholomorphic to $U \cap w(U_P^-) = w\left(\prod_{\beta\in\Delta(w)} U_{-\beta}\right) = \prod_{\alpha\in\Delta(w^{-1})} U_\alpha$ and $B^-.x_w = (U^- \cap w(U_P^-)).x_w$ is an open neighborhood of x_w in $T(w)$ biholomorphic to $U^- \cap w(U_P^-)$.

Let A be a set consisting of roots. By taking the product of the isomorphisms u_α where $\alpha \in A$, we get an isomorphism

$$u = \prod_{\alpha\in A} u_\alpha : \mathbb{C}^{|A|} \to \prod_{\alpha\in A} U_\alpha$$

and u defines coordinates $(x_\alpha)_{\alpha\in A}$ on $\prod_{\alpha\in A} U_\alpha$. We remark that the coordinates depend on the order of roots in A. By taking an appropriate set A of roots, we get coordinates systems on $w(U_P^-).x_w = wB.x_0$, $B.x_w$, and $B^-.x_w$.

For a subset I of Φ let W_I be the Weyl group of L_I. Then the left coset space $W_I\backslash W^P$ parameterizes $P_I^{\pm}$-orbits in $S = G/P$ so that we have

$$S = \coprod_{[\sigma]\in W_I\backslash W^P} P_I^{\pm}.x_\sigma$$

(8.4.6 of [36]). The closure of a P_I-orbit in S is a Schubert variety and the closure of a P_I^--orbit in S is an opposite Schubert variety (For, an irreducible B-invariant subvariety of S is the closure of a B-orbit and an irreducible B^--invariant subvariety of S is the closure of a B^--orbit).

Conversely, the stabilizer of a Schubert variety in G is a parabolic subgroup P_I for some $I \subset \Phi$.

Proposition 2.1. *For $w \in W^P$, let $I = \Phi \cap w(\Delta(P^-))$. Then*

(1) *the stabilizer of $S(w)$ in G is P_I;*
(2) *$L_I.x_w$ is closed, and has positive dimension if $w \neq Id$.*

Proof. (1) (Section 2.6 of [3]). For a simple root α, the minimal parabolic subgroup P_α acts invariantly on BwB if and only if $w^{-1}(\alpha) < 0$, or equivalently, $\ell(s_\alpha w) = \ell(w) - 1$. From $s_\alpha w = ws_{w^{-1}(\alpha)}$ it follows that P_α acts invariantly on $S(w)$ for any simple root α with $w^{-1}(\alpha) \in \Delta(L_P)$. Thus the stabilizer group of $S(w)$ is P_I where $I = \{\alpha_1, \cdots, \alpha_\ell\} \cap w(\Delta(P^-))$.

(2) $L_I \cap B$ is contained in $L_I \cap wP^-w^{-1}$ and thus $L_I \cap B^-$ is contained in the isotropy $L_I \cap wPw^{-1}$ of L_I at $x_w = wx_0$. Therefore, the L_I-orbit $L_I.x_w$ is closed.

If $w \neq id$, there is a simple root α such that $w^{-1}(\alpha) < 0$. Then α is contained in $I = \{\alpha_1, \cdots, \alpha_\ell\} \cap w(\Delta(P^-))$. Then, $w^{-1}(\alpha)$ is contained in $\Delta(U_P^-)$ because any $\beta < 0$ with $w(\beta) > 0$ is contained in $\Delta(U_P^-)$ by definition of W^P. Thus $U_\alpha \subset (L_I \cap U) \cap wU_P^-w^{-1}$ and $x_w \neq U_\alpha.x_w \subset L_I.x_w$. □

2.3. **$\mathbb{C}^*$-actions.** Let $\lambda : \mathbb{C}^* \to T$ be a cocharacter of T, i.e., a homomorphism from the multiplicative group $\mathbb{C}^*$ to T. Then λ induces a $\mathbb{C}^*$-action on G: $t.g = \lambda(t)g\lambda(t)^{-1}$ for

$t \in \mathbb{C}^*$ and $g \in G$, which again induces a $\mathbb{C}^*$-action on G/P because T normalizes P. Then for a root α and the isomorphism $u_\alpha : \mathbb{C} \to U_\alpha$, we have

$$t.u_\alpha(x) = \lambda(t)u_\alpha(x)\lambda(t)^{-1} = u_\alpha(t^{\langle \alpha,\lambda\rangle}x)$$

where $t \in \mathbb{C}^*$ and $x \in \mathbb{C}$.

Let A be a set consisting of roots. In coordinates $(x_\alpha)_{\alpha\in A}$ on $\prod_{\alpha\in A} U_\alpha$, the $\mathbb{C}^*$-action induced by λ on $\prod_{\alpha\in A} U_\alpha$ can be written as

$$t.(x_\alpha)_{\alpha\in A} = (t^{\langle\alpha,\lambda\rangle}x_\alpha)_{\alpha\in A}$$

for $t \in \mathbb{C}^*$ and $(x_\alpha)_{\alpha\in A} \in \mathbb{C}^{|A|}$.

Let $P(\lambda)$ be the subgroup of G consisting of $g \in G$ such that $\lim_{t\to 0} t.g$ exists. Then, $P(\lambda) = P_I$ for some choice of a Borel subgroup of G where I is the set of simple roots orthogonal to λ, and thus is a parabolic subgroup of G (8.4.5 of [36]). Conversely, any parabolic subgroup of G is of the form $P(\lambda)$ for some cocharacter λ. Define $P(-\lambda)$ by the subgroup of G consisting of $g \in G$ such that $\lim_{t\to\infty} t.g$ exists. If $P(\lambda) = P_I$, then $P(-\lambda) = P_I^-$.

One way to understand the $P_I^\pm$-orbit decomposition on $S = G/P$ is as the $(\pm)$-decomposition for the associated $\mathbb{C}^*$-action λ on S. Before giving this relation, we state Bialynicki-Birula decomposition theorem on an arbitrary projective manifold. For a fixed point x of a $\mathbb{C}^*$-action on a projective manifold X, noting that T_xX is a $\mathbb{C}^*$-module, let $(T_xX)^\pm$ denote the $\mathbb{C}^*$-submodule of T_xX spanned by all vectors $v \in T_xX$ such that for $t \in \mathbb{C}^*$ we have $t.v = t^m v$ for some $m \in \mathbb{Z}_\pm$.

Proposition 2.2 (Theorem 4.3 of [1]). *Given a $\mathbb{C}^*$-action on a nonsingular projective variety X let $\coprod_{i=1}^r X_i$ be the decomposition of its fixed point set into connected components. Then there are canonical decompositions $X = \coprod_{i=1}^r X_i^+$ and $X = \coprod_{i=1}^r X_i^-$ of X into locally closed nonsingular $\mathbb{C}^*$-invariant subvarieties and $\mathbb{C}^*$-fibrations $\gamma_i^+ : X_i^+ \to X_i$ and $\gamma_i^- : X_i^- \to X_i$ for $i = 1, \dots, r$ such that the fixed point set of $X_i^\pm$ is X_i and for $x \in X_i$, $T_xX_i^\pm = T_xX_i \oplus (T_xX)^\pm$ for $i = 1, \dots, r$. Furthermore, such decompositions are unique.*

The decomposition $X = \coprod_{i=1}^r X_i^+$ (respectively, $X = \coprod_{i=1}^r X_i^-$) is called the $(+)$-decomposition (respectively, the $(-)$-decomposition). For any $i = 1, \dots, r$

$$\begin{aligned} X_i^+ &= \{x \in X : \lim_{t\to 0} t.x \in X_i\} \\ X_i^- &= \{x \in X : \lim_{t\to\infty} t.x \in X_i\} \end{aligned}$$

and X_i^+ (respectively, X_i^-) is the stable (respectively, unstable) subvariety of X corresponding to X_i ([2]).

Proposition 2.3. *Let λ be a cocharacter of T with $P(\lambda) = P_I$. The $(\pm)$-decomposition of $S = G/P$ under the $\mathbb{C}^*$-action induced by λ is*

$$S = \coprod_{[\sigma]\in W_I\backslash W^P} P_I^\pm.x_\sigma$$

together with the projections $\gamma_{[\sigma]}^\pm : P_I^\pm.x_\sigma \to L_I.x_\sigma$. Furthermore, there is a unique $[\sigma^+] \in W_I\backslash W^P$ such that $P_I^-.x_{\sigma^+}$ is closed. In this case, $P_I^+.x_{\sigma^+}$ is open in S and it intersects $P_I^-.x_\tau$ nontrivially for any $\tau \in W^P$.

Proof. The decompositions $S = \coprod_{[\sigma]\in W_I\backslash W^P} P_I^+.x_\sigma$ and $S = \coprod_{[\sigma]\in W_I\backslash W^P} P_I^-.x_\sigma$ and projections $\gamma^+_{[\sigma]} : P_I^+.x_\sigma \to L_I.x_\sigma$ and $\gamma^-_{[\sigma]} : P_I^-.x_\sigma \to L_I.x_\sigma$ satisfy the conditions in Proposition 2.2. The first statement follows from the uniqueness of the $(\pm)$-decomposition.

If $P_I^-.x_\sigma$ is closed, then U_I^- acts on $P_I^-.x_\sigma$ trivially and thus we have $P_I^-.x_\sigma = L_I.x_\sigma$ and $\dim(T_xS)^- = 0$ for $x \in L_I.x_\sigma$. Therefore, $P_I^+.x_\sigma$ is open in S, and since the decomposition is locally closed, there is a unique $[\sigma^+] \in W_I\backslash W^P$ such that $P_I^+.x_{\sigma^+}$ is open (cf. Corollary 1 of [1]). Therefore, $P_I^-.x_{\sigma^+}$ is the unique closed P_I^--orbit in S.

Suppose that $P_I^+.x_{\sigma^+}$ is disjoint from $P_I^-.x_\tau$ for some $\tau \in W^P$. Then $P_I^-.x_\tau$ is contained in the complement Z of $P_I^+.x_{\sigma^+}$ in S. Since Z is P_I^+-invariant, $P_I^+P_I^-.x_\tau$ is contained in Z. However, $P_I^+P_I^-$ is open in G, and thus $P_I^+P_I^-.x_\tau$ is open in S, contradicting to the fact that Z is a proper subvariety of S. Therefore, $P_I^+.x_{\sigma^+}$ intersects $P_I^-.x_\tau$ for every $\tau \in W^P$. □

Remark 2.1. For $\sigma, \tau \in W^P$, $P_I^-.x_\sigma$ is in the closure of $P_I^-.x_\tau$ if and only if $P_I^+.x_\sigma \cap P_I^-.x_\tau$ is nonempty (Corollary 1.2 of [10], Section 1.3 of [4]).

