ON THE EXCEPTIONAL SET OF THE GENERALIZED RAMANUJAN CONJECTURE FOR $GL(3)$

YUK-KAM LAU, MING HO NG AND YINGNAN WANG

Abstract. Recently Luo and Zhou showed that for a fixed $GL(2)$ Hecke-Maass cusp form, the natural density of primes at which the Satake parameters fail the Ramanujan Conjecture does not exceed 1/35. In this short note, we investigate the $GL(3)$ case and obtain two similar (conditional) results.

1. Introduction

The classical Ramanujan conjecture, due to Ramanujan ([22]), is about the size of the Fourier coefficients $\tau(n)$ of the discriminant modular form $\Delta(z)$, asserting that $|\tau(n)| \leq n^{11/2}d(n)$ where $d(n) = \sum_{d|n} 1$. Deligne [5] confirmed the Ramanujan conjecture for $\Delta(z)$ as well as holomorphic Hecke cusp forms of weight $k \geq 2$. However the classical Ramanujan conjecture for Maass forms is still open. Nevertheless, it is believed to be true in a wider context. The Generalized Ramanujan Conjecture (GRC) for generic cuspidal automorphic representations of $GL(n)$ is formulated as that all the Satake parameters of the corresponding representation are of modulus 1. The GRC is deep and currently beyond our reach. In the literature, there has been some work on estimating the number of forms that fail GRC at a fixed prime p. Readers are referred to [23, 11, 25] for $GL(2)$, [2] for $GL(3)$ and [12, 10, 16] for $GL(n)$ with $n \geq 3$.

Naturally, one is interested in knowing the density of primes at which the Ramanujan conjecture holds for a given Hecke-Maass cusp form on $GL(n)$. When $n = 2$, Ramakrishnan [18] proved that the set of such primes has a lower Dirichlet density of at least 9/10. Later Kim and Shahidi [9] improved Ramakrishnan's result to 34/35 by using the cuspidality of third and fourth symmetric powers of a cusp form on $GL(2)$. Recently, Luo and Zhou [15] refined Kim and Shahidi's result and showed that the natural density of such primes (at which the Ramanujan conjecture holds) should be at least 34/35.

When $n \geq 3$, let ϕ be any element of the orthonormal basis $\mathcal{H}$ of Hecke-Maass cusp forms for $SL_n(\mathbb{Z})$ and let p be any prime. Denote $\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \ldots, \alpha_{\phi,n}(p)$ the corresponding Satake parameters of ϕ at p. It is known that

$$\alpha_{\phi,1}(p)\alpha_{\phi,2}(p)\cdots\alpha_{\phi,n}(p) = 1 \tag{1.1}$$

and (the unitary condition)

$$\{\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \ldots, \alpha_{\phi,n}(p)\} = \left\{\overline{\alpha_{\phi,1}(p)^{-1}}, \overline{\alpha_{\phi,2}(p)^{-1}}, \ldots, \overline{\alpha_{\phi,n}(p)^{-1}}\right\}. \tag{1.2}$$

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GRC asserts that

$$|\alpha_{\phi,1}(p)| = |\alpha_{\phi,2}(p)| = \cdots = |\alpha_{\phi,n}(p)| = 1, \tag{1.3}$$

which is out of our reach. The best bound towards GRC is

$$|\alpha_\ell(p)| \le p^{\theta_n} \text{ for } \ell = 1, 2, \ldots, n, \tag{1.4}$$

where (see [8] for $n = 3, 4$ and [14] for $n \ge 5$)

$$\theta_3 = \frac{5}{14}, \qquad \theta_4 = \frac{9}{22}, \qquad \theta_n = \frac{1}{2} - \frac{1}{n^2+1} \quad (n \ge 5).$$

Let $S(\phi)$ denote the set of primes p that $|\alpha_{\phi,1}(p) + \alpha_{\phi,2}(p) + \cdots + \alpha_{\phi,n}(p)| \le n$. Ramakrishnan [18] proved that $S(\phi)$ has a lower Dirichlet density of at least $1 - 1/n^2$. However, unlike the $GL(2)$ case, this gives no information on the set of primes at which GRC holds.

