A LARGE SIEVE INEQUALITY OF ELLIOTT-MONTGOMERY-VAUGHAN TYPE FOR MAASS FORMS ON $GL(n,\mathbb{R})$ WITH APPLICATIONS

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Abstract. In this paper, we establish a large sieve inequality of Elliott-Montgomery-Vaughan type for Maass forms on $GL(n,\mathbb{R})$ and explore three applications.

1. Introduction

Elliott [2], [1], and Montgomery and Vaughan [11] independently developed some sort of large sieve inequalities to study Linnik's problem, which may yield a more general result than the classical Vinogradov's result, cf. [9]. This device, known as the large sieve inequalities of Elliott-Montgomery-Vaughan (E-M-V) type, was generalized to the setting of primitive holomorphic cusp forms on $GL(2,\mathbb{R})$ and applied to obtain some statistical results on Hecke eigenvalues of primitive holomorphic cusp forms in [8]. Later, Wang [15] generalized the results to the case of Maass forms on $GL(2,\mathbb{R})$.

It is natural to ask for a generalization of large sieve inequalities of E-M-V type to Maass forms on $GL(n,\mathbb{R})$ ($n \geq 3$). There are two main difficulties: the first one is that for $n \geq 3$ the Hecke relations for $GL(n,\mathbb{R})$ is much more complicated than that of $GL(2,\mathbb{R})$, and the trace formula for $GL(n,\mathbb{R})$ with $n \geq 3$ is not as simple as the trace formula (say Kuznetsov's and Petersson's trace formulas) on $GL(2,\mathbb{R})$. Recently, Xiao and Xu [16], using Kuznetsov's trace formula and Hecke's relations, made a breakthrough and obtained a large sieve inequality of E-M-V type to Maass forms on $GL(3,\mathbb{R})$. Moreover, they also applied their large sieve inequality to get a statistical result of sign changes on the Hecke eigenvalues for $GL(3,\mathbb{R})$.

In this paper, we generalize the large sieve inequalities of E-M-V type to Maass forms on $GL(n,\mathbb{R})$ for all $n \geq 3$ which is of the same strength as the case of automorphic forms on $GL(2,\mathbb{R})$ (see [8, 15]). Our main tool is the automorphic Plancherel density theorem - a recent great progress due to Matz and Templier [10]. We remark the use of properties of (degenerated) Schur's polynomials instead of Hecke's relations to avoid the complicated calculations as in [16]. More precisely the (degenerated) Schur polynomial is employed to evaluate the main term when applying the truncated trace formula [6, Corollary 3.3] since the main term in [6, Corollary 3.3] is expressed in the form of orbital integral involving the (degenerated) Schur polynomial by the work of Matz and Templier [10]. Moreover, we apply our large sieve inequality - Theorem 1.1 on the $GL(n,\mathbb{R})$ analogue of Linnik's problem, the sign change problems, and the Montgomery-Vaughan conjecture.

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Let $\mathcal{H}^\natural = \{\phi_j\}$ be an orthogonal basis consisting of Hecke-Maass cusp forms for $SL(n,\mathbb{R})$. Each ϕ_j is associated with a Langlands parameter $\mu_j \in \mathfrak{a}^*_{\mathbb{C}}/W$ where $\mathfrak{a}^*_{\mathbb{C}} \cong \{\underline{z} \in \mathbb{C}^n : \sum_i z_i = 0\}$ and W is the Weyl group of $GL(n,\mathbb{R})$. For $t \geq 1$, we let

$$\mathcal{H}_t := \{\phi_j \in \mathcal{H}^\natural : \ \|\mu_j\|_2 \leq t, \ \mu_j \in \mathtt{i}\mathfrak{a}^*\} \tag{1.1}$$

where $\|\cdot\|_2$ is the standard Euclidean norm, and $\mathtt{i}\mathfrak{a}^* \subset \mathfrak{a}^*_{\mathbb{C}}$ is isomorphic to $\mathtt{i}\mathbb{R}^{n-1}$. It is known that $|\mathcal{H}_t| \asymp t^d$ with $d = n(n+1)/2$.

Let $A_\phi(m_1, m_2, \ldots, m_{n-1})$ be the Fourier coefficient of $\phi \in \mathcal{H}_t$. In this paper, we normalize each $\phi \in \mathcal{H}_t$ such that

$$A_\phi(1,1,\ldots,1) = 1.$$

It is well-known that

$$A_\phi(m_1, m_2, \ldots, m_{n-1}) = \overline{A_\phi(m_{n-1}, m_{n-2}, \ldots, m_1)}.$$

Moreover, for any $\kappa = (\kappa_1, \cdots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$ and any prime p,

$$\begin{aligned} A_\phi(p^\kappa) &:= A_\phi(p^{\kappa_1}, p^{\kappa_2}, \cdots, p^{\kappa_{n-1}}) \\ &= S_\kappa(\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \cdots, \alpha_{\phi,n}(p)) \end{aligned} \tag{1.2}$$

where S_κ is the (degenerate) Schur polynomial (see Section 2 for definition and refer to [3] or [7] for a detailed exposition) and $\alpha_\phi(p) := (\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \cdots, \alpha_{\phi,n}(p))$ is (a representative of) the Satake parameter associated to ϕ at p. Every Satake parameter $\alpha_\phi(p)$ satisfies $\prod_{i=1}^n \alpha_{\phi,i}(p) = 1$ and

$$\alpha_{\phi,1}(p) + \cdots + \alpha_{\phi,n}(p) = A_\phi(p, 1, \ldots, 1).$$

Put $\kappa^\iota = (\kappa_{n-1}, \cdots, \kappa_1)$ if $\kappa = (\kappa_1, \cdots, \kappa_{n-1})$. Then we have

$$A_\phi(p^{\kappa^\iota}) = A_\phi(p^{\kappa_{n-1}}, \cdots, p^{\kappa_1}) = \overline{A_\phi(p^\kappa)},$$

and $A_\phi(p^\kappa) \in \mathbb{R}$ if $\kappa = \kappa^\iota$.