3. Transversality with respect to a $\mathbb{C}^*$-action

Let $\mathcal{O}$ be a P_I^--orbit in $S = G/P$ and let $\overline{\mathcal{O}}$ be its closure in S. By the transversality of a general translate ([26]), for any subvariety Z of S of complementary dimension, there is a translate gZ of Z by an element $g \in G$ which intersects $\overline{\mathcal{O}}$ transversely. Then the intersection multiplicity of gZ and $\overline{\mathcal{O}}$ at each intersection point is one, so that we have $gZ \cap \overline{\mathcal{O}} = p_1 + \cdots + p_r$ where all p_i are distinct. Take the limit $\lim_{t\to\infty} t.p_i := x_i$ for the $\mathbb{C}^*$-action λ associated to P_I^- (the one induced by a cocharacter λ of T such that $P(-\lambda) = P_I^-$). Then x_i's are not necessarily distinct any more. The goal of this section is to prove that there is a translate Z' of Z such that

(1) Z' intersects $\mathcal{O}$ transversely and
(2) the limits $\lim_{t\to 0} t.p'_i$ are all distinct, where $p'_1 + \cdots + p'_r = Z' \cap \mathcal{O}$

(see Proposition 3.3). Let $\gamma : \mathcal{O} \to \mathcal{H}$ be the projection map to the fixed point set $\mathcal{H}$ in $\mathcal{O}$ of the $\mathbb{C}^*$-action λ. Then the condition (2) is equivalent to the condition that $\gamma_*(Z' \cap \mathcal{O}) = x_1 + \cdots + x_r$ with all x_i being distinct.

Because the complement of $\mathcal{O}$ in $\overline{\mathcal{O}}$ has dimension less than $\dim \mathcal{O}$, we can find a translate of Z satisfying the condition (1), if we apply the transversality Theorem of the following form. For subvarieties Y, Z of S, we say that Y meets Z *properly* if for each irreducible component C of $Y \cap Z$, $\operatorname{codim}(C) = \operatorname{codim} Y + \operatorname{codim} Z$. Recall that as a linear algebraic group G is an affine algebraic manifold.

Proposition 3.1 ([26], Lemma 1.3.1 of [4])**.** *Let Y, Z be subvarieties of a rational homogeneous manifold $S = G/P$. Let $Y_0 \subset Y$ (resp. $Z_0 \subset Z$) be the nonsingular locus of Y (resp. Z). Then, there exists a nonempty Zariski open subset W of G such that for any group element g belonging to W, Y meets gZ properly, and $Y_0 \cap gZ_0$ is nonsingular and dense in $Y \cap gZ$. In particular, if $\dim(Y) + \dim(Z) = \dim(S)$, then Y meets gZ transversely for all g belonging to some dense Zariski open subset of G.*

To get a translate of Z satisfying the condition (2) we use a birational morphism $f : \widetilde{\mathcal{O}} \to \overline{\mathcal{O}}$ constructed in the following way. Let $\mathcal{H}$ be the fixed point set of the $\mathbb{C}^*$-action λ in $\mathcal{O}$. Then $\mathcal{H}$ is an L_I-orbit. Take a base point o of $\mathcal{H}$ and put F to be the U_I^--orbit of o. Then $\gamma : \mathcal{O} \to \mathcal{H}$ is the L_I-homogeneous fiber bundle over $\mathcal{H}$ with typical fiber F. Take the closure $\overline{F}$ of F in S. Then the homogeneous fiber bundle $\widetilde{\mathcal{O}}$ over $\mathcal{H}$ with fiber $\overline{F}$ has $\mathcal{O}$ as a dense open subset. Define a map

$$\begin{array}{ccc} \widetilde{\mathcal{O}} = L_I \times_{P_1} \overline{F} & \xrightarrow{f} & \overline{\mathcal{O}} \\ \downarrow \widetilde{\gamma} & & \\ \mathcal{H} = L_I/P_1 & & \end{array}$$

by $f([\ell, x]) = \ell x$ for $\ell \in L_I$ and $x \in \overline{F}$, where P_1 is the isotropy group of L_I at o. Then f is a birational morphism. For $z \in \mathcal{H}$, the restriction of f to the fiber $\widetilde{\gamma}^{-1}(z)$ of $\widetilde{\gamma}$ is a closed embedding into S. We denote the image $f(\widetilde{\gamma}^{-1}(z))$ by $\overline{F}_z$. Then $\overline{\mathcal{O}}$ is the union of $\overline{F}_z$, where $z \in \mathcal{H}$. Thus, for $y \in \overline{\mathcal{O}}$, $f^{-1}(y)$ is biholomorphic to the subvariety of $\mathcal{H}$ consisting of points $z \in \mathcal{H}$ with $y \in \overline{F}_z$. Therefore, f is proper.

Remark 3.1. The proper morphism $f : \widetilde{\mathcal{O}} \to \overline{\mathcal{O}}$ is a generalization of a collapsing of a homogeneous vector bundle in [25].

Set $\mathcal{E}$ to be the subset $\{y \in \overline{\mathcal{O}} : \dim f^{-1}(y) > 0\}$ of $\overline{\mathcal{O}}$. Since $\overline{\mathcal{O}}$ is a Schubert variety, it is normal, and thus the restriction of f to $\widetilde{\mathcal{O}} - f^{-1}(\mathcal{E})$ is a biholomorphism onto $\overline{\mathcal{O}} - \mathcal{E}$. Therefore, $f^{-1}(\mathcal{E})$ is the exceptional locus of f and $\mathcal{E}$ is the image $f(\mathrm{Exc}(f))$ of the exceptional locus of f.

Lemma 3.2. *Let $\overline{F}$ and $\mathcal{E}$ be as in the above. For any $x \in \overline{F} \cap \mathcal{E}$ and $y \in F$, there is a vector field X of TS such that $X(x) = 0$ and $X(y) \neq 0 \in T_y(L_I.y)$.*

Proof. For $x \in \overline{F}$, let P_x be the isotropy group of L_I at x. Then x is contained in $\mathcal{E}$ if and only if $P_x.o$ has positive dimension, because $\mathcal{E}$ consists of $x \in \overline{\mathcal{O}}$ such that $\dim f^{-1}(x) > 0$. Thus for $x \in \mathcal{E} \cap \overline{F}$, there is a one parameter subgroup (ℓ_ϵ) of P_x such that $\frac{d}{d\epsilon}|_{\epsilon=0} \ell_\epsilon.o \neq 0$. Furthermore, for any subgroup P' of L_I and for any $y \in F$, $P'.o$ has positive dimension if and only if $P'.y$ has positive dimension. Therefore, for $x \in \mathcal{E} \cap \overline{F}$ and $y \in F$, there is a one parameter subgroup (ℓ_ϵ) of P_x such that $\frac{d}{d\epsilon}|_{\epsilon=0} \ell_\epsilon.y \neq 0$. □

Proposition 3.3. *Let $S = G/P$ be a rational homogeneous manifold and let $\mathcal{O}$ be a P_I^--orbit in S. Assume that the exceptional locus of $f : \widetilde{\mathcal{O}} \to \overline{\mathcal{O}}$ associated to $\mathcal{O}$ is not empty. Let Z be a reduced irreducible subvariety of S of dimension* $\dim S - \dim \mathcal{O}$ *with nonzero intersection number with $\overline{\mathcal{O}}$. Then there exists $g \in G$ such that*

(1) *gZ intersects $\mathcal{O}$ transversely and*
(2) *$\gamma_*(gZ \cap \mathcal{O}) = x_1 + \cdots + x_r$ with all x_i being distinct,*

where $\gamma : \mathcal{O} \to \mathcal{H}$ is the projection map to the fixed point set $\mathcal{H}$

Proof. Without loss of generality, up to the action of G, we may assume that Z intersects $\overline{\mathcal{O}}$ transversely and the intersection $Z \cap \overline{\mathcal{O}}$ is contained in $\mathcal{O}$ (Proposition 3.1). Then Z intersects $\mathcal{O}$ at r distinct points. If $\mathcal{O}$ is closed, then $\mathcal{O} = \mathcal{H}$ and there is nothing to prove. From now one we assume that $\mathcal{O}$ is not closed and thus $\gamma : \mathcal{O} \to \mathcal{H}$ has positive fiber dimension. Let F be the U_I^--orbit of the base point o of $\mathcal{H}$ and $\mathcal{E}$ be the image $f(\mathrm{Exc}(f))$ of the exceptional locus of f as in Lemma 3.2.

Write $\gamma_*(Z \cap \mathcal{O}) = n_1 x_1 + \cdots + n_s x_s$ (where all x_i are distinct). Put $m_0 = 0$ and $m_k = m_{k-1} + n_k$ for $k = 1, \ldots, s$. We order points $p_1, \ldots, p_r$ in $Z \cap \mathcal{O}$ in such a way that

p_i lies over x_k for $m_{k-1}+1 \le i \le m_k$, $k = 1, \dots, s$. Take a neighborhood $\mathcal{U}_k$ of x_k in $\mathcal{H}$ for $1 \le k \le s$ such that $\mathcal{U}_k \cap \mathcal{U}_l = \emptyset$ for any $1 \le k \neq l \le s$. For $m_{k-1}+1 \le i \le m_k$, take a neighborhood $\mathcal{B}_i$ of p_i such that $\gamma(\mathcal{B}_i) \subset \mathcal{U}_k$ for any $m_{k-1}+1 \le i \le m_k$ and $\mathcal{B}_i \cap \mathcal{B}_j = \emptyset$ for any $m_{k-1}+1 \le i \neq j \le m_k$.

Let $\{g_\epsilon\}$ be a one parameter subgroup of automorphisms of S. Then, for sufficiently small ϵ, the intersection $g_\epsilon Z \cap \mathcal{O}$ is contained in the union $\coprod_{i=1}^r \mathcal{B}_i$. We order the points $p_{1,\epsilon}, p_{2,\epsilon}, \dots, p_{r,\epsilon}$ in the intersection $g_\epsilon Z \cap \mathcal{O}$ in such a way that $p_{i,\epsilon} \in \mathcal{B}_i$ for all $i = 1, \dots, r$. By the construction of $\mathcal{B}_i$, $p_{i,\epsilon}$ and $p_{j,\epsilon}$ lie in different fibers if p_i and p_j lie in different fibers. Therefore, it suffices to show that for any two points p_i and p_j in the same fiber, there is g_ϵ such that $p_{i,\epsilon}$ and $p_{j,\epsilon}$ lie in different fibers.

If p_i were in $\mathcal{E}$, then, by Lemma 3.2, there would be a one-parameter subgroup (ℓ_ϵ) of L_I such that ℓ_ϵ fixes p_i and ℓ_ϵ sends p_j to a different fiber. However, our p_i is not in $\mathcal{E}$ and we cannot apply this lemma directly.

If $n_k = 1$ for all $k = 1, \dots, s$, then there is nothing to prove. Suppose that $n_1 > 1$. We will use the same notations as in Lemma 3.2. Take two points p_1, p_2 in $(Z \cap \overline{\mathcal{O}}) \cap \gamma^{-1}(x_1)$. We may assume that p_1, p_2 lie in F. Consider the limit $p_0 := \lim_{t\to 0} t.p_1$, where the action of $t \in \mathbb{C}^*$ is given by the $\mathbb{C}^*$-action λ associated to P_I^-.