When $n = 3$, more information is particularly known. Ramakrishnan [19] proved that there are infinitely many primes at which GRC holds. Later, Walji [24] proved that for any $\alpha \ge 1$, the set

$$S(\phi, \alpha) = \left\{ p \text{ primes} : \max_{1 \le i \le 3} |\alpha_{\phi,i}(p)| \le \alpha \right\}$$

has a lower Dirichlet density of $1 - 1/(\alpha + 1/\alpha - 1)^2$. Below we refine Walji's result and show that the natural density of $S(\phi, \alpha)$ should be at least $1 - 1/(\alpha + 1/\alpha - 1)^2$.

Theorem 1.1. *For any $\alpha \ge 1$, we have*

$$\limsup_{x \to \infty} \frac{1}{\pi(x)} \left| \left\{ p \le x : \max_{1 \le i \le 3} |\alpha_{\phi,i}(p)| \le \alpha \right\} \right| \ge 1 - \frac{1}{(\alpha + \alpha^{-1} - 1)^2}$$

where $\pi(x)$ is the number of primes not exceeding x.

Remark 1.1. Our method for Theorem 1.1 works well in the $GL(2)$ case and yields that

$$\limsup_{x \to \infty} \frac{1}{\pi(x)} \left| \left\{ p \le x : \max_{1 \le i \le 2} |\alpha_{f,i}(p)| \le \alpha \right\} \right| \ge 1 - \frac{1}{(\alpha + \alpha^{-1})^2}$$

for any $\alpha \ge 1$ and for any Hecke-Maass cusp form f for $SL_2(\mathbb{Z})$. This result can be viewed as a supplement to Theorem 1.2 in [15].

Theorem 1.1 becomes trivial when $\alpha = 1$. It is plausible that one needs more information so as to get a non-trivial result for the marginal case (of $\alpha = 1$). Hence we are led to establish a conditional result, which is done under the mild Hypothesis (MS) below.

To state the hypothesis, let us firstly recall the Rankin-Selberg L-function

$$L(s, \phi \times \tilde{\phi}) = \prod_p \prod_{i=1}^{3} \prod_{j=1}^{3} (1 - \alpha_{\phi,i}(p) \alpha_{\phi,j}(p)^{-1} p^{-s})^{-1} =: \sum_{n \ge 1} \lambda_{\phi \times \tilde{\phi}}(n) n^{-s}$$

($\tilde{\phi}$ denotes the dual form of ϕ) and the symmetric square L-function

$$L(s, \mathrm{sym}^2 \phi) = \prod_p \prod_{1 \le i \le j \le 3} (1 - \alpha_{\phi,i}(p) \alpha_{\phi,j}(p) p^{-s})^{-1} =: \sum_{n \ge 1} \lambda_{\mathrm{sym}^2 \phi}(n) n^{-s}$$

for $\Re e\, s \gg 1$.[1] Both L-functions have meromorphic continuation to the whole complex plane and satisfy functional equations of Riemann type, cf. [7] and [4] for instance. The

[1]Here we consider the case that the automorphic representation π_ϕ associated to ϕ is unramified at all finite primes. The Satake parameter $\alpha_{\phi,i}(p)$ at ramifed primes may be zero. The local factors of $L(s, \phi \times \widetilde{\phi})$ at ramified primes will be different, see [3, Section 1].