Notation: For $\kappa = (\kappa_1, \ldots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$, we denote $\|\kappa\| := \sum_{j=1}^{n-1}(n-j)\kappa_j$ and $|\kappa| = \sum_{j=1}^{n-1} \kappa_j$.

Theorem 1.1. *Let $0 \neq \kappa = (\kappa_1, \ldots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$. Let $j \geq 1$ be any integer and $\{b_p\}_p$ be a sequence of complex numbers indexed by prime numbers such that $|b_p| \leq B$ for some constant $B > 0$ and for all primes p. Then*

$$\begin{aligned} &\frac{1}{|\mathcal{H}_t|} \sum_{\phi \in \mathcal{H}_t} \left| \sum_{P < p \leq Q} b_p \frac{A_\phi(p^{\kappa_1}, \ldots, p^{\kappa_{n-1}})}{p} \right|^{2j} \\ &\ll t^{-1/2} \left(\frac{BC_\kappa Q^{L\|\kappa\|}}{\log P} \right)^{2j} + \left(\frac{(BC_\kappa)^2 j}{P \log P} \right)^j \left\{ 1 + \left(\frac{40 j \log P}{P} \right)^{j/3} \right\} \end{aligned} \tag{1.3}$$

holds uniformly for

$$B > 0, \qquad j \geq 1, \qquad 2 \leq P < Q \leq 2P,$$

where L is a positive constant, $1 \leq C_\kappa := 10(1+|\kappa|)^{n^2-n}$ and the implied constant depends on κ only.

Let $q \geq 2$ be an integer and χ be a non principal Dirichlet character modulo q. Then the evaluation of the least integer n_χ among all positive integers n for which $\chi(n) \neq 0, 1$ is referred as Linnik's problem. One generalization formulated to Maass forms on $GL(n, \mathbb{R})$ is the evaluation of the smallest integer n for which

$$A_{\phi_1}(n, 1, \ldots, 1) \neq A_{\phi_2}(n, 1, \ldots, 1),$$

where $\phi_1 \neq \phi_2$. We denote this smallest integer by $n_{1,2}$. The first application uses Theorem 1.1 to investigate an analogue of Linnik's problem.

Suppose $\mathcal{P}$ is a set of prime numbers of positive density in the sense,

$$\sum_{\substack{z < p \leq 2z \\ p \in \mathcal{P}}} \frac{1}{p} \geq \frac{\Delta}{\log z} \quad (\forall\ z \geq z_0), \tag{1.4}$$

with some fixed constants $\Delta > 0$ and $z_0 > 0$.

Theorem 1.2. *Let $0 \neq \kappa = (\kappa_1, \ldots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$ and assume the set $\mathcal{P}$ (of primes) satisfies (1.4). Let $\Lambda = \{\lambda(p)\}_p$ be a fixed complex sequence indexed by prime numbers. For any $\delta > 0$, there is a positive constant $C = C(\delta, \kappa, \mathcal{P})$ such that the number of $\phi \in \mathcal{H}_t$ verifying*

$$A_\phi(p^{\kappa_1}, \ldots, p^{\kappa_{n-1}}) = \lambda(p) \quad \textit{for} \quad p \in \mathcal{P}, \quad \textit{and} \quad \delta \log t < p \leq 2\delta \log t$$

is bounded by

$$\ll t^d e^{-C \log t / \log_2 t}$$

where $\log_r$ is the r-fold iterated logarithm. The implied constant depends at most on $\delta, \kappa, \mathcal{P}$.

Remark 1.1. *Refer to [8] and [15] for the case of $GL(2, \mathbb{R})$.*

Corollary 1.1. *Let $\phi_0 \in \mathcal{H}_t$ be fixed and $\mathcal{P}$ be stated as in Theorem 1.2. Let $\ell \in \mathbb{N}$ and $\delta > 0$ be any number. Then there is a positive constant $C = C(\delta, \ell, \mathcal{P})$ such that the number of $\phi \in \mathcal{H}_t$ verifying*

$$A_\phi(p^\ell, 1, \ldots, 1) = A_{\phi_0}(p^\ell, 1, \ldots, 1) \quad \textit{for} \quad p \in \mathcal{P}, \quad \textit{and} \quad \delta \log t < p \leq 2\delta \log t$$

is bounded by

$$\ll_{\delta, \ell, \mathcal{P}} t^d e^{-C \log t / \log_2 t}.$$

By the corollary, we see that for any fixed ϕ_1, the number of $\phi_2 \in \mathcal{H}_t$ for which

$$n_{1,2} \ll \log t$$

does not hold is

$$\ll |\mathcal{H}_t| e^{-C \log t / \log_2 t}.$$

The second application concerns the sign changes of Maass forms on $GL(n, \mathbb{R})$. In case of $GL(2, \mathbb{R})$, there are fruitful results (for example, see [5], [12], [13]). In case of $GL(3, \mathbb{R})$, Steiger [14] proved that there is a positive proportion of Hecke-Maass forms ϕ with positive real part of $A_\phi(p, 1)$ for a fixed prime p and Xiao and Xu [16] gave a statistical result on the signs of $A_\phi(p^{\kappa_1}, p^{\kappa_2}) + A_\phi(p^{\kappa_2}, p^{\kappa_1})$. Applying Theorem 1.1, we obtain the following result.