Assume that p_0 belongs to $\mathcal{E}$. Then, by Lemma 3.2, for each $t \in \mathbb{C}^*$, there is a vector field X of TS such that $X(p_0) = 0$ and $X(t.p_2) \notin T_{t.p_2} t.Z + T_{t.p_2} F$. Thus, for sufficiently small t, there is a vector field X' of TS such that $X'(t.p_1) = 0$ and $X'(t.p_2) \notin T_{t.p_2} t.Z + T_{t.p_2} F$. By taking such a $t \in \mathbb{C}^*$ and by replacing Z by $t.Z$, we may assume that there is a vector field Y of TS such that $Y(p_1) = 0$ and $Y(p_2) \notin T_{p_2} Z + T_{p_2} F$.

Let $\{g_\epsilon\}_{\epsilon\in\mathbb{C}}$ denote the one parameter subgroup of automorphisms of S associated to Y. Let $p_{1,\epsilon}, p_{2,\epsilon}, \dots, p_{r,\epsilon}$ be points in the intersection $g_\epsilon Z \cap \mathcal{O}$, ordered in such a way as at the beginning of the proof. Then, $p_{1,\epsilon} = p_1 \in F$ for all ϵ. We will show that $p_{2,\epsilon} \notin F$ for some ϵ.

Suppose that $p_{2,\epsilon} \in F$ for all ϵ. Then $\frac{d}{d\epsilon}|_{\epsilon=0} p_{2,\epsilon} \in T_{p_2} F$. Write $p_{2,\epsilon} = g_\epsilon z_\epsilon$, where $z_\epsilon \in Z$. Since $g_0 =$ the identity and $p_{2,0} = p_2$, we have $z_0 = p_2$. From

$$\frac{d}{d\epsilon}|_{\epsilon=0} p_{2,\epsilon} = \frac{d}{d\epsilon}|_{\epsilon=0} g_\epsilon p_2 + \frac{d}{d\epsilon}|_{\epsilon=0} z_\epsilon \in T_{p_2} F$$

and $\frac{d}{d\epsilon}|_{\epsilon=0} z_\epsilon \in T_{p_2} Z$, it follows that we have $Y(p_2) = \frac{d}{d\epsilon}|_{\epsilon=0} g_\epsilon p_2 \in T_{p_2} Z + T_{p_2} F$, which contradicts to the assumption $Y(p_2) \notin T_{p_2} Z + T_{p_2} F$. Therefore, $p_{2,\epsilon} \notin F$ for some ϵ.

If $\mathcal{E}$ is equal to the boundary of $\mathcal{O}$, then any limit $p_0 = \lim_{t\to 0} t.p_1$ lies in $\mathcal{E}$ and by the arguments in the above there is $g_\epsilon \in G$ such that $p_{1,\epsilon}$ and $p_{2,\epsilon}$ lie in different fibers. This is the case if, for example, $\overline{\mathcal{O}}$ is an odd symplectic Grassmannian embedded into a symplectic Grassmannian. However, $\mathcal{E}$ is not necessarily equal to the boundary of $\mathcal{O}$.

Lemma 3.4. *Assume that $\mathcal{E}$ is nonempty. Then the set $\mathcal{E}_F^+$ of points $p \in F$ such that the limit $\lim_{t\to 0} t.p$ is contained in $\mathcal{E}$ is nonempty.*

Proof. The $P_I^{\pm}$-orbit decomposition of $S = G/P$ is given by

$$S = \coprod_{[\sigma]\in W_I\backslash W^P} \mathcal{O}_\sigma^{\pm}$$

where $\mathcal{O}_\sigma^{\pm}$ is $P_I^{\pm}.x_\sigma$. Our P_I^--orbit $\mathcal{O}$ is $\mathcal{O}_w^-$ for some $w \in W^P$. Let $\Sigma := \mathcal{O}_\sigma^-$ be the unique closed P_I^--orbit in S (Proposition 2.3). Since Σ is closed, it is in fact equal to $\mathcal{H}_\sigma$, the fixed point set in $\mathcal{O}_\sigma^-$. Thus any point in $\mathcal{O}_\sigma^+$ has limit in $\Sigma = \mathcal{H}_\sigma$ as t goes to 0. By Proposition 2.3, $\mathcal{O}_\sigma^+$ has nonempty intersection with $\mathcal{O}_w^-$. Since $\mathcal{O}_\sigma^+ \cap \mathcal{O}_w^-$ is invariant under the action of L_I, $\mathcal{O}_\sigma^+$ has nonempty intersection with F, too.

Since $\mathcal{E}$ is closed and invariant under the action of P_I^-, and is nonempty, the unique closed P_I^--orbit Σ is contained in $\mathcal{E}$ and hence $\mathcal{E}_F^+$ is nonempty. □

By Lemma 3.4, there is an element u of U_I^- such that $u.p_1$ belongs to $\mathcal{E}_F^+$. By replacing Z by $u.Z$, we may assume that p_1 has the property that the limit $p_0 := \lim_{t\to 0} t.p_1$ belongs to $\mathcal{E}$. The proof goes in the same line as before, so that we get $g_\epsilon \in G$ such that $p_{1,\epsilon}$ and $p_{2,\epsilon}$ lie in different fibers.

Now the number of intersection points in $g_\epsilon Z \cap \mathcal{O}$ lying over the fiber F is less than n_1. Repeat this process until we get $g \in G$ such that any two points in $gZ \cap \mathcal{O}$ lie in different fibers of γ. □

Remark 3.2. (1) The closure $\overline{\mathcal{O}}$ of a P_I^--orbit in $S = G/P$ is a Schubert variety, and if S has Picard number one, then $\overline{\mathcal{O}}$ has Picard number one, too ([30], [4]), and therefore, the exceptional locus of f is always nonempty (See Proposition 3.5).

(2) Applying Proposition 3.1 to Z and Borel group orbits in the complement of the open Borel orbit in $\mathcal{O}$, we may assume that Z intersects only the open B^--orbit in $\mathcal{O}$ at the beginning of the proof. In the rest of the proof we can still maintain that the intersection points are contained in the open Borel group orbit, so that there is a translate of Z satisfying conditions (1) - (3) in Proposition 3.6.

Proposition 3.5. *Let $S = G/P$ be the rational homogeneous manifold of Picard number one and let $T(w)$ be an opposite Schubert variety of S so that the stabilizer of $T(w)$ is P_I^-. Let $f : \widetilde{\mathcal{O}} \to \overline{\mathcal{O}}$ be the birational morphism associated with the open P_I^--orbit $\mathcal{O}$ in $T(w)$. If P_I^- has more than one orbits in $T(w)$, then the exceptional locus of f is nonempty.*

Proof. The open P_I^--orbit $\mathcal{O}$ is an L_I-homogeneous fiber bundle over the fixed point set $\mathcal{H}$ in $\mathcal{O}$. By the same arguments as in the proof of Proposition 2.1, $\mathcal{H}$ is of positive dimension. If P_I^- has more than one orbit in $T(w)$, then $\mathcal{O}$ is a nontrivial fiber bundle over $\mathcal{H}$ and so is $\widetilde{\mathcal{O}}$. Hence $\widetilde{\mathcal{O}}$ has Picard number ≥ 2. If S has Picard number one, then so does $T(w)$ ([30]). Since $\mathcal{O}$ is an open P_I^--orbit in $T(w)$, its closure $\overline{\mathcal{O}}$ is $T(w)$, and thus $\overline{\mathcal{O}}$ has Picard number one.

Suppose that the exceptional locus of f is empty. Then the proper birational morphism $f : \widetilde{\mathcal{O}} \to \overline{\mathcal{O}}$ is a biholomorphism because $\overline{\mathcal{O}}$ is normal. Thus $\widetilde{\mathcal{O}}$ and $\overline{\mathcal{O}}$ have the same Picard number, a contradiction. Therefore, the exceptional locus of f is nonempty. □

Proposition 3.6. *Let $S = G/P$ be the rational homogeneous manifold and let $T(w)$ be an opposite Schubert variety of S whose stabilizer in G is P_I^-. Let $Z \subset S$ be a reduced irreducible subvariety of S having homology class $[Z] = r[S(w)]$ and*

1. *Z intersects $T(w)$ transversely and*
2. *$Z \cap T(w)$ is contained in $B^-.x_w$ and*
3. *$\gamma_*(Z \cap T(w)) = x_1 + \cdots + x_r$ with all x_i being distinct,*

where γ is the projection map from $\mathcal{O} := P_I^-.x_w$ to $\mathcal{H} := L_I.x_w$. Consider the $\mathbb{C}^*$-action associated with P_I^-. Let $\mathcal{Z} \subset S \times \mathbb{P}^1$ be the closure of the union $\mathcal{Z}^0 := \bigcup_{t\in\mathbb{C}} t.Z \times \{t\}$ in $S \times \mathbb{P}^1$ and $\pi : \mathcal{Z} \to \mathbb{P}^1$ be the restriction of the second projection map $S \times \mathbb{P}^1 \to \mathbb{P}^1$.

Write $\pi^{-1}(\infty) = Z_\infty \times \{\infty\}$. *Then* $\mathcal{Z}$ *is irreducible and* Z_∞ *is* $g_1 S_0 + \cdots + g_r S_0$, *where* $g_i \in G$ *is such that* $x_i = g_i.x_w$ *for* $i = 1, \ldots, r$.

Remark 3.3. It may hold under a weaker condition (2)' than the condition (2), that $Z \cap T(w)$ is contained in $\mathcal{O}$. For simplicity of the proof, we put the condition (2) instead of (2)'. As we remark after Proposition 3.3, this will not cause any problem in later use.