Rankin-Selberg L-function $L(s,\phi\times\tilde{\phi})$ has a pole at $s=1$ and one may derive an analogue of Prime Number Theorem (for example, see [13, (1.3)])

$$\sum_{p\leq x}|A_\phi(p,1)|^2\sim\pi(x). \tag{1.5}$$

The continuation of $L(s,\mathrm{sym}^2\phi)$ is meromorphic with a simple pole at $s=1$ or entire depending on whether ϕ is self-dual or not, cf. [4, p.139]. It is known that ϕ is a self-dual form for $GL(3)$ if and only if ϕ is a symmetric square lift of a $GL(2)$ form, cf. [21]. Hence the Satake parameters for a self-dual ϕ are $\alpha^2,1,\beta^2$ where α,β are Satake parameters of a $GL(2)$ form. The density of exceptional primes (for which $\max\{|\alpha|,|\beta|\}\geq 1\}$) is at most $1/35$ by [15]. Actually the self-dual forms are of density zero amongst all Hecke-Maass forms for $SL_3(\mathbb{Z})$, cf. [17].

We may confine to the non-self dual case, and state now our hypothesis and result.

Hypothesis (MS).

$$\sum_{p\leq x}\lambda_{\phi\times\tilde{\phi}}(p)^2\sim 2\pi(x) \tag{1.6}$$

and

$$\sum_{p\leq x}|\lambda_{\mathrm{sym}^2\phi}(p)|^2\sim\pi(x). \tag{1.7}$$

Theorem 1.2. *Let ϕ be a non-self dual Hecke-Maass cusp form for $SL_3(\mathbb{Z})$. Under Hypothesis (MS), we have*

$$\limsup_{x\to\infty}\frac{1}{\pi(x)}\left|\left\{p\leq x:\max_{1\leq i\leq 3}|\alpha_{\phi,i}(p)|>1\right\}\right|\leq\frac{12}{25}. \tag{1.8}$$

Remark 1.2. If ϕ is a $GL(2)$ non-CM form, then the corresponding statements in Hypothesis (MS) are already proved. Both (1.6) and (1.7) are expected by Langlands functoriality. Moreover the constants in front of $\pi(x)$ are predicted by the (unsettled) Sato-Tate distribution of the Satake parameters, cf. Section 2. From the proof, we still can get a non-trivial but weaker bound 14/25 for (1.8) if only (1.7) is assumed.

2. Satake parameters and Sato-Tate distribution

A Hecke-Maass cusp form $\phi\in\mathcal{H}$ for $SL_3(\mathbb{Z})$ admits a Fourier series (as a double sum over $(m,n)\in\mathbb{N}\times\mathbb{N}$) whose normalized coefficients $A_\phi(m,n)$ are multiplicative. Moreover, let $\mathbf{k}=(k_1,k_2)\in\mathbb{N}_0^2$. The coefficients $A_\phi(p^{\mathbf{k}}):=A_\phi(p^{k_1},p^{k_2})$ are (Hecke) eigenvalues of Hecke operators and can be expressed in terms of the Satake parameters of ϕ. For instance,

$$A_\phi(p,1)=\sum_{1\leq i\leq 3}\alpha_{\phi,i}(p)\quad\text{and}\quad A_\phi(1,p)=\sum_{1\leq i<j\leq 3}\alpha_{\phi,i}(p)\alpha_{\phi,j}(p).$$

In fact, define the degenerate Schur polynomial $S_{\mathbf{k}}$ (see [6, p.233])

$$S_{\mathbf{k}}(x_1,x_2,x_3):=\frac{\det\left(x_j^{\sum_{l=1}^{3-i}(k_l+1)}\right)_{1\leq i,j\leq 3}}{\det\left(x_j^{\sum_{l=1}^{3-i}1}\right)_{1\leq i,j\leq 3}}.$$

We have, for any prime p,

$$A_\phi(p^{\mathbf{k}}) = S_{\mathbf{k}}(\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \alpha_{\phi,3}(p)). \tag{2.1}$$

Using (1.1) and (1.2), we see that $\overline{A_\phi(p^{k_1}, p^{k_2})} = A_\phi(p^{k_2}, p^{k_1})$ and hence $A_\phi(m,n) = \overline{A_\phi(n,m)}$. Also it follows from the definitions that

$$\lambda_{\phi\times\tilde\phi}(p) = |A_\phi(p,1)|^2 \qquad \text{and} \qquad \lambda_{\mathrm{sym}^2\phi}(p) = A_\phi(p,1)^2 - A_\phi(1,p). \tag{2.2}$$