Theorem 1.3. *Let $0 \neq \kappa = (\kappa_1, \dots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$. Let $\{\varepsilon_p\}_{p\in\mathcal{P}}$ be a sequence of real numbers with $\varepsilon_p \in \{\pm 1\}$ where the set of primes $\mathcal{P}$ satisfies (1.4). For any $\delta > 0$, there is a positive constant $C = C(\delta, \kappa, \mathcal{P})$ such that the number of $\phi \in \mathcal{H}_t$ verifying*

$$\varepsilon_p(A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})) > 0$$

for $p \in \mathcal{P}$ and $\delta \log t < p \le 2\delta \log t$ is bounded by

$$\ll t^d e^{-C \log t/\log_2 t}.$$

The implied constant depends at most on $\delta, \kappa, \mathcal{P}$.

Remark 1.2. *Refer to [8] and [15] for the case of $GL(2,\mathbb{R})$ and [16] for $GL(3,\mathbb{R})$.*

The size of $L(1, f)$ for L-functions over a family of f has attracted much interest. For $\phi \in \mathcal{H}_t$, its associated L-function is defined as

$$L(s, \phi) := \sum_{m\ge 1} A_\phi(m, 1, \cdots, 1) m^{-s},$$

for $\Re\mathrm{e}\, s > (n + 1)/2$, and factors into the Euler product

$$L(s, \phi) = \prod_p \prod_{i=1}^n (1 - \alpha_{\phi,i}(p)p^{-s})^{-1}$$

where $\alpha_{\phi,i}(p)$, $1 \le i \le n$, are the Satake parameters. It is well-known that $L(s, \phi)$ can be analytically continued to the whole complex plane.

Recently, Lau and Wang [7] proved that for all $\phi \in \mathcal{H}_t$, we have

$$\{1 + o(1)\}(2B_n^- \log_2 t)^{-A_n^-} \le |L(1, \phi)| \le \{1 + o(1)\}(2B_n^+ \log_2 t)^{A_n^+}.$$

under the Generalized Ramanujan Conjecture and the Generalized Riemann Hypothesis. Here $B_n^\pm$ are the positive constants in [7, Lemma 5.3] and

$$A_n^+ := n \qquad \text{and} \qquad A_n^- := \begin{cases} n & \text{if } n \text{ is even,} \\ n\cos(\pi/n) & \text{if } n \text{ is odd.} \end{cases}$$

On the other hand, Lau and Wang [7] also proved that there exist $\phi^\pm \in \mathcal{H}_t$ such that

$$|L(1, \phi^-)| \le \{1 + o(1)\}(B_n^- \log_2 t)^{-A_n^-} \quad , \quad |L(1, \phi^+)| \ge \{1 + o(1)\}(B_n^+ \log_2 t)^{A_n^+}.$$

The proportion of such exceptional $\phi^\pm$ in $\mathcal{H}_t$ is at least $\exp\big(-(\log t)/(\log_2 t)^{3+o(1)}\big)$. In fact, alongside the Montgomery-Vaughan conjecture (cf. Conjecture 1 in [4]), the proportion of $\phi^\pm$ in $\mathcal{H}_T$ satisfying $|L(1, \phi^\pm)|^{\pm 1} \ge (B_n^\pm \log_2 T)^{A_n^\pm}$ is predicted to be $>$ $\exp(-C \log t/\log_2 t)$ and $< \exp(-c \log t/\log_2 t)$ respectively for some constants $C > c > 0$.

Theorem 1.1 gives an upper bound towards the Montgomery-Vaughan conjecture. Define

$$F_t^+(s) = \frac{1}{|\mathcal{H}_t|} \sum_{\substack{\phi\in\mathcal{H}_t \\ |L(1,\phi)|>(B_n^+ s)^{A_n^+}}} 1$$

and

$$F_t^-(s) = \frac{1}{|\mathcal{H}_t|} \sum_{\substack{\phi\in\mathcal{H}_t \\ |L(1,\phi)|<(B_n^- s)^{A_n^-}}} 1.$$

Theorem 1.4. *For any $\varepsilon > 0$, there are two positive constants $c = c(\varepsilon)$ and $t_0 = t_0(\varepsilon)$ such that*

$$F_t^{\pm}(\log_2 t + r) \le \exp\left(-c(|r|+1)\frac{\log t}{(\log_2 t)(\log_3 t)(\log_4 t)}\right)$$

for $t \ge t_0$ and $\log \varepsilon \le r \le (9-\varepsilon)\log_2 t$.

Remark 1.3. *Refer to [8] and [15] for the case of $GL(2,\mathbb{R})$.*

2. Preliminaries

The Fourier coefficients $A_\phi(p^\kappa)$ can be expressed in terms of the (degenerate) Schur polynomials and Satake parameters as in (1.2). The degenerate Schur polynomial is defined as

$$S_\kappa(x_1, x_2, \cdots, x_n) := \frac{\det\left(x_j^{\sum_{l=1}^{n-i}(\kappa_l+1)}\right)_{1\le i,j\le n}}{\det\left(x_j^{\sum_{l=1}^{n-i}1}\right)_{1\le i,j\le n}} \tag{2.1}$$

for $\kappa = (\kappa_1, \cdots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$. Matz and Templier established an automorphic equidistribution of the family $\{A_\phi(p^\kappa) : \phi \in \mathcal{H}^\natural\}$ – the vertical Sato-Tate law for Hecke-Maass forms. Now we explain a consequence of the equidistribution result.