Proof. We will first consider the closure of $\mathcal{Z}^0 \cap (\mathcal{U} \times \mathbb{P}^1)$, where $\mathcal{U} := w(U_P^-).x_w$ is the open big cell in S containing x_w. Write $w(U_P^-) = \prod_{\alpha \in A} U_\alpha$ for a set A consisting of roots. Then

$$w(U_P^-) = \left(\prod_{\alpha \in A^+} U_\alpha\right)\left(\prod_{\beta \in A^0} U_\beta\right)\left(\prod_{\gamma \in A^-} U_\gamma\right),$$

where

$$\begin{aligned} A^+ &= \{\alpha \in A : \langle \alpha, \lambda \rangle > 0\} \\ A^0 &= \{\alpha \in A : \langle \alpha, \lambda \rangle = 0\} \\ A^- &= \{\alpha \in A : \langle \alpha, \lambda \rangle < 0\} \end{aligned}$$

Choose coordinates $(z_\alpha, z_\beta, z_\gamma)_{\alpha \in A^+, \beta \in A^0, \gamma \in A^-}$ on $\mathcal{U} \simeq w(U_P^-)$ keeping this order. Putting $n_\alpha = \langle \alpha, \lambda \rangle$ for $\alpha \in A^+$ and $n_\gamma = -\langle \gamma, \lambda \rangle$ for $\gamma \in A^-$, we get that the $\mathbb{C}^*$-action induced by λ can be expressed as:

$$t.(z_\alpha, z_\beta, z_\gamma) = (t^{n_\alpha} z_\alpha, z_\beta, t^{-n_\gamma} z_\gamma)$$

for $t \in \mathbb{C}^*$. The closure of $\{(0, z_\beta, z_\gamma) : z_\beta \in \mathbb{C}^{|A^0|}, z_\gamma \in \mathbb{C}^{|A^-|}\}$ in S is $T(w)$. For each fixed $z_{\beta,0} \in \mathbb{C}^{|A^0|}$, the closure of $\{(z_\alpha, z_{\beta,0}, 0) : z_\alpha \in \mathbb{C}^{|A^+|}\}$ in S is $gS(w)$, where $g \in L_I$ is such that $g.x_w$ has coordinates $(0, z_{\beta,0}, 0)$.

Let $Z \subset X$ be a subvariety satisfying the assumptions (1) – (3). Write $Z \cap T(w) = \{p_1, \ldots, p_r\}$. By (1) and (2), p_i are all distinct and are contained in $\mathcal{U}$. Let $(p_{i,\alpha}, p_{i,\beta}, p_{i,\gamma})$ be the coordinates of p_i for $i = 1, \ldots, r$. We will show that for each p_i there is a neighborhood V_i of p_i in Z such that $t.V_i$ converges to $\{(z_\alpha, p_{i,\beta}, 0) : z_\alpha \in \mathbb{C}^{|A^+|}\}$ as $t \to \infty$.

Fix i. By (1) and (2), there is a neighborhood V of p_i in $Z \cap \mathcal{U}$ of the form

$$V = \left\{(z_\alpha, G_\beta(z_\alpha), H_\gamma(z_\alpha)) : (z_\alpha)_{\alpha \in A^+} \in V'\right\}$$

where V' is an open ball centered at 0 in $\mathbb{C}^{|A^+|}$, and $G = (G_\beta), H = (H_\gamma)$ are holomorphic maps defined on V' with values in $\mathbb{C}^{|A^0|}$, $\mathbb{C}^{|A^-|}$ such that $G_\beta(0) = p_{i,\beta}$.

Now

$$t.V = \left\{(t^{n_\alpha} z_\alpha, G_\beta(z_\alpha), t^{-n_\gamma} H_\gamma(z_\alpha)) : (z_\alpha)_{\alpha \in A^+} \in V'\right\}$$

for $t \in \mathbb{C}^*$. Put $y_\alpha = t^{n_\alpha} z_\alpha$. Then

$$t.V = \left\{(y_\alpha, G_\beta(t^{-n_\alpha} y_\alpha), t^{-n_\gamma} H_\gamma(t^{-n_\alpha} y_\alpha)) : (y_\alpha)_{\alpha \in A^+} \in t.V'\right\}.$$

As $t \to \infty$, $G_\beta(t^{-n_\alpha} y_\alpha)$ tends to $G_\beta(0)$ and $t^{-n_\gamma} H_\gamma(t^{-n_\alpha} y_\alpha)$ tends to zero, so that $t.V$ converges to

$$\left\{(y_\alpha, p_{i,\beta}, 0) : (y_\alpha)_{\alpha \in A^+} \in \mathbb{C}^{|A^+|}\right\},$$

which is an open dense subset of $g_i S(w)$.

Consequently, the support of Z_∞ contains the support of the cycle $g_1 S(w) + \cdots + g_r S(w)$. Since $\mathcal{Z}^0$ is irreducible, so is $\mathcal{Z}$. Thus Z_∞ has the same homology class as Z (Section 1.6 of [11]). Therefore, Z_∞ is equal to $g_1 S(w) + \cdots + g_r S(w)$. □

4. Rigidity

We review the geometric theory of uniruled projective manifolds modeled on varieties of minimal rational tangents and its application to the homological rigidity of smooth Schubert varieties ([15], for an expository survey see [32]) and then we generalize the method to prove the Schur rigidity.

4.1. **Varieties of minimal rational tangents.** Let $(X, \mathcal{L})$ be a polarized projective manifold. A (parameterized) rational curve on X is a nonconstant holomorphic map $f : \mathbb{P}^1 \to X$. A rational curve f is said to be *free* if the pull-back bundle f^*TX on $\mathbb{P}^1$ is semipositive. A free rational curve f such that the degree $f^*\mathcal{L}$ is minimum among all free rational curves is called a *minimal rational curve.* Let $\mathcal{H}$ be a connected component of $\mathrm{Hom}(\mathbb{P}^1, X)$ containing a minimal rational curve and let $\mathcal{H}^0$ be the subset consisting of free rational curves. Then the quotient space $\mathcal{K} = \mathcal{H}^0 / \mathrm{Aut}(\mathbb{P}^1)$ of (unparameterized) minimal rational curves is called a *minimal rational component.*

Fix a minimal rational component $\mathcal{K}$. By a minimal rational curve we implicitly mean a minimal rational curve belonging to $\mathcal{K}$. For a general $x \in X$ the space $\mathcal{K}_x$ of minimal rational curves passing through x is a projective manifold. Define a rational map from $\mathcal{K}_x$ to $\mathbb{P}(T_xX)$ by associating a minimal rational curve immersed at x to the tangent line at x, which is called the tangent map. The strict transformation $\mathcal{C}_x(X)$ of the tangent map is called the *variety of minimal rational tangents of X at x.* The union of $\mathcal{C}_x(X)$ over general $x \in X$ forms a fibered space $\mathcal{C}(X)$ over X.

Let $S = G/P$ be a rational homogeneous manifold associated to a simple root. Then the Picard number of S is one and the ample generator of the Picard group defines a G-equivariant embedding of S into a projective space $\mathbb{P}^N$. Lines $\mathbb{P}^1$ lying on S are minimal rational curves, and we will choose the family $\mathcal{K}$ of lines lying on S as our minimal rational component, so that the variety $\mathcal{C}_x(S)$ of minimal rational tangents of S at any x in S is defined by the space of all tangent directions of lines lying on S passing through x. If S is associated to a long root, then G acts on $\mathcal{K}$ transitively. If S is associated to a short root, then $\mathcal{K}$ has two G-orbits. In any case, by a general line we mean a line corresponding to a point in the open G-orbit in $\mathcal{K}$, and by a general point in $\mathcal{C}_x(S)$ we mean the tangent direction of a general line.

Let Z be an irreducible (possibly singular) subvariety of S uniruled by lines contained in the smooth locus of Z. By a general point in Z we mean a point z in the smooth locus of Z such that there is a line passing through z contained in the smooth locus of Z and the deformation of any line passing through z covers an open dense subset of Z. By the variety $\mathcal{C}_x(Z)$ of minimal rational tangents of Z at a general $x \in Z$ we mean the variety of minimal rational tangents associated to the family $\mathcal{K}_Z$ of lines lying on Z.

A smooth Schubert variety S_0 of S is uniruled by lines lying on S_0. By a general point of S_0 we mean a point in the open orbit in S_0 of the stabilizer of S_0 in G. When there is a general line contained in S_0, by a general point in $\mathcal{C}_x(S_0)$ at a general point $x \in S_0$ we mean the tangent direction of a general line passing through x. For descriptions of $\mathcal{C}_x(S_0)$ see [15].

4.2. **Local characterizations.** Let $S = G/P$ be a rational homogeneous manifold associated to a simple root, and $S_0 \subset S$ be a smooth Schubert variety. Consider the following two conditions:

(I) at a general point $\alpha \in \mathcal{C}_x(S_0)$, for any $h \in P_x$ sufficiently close to the identity element $e \in P_x$ and satisfying $T_\alpha(h\mathcal{C}_x(S_0)) = T_\alpha(\mathcal{C}_x(S_0))$ we must have $h\mathcal{C}_x(S_0) = \mathcal{C}_x(S_0)$;

(II) any local deformation of $\mathcal{C}_x(S_0)$ in $\mathcal{C}_x(S)$ is induced by the action of P_x,

where P_x is the isotropy of G at a general point $x \in S_0$.

Proposition 4.1 (Proposition 3.2 and Proposition 3.6 of [15])**.** *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root, and $S_0 \subset S$ be a smooth Schubert variety. Assume that $\mathcal{C}_x(S_0)$ satisfies* (I) *and* (II) *at a general point $x \in S_0$. Then, the following holds true.*

(1) *If a smooth subvariety Z of S is uniruled by lines of S lying on Z and contains x as a general point with $\mathcal{C}_x(Z) = \mathcal{C}_x(S_0)$, then S_0 is contained in Z.*

(2) *Any local deformation of S_0 in S is induced by the action of G.*

Proposition 4.1 (2), together with Proposition 2.2 of [15], implies that (I) and (II) are sufficient conditions for a smooth Schubert variety to be homologically rigid.

Theorem 4.2 ([15])**.** *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root, and $S_0 \subset S$ be a smooth Schubert variety. Assume that $\mathcal{C}_x(S_0)$ satisfies* (I) *and* (II) *at a general point $x \in S_0$. Then any subvariety $Z \subset S$ having homology class $[Z] = [S_0]$ is gS_0 for some $g \in G$.*

We will modify the proof of Proposition 4.1 to get sufficient conditions for a smooth Schubert variety to be Schur rigid (Theorem 4.5). Proposition 3.3 and Proposition 3.6 will replace arguments in the proof of Proposition 2.2 of [15]. In what follows for a smooth Zariski open subset W of an irreducible projective subvariety $\Sigma \subset \mathbb{P}^N$, $N \geq 2$, we say that W is uniruled by lines to mean that there exists a projective line ℓ lying on W with semipositive normal bundle. Equivalently, W is uniruled by lines if and only if the union of lines lying on W covers a nonempty Zariski open subset of W.

Proposition 4.3. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root, and $S_0 \subset S$ be a smooth Schubert variety. Assume that $\mathcal{C}_x(S_0)$ satisfies* (I) *and* (II) *at a general point $x \in S_0$ as in Proposition 4.1. Then, the following holds true.*

(1) *Let Z be a reduced and irreducible* (*possibly singular*) *subvariety of S uniruled by lines contained in the smooth locus of Z. If Z contains x as a general point with $\mathcal{C}_x(Z) = \mathcal{C}_x(S_0)$, then S_0 is contained in Z.*

(2) *Let $\mathcal{Z} \subset S \times \mathbb{C}$ be an irreducible* (*possibly singular*) *subvariety with the restriction $\pi : \mathcal{Z} \to \mathbb{C}$ of the second projection map $S \times \mathbb{C} \to \mathbb{C}$. If there is a reduced and smooth quasi-projective subvariety $\widehat{S}_0 \times \{0\}$ of $\pi^{-1}(0) \cap (S_0 \times \{0\})$ with $\dim \widehat{S}_0 = \dim S_0$ which is uniruled by lines contained in $\widehat{S}_0 \times \{0\}$, then for $t \in \mathbb{C}$ sufficiently small, there is $g_t \in G$ such that $g_t S_0 \times \{t\}$ is contained in $\pi^{-1}(t)$.*

Proof. (1) By applying the same arguments as in the proof of Proposition 4.1 (1) (Proposition 3.2 of [15]), using the assumption (I) and (II) and deformation theory of minimal rational curves, we get that S_0 is contained in Z.

