Hypertranscendency of Perturbations of Hypertranscendental Functions

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Abstract

Inspired by the work of Bank on the hypertranscendence of Γe^h where Γ is the Euler gamma function and h is an entire function, we investigate when a meromorphic function fe^g cannot satisfy any algebraic differential equation over certain field of meromorphic functions, where f and g are meromorphic and entire on the complex plane, respectively. Our results (Theorem 1 and 2) give partial solutions to Bank's Conjecture (1977) on the hypertranscendence of Γe^h. We also give some sufficient conditions for hypertranscendence of meromorphic function of the form $f+g$, $f\cdot g$ and $f\circ g$ in Theorem 3 and 4.

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1. Introduction and main results

A meromorphic function f on the complex plane is said to be *hypertranscendental* over a field $\mathcal{K}$ of meromorphic functions, if f does not satisfy any nontrivial algebraic differential equation whose coefficients are in the field $\mathcal{K}$. We are interested in those $\mathcal{K}$ which are related to the growth of f. Let $T(r,f)$ be the Nevanlinna characteristic function of f (see Section 2 for the definitions and notations in Nevanlinna theory). We denote by $S(r,f)$ any quantity which is of growth $o(T(r,f))$ as $r\to\infty$ outside a set of finite measure $E\subset(0,\infty)$. By $\mathcal{M}_0$ we mean the field of meromorphic functions y with $T(r,y)=o(r)$ as $r\to\infty$ outside a set of finite measure and $\mathcal{S}_f$ (resp. $\mathcal{S}^f$) the field of meromorphic functions y satisfying the growth condition $T(r,y)=S(r,f)$ (resp. $T(r,y)=O(T(r,f))$ as $r\to\infty$ outside a set of finite measure).

In 1887, Hölder [1] established the hypertranscendence of the Euler gamma function Γ over the field of rational functions, i.e., Γ cannot satisfy any nontrivial alge-

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braic differential equation whose coefficients are rational functions. Hilbert [2], in 1901, proved the hypertranscendence of Riemann zeta function using the functional equation of ζ and Γ. In 1976, Bank and Kaufman [3] extended the famous theorems of Hölder and Hilbert by showing that Γ and ζ are hypertranscendental over the field $\mathcal{M}_0$. One year later, Bank [4] asked to what extend the hypertranscendence of Γ is due to the nature of its poles and zeros. In particular, he posed the following conjecture.

Bank's Conjecture ([4])**.** *For every entire function h, Γe^h is hypertranscendental over $\mathcal{M}_0$.*

Bank [4] gave an affirmative answer to the above conjecture when either h or h' has only finitely many zeros. In 1980, he [5] generalized this result to the following.

Theorem A ([5])**.** *Let h be an entire function with the property that for some nonnegative integer j, and some complex number a, the following condition holds :*

$$\overline{N}\big(r, 1/(h^{(j)} - a)\big) = S(r, h^{(j)}), \tag{1}$$

where as usual, $h^{(0)}$ denotes h. Then the function Γe^h is hypertranscendental over $\mathcal{M}_0$.

Related to Theorem A, we obtained the following.

Theorem 1. *Let h be an entire function such that $T(r, \Gamma'/\Gamma) = S(r, h^{(j)})$ and*

$$\delta(a, h^{(j)}) > 0, \tag{2}$$

for some $a \in \mathcal{M}_0$ and some nonnegative integer j. Then Γe^h is hypertranscendental over $\mathcal{M}_0$.

Related to Bank's Conjecture, we have the following partial result.

Theorem 2. *For any entire function h, $P(z, \Gamma e^h \ldots, (\Gamma e^h)^{(n)}) \not\equiv 0$ for any nontrivial distinguished polynomial $P(z, u_0, \ldots, u_n)$ over $\mathcal{M}_0$.*

Remark 1. The notion of distinguished polynomial was first introduced by B. Q. Li and Z. Ye in [6]. The definition is given as follow.

Let $I = (i_0, i_1, \ldots, i_k)$ be a multi-index with $|I| = i_0 + i_1 + \cdots + i_k$. A polynomial in the variables $u_0, u_1, \ldots, u_k$ with meromorphic function coefficients in a set S can always be written as

$$P(z, u_0, u_1, \ldots, u_k) = \sum_{I \in \Lambda} a_I(z) u_0^{i_0} u_1^{i_1} \cdots u_k^{i_k},$$

where the coefficients a_I are meromorphic functions in S and Λ is an index set. We call P a *distinguished polynomial* in $u_0, u_1, \ldots, u_k$ with coefficients in S, or simply an S-*distinguished polynomial*, if the index set Λ satisfies $|I_i| \neq |I_j|$ for any distinct indices I_i, I_j in Λ. In other words, each homogeneous part of the distinguished polynomial P contains one term only.

If $\mathcal{K}$ is a field of meromorphic functions, we denote by $A(\mathcal{K})$ the set of all meromorphic functions which satisfy some algebraic differential equation over $\mathcal{K}$. It is well known (see Chapter 14 of [7]) that $A(\mathcal{K})$ is a differential field, i.e., a field with an additional map $D : A(\mathcal{K}) \to A(\mathcal{K})$ such that $D(a \cdot b) = (Da) \cdot b + a \cdot Db$ for any $a, b \in A(\mathcal{K})$.

To explain the difference between Theorem