Let $\mathfrak{S}_n$ be the symmetric group and

$$T_0 = \left\{(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \in (S^1)^n : e^{i(\theta_1+\theta_2+\cdots+\theta_n)} = 1\right\}.$$

We define two measures $d\mu_{\mathrm{ST}}$ and $d\mu_p$ on $T_0/\mathfrak{S}_n$ whose integration formulas (over $[0,2\pi]^{n-1}$) are given by

$$d\mu_{\mathrm{ST}} = \frac{1}{n!}\frac{1}{(2\pi)^{n-1}}\prod_{1\le i<j\le n}|e^{\mathrm{i}\theta_i} - e^{\mathrm{i}\theta_j}|^2\, d\theta_1\cdots d\theta_{n-1}$$

and

$$d\mu_p = \frac{1}{n!}\prod_{i=2}^{n}\frac{1-p^{-i}}{1-p^{-1}}\cdot\prod_{1\le i<j\le n}\left|\frac{e^{\mathrm{i}\theta_i} - p^{-1}e^{\mathrm{i}\theta_j}}{e^{\mathrm{i}\theta_i} - e^{\mathrm{i}\theta_j}}\right|^{-2}\cdot\frac{1}{(2\pi)^{n-1}}\, d\theta_1\cdots d\theta_{n-1}.$$

Define $S_\kappa(1,\cdots,1)$ by taking $x_i \to 1$. By [7, Lemma 7.1 (2)], we have for any $X \ge 1$ and $\kappa \in \mathbb{N}_0^{n-1}$,

$$\max_{|x_i|\le X,\forall i}|S_\kappa(x_1,\cdots,x_n)| \le X^{\|\kappa\|}S_\kappa(1,\cdots,1) \le X^{\|\kappa\|}(1+|\kappa|)^{n^2-n}. \tag{2.2}$$

A consequence of Matz and Templier's work on the vertical Sato-Tate is the following, cf. [6, Corollary 3.3].

Lemma 2.1. *Let $\kappa = (\kappa_1, \cdots, \kappa_{n-1}) \in \mathbb{N}_0^{n-1}$, $\mathcal{H}_t$ and $A_\phi(p^\kappa) = A_\phi(p^{\kappa_1}, \cdots, p^{\kappa_{n-1}})$ be defined as above, cf. (1.1), (1.2) and (2.1). Then for any $\ell, m \in \mathbb{N}$,*

$$\begin{aligned}&\frac{1}{|\mathcal{H}_t|}\sum_{\phi\in\mathcal{H}_t}\prod_{p^{u_p}\|\ell, p^{v_p}\|m} A_\phi(p^{\kappa_1},\ldots,p^{\kappa_{n-1}})^{u_p}\overline{A_\phi(p^{\kappa_1},\ldots,p^{\kappa_{n-1}})}^{v_p}\\ &= \prod_{p^{u_p}\|\ell, p^{v_p}\|m}\int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p}\overline{S_\kappa^{v_p}}\,d\mu_p + O\Bigg(t^{-1/2}\prod_{p^{u_p}\|\ell, p^{v_p}\|m}\left(c_\kappa p^{L\|\kappa\|}\right)^{u_p+v_p}\Bigg)\end{aligned}$$

where L is a positive constant, $1 \le c_\kappa := (1+|\kappa|)^{n^2-n}$.

The product of two Schur polynomials S_κ and $S_{\kappa'}$ may be evaluated with the Littlewood-Richardson rule:

$$S_\kappa S_{\kappa'} = S_\kappa \cdot S_{\kappa'} = \sum_\xi d^\xi_{\kappa\kappa'} S_\xi \tag{2.3}$$

where $d^\xi_{\kappa\kappa'}$'s are nonnegative integers and the summation runs over $\xi \in \mathbb{N}_0^{n-1}$ satisfying $\|\xi\| \le \|\kappa\| + \|\kappa'\|$ and $\|\xi\| \equiv \|\kappa\| + \|\kappa'\| \bmod n$. (Recall $\|\kappa\| := \sum_i (n-i)\kappa_i$.) Moreover $\{S_\kappa\}$ form an orthonormal set under the inner product induced by the measure $d\mu_{\mathrm{ST}}$,

$$\langle S_\kappa, S_{\kappa'}\rangle \;=\; \int_{[0,2\pi]^{n-1}} S_\kappa(\underline{\theta})\overline{S_{\kappa'}(\underline{\theta})} d\mu_{\mathrm{ST}} = \delta_{\kappa=\kappa'}. \tag{2.4}$$

As well, by [7, Proposition 7.4 (1)] we have

$$\int_{T_0/\mathfrak{S}_n} S_\kappa d\mu_p = \prod_{i=1}^{n-1}(1-p^{-i}) \cdot \sum_{\eta\in\mathbb{N}_0^{n-1}} d^\eta_{\kappa\eta} \cdot p^{-\|\eta\|}$$

where the sum over η is supported on $|\eta| \ge \|\kappa\|/n$ and with (2.2) and (2.4),

$$0 \le d^\eta_{\kappa\eta} = \int_{T_0/\mathfrak{S}_n} S_\kappa |S_\eta|^2 d\mu_{\mathrm{ST}} \le (1+|\kappa|)^{(n^2-n)}.$$

Consequently, for $\|\kappa\| \ne 0$ we have

$$\begin{aligned}
&\Big|\int_{T_0/\mathfrak{S}_n} S_\kappa d\mu_p\Big| \\
&\le (1+|\kappa|)^{(n^2-n)} \prod_{i=1}^{n-1}(1-p^{-i}) \max_{\sum_i \eta_i = \lceil \frac{\|\kappa\|}{n}\rceil} \Big(\prod_{1\le i\le n-1}\sum_{\ell\ge\eta_i} p^{-i\ell}\Big) \\
&\le (1+|\kappa|)^{(n^2-n)} \max_{|\eta|=\lceil\frac{\|\kappa\|}{n}\rceil} p^{-\|\eta\|} \\
&\le (1+|\kappa|)^{(n^2-n)} p^{-1}
\end{aligned} \tag{2.5}$$

where $\lceil x\rceil$ denotes the smallest integer greater than or equal to x. (Note $|\eta| \le \|\eta\|$.)