(2) From the smoothness of $\widehat{S}_0$ it follows that there is a restriction $\widehat{\pi} : \widehat{\mathcal{Z}} \to \mathbb{C}$ of $\pi : \mathcal{Z} \to \mathbb{C}$ to a smooth quasi projective Zariski dense open subset $\widehat{\mathcal{Z}}$ of $\mathcal{Z}$ with $\widehat{\pi}$ being smooth and $\widehat{\pi}^{-1}(0) = \widehat{S}_0 \times \{0\}$, and $\widehat{\pi}^{-1}(t)$ is contained in the smooth locus of $\pi^{-1}(t)$ for any t (See the proof of Lemma 1.1 of [31]). For $\epsilon > 0$ sufficiently small there exists

a holomorphic section $\sigma : \Delta(\epsilon) \to \widehat{\mathcal{Z}}$ of $\widehat{\pi} : \widehat{\mathcal{Z}} \to \mathbb{C}$ over $\Delta(\epsilon)$ such that $\sigma(0)$ is a general point of S_0. By Kodaira stability [27] any line contained in $\widehat{\pi}^{-1}(0)$ passing through $\sigma(0)$ can be deformed to a line in $\widehat{\pi}^{-1}(t)$ passing through $\sigma(t)$. By putting together all such lines we get $\bigcup_{t\in\Delta(\epsilon)} \mathcal{C}^0_{\sigma(t)}(\widehat{\pi}^{-1}(t)) \subset \bigcup_{t\in\Delta(\epsilon)} \left(\mathcal{C}_{\sigma(t)}(S) \times \{t\}\right)$, where $\mathcal{C}^0_{\sigma(t)}(\widehat{\pi}^{-1}(t))$ is a Zariski open subset of some subvariety of $\mathcal{C}_{\sigma(t)}(S) \times \{t\}$. Taking topological closure in $\bigcup_{t\in\Delta(\epsilon)} \left(\mathcal{C}_{\sigma(t)}(S) \times \{t\}\right)$ we obtain $\bigcup_{t\in\Delta(\epsilon)} \mathcal{C}_{\sigma(t)}(\widehat{\pi}^{-1}(t)) \subset \bigcup_{t\in\Delta(\epsilon)} \left(\mathcal{C}_{\sigma(t)}(S) \times \{t\}\right)$. Since $\mathcal{C}_{\sigma(0)}(\widehat{\pi}^{-1}(0)) = \left(\mathcal{C}_{\sigma(t)}(S) \times \{0\}\right)$ is smooth, $\bigcup_{t\in\Delta(\epsilon)} \mathcal{C}_{\sigma(t)}(\widehat{\pi}^{-1}(t)) \to \mathbb{C}$ can be regarded as a regular family of submanifolds of $\mathcal{C}_x(S)$. By the assumption (II), there is $g_t \in G$ such that $\mathcal{C}_{\sigma(t)}(\widehat{\pi}^{-1}(t)) = \mathcal{C}_{\sigma(t)}(g_t S_0) \times \{t\}$. By (1), $g_t S_0 \times \{t\}$ is contained in $\pi^{-1}(t)$. □

Remark 4.1. As mentioned in the Introduction, in order to adapt the proof of Proposition 4.3 to a singular Schubert variety $S_0 \subset S$ we need at least to have an ample supply of minimal rational curves on the smooth locus of S_0 so that Kodaira Stability Theorem can be applied. However, there exist singular Schubert varieties S_0 on certain S such that all projective lines on S lying on S_0 must pass through the singular locus of S_0. This is the case, for instance, when S_0 is the Lagrangian Grassmannian of rank ≥ 2 and $S_0 \subset S$ is the subvariety swept out by minimal rational curves on S passing through a fixed base point $x_0 \in S$. In this case $S_0 \subset S$ is a Schubert variety with a unique isolated singularity x_0, and all minimal rational curves on S lying on S_0 must pass through the base point x_0.

Proposition 4.4. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root, and $S_0 \subset S$ be a smooth Schubert variety. Assume that $\mathcal{C}_x(S_0)$ satisfies* (I) *and* (II) *at a general point $x \in S_0$, and S_0 intersects gS_0 in codimension 2 for any $g \in G$. Then there is no reduced irreducible subvariety $Z \subset S$ having homology class $[Z] = r[S_0]$ for any $r \geq 2$.*

Proof. Assume that there exists a reduced irreducible subvariety Z of S having homology class $[Z] = r[S_0]$ for some $r \geq 2$. Let T_0 be the Schubert variety opposite to S_0 and P_I^- be the stabilizer of T_0 in G. Denote by $\gamma : \mathcal{O} \to \mathcal{H}$ the projection map from the open P_I^--orbit $\mathcal{O}$ in T_0 to the fixed point set $\mathcal{H}$ in $\mathcal{O}$ of the $\mathbb{C}^*$-action associated to P_I^-. Then by Proposition 3.3 and Remark 3.2, there exists a reduced irreducible subvariety Z' of S having homology class $[Z'] = r[S_0]$ and

(1) Z' intersects T_0 transversely and
(2) $Z' \cap T_0$ is contained in $B^-.x_w$ and
(3) $\gamma_*(Z' \cap T_0) = x_1 + \cdots + x_r$, where all x_i are distinct.

As in Proposition 3.6, take the closure $\mathcal{Z}$ in $S \times \mathbb{P}^1$ of the union $\bigcup_{t\in\mathbb{C}^*} t.Z' \times \{t\}$. By Proposition 3.6, $\mathcal{Z}$ is irreducible and $\pi^{-1}(\infty) = (g_1 S_0 + \cdots + g_r S_0) \times \{\infty\}$. By the assumption that S_0 intersects gS_0 in codimension 2, up to the action of G, there is a smooth quasi-projective subvariety $\widehat{S}_0 \times \{\infty\}$ of $\pi^{-1}(\infty) \cap (S_0 \times \{\infty\})$ with $\dim \widehat{S}_0 = \dim S_0$ which is uniruled by lines in $\widehat{S}_0 \times \{\infty\}$. Applying Proposition 4.3 (2) with t replaced by $\frac{1}{t}$, for t sufficiently large, we have $t.Z' = g_t S_0$ for some $g_t \in G$. But, then, $[Z'] = [S_0]$ while $[Z'] = r[S_0]$ with $r \geq 2$, a contradiction. Consequently, there is no reduced irreducible subvariety Z of S having homology class $[Z] = r[S_0]$ for some $r \geq 2$. □

Theorem 4.5. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root, and $S_0 \subset S$ be a smooth Schubert variety. Assume that $\mathcal{C}_x(S_0)$ satisfies* (I) *and* (II) *at a general point $x \in S_0$, and S_0 intersects gS_0 in codimension 2 for any $g \in G$. Then for any subvariety $Z \subset S$ with homology class $[Z] = r[S_0]$ we have $Z = n_1 g_1 S_0 + \cdots + n_s g_s S_0$, where $g_i \in G$ and $n_i \in \mathbb{Z}_+$.*

Proof. Consider the decomposition $Z = m_1 Z_1 + \cdots + m_l Z_l$ of Z by its irreducible components, where Z_i are reduced and $m_i \in \mathbb{Z}_+$ for all i. From $[Z] = r[S_0]$ it follows that $[Z_i] = r_i[S_0]$ for some r_i. By Proposition 4.4 we have $r_i = 1$. By Theorem 4.2, Z_i is $g_i S_0$ for some $g_i \in G$. Therefore, $Z = m_1 g_1 S_0 + \cdots + m_l g_l S_0$. □

4.3. **Intersections of translates.** It remains to confirm the validity of the conditions in Theorem 4.5 for non-linear smooth Schubert varieties $S_0 \subset S$.

Let $\{\alpha_1, \ldots, \alpha_n\}$ be the system of simple roots of a simple Lie group L and let $\{\varpi_1, \ldots, \varpi_n\}$ be the system of fundamental weights of L. Let $V_L(\varpi_i)$ denote the irreducible L-representation space with the highest weight ϖ_i. Take a highest weight vector v_i in $V_L(\varpi_i)$ for $i = 1, \ldots, n$. We denote the closure of the L-orbit of $[v_i + v_j]$ in $\mathbb{P}(V_L(\varpi_i) \oplus V_L(\varpi_j))$ by (L, α_i, α_j).

Proposition 4.6 (Proposition 3.7 of [15], Theorem 1.3 of [13])**.** *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root. Then a non-linear smooth Schubert variety S_0 of S is either a homogeneous submanifold associated to a subdiagram of S or one of the following.*

1. $S_0 = (C_n, \alpha_{i+1}, \alpha_i)$ *and* $S = (C_m, \alpha_k)$, $2 \leq n \leq m$ *and* $1 \leq i \leq n-1$ *and* $m - k = n - i$;
2. $S_0 = (C_2, \alpha_2, \alpha_1)$ *and* $S = (F_4, \alpha_3)$;
3. $S_0 = (B_3, \alpha_2, \alpha_3)$ *and* $S = (F_4, \alpha_3)$.

Example 4.7. We describe embedding of S_0 in S for the pairs (S, S_0) in the list (1) - (3) of Proposition 4.6.