(We have suitably applied (1.1), (1.2) and $A_\phi(1,p) = \overline{A_\phi(p,1)}$.) The Hecke relation (see [6, p.173]) yields $A_\phi(p,1)^2 = A_\phi(p^2,1) + A_\phi(1,p)$, and thus

$$\lambda_{\phi\times\tilde\phi}(p)^2 = (A_\phi(p^2,1) + A_\phi(1,p))(A_\phi(1,p^2) + A_\phi(p,1)) \tag{2.3}$$

and

$$|\lambda_{\mathrm{sym}^2\phi}(p)|^2 = A_\phi(p^2,1)A_\phi(1,p^2). \tag{2.4}$$

Now we turn to some conjectural properties of the Satake parameters. In light of (1.1) and (1.2), we may write

$$\alpha_{\phi,j}(p) = e^{i\theta_{\phi,j}(p)} \qquad \text{for } 1 \le j \le 3,$$

where $\theta_{\phi,j}(p) \in [0,2\pi) \cup \{a+bi : a \in [0,2\pi), b \in \mathbb{R}^\times\}$. This $\theta_{\phi,j}(p)$ is uniquely determined. The conjecture (GRC) asserts that

$$\underline{\theta}_\phi(p) := (\theta_{\phi,1}(p), \theta_{\phi,2}(p), \theta_{\phi,3}(p)) \in [0,2\pi)^3$$

or $\underline{\alpha}_\phi(p) := (\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \alpha_{\phi,3}(p)) \in S^{1^3}$ where S^{1^3} is the product of 3 unit circles. As the Satake parameters parametrize conjugacy classes, the triples under permutation of entries by the symmetric group $\mathfrak{S}_3$ of degree 3 are identified. Hence under GRC, we may view $\underline{\alpha}_\phi(p)$ (or $\underline{\theta}_\phi(p)$) as a point $S^{1^3}/\mathfrak{S}_3$ (or $[0,2\pi)^3/\mathfrak{S}_3$), and moreover, for a *primitive*[2] form ϕ, it is believed that the Satake parameter $\underline{\alpha}_\phi(p)$ satisfies an analogue of the Sato-Tate conjecture: for any fixed non-self dual ϕ, the sets $\{\underline{\alpha}_\phi(p) : p \le x\}$ become equidistributed on $S^{1^3}/\mathfrak{S}_3$ with respect to the Sato-Tate measure

$$d\mu_{\mathrm{ST}} = \frac{1}{3!(2\pi)^2} \prod_{1\le \ell < m \le 3} |e^{i\theta_\ell} - e^{i\theta_m}|^2 d\theta_1 d\theta_2,$$

where $\sum_{\ell=1}^3 \theta_\ell = 0$, as $x \to \infty$. (Recall that the Sato-Tate conjecture for $GL(2)$ is expected unless π_ϕ is solvable polyhedral or icosahedral[3], cf. [20].) It is known that Schur polynomials $S_{\mathbf{k}}$ are orthonormal with respect to the Sato-Tate measure

$$\langle S_{\mathbf{k}}, S_{\mathbf{k}'} \rangle := \int_{[0,2\pi]^2} S_{\mathbf{k}} \overline{S_{\mathbf{k}'}}\, d\mu_{\mathrm{ST}} = \delta_{\mathbf{k}=\mathbf{k}'}. \tag{2.5}$$

Consequently, we *expect* via the Sato-Tate conjecture that

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} \lambda_{\phi\times\tilde\phi}(p)^2 = \int_{[0,2\pi]^2} (S_{(2,0)} + S_{(0,1)})\overline{(S_{(2,0)} + S_{(0,1)})}\, d\mu_{\mathrm{ST}} = 2$$

and

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\le x} |\lambda_{\mathrm{sym}^2\phi}(p)|^2 = \int_{[0,2\pi]^2} S_{(2,0)}\overline{S_{(2,0)}}\, d\mu_{\mathrm{ST}} = 1$$

[2]Here we mean the associated automorphic representation π_ϕ is primitive in the sense of Arthur [1], see §7.2 (iv) and §7.5.