By Cauchy-Schwarz's inequality and (2.2), we have

$$\begin{aligned}
\sum_{\|\xi\|\le n|\kappa|} (d^\xi_{\kappa\kappa'})^2 &= \langle S_\kappa S_{\kappa'}, S_\kappa S_{\kappa'}\rangle \\
&\le S_\kappa(1,\ldots,1) S_{\kappa'}(1,\ldots,1) \langle S_\kappa, S_\kappa\rangle^{1/2} \langle S_{\kappa'}, S_{\kappa'}\rangle^{1/2} \\
&\le ((1+|\kappa|)(1+|\kappa'|))^{n^2-n} = c_\kappa c_{\kappa'}.
\end{aligned} \tag{2.6}$$

We need an arithmetic function and a result in [8].

Lemma 2.2. *Let $2 \le P < Q \le 2P$, $j \ge 1$ and $n \ge 1$. Define*

$$a_j(n) = a_j(n; P, Q) = |\{(p_1,\ldots,p_j) : p_1\cdots p_j = n,\ P < p_1,\ldots,p_j \le Q\}|.$$

For any $d > 0$, $\sum_n a_j(n) d^{\Omega(n)}/n \ll (3d/\log P)^j$; moreover,

$$\begin{aligned}\sum_n a_j(n^2)\frac{d^{\Omega(n)}}{n^2} &\leq \delta_{2|j}\Big(\frac{3dj}{P\log P}\Big)^{j/2}\\ \sum_n{}^{\natural} a_j(n)\frac{d^{\Omega(n)}}{n} &\leq \Big(\frac{12d^2j}{P\log P}\Big)^{j/2}\Big\{1+\Big(\frac{j\log P}{54P}\Big)^{j/6}\Big\}\\ \sum_m{}^{\flat}\sum_{(m,n)=1}{}^{\natural} a_j(mn)\frac{d^{\Omega(mn)}}{m^2n} &\leq \Big(\frac{48d^2j}{P\log P}\Big)^{j/2}\Big\{1+\Big(\frac{20j\log P}{P}\Big)^{j/6}\Big\}\end{aligned}$$

where $\Omega(n)$ counts the number of (not necessarily distinct) prime divisors, $\delta_{2|j} = 1$ if $2|j$ or 0 otherwise, $\sum^{\flat}$ and $\sum^{\sharp}$ run over squarefree and squarefull integers respectively.

3. Proof of Theorem 1.1

Let $a_j(\cdot)$ be defined as in Lemma 2.2. Squaring out, we have

$$\begin{aligned}&\Big|\sum_{P<p\leq Q} b_p\frac{A_\phi(p^{\kappa_1},\dots,p^{\kappa_{n-1}})}{p}\Big|^{2j}\\ &=\Big(\sum_{P<p\leq Q} b_p\frac{A_\phi(p^{\kappa_1},\dots,p^{\kappa_{n-1}})}{p}\Big)^j\Big(\sum_{P<p\leq Q} \overline{b_p}\frac{\overline{A_\phi(p^{\kappa_1},\dots,p^{\kappa_{n-1}})}}{p}\Big)^j\\ &=\Big(\sum_{P^j<\ell\leq Q^j} a_j(\ell)\frac{b_\ell}{\ell}\prod_{p^u\|\ell} A_\phi(p^{\kappa_1},\dots,p^{\kappa_{n-1}})^u\Big)\\ &\quad\times\Big(\sum_{P^j<m\leq Q^j} a_j(m)\frac{\overline{b_m}}{m}\prod_{q^v\|m} \overline{A_\phi(p^{\kappa_1},\dots,p^{\kappa_{n-1}})}^v\Big)\\ &=\sum_{P^j<\ell,m\leq Q^j} a_j(\ell)a_j(m)\frac{b_\ell\overline{b_m}}{\ell m}\\ &\quad\times\prod_{p^{u_p}\|\ell,p^{v_p}\|m} A_\phi(p^{\kappa_1},\dots,p^{\kappa_{n-1}})^{u_p}\overline{A_\phi(p^{\kappa_1},\dots,p^{\kappa_{n-1}})}^{v_p}\end{aligned}$$

Averaging over $\phi \in \mathcal{H}_t$, it follows from Lemma 2.1 that

$$\begin{aligned}\frac{1}{|\mathcal{H}_t|}\sum_{\phi\in\mathcal{H}_t}\prod_{p^{u_p}\|\ell,p^{v_p}\|m}\cdots\\ =\prod_{p^{u_p}\|\ell,p^{v_p}\|m}\int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p}\overline{S_\kappa^{v_p}}d\mu_p + O\left(t^{-1/2}\big(c_\kappa Q^{L\|\kappa\|}\big)^{2j}\right).\end{aligned}\tag{3.1}$$

Thus the left side of (1.3) can be expressed as follows:

$$\frac{1}{|\mathcal{H}_t|}\sum_{\phi\in\mathcal{H}_t}\Big|\sum_{P<p\leq Q} b_p\frac{A_\phi(p^{\kappa_1},\dots,p^{\kappa_{n-1}})}{p}\Big|^{2j} = M+E.\tag{3.2}$$

The error term E is

$$\ll t^{-1/2}\big(c_\kappa Q^{L\|\kappa\|}\big)^{2j} \sum_{P^j<\ell,m\le Q^j} a_j(\ell)a_j(m)\frac{b_\ell \overline{b_m}}{\ell m}$$

$$\ll t^{-1/2}\big(c_\kappa Q^{L\|\kappa\|}\big)^{2j}\left(\frac{3B}{\log P}\right)^{2j} \tag{3.3}$$

by Lemma 2.2.