(1) ([33]) Take an isotropic basis $\{e_1, \cdots, e_{2n}\}$ of $\mathbb{C}^{2n}$ and extend it to an isotropic basis $\{e_0, e_1, \cdots, e_{2n}, e_{2n+1}\}$ of $\mathbb{C}^{2n+2}$. The embedding $V_{C_n}(\varpi_1) = \mathbb{C}^{2n} \to V_{C_{n+1}}(\varpi_1) = \mathbb{C}^{2n+2}$ induces an embedding $V_{C_n}(\varpi_{i+1}) \subset \wedge^{i+1}\mathbb{C}^{2n} \to V_{C_{n+1}}(\varpi_{i+1}) \subset \wedge^{i+1}\mathbb{C}^{2n+2}$ and an embedding $V_{C_n}(\varpi_i) \subset \wedge^i \mathbb{C}^{2n} \to V_{C_{n+1}}(\varpi_{i+1}) \subset \wedge^{i+1}\mathbb{C}^{2n+2}$ given by $v_1 \wedge \cdots \wedge v_i \mapsto v_1 \wedge \cdots \wedge v_i \wedge e_0$. Combining these two embeddings we get an embedding

$$\begin{array}{rcl} V_{C_n}(\varpi_{i+1}) \oplus V_{C_n}(\varpi_i) & \to & V_{C_{n+1}}(\varpi_{i+1}) \\ (e_1 \wedge \cdots \wedge e_i \wedge e_{i+1}, e_1 \wedge \cdots \wedge e_i) & \mapsto & e_1 \wedge \cdots \wedge e_i \wedge e_{i+1} + e_1 \wedge \cdots \wedge e_i \wedge e_0 \\ & & = e_1 \wedge \cdots \wedge e_i \wedge (e_0 + e_{i+1}). \end{array}$$

By taking the closure of $Sp(2n)$-orbit of $e_1 \wedge \cdots \wedge e_i \wedge (e_0 + e_{i+1})$ in $V_{C_{n+1}}(\varpi_{i+1})$, we get an embedding of $S_0 = (C_n, \alpha_{i+1}, \alpha_i)$ into $S = (C_{n+1}, \alpha_{i+1})$. Geometrically, S_0 is the space of isotropic $(i+1)$-subspaces of $\mathbb{C}^{2n+2}$ contained in the subspace $H := \mathbb{C}e_0 + \mathbb{C}^{2n}$ of $\mathbb{C}^{2n+2}$.

(2) After taking the composition of the embedding of $(C_2, \alpha_2, \alpha_1)$ into (C_3, α_2) as in the case of (1) and the embedding of (C_3, α_2) into (F_4, α_3) as a homogeneous submanifold, we get an embedding of $(C_2, \alpha_2, \alpha_1)$ into (F_4, α_3)

(3) We recall some facts on the projective geometry of the rational homogeneous manifold $\mathbb{OP}^2_0$ of type (F_4, α_4) (Proposition 6.6 and Proposition 6.7 of [29]). Let $\mathcal{J}_3(\mathbb{O})$ be the space of 3×3 $\mathbb{O}$-Hermitian symmetric matrices

$$\mathcal{J}_3(\mathbb{O}) = \left\{ B = \begin{pmatrix} r_1 & \overline{x_3} & \overline{x_2} \\ x_3 & r_2 & \overline{x_1} \\ x_2 & x_1 & r_3 \end{pmatrix}, r_i \in \mathbb{C}, x_j \in \mathbb{O} \right\}$$

and let $\mathcal{J}_3(\mathbb{O})_0 := \{B \in \mathcal{J}_3(\mathbb{O}) : \mathrm{tr} B = 0\}$. Then we have $V_{E_6}(\omega_1) = \mathcal{J}_3(\mathbb{O})$ and $V_{F_4}(\omega_4) = \mathcal{J}_3(\mathbb{O})_0$ and

$$\mathbb{OP}_0^2 := \mathbb{P}\{A \in \mathcal{J}_3(\mathbb{O})_0 : A^2 = 0\} \subset \mathbb{P}(V_{F_4}(\omega_4))$$

is the rational homogeneous manifold of type (F_4, α_4).

Let $A \in \mathcal{J}_3(\mathbb{O})_0$ be such that $A^2 = 0$. Then the affine tangent space $\widehat{T}_{[A]}\mathbb{OP}_0^2$ of $\mathbb{OP}_0^2$ at $[A]$ has a filtration

$$\mathbb{C}A \subset \widehat{T}_{[A],1}\mathbb{OP}_0^2 \subset \widehat{T}_{[A]}\mathbb{OP}_0^2$$

invariant under the isotropy group of G at $[A]$, where

$$\begin{aligned} \widehat{T}_{[A]}\mathbb{OP}_0^2 &= \{B \in \mathcal{J}_3(\mathbb{O})_0 : AB + BA = 0\} \\ \widehat{T}_{[A],1}\mathbb{OP}_0^2 &= \{B \in \mathcal{J}_3(\mathbb{O})_0 : AB = 0\}. \end{aligned}$$

Furthermore, as a representation of $Spin(7)$, $\widehat{T}_{[A]}\mathbb{OP}_0^2$ is decomposed as $\mathbb{C}A \oplus V_{B_3}(\omega_3) \oplus V_{B_3}(\omega_1)$ and $\widehat{T}_{[A],1}\mathbb{OP}_0^2$ is decomposed as $\mathbb{C}A \oplus V_{B_3}(\omega_3)$. Therefore, we have an embedding of $\mathbb{C}A \oplus V_{B_3}(\omega_3)$ into $V_{F_4}(\omega_4)$.

From this embedding we get an embedding of $V_{B_3}(\omega_3)$ into $V_{F_4}(\omega_3)$ defined by

$$B \in V_{B_3}(\omega_3) \mapsto A \wedge B \in V_{F_4}(\omega_3) \subset \wedge^2(V_{F_4}(\omega_4))$$

and an embedding of $V_{B_3}(\omega_2) \subset \wedge^2(V_{B_3}(\omega_3))$ into $V_{F_4}(\omega_3) \subset \wedge^2(V_{F_4}(\omega_4))$. Combining these two embeddings we get an embedding

$$\begin{aligned} V_{B_3}(\omega_2) \oplus V_{B_3}(\omega_3) &\to V_{F_4}(\omega_3) \subset \wedge^2 V_{F_4}(\omega_4) \\ (C \wedge B, B) &\mapsto C \wedge B + A \wedge B = (C + A) \wedge B. \end{aligned}$$

By taking the closure of the $Spin(7)$-orbit of $(C+A) \wedge B$ in $V_{F_4}(\omega_3)$ we get an embedding of $S_0 = (B_3, \alpha_2, \alpha_3)$ into $S = (F_4, \alpha_3)$. Geometrically, S_0 is the space of 2-subspaces $\langle C_1, C_2 \rangle$ of $\mathcal{J}_3(\mathbb{O})_0$ contained in the subspace $H := \{B \in \mathcal{J}_3(\mathbb{O})_0 : AB = 0\}$ of $\mathcal{J}_3(\mathbb{O})_0$, where $C_1^2 = C_2^2 = C_1C_2 = 0$.

Proposition 4.8. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root, and $S_0 \subset S$ be a smooth non-linear Schubert variety. Then $\mathcal{C}_x(S_0)$ satisfies* (I) *and* (II) *at a general point $x \in S_0$, and S_0 intersects gS_0 in codimension 2 for any $g \in G$ with $S_0 \neq gS_0$.*

Proof. By Proposition 4.6, a non-linear smooth Schubert variety S_0 of S is either a homogeneous submanifold associated to a subdiagram of S or one of the followings.

(1) $S_0 = (C_n, \alpha_{i+1}, \alpha_i)$ and $S = (C_m, \alpha_k)$, $2 \leq n \leq m$ and $1 \leq i \leq n-1$ and $m - k = n - i$;
(2) $S_0 = (C_2, \alpha_2, \alpha_1)$ and $S = (F_4, \alpha_3)$;
(3) $S_0 = (B_3, \alpha_2, \alpha_3)$ and $S = (F_4, \alpha_3)$.

In the proof of Proposition 4.6 we have already proved that properties (I) and (II) hold for a non-linear smooth Schubert variety S_0 (See Proposition 3.3 and Proposition 3.5 of [15] for a homogeneous submanifold associated to a subdiagram of S, Proposition 4.3 and Lemma 4.4 of [15] for (1), and Proposition 3.4 of [13] for (2) and (3)). This completes the proof of the first statement.

To prove the second statement, we first consider the case when S_0 is a homogeneous submanifold associated to a subdiagram $\mathcal{D}_0$ of $\mathcal{D}(S)$. Then the stabilizer of S_0 in G is the parabolic subgroup P_I, where Λ is the set of simple roots in $\mathcal{D}(S) - \mathcal{D}_0$ which are adjacent to $\mathcal{D}_0$ and I is the complement of Λ in the set of simple roots in $\mathcal{D}(S)$. Therefore, we have an isomorphism $\{gS_0 : g \in G\} \simeq G/P_I$.

Let s_γ denote the element in $\mathcal{W}^{P_I}$ given by the simple reflection with respect to γ. We claim that if $s_\gamma S_0 \cap S_0$ has dimension $\leq \dim S_0 - 2$ for any simple root γ in Λ, then $gS_0 \cap S_0$ has dimension $\leq \dim S_0 - 2$ for any $g \in G$ with $S_0 \neq gS_0$.

To prove the claim, put

$$\begin{aligned} G_1 &:= \{g \in G : \dim(S_0 \cap gS_0) \geq \dim S_0 - 1\} \\ \mathcal{K}_1 &:= \{gP_I \in G/P_I : g \in G_1\}. \end{aligned}$$

Then G_1 contains the stabilizer P_I of S_0. Since B stabilizes S_0, for $b, b' \in B$ and $g \in G$, we have $S_0 \cap bgb'S_0 = bS_0 \cap bgb'S_0 = b(S_0 \cap gb'S_0) = b(S_0 \cap gS_0)$ so that $\dim(S_0 \cap bgb'S_0) = \dim(S_0 \cap gS_0)$. As a consequence, G_1 is a union of double B-cosets.

If G_1 contains P_I properly, then $\mathcal{K}_1$ is a positive-dimensional subvariety of G/P_I which is invariant by B. By the Bruhat decomposition of G/P_I, the closure of a B-orbit in G/P_I of dimension k has at least one B-orbit of dimension j for each $j \leq k$. Hence there is a B-orbit in $\mathcal{K}_1$ of dimension one. A one-dimensional B-orbit in G/P_I corresponds to a simple root in Λ, and thus there is $\gamma \in \Lambda$ such that $\dim(S_0 \cap s_\gamma S_0) \geq \dim S_0 - 1$. This completes the proof of the claim. In the remaining part of the proof we will show that $s_\gamma S_0 \cap S_0$ has dimension $\leq \dim S_0 - 2$ for any simple root γ in Λ.

Let x_0 denote the base point of G/P with the isotropy group P and let α_k be the simple root associated to P. Then the tangent space $T_{x_0}S_0$ is generated by the root vectors of roots with zero coefficients in the simple roots in Λ and a negative coefficient in α_k. The isotropy action of s_γ on $T_{x_0}S$ is given by mapping an element in the root space $\mathfrak{g}_\alpha$ to an element in the root space $\mathfrak{g}_{s_\gamma(\alpha)}$. If S_0 is not linear, then for any $\gamma \in \Lambda$, there are at least two roots α with $\mathfrak{g}_\alpha \subset T_{x_0}S_0$ such that $\langle \alpha, \gamma \rangle \neq 0$. Furthermore, for $\gamma \in \Lambda$ with $\langle \alpha, \gamma \rangle \neq 0$, we have $\mathfrak{g}_{s_\gamma(\alpha)} = \mathfrak{g}_{\alpha - \langle \alpha, \gamma \rangle \gamma} \not\subset T_{x_0}S_0$. Thus, for any $\gamma \in \Lambda$, $T_{x_0}S_0 \cap T_{x_0}(s_\gamma S_0)$ has dimension $\leq \dim T_{x_0}S_0 - 2$ and $s_\gamma S_0 \cap S_0$ has dimension $\leq \dim S_0 - 2$.