[3]The representations $\mathrm{sym}^m \pi_\phi$, $m \in \mathbb{N}$, are not all cuspidal assuming π_ϕ has trivial central character.

by (2.3), (2.4), (2.1) and (2.5). This explains the constants proposed in Hypothesis (MS).

3. Proof of Theorem 1.1

If GRC fails at p, then by (1.2), we put $\alpha_{\phi,1}(p) = \rho e^{i\theta}$, $\alpha_{\phi,2}(p) = \rho^{-1}e^{i\theta}$ and $\alpha_{\phi,3}(p) = e^{-2i\theta}$, where ρ is a non-zero real number with $\rho > 1$ and $\theta \in [0, 2\pi)$. (Here we suppress the dependence of ϕ and p in the notation ρ and θ for simplicity.) Then $A_\phi(p,1) = e^{i\theta}(\rho + \rho^{-1} + e^{-i3\theta})$, and by (2.2) and $\rho > \alpha$,

$$\lambda_{\phi\times\tilde{\phi}}(p) \;=\; \left|\rho + \rho^{-1} + e^{-i3\theta}\right|^2 \geq (\alpha + \alpha^{-1} - 1)^2,$$

as $\rho + \rho^{-1}$ lies on the real axis and is at least $\alpha + \alpha^{-1}$ while $e^{i3\theta}$ lies on the unit circle. With the non-negativity of $\lambda_{\phi\times\tilde{\phi}}(p)$, we infer that

$$(\alpha + \alpha^{-1} - 1)^2 \frac{|\{p \leq x : \rho_\phi(p) > \alpha\}|}{\pi(x)} \leq \frac{\sum_{p\leq x} \lambda_{\phi\times\tilde{\phi}}(p)}{\pi(x)}.$$

By (1.5), we have

$$\limsup_{x\to\infty} \frac{1}{\pi(x)} \sum_{p\leq x} \lambda_{\phi\times\tilde{\phi}}(p) = 1$$

and Theorem 1.1 follows plainly.

4. Proof of Theorem 1.2

The key is to construct a suitable symmetric function. Consider

$$\begin{aligned} &S(x_1, x_2, x_3) \\ &= \frac{1}{16} \sum_{1\leq i<j\leq 3} (x_i + x_j)^2 (x_i^{-1} + x_j^{-1})^2 + \frac{1}{32} \prod_{1\leq i<j\leq 3} (x_i + x_j)(x_i^{-1} + x_j^{-1}) \\ &= \frac{1}{16} U(x_1, x_2, x_3) + \frac{1}{32} V(x_1, x_2, x_3), \text{ say.} \end{aligned}$$

If $\alpha_{\phi,1}(p) = \rho e^{i\theta}$, $\alpha_{\phi,2}(p) = \rho^{-1}e^{i\theta}$ and $\alpha_{\phi,3}(p) = e^{-2i\theta}$ for some $\rho \in (1, \infty)$ and $\theta \in [0, 2\pi)$, then with $\underline{\alpha}_\phi(p) := (\alpha_{\phi,1}(p), \alpha_{\phi,2}(p), \alpha_{\phi,3}(p))$ and $z^2 = \rho e^{i3\theta}$ $(z \in \mathbb{C})$,

$$U(\underline{\alpha}_\phi(p)) = (\rho + \rho^{-1})^4 + 2\Re e\,(z + z^{-1})^4$$

and

$$V(\underline{\alpha}_\phi(p)) = (\rho + \rho^{-1})^2 \left|z + z^{-1}\right|^4 \geq 4\left|z + z^{-1}\right|^4.$$

Hence $S(\underline{\alpha}_\phi(p)) \geq 1$ if GRC for ϕ fails at p.