Next we evaluate the main term

$$M=\sum_{P^j<\ell,m\le Q^j} a_j(\ell)a_j(m)\frac{b_\ell\overline{b_m}}{\ell m}\prod_{p^{u_p}\|\ell,p^{v_p}\|m}\int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p}\overline{S_\kappa^{v_p}}\,d\mu_p. \tag{3.4}$$

Write $\ell=\ell_1\ell'$ and $m=m_1m'$ such that ℓ_1m_1 is squarefree, $\ell'm'$ is squarefull and $(\ell_1m_1,\ell'm')=1$.[1] (Note $\ell_1m_1=1$ when ℓm is squarefull.) Set $h=\ell_1m_1$ and $r=\ell'm'$. We split the product over prime divisors of ℓm in (3.4) into a product of two pieces over prime divisors of ℓ_1m_1 and $\ell'm'$ respectively:

$$\prod_{p^{u_p}\|\ell,p^{v_p}\|m}\cdots=\prod_{p^{u_p}\|\ell_1,p^{v_p}\|m_1}\int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p}\overline{S_\kappa^{v_p}}\,d\mu_p\prod_{p^{u_p}\|\ell',p^{v_p}\|m'}\int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p}\overline{S_\kappa^{v_p}}\,d\mu_p.$$

Inside the second product, we invoke the trivial bound (2.2) and for the first product, (as ℓ_1m_1 is squarefree) we have $u_p+v_p=1$ and thus apply (2.5). This leads to

$$\begin{aligned}&\Big|\prod_{p^{u_p}\|\ell,p^{v_p}\|m}\int_{T_0/\mathfrak{S}_n} S_\kappa^{u_p}\overline{S_\kappa^{v_p}}\,d\mu_p\Big|\\ &\le(1+|\kappa|)^{\Omega(\ell'm')(n^2-n)}\prod_{p^{u_p}\|\ell_1,p^{v_p}\|m_1}(1+|\kappa|)^{n^2-n}p^{-1}\\ &\le(1+|\kappa|)^{2j(n^2-n)}h^{-1},\end{aligned}$$

and

$$\begin{aligned}|M|&\le(1+|\kappa|)^{2j(n^2-n)}\sum_{P^j<\ell_1\ell',m_1m'\le Q^j} a_j(\ell_1\ell')a_j(m_1m')\frac{|b_{\ell_1\ell'}\overline{b_{m_1m'}}|}{(\ell_1m_1)^2\ell'm'}\\ &\le(1+|\kappa|)^{2j(n^2-n)}B^{2j}\sum_h{}^{\flat}\sum_r{}^{\natural}\frac{1}{h^2r}\sum_{\substack{P^j<\ell_1\ell',m_1m'\le Q^j\\ \ell_1m_1=h,\ \ell'm'=r}} a_j(\ell_1\ell')a_j(m_1m')\\ &\le(1+|\kappa|)^{2j(n^2-n)}B^{2j}\sum_h{}^{\flat}\sum_r{}^{\natural}\frac{a_{2j}(hr)}{h^2r}\\ &\ll(1+|\kappa|)^{2j(n^2-n)}B^{2j}\left(\frac{96j}{P\log P}\right)^j\left\{1+\left(\frac{40j\log P}{P}\right)^{j/3}\right\}\end{aligned}$$

where the implied constant is independent of j.

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[1]The decomposition is unique. Assume $\ell=\ell_1\ell'=\ell_2\ell''$ and $m=m_1m'=m_2m''$ are two such decomposition. Every positive integer decomposes uniquely into a product of a squarefree integer and a squarefull integer. From $(\ell_1m_1)(\ell'm')=(\ell_2m_2)(\ell''m'')$, we get $(*)$: $\ell_1m_1=\ell_2m_2$ and $\ell'm'=\ell''m''$. As ℓ_1m_1 is squarefree, we have $(\ell_1,m_1)=1$; with $(\ell_1m_1,\ell'm')=1$, we infer $(\ell_1,m)=1$. So $(\ell_1,m_2)=1$, and $(\ell_2,m_1)=1$ by symmetry. By $(*)$, $\ell_1=\ell_2$ and $m_1=m_2$.

4. Proof of Theorem 1.2

Let $\delta \log t \leq P \leq (\log t)^{10}$ and write $\mathcal{P}_P := \mathcal{P} \cap (P, 2P]$. Define

$$E(t;P) = \{\phi \in \mathcal{H}_t : A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) = \lambda(p) \text{ for } p \in \mathcal{P} \cap (P, 2P]\}.$$

As the Ramanujan Conjecture is open, we consider the exceptional set over each prime

$$\mathcal{E}(t,p) = \left\{ \phi \in \mathcal{H}_t : \log \max_{1 \leq i \leq n} |\alpha_{\phi,i}(p)| > 1 \right\}$$

whose size is under control. Indeed, analogously to Sarnak's bound for the $GL(2)$ Maass forms, we have $|\mathcal{E}(t,p)| \ll t^{d - c_0/\log p}$ where $c_0 > 0$ is a constant, cf. [7, Theorem 7.3]. Hence

$$\Big| \bigcup_{p \in \mathcal{P}_P} \mathcal{E}(t,p) \Big| \ll t^{d - c'/\log P}$$

for some constant c'. Set

$$E^*(t;P) = E(t;P) \setminus \bigcup_{p \in \mathcal{P}_P} \mathcal{E}(t,p).$$

It remains to prove that

$$E^*(t;P) \ll_{\delta,\kappa,\mathcal{P}} t^d e^{-C \log t / \log_2 t}$$

for all $t > T_0$, where $T_0 = T_0(\delta, \kappa, \mathcal{P})$ is a sufficiently large number. We may assume

$$|\lambda(p)| < e^{\|\kappa\|}(1 + |\kappa|)^{n^2 - n} \tag{4.1}$$

for all $P \leq p \leq 2P$; otherwise the set $E(t;P)$ is empty by (2.2). Suppose $j \in \mathbb{N}$ is chosen such that

$$j \leq \frac{P}{40 \log P}. \tag{4.2}$$

We apply Theorem 1.1 with

$$b_p = \begin{cases} \overline{\lambda(p)} & \text{if } p \in \mathcal{P}_P, \\ 0 & \text{otherwise.} \end{cases} \tag{4.3}$$