If (S_0, S) is of type (1) in Proposition 4.6, then S_0 consists of isotropic k-subspaces of $V = \mathbb{C}^{2n+2}$ which are contained in a hyperplane H of V. Thus $S_0 \cap gS_0$ consists of isotropic k-subspaces of $V = \mathbb{C}^{2n+2}$ which are contained in the intersection $H \cap gH$. If $S_0 \neq gS_0$, then $H \cap gH$ is a proper subspace of H and thus we have $\dim(S_0 \cap gS_0) \leq \dim S_0 - 2$. The proof is similar when (S_0, S) is of type (2) in Proposition 4.6

If (S_0, S) is of type (3) in Proposition 4.6, then S_0 consists of $C_1 \wedge C_2$, where $C_1, C_2 \in H := \{B \in \mathcal{J}_3(\mathbb{O})_0 : AB = 0\}$ are such that $C_1^2 = C_2^2 = C_1C_2 = 0$. If $S_0 \neq gS_0$, then $H \cap gH = \{B \in \mathcal{J}_3(\mathbb{O})_0 : AB = (gA)B = 0\}$ is a proper subspace of H and thus we have $\dim(S_0 \cap gS_0) \leq \dim S_0 - 2$. □

Proof of Theorem 1.1. From Theorem 4.5 and Proposition 4.8 it follows that for any non-linear smooth Schubert variety S_0 of a rational homogeneous manifold S of Picard number one, the pair (S, S_0) is Schur rigid. □

4.4. **Maximal linear spaces.** For obvious reasons linear Schubert variety is not Schur rigid if it is not a maximal linear space. In this section we will show that a maximal linear Schubert variety is Schur rigid with some trivial exceptions.

Let (S, S_0) be a pair consisting of a rational homogeneous manifold $S = G/P$ of Picard number one and a smooth Schubert variety $S_0 \subset S$. We denote by $0 \in S_0$ a reference point lying on the unique open B-orbit. Recall that (S, S_0) is *Schubert rigid* if and only if the following holds. Any complex submanifold $Z \subset W$ of some connected open subset $W \subset S$ must necessarily be an open subset of a translate gS_0 of S_0 by some $g \in G$ whenever for a point x belonging to some nonempty open subset $U \subset W$ there exists some $g \in \text{Aut}(S)$, where g depends on x, such that $g(0) = x$ and such that $dg(T_0S_0) = T_xZ$. The latter condition may also be formulated more geometrically by stating that Z is tangent at every point $x \in U$ to some translate gS_0, $g \in G$, at the point $g(0) = x$. Here "open" means "open with respect to the complex topology". In what follows we are concerned solely with the cases where $S_0 \subset S$ is a maximal linear subspace. From the discussion in the Introduction it follows that Schubert rigidity is a necessary condition for Schur rigidity.

Theorem 4.9 (cf. Theorem 1.3 of [16])**.** *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root and let S_0 be a maximal linear space in S. Then (S, S_0) is Schubert rigid whenever (S, S_0) is not of the type belonging to any of the classes given by*

(1) $S = (B_n, \alpha_k)$ $(k \leq n-2)$ *and* $S_0 = \mathbb{P}^{n-k}$;
(2) $S = (C_n, \alpha_n)$ *and* $S_0 = \mathbb{P}^1$;
(3) $S = (F_4, \alpha_1)$ *and* $S_0 = \mathbb{P}^2$;
(4) $S = (G_2, \alpha_2)$ *and* $S_0 = \mathbb{P}^1$.

Furthermore, when (S, S_0) is of the type belonging to any one of the classes (1)–(3) in the above, (S, S_0) is not Schubert rigid. More precisely, there exists a nonlinear irreducible projective subvariety $Z \subset S$ such that Z is tangent at every smooth point to some maximal linear subspace gS_0, $g \in G$, of S.

Theorem 4.9 is a restatement of Theorem 1.3 of [16] with corrections. When $(S, S_0) = ((B_n, \alpha_k), \mathbb{P}^{n-k})$ for $k = n-1$, S_0 is not a maximal linear space and this case was included mistakenly in the list of exceptional cases in Theorem 1.3 of [16]. When $(S, S_0) = ((G_2, \alpha_2), \mathbb{P}^1)$, the arguments in the proof of Lemma 5.1 of [16] do not work.

The method of constructing counterexamples explained in p. 2354 of [16] works for the cases (1)–(3) in Theorem 4.9 but does not work for the case (4) in Theorem 4.9. It remains open whether this is Schubert rigid or not.

We note that Corollary 1.2 of [31], which is logically a consequence of Main Theorem of [31] and Theorem 1.3 of [16], should also be amended accordingly.

Example 4.10. The odd Spinor variety $\mathbb{S}_{B_3}$, the rational homogeneous manifold of type (B_3, α_3), has two types of $\mathbb{P}^3$'s: one is a homogeneous submanifold associated to a subdiagram of the Dynkin diagram of $\mathbb{S}_{B_3}$ and the other is not. We denote the first by $\mathbb{P}^3_{B_2}$ and the second by $\mathbb{P}^3_{A_3}$.

In fact, the odd Spinor variety $\mathbb{S}_{B_3}$ is biholomorphic to the even Spinor variety $\mathbb{S}_{D_4}$, the rational homogeneous manifold of type (D_4, α_4). There are two types of $\mathbb{P}^3$'s in $\mathbb{S}_{D_4}$, each of which is a homogeneous submanifold associated to a subdiagram $\mathcal{D}_0$ of the Dynkin diagram $\mathcal{D}$ of $\mathbb{S}_{D_4}$ (Theorem 4.9 of [29]). In both cases, the subdiagram $\mathcal{D}_0$ is of type A_3, but only the first one still is associated to a subdiagram of the Dynkin diagram of $\mathbb{S}_{B_3}$, which is of type B_2. Under the biholomorphism from $\mathbb{S}_{B_3}$ to $\mathbb{S}_{D_4}$, the first corresponds to $\mathbb{P}^3_{B_2}$ and the second corresponds to $\mathbb{P}^3_{A_3}$.

The odd Spinor variety $\mathbb{S}_{B_3}$ is a homogeneous submanifold of the rational homogeneous manifold S of type (F_4, α_3). Thus $\mathbb{P}^3_{B_2}$ and $\mathbb{P}^3_{A_3}$ are linear spaces in S, too, and both of them are maximal.

The odd Spinor variety $\mathbb{S}_{B_3}$ is isomorphic to the space of non-generic lines on the rational homogeneous manifold $\mathbb{OP}^2_0$ of type (F_4, α_4) passing through a point (Proposition 6.5 of [29]). Thus $\mathbb{P}^3_{B_2}$ and $\mathbb{P}^3_{A_3}$ in $\mathbb{S}_{B_3}$ correspond to two types of $\mathbb{P}^4$'s in $\mathbb{OP}^2_0$, the first is not maximal and the second is maximal. We denote the second by $\mathbb{P}^4_{A_4}$.

Proposition 4.11. *Let (S, S_0) be one of the following:*

(1) $S = (F_4, \alpha_3)$ *and* $S_0 = \mathbb{P}^3_{B_2}$
(2) $S = (F_4, \alpha_3)$ *and* $S_0 = \mathbb{P}^3_{A_3}$
(3) $S = (F_4, \alpha_4)$ *and* $S_0 = \mathbb{P}^4_{A_4}$,

where $\mathbb{P}^3_{B_2}$ and $\mathbb{P}^3_{A_3}$ and $\mathbb{P}^4_{A_4}$ are given in Example 4.10. Then (S, S_0) is not homologically rigid.

Proof. The space of $\mathbb{P}^3$'s in $\mathbb{S}_{D_4} = \mathbb{S}_{B_3}$ has two connected components, $\mathcal{H}_1$ and $\mathcal{H}_2$, the former containing $\mathbb{P}^3_{B_2}$ and the latter containing $\mathbb{P}^3_{A_3}$. Each $\mathcal{H}_i$ is an orbit of D_4 and is the union of two B_3-orbits, one is closed and the other is open (Theorem 4.9 and Remark 5.10 of [29]). The closed B_3-orbit in $\mathcal{H}_1$ consists of translates of $\mathbb{P}^3_{B_2}$ by the group B_3. Thus $\mathbb{P}^3$'s in the open B_3-orbit in $\mathcal{H}_1$ have the same homology class as $\mathbb{P}^3_{B_2}$ but are not translates of $\mathbb{P}^3_{B_2}$ by the group B_3.

Consider $\mathbb{S}_{B_3}$ as a homogeneous submanifold of the rational homogeneous manifold S of type (F_4, α_3) as in Example 4.10. Then we have $\mathbb{P}^3_{B_2} \subset \mathbb{S}_{B_3} \subset S$ and we may regard $\mathcal{H}_1$ as the space of $\mathbb{P}^3$'s in S contained in $\mathbb{S}_{B_3}$. We claim that $\mathbb{P}^3$'s in the open B_3-orbit in $\mathcal{H}_1$ are not translates of $\mathbb{P}^3_{B_2}$ by the group F_4.

For the proof of the claim observe first of all that the holomorphic tangent bundle TS is equipped with a nontrivial F_4-invariant filtration corresponding to the gradation of the Lie algebra of F_4 defined by the marked Dynkin diagram (F_4, α_3). Let $0 \neq D \subset TS$ be the minimal proper distribution in the filtration. Taking intersection of distributions in the filtration with the holomorphic tangent bundle $T\mathbb{S}_{B_3}$ we obtain a 2-step filtration corresponding to the marked Dynkin diagram (B_3, α_3). Let $0 \neq D' \subset T\mathbb{S}_{B_3}$ be the minimal proper distribution in the filtration. We have $D' = D|_{\mathbb{S}_{B_3}} \cap T\mathbb{S}_{B_3}$.

Let $\mathbb{P}$ be an element of $\mathcal{H}_1$. Then, $\mathbb{P}$ is in the closed B_3-orbit of $\mathcal{H}_1$ if and only if $\mathbb{P}$ is tangent to D' at every point $x \in \mathbb{P}$. Let $\mathbb{Q} \in \mathcal{H}_1$ be an element of the open B_3-orbit in $\mathcal{H}_1$. Then, $\mathbb{Q}$ is not tangent to D', hence it is not tangent to D. However, since $\mathbb{P}^3_{B_2}$ is tangent to D' and hence *a fortiori* to D everywhere on $\mathbb{P}^3_{B_2} \subset S$, and since $\mathrm{Aut}(S) = F_4$ preserves D, for any $\varphi \in F_4$, $\varphi(\mathbb{P}^3_{B_2})$ must be everywhere tangent to D, and it follows that $\mathbb{Q} \neq \varphi(\mathbb{P}^3_{B_2})$ for any $\varphi \in F_4$, proving the claim. Hence, $\mathbb{P}^3_{B_2}$ is not homologically rigid in S. By the same arguments $\mathbb{P}^3_{A_3}$ is not homologically rigid in S.