Next by expanding out directly, one checks that

$$U(\alpha_1, \alpha_2, \alpha_3) = 3 + 4\Big(\sum_{1\leq i\leq 3} \alpha_i\Big)\Big(\sum_{1\leq i\leq 3} \alpha_i^{-1}\Big) + \Big(\sum_{1\leq i\leq 3} \alpha_i^2\Big)\Big(\sum_{1\leq i\leq 3} \alpha_i^{-2}\Big);$$

when $\alpha_1, \alpha_2, \alpha_3$ satisfy the unitary condition (1.2), we have

$$U(\alpha_1, \alpha_2, \alpha_3) = 3 + 4\Big|\sum_{1\leq i\leq 3} \alpha_i\Big|^2 + \Big|\sum_{1\leq i\leq 3} \alpha_i^2\Big|^2.$$

Moreover, $1+(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)(\alpha_2+\alpha_3) = \big(\sum_{1\le i\le 3}\alpha_i\big)\big(\sum_{1\le i\le 3}\alpha_i^{-1}\big)$ if $\alpha_1\alpha_2\alpha_3=1$. Hence if $\alpha_1,\alpha_2,\alpha_3$ satisfy (1.1) and (1.2), then

$$V(\alpha_1,\alpha_2,\alpha_3) = \Big(\Big|\sum_{1\le i\le 3}\alpha_i\Big|^2 - 1\Big)^2.$$

This follows that $S(\underline{\alpha}_\phi(p)) = \frac{7}{32} +$ nonnegative terms for all p. As before, we deduce that

$$\frac{|\{p\le x : \rho_\phi(p)>1\}|}{\pi(x)} + \frac{7}{32}\Big(1-\frac{|\{p\le x : \rho_\phi(p)>1\}|}{\pi(x)}\Big) \le \frac{1}{\pi(x)}\sum_{p\le x} S(\underline{\alpha}_\phi(p)). \tag{4.1}$$

By the definitions of $\lambda_{\phi\times\tilde\phi}(p)$ and $\lambda_{\mathrm{sym}^2\phi}(p)$, we see that

$$\begin{aligned} U(\underline{\alpha}_\phi(p)) =& 3+4|A(p,1)|^2 + \big|\lambda_{\mathrm{sym}^2\phi}(p) - A(1,p)\big|^2 \\ V(\underline{\alpha}_\phi(p)) =& \big(\lambda_{\phi\times\tilde\phi}(p)-1\big)^2. \end{aligned}$$

Since $\lambda_{\mathrm{sym}^2\phi}(p) + A(1,p) = A(p,1)^2$, we obtain with (2.2) that

$$|\lambda_{\mathrm{sym}^2\phi}(p)|^2 + |A(p,1)|^2 + 2\Re e\,\lambda_{\mathrm{sym}^2\phi}(p)\overline{A(1,p)} = \lambda_{\phi\times\tilde\phi}(p)^2$$

and thus

$$\begin{aligned} S(\underline{\alpha}_\phi(p)) =& \frac{3+6|A_\phi(p,1)|^2 + 2|\lambda_{\mathrm{sym}^2\phi}(p)|^2 - \lambda_{\phi\times\tilde\phi}(p)^2}{16} \\ &+ \frac{\lambda_{\phi\times\tilde\phi}(p)^2 - 2|A_\phi(p,1)|^2+1}{32}. \end{aligned}$$

Theorem 1.2 follows readily from Hypothesis (MS), (1.5) and (4.1).

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Yuk-Kam Lau, Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong

Email address: yklau@maths.hku.hk

Ming Ho Ng, Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong

Email address: minghong515@gmail.com

Yingnan Wang, Shenzhen Key Laboratory of Advanced Machine Learning and Applications, College of Mathematics and Statistics, Shenzhen University, Shenzhen, Guangdong 518060, P.R. China

Email address: ynwang@szu.edu.cn