Since $\overline{\lambda(p)} A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) = |A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})|^2$ for $\phi \in E^*(t;P)$, it follows that

$$\begin{aligned} &\sum_{\phi \in E^*(t;P)} \Big| \sum_{p \in \mathcal{P}_P} \frac{|A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})|^2}{p} \Big|^{2j} \\ &\leq \sum_{\phi \in \mathcal{H}_t} \Big| \sum_{P < p \leq 2P} b_p \frac{A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})}{p} \Big|^{2j} \\ &\ll t^d \left(\frac{(B_1 C_\kappa)^2 j}{P \log P} \right)^j + t^{d-1/2} \left(\frac{B_1 C_\kappa Q^{L\|\kappa\|}}{\log P} \right)^{2j} \end{aligned} \tag{4.4}$$

where $B_1 = e^{\|\kappa\|}(1 + |\kappa|)^{n^2 - n}$ and $Q = 2P$, in view of (4.1).

The size of $|A_\phi(p^{\kappa_1},\dots,p^{\kappa_{n-1}})|^2$ is about 1 on average. To see it, we firstly deduce from (1.2) and (2.3) that

$$\begin{aligned}|A_\phi(p^{\kappa_1},\dots,p^{\kappa_{n-1}})|^2 &= A_\phi(p^{\kappa_1},\dots,p^{\kappa_{n-1}})A_\phi(p^{\kappa_{n-1}},\dots,p^{\kappa_1})\\ &= 1+\sum_{\substack{\xi\neq\mathbf{0}\\ \|\xi\|\le n|\kappa|}} d^{\xi}_{\kappa\kappa^\iota}A_\phi(p^{\xi_1},\dots,p^{\xi_{n-1}})\end{aligned}\tag{4.5}$$

where $\kappa^\iota=(\kappa_{n-1},\dots,\kappa_1)$. (Then $\|\kappa^\iota\|=n|\kappa|-\|\kappa\|$.)

Secondly, we exploit the oscillation among $A_\phi(p^{\xi_1},\dots,p^{\xi_{n-1}})$ by Theorem 1.1 (again). For $\xi=(\xi_1,\dots,\xi_{n-1})$ with $1\le\|\xi\|\le n|\kappa|=|n\kappa|$, we define

$$E^{\xi}(t;P)=\left\{\phi\in\mathcal{H}_t:\left|\sum_{\substack{P<p\le 2P\\ p\in\mathcal{P}}}\frac{A_\phi(p^{\xi_1},\dots,p^{\xi_{n-1}})}{p}\right|\ge\frac{\Delta'}{\log P}\right\}$$

where $\Delta':=\Delta/(2c_\kappa c_{\kappa^\iota})<\Delta/2$. Taking $b_p=1$ if $p\in\mathcal{P}_P$ or 0 otherwise, we get from Theorem 1.1 with $C_\xi\le C_{n\kappa}$ that

$$|E^{\xi}(t;P)|\ll t^d\left(\frac{C_{n\kappa}^2 j\log P}{\Delta'^2P}\right)^j+t^{d-1/2}\left(\frac{C_{n\kappa}Q^{L\|\xi\|}}{\Delta'}\right)^{2j}.\tag{4.6}$$

For $\phi\in E^*(t;P)\setminus\bigcup_{\substack{\xi\neq\mathbf{0}\\ \|\xi\|\le n|\kappa|}}E^{\xi}(t;P)$, the inner sum (over p) in (4.4) is, by (4.5),

$$\ge\sum_{\substack{P<p\le 2P\\ p\in\mathcal{P}}}\frac1p-\sum_{\substack{\xi\neq\mathbf{0}\\ \|\xi\|\le n|\kappa|}}d^{\xi}_{\kappa\kappa^\iota}\left|\sum_{\substack{P<p\le 2P\\ p\in\mathcal{P}}}\frac{A_\phi(p^{\xi_1},\dots,p^{\xi_{n-1}})}{p}\right|\ge\frac{\Delta}{2\log P}\tag{4.7}$$

Here we have applied that $c_\kappa c_{\kappa^\iota}\Delta'\le\Delta/2$ and

$$\sum_{\substack{\xi\neq\mathbf{0}\\ \|\xi\|\le n|\kappa|}}d^{\xi}_{\kappa\kappa^\iota}\le\sum_{\|\xi\|\le n|\kappa|}(d^{\xi}_{\kappa\kappa^\iota})^2\le c_\kappa c_{\kappa^\iota}\tag{4.8}$$

by (2.6).

Applying the lower bound (4.7) to the left-hand side of (4.4), we thus infer

$$\begin{aligned}&\left(\frac{\Delta}{2\log P}\right)^{2j}\left|E^*(t;P)\setminus\bigcup_{\substack{\xi\neq\mathbf{0}\\ \|\xi\|\le n|\kappa|}}E^{\xi}(t;P)\right|\\ &\ll t^d\left(\frac{(B_1C_\kappa)^2j}{P\log P}\right)^j+t^{d-1/2}\left(\frac{B_1C_\kappa Q^{L\|\kappa\|}}{\log P}\right)^{2j}\end{aligned}$$

and, together with (4.6),

$$|E^*(t;P)|\ll t^d\left(\frac{(B_1C_{n\kappa})^2j\log P}{\Delta'^2P}\right)^j+t^{d-1/2}\left(\frac{B_1C_{n\kappa}Q^{L\|\kappa\|}}{\Delta'}\right)^{2j}.\tag{4.9}$$

Recall $\delta\log t\le P\le(\log t)^{10}$. Take

$$j=\left\lceil\Delta^*\frac{\log t}{\log P}\right\rceil$$

with

$$\Delta^* = \min\Big(\frac{\delta}{40}, \frac{\delta\Delta'^2}{(2B_1C_{n\kappa})^2}, \frac{1}{8L\|\kappa\|}\Big).$$

Thus (4.2) is valid and the term inside the first bracket of (4.9) is bounded by $1/4$. Let T_0 be large enough so that $1 < j < \delta(\log t)/(\log_2 t)$ and the second term in the right-side of (4.9) is less than $t^{d-1/6}$ whenever $t > T_0$. Then we conclude that

$$|E^*(t;P)| \ll t^d e^{-C\log t/\log_2 t}$$

for some constant $C > 0$ depending on δ, κ and $\mathcal{P}$. The proof of Theorem 1.2 is complete.