Similarly, $\mathbb{P}^3_{A_4}$ is not homologically rigid in the rational homogeneous manifold of type (F_4, α_4). □

Proposition 4.12. *Let $S = G/P$ be a rational homogeneous manifold associated to a simple root and let S_0 be a maximal linear space in S. Then (S, S_0) is Schur rigid except when S is associated to a long root and S_0 is a homogeneous submanifold of S associated to a subdiagram $\mathcal{D}(S_0)$ of the marked Dynkin diagram $\mathcal{D}(S)$ of the following type:*

(1) $S = (B_n, \alpha_k)$ *and* $S_0 = \mathbb{P}^{n-k}$ *with* $\Lambda = \{\alpha_n\}$ *for* $k = 1$ *and with* $\Lambda = \{\alpha_{k-1}, \alpha_n\}$ *for* $2 \leq k \leq n-2$;
(2) $S = (C_n, \alpha_n)$ *and* $S_0 = \mathbb{P}^1$ *with* $\Lambda = \{\alpha_{n-1}\}$;
(3) $S = (F_4, \alpha_1)$ *and* $S_0 = \mathbb{P}^2$ *with* $\Lambda = \{\alpha_3\}$;

(4) *$S = (G_2, \alpha_2)$ and $S_0 = \mathbb{P}^1$ with $\Lambda = \{\alpha_1\}$,*

where Λ denotes the set of simple roots in $\mathcal{D}(S) - \mathcal{D}(S_0)$ which are adjacent to $\mathcal{D}(S_0)$, or, when S is associated to a short root and S_0 is of the following form:

(5) *$S = (F_4, \alpha_3)$ and $S_0 = \mathbb{P}^3_{B_2}$;*
(6) *$S = (F_4, \alpha_3)$ and $S_0 = \mathbb{P}^3_{A_3}$;*
(7) *$S = (F_4, \alpha_4)$ and $S_0 = \mathbb{P}^4_{A_4}$,*

where $\mathbb{P}^3_{B_2}$ and $\mathbb{P}^3_{A_3}$ and $\mathbb{P}^4_{A_4}$ are given in Example 4.10.

Proof. Let $S = G/P$ be a rational homogeneous manifold associated to a simple root and let S_0 be a maximal linear space in S. As in the proof of Theorem 1.1, we will apply Theorem 4.5 to get the Schur rigidity. The condition (I) in Section 4.2 is satisfied because $\mathcal{C}_x(S_0)$ is a linear space. It suffices to show that if (S, S_0) is not in the list (1)–(7), then

- $\mathcal{C}_x(S_0)$ satisfies the condition (II) in Section 4.2 at a general point $x \in S_0$ and
- S_0 intersects gS_0 in codimension 2 for any $g \in G$.

Assume that S is associated to a long simple root. Then a linear Schubert variety S_0 of S is a homogeneous submanifold associated to a subdiagram $\mathcal{D}_0$ of $\mathcal{D}(S)$, and $\mathcal{C}_x(S_0)$ satisfies (II) (Lemma 4.1 of [15]).

If (S, S_0) is not of the form (1) - (4), then, for any $\gamma \in \Lambda$, there are at least two roots α with $\mathfrak{g}_\alpha \subset T_{x_0} S_0$ such that $\langle \alpha, \gamma \rangle \neq 0$. Therefore, as in the proof of Proposition 4.8, $T_x S_0 \cap T_x(s_\gamma S_0)$ has codimension ≥ 2 in $T_x S_0$ and hence $S_0 \cap gS_0$ has codimension ≥ 2 for any $g \in G$ with $S_0 \neq gS_0$. By Theorem 4.5, (S, S_0) is Schur rigid.

If (S, S_0) is one of the forms in (1) - (3), it is not Schubert rigid by Theorem 4.9 and thus is not Schur rigid. If (S, S_0) is of the form (4), then S_0 is a minimal rational curve in S. Schur rigidity must fail for (S, S_0) whenever S is not linear and S_0 is a minimal rational curve (and a Schubert cycle) for the following elementary observation. Embed S into a projective space $\mathbb{P}^N$ by the minimal canonical embedding (i.e., by $\mathcal{O}(1)$) and suppose the homology class of S in $\mathbb{P}^N$ is k times the positive generator of the $H_{2n}(S, \mathbb{Z})$, where $n = \dim(S)$. Then, a general linear section of dimension 1 is a reduced and irreducible smooth curve of degree $k > 1$. Hence the pair (S, S_0) is not Schur rigid. This completes the proof for the case where S is associated to a long root.

Assume that S is associated to a short simple root. Then $\mathcal{C}_x(S)$ and its maximal linear spaces are given as follows ([29], Lemma 5.1 of [16]):

1. $(C_\ell, \alpha_k), k \geq 2$: $\mathcal{C}_x(S)$ is $\mathbb{P}(\{u \otimes q + cu^2 : u \in U, q \in Q, c \in \mathbb{C}\}) \subset \mathbb{P}((U \otimes Q) \oplus S^2 U)$ and its maximal linear space is

   1.a $\mathbb{P}(\{u \otimes q + cu^2 : q \in Q, c \in \mathbb{C}\}) \simeq \mathbb{P}^{2m}$ for some $u \in U$ or
   1.b $\mathbb{P}(\{u \otimes q : u \in U\}) \simeq \mathbb{P}^{k-1}$ for some $q \in Q$

   where U is a vector space of dimension k and Q is a vector space of dimension $2m := 2\ell - 2k$.

2. (F_4, α_3): $\mathcal{C}_x(S)$ is $\mathbb{P}(\{e^* \otimes q + (e_1^* \wedge e_2^*) \otimes q^2 : e \wedge e_1 \wedge e_2 = 0, e, e_1, e_2 \in E, q \in Q\})$ and its maximal linear space is

2.a $\mathbb{P}^2$'s in $\mathbb{P}(\{e^* \otimes q + (e_1^* \wedge e_2^*) \otimes q^2 : e \wedge e_1 \wedge e_2 = 0, e, e_1, e_2 \in E\}) \simeq \mathbb{Q}^4$ for some $q \in Q$ or

2.b $\mathbb{P}(\{e^* \otimes q : q \in Q\}) \simeq \mathbb{P}^1$ for some $e^* \in E^*$

where E is a vector space of dimension 3 and Q is a vector space of dimension 2.

3. (F_4, α_4): $\mathcal{C}_x(S)$ is the closure of L-orbit of $[v_1 + v_2]$ in $\mathbb{P}(V_1 \oplus V_2)$ and its maximal linear space is

3.a $\mathbb{P}^4$ = the cone over $\mathbb{P}^3_{B_2}$ in $\mathbb{S}_{B_3} \subset \mathbb{P}(V_1)$ with the vertex at $[v_2]$ and its L-translates, or

3.b $\mathbb{P}^3$ = the cone over $\mathbb{P}^2$ in $\mathbb{Q}^5 \subset \mathbb{P}(V_2)$ with the vertex at $[v_1]$ and its L-translates, or

3.c $\mathbb{P}^3_{A_3}$'s in $\mathbb{S}_{B_3} \subset \mathbb{P}(V_1)$,

where L, the simple group of type B_3, is the semisimple part of the isotropy group of G at x, and V_1 is the spin representation of L and V_2 is the standard representation of L, and v_1 is a highest weight vector of V_1 and v_2 is a highest weight vector of V_2, and $\mathbb{S}_{B_3}$ is the highest weight orbit in $\mathbb{P}(V_1)$ and $\mathbb{Q}^5$ is the highest weight orbit in $\mathbb{P}(V_2)$.

If $\mathcal{C}_x(S_0)$ is of the form 1.b or 2.b, then S_0 does not contain a general line and Theorem 4.5 cannot apply directly. However, there is an embedding of S into another rational homogeneous manifold S' in such a way that S_0 is a Schubert variety of S', and the homological rigidity of (S, S_0) follows from the homological rigidity of (S', S_0) (See the proof of Proposition 3.6 of [15]). Since (S', S_0) is Schur rigid, so is (S, S_0).

If $\mathcal{C}_x(S_0)$ is of the form 1.a, then any local deformation of $\mathcal{C}_x(S_0)$ is of the same form $\mathbb{P}(\{u' \otimes q + cu'^2 : q \in Q, c \in \mathbb{C}\}) \simeq \mathbb{P}^{2m}$ but for a different $u' \in U$. Thus $\mathcal{C}_x(S_0)$ satisfies the condition (II). Since $\mathcal{C}_x(S_0) \cap \mathcal{C}_x(gS_0)$ is empty, $S_0 \cap gS_0$ has dimension ≤ 0 because S_0 is linear. From $\dim S_0 \geq 2$ it follows that $S_0 \cap gS_0$ has codimension ≥ 2 for any $g \in G$.

If $\mathcal{C}_x(S_0)$ is of the form 3.a or 3.b, any deformation of $\mathcal{C}_x(S_0)$ in $\mathcal{C}_x(S)$ is again a maximal linear space in $\mathcal{C}_x(S_0)$ and thus is an L-translate of $\mathcal{C}_x(S_0)$. Hence $\mathcal{C}_x(S_0)$ satisfies the condition (II). The space of $\mathbb{P}^n$'s in $\mathbb{Q}^{2n}$ has two connected components, and any two $\mathbb{P}^n$ in the same connected component intersects in codimension two. Thus, the intersection $\mathcal{C}_x(S_0) \cap \mathcal{C}_x(gS_0)$ has codimension ≥ 3 in $\mathcal{C}_x(S_0)$. It follows that $S_0 \cap gS_0$ has codimension ≥ 3 in S_0 because S_0 is linear.

Therefore, if $\mathcal{C}_x(S_0)$ is neither of the form 2.a nor of the form 3.c, then (S, S_0) is Schur rigid by Theorem 4.5.

If $\mathcal{C}_x(S_0)$ is of the form 2.a, then S_0 is contained in the homogeneous submanifold S_1 of S associated to the subdiagram of type (B_3, α_3) obtained by $\Lambda = \{\alpha_4\}$. Thus, S_0 is either $\mathbb{P}^3_{B_2}$ or $\mathbb{P}^3_{A_3}$ embedded in $S_1 \simeq \mathbb{S}_{B_3}$. If $\mathcal{C}_x(S_0)$ is of the form 3.c, then S_0 is $\mathbb{P}^4_{A_4}$. Therefore, if $\mathcal{C}_x(S_0)$ is of the form 2.a or 3.c, i.e., (S, S_0) is one of the forms (5) - (7), then it is not homologically rigid by Proposition 4.11 and thus it is not Schur rigid. □

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Center for Mathematical Challenges, Korea Institute for Advanced Study, 85 Hoegiro, Dongdaemun-gu, Seoul 02455, Korea
Email address: `jhhong00@kias.re.kr`

Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong
Email address: `nmok@hku.hk`