5. Proof of Theorem 1.3

The method of proof is the same as Theorem 1.2, starting with the set

$$F(t;P) = \big\{\phi \in \mathcal{H}_t : \varepsilon_p\big(A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})\big) > 0 \text{ for } p \in \mathcal{P}_P\big\}.$$

The task is to evaluate

$$F^*(t;P) = F(t;P) \setminus \bigcup_{p\in\mathcal{P}_P} \mathcal{E}(t,p).$$

Using the positivity of $\varepsilon_p\big(A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})\big)$ for $\phi \in F^*(t;P)$, we have

$$\begin{aligned}
&|A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})|^2 \\
&\le 2e^{\|\kappa\|}(1+|\kappa|)^{n^2-n}\varepsilon_p\big(A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})\big).
\end{aligned}$$

by (2.2), and the analogue of (4.5) follows from (2.3) and (2.4):

$$\begin{aligned}
&|A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}}) + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})|^2 \\
&= 2A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1}) \\
&\quad + A_\phi(p^{\kappa_1}, \dots, p^{\kappa_{n-1}})^2 + A_\phi(p^{\kappa_{n-1}}, \dots, p^{\kappa_1})^2 \\
&= 2(1+\delta_{\kappa,\kappa^\iota}) + \sum_{\substack{\xi\neq\mathbf{0}\\ \|\xi\|\le 2n|\kappa|}} (d^\xi_{\kappa\kappa} + 2d^\xi_{\kappa\kappa^\iota} + d^\xi_{\kappa^\iota\kappa^\iota})A_\phi(p^{\xi_1}, \dots, p^{\xi_{n-1}})
\end{aligned}$$

where $\delta_{\kappa,\kappa^\iota}$ if $\kappa = \kappa^\iota$ or 0 otherwise, and $\kappa^\iota = (\kappa_{n-1}, \dots, \kappa_1)$.

6. Proof of Theorem 1.4

Let $\varepsilon \in (0, 10^{-10}]$ be fixed. We need a short Euler product approximation for a bulk of $L(1,\phi)$'s.

Proposition 6.1. *There are a constant $c' > 0$ and a subset $E^1(z)$ of $\mathcal{H}_t$ such that*

$$L(1,\phi) = \left\{1 + O\left(\frac{1}{\log_2 t}\right)\right\}\prod_{p\le z}\prod_{i=1}^n\left(1 - \frac{\alpha_{\phi,i}(p)}{p}\right)^{-1}$$

uniformly for $\varepsilon\log t \le z \le (\log t)^{10}$ and all Maass forms $\phi \in \mathcal{H}_t \setminus E^1(z)$, where the implied constant in the O-term is absolute and

$$|E^1(z)| = O_\varepsilon\left(t^d\exp\left(-c'\frac{\log t}{(\log_2 t)(\log_3 t)(\log_4 t)}\right)\right).$$

Proof. We follow the same approach as in the proof of [8, Proposition 8.1]. A crucial difference is without the Ramanujan bound now, and thus we exclude the forms outside the set

$$\mathcal{K}_t = \mathcal{K}_t(\eta) := \left\{ \phi \in \mathcal{H}_t : \log \max_{1\le i\le n} |\alpha_{\phi,i}(p)| \le 1/(\log_3 t)(\log_4 t), \forall p \le (\log t)^{1/\eta} \right\}$$

where $\eta > 0$ is any number. The size of the exceptional set, i.e. $\mathcal{H}_t^- = \mathcal{H}_t \backslash \mathcal{K}_t$, is small:

$$\mathcal{H}_t^- \ll t^d \exp\left(-c \frac{\eta \log t}{(\log_2 t)(\log_3 t)(\log_4 t)} \right) \tag{6.1}$$

for some constant $c > 0$, by [7, Theorem 7.3] (see also [7, (6.1)]). We work on $\mathcal{K}_t$ with the argument in [8] to complete the proof. □

Now we prove Theorem 1.4. For $\phi \in \mathcal{H}_t \backslash E^1(z)$, we have

$$\begin{aligned} |L(1,\phi)| &\le \left\{ 1 + O\left(\frac{1}{\log_2 t} \right) \right\} \prod_{p\le z} \left(1 - \frac{\alpha'}{p} \right)^{-n} \\ &\le \left\{ 1 + O\left(\frac{1}{\log_2 t} \right) \right\} (e^\gamma \log z)^{\alpha' n} \\ &\le \left\{ e^\gamma \left((e^{\gamma(1-1/\alpha')} \log z)^{\alpha'} + C_0 (\log_2 t)^{\alpha'-1} \right) \right\}^n, \end{aligned}$$

where C_0 is an absolute constant and $\alpha' = \exp(1/(\log_3 t)(\log_4 t))$. Taking

$$\begin{aligned} z &= e^{e^{-\gamma(1-1/\alpha')} (\log_2 t + r - C_0 (\log_2 t)^{\alpha'-1})^{1/\alpha'}} \\ &= e^{(1+O((\log_4 t)^{-1})(\log_2 t + r - C_0(\log_2 t)^{\alpha'-1})}, \end{aligned}$$

the proof is complete for F_t^+. The case of F_t^- is treated in the same fashion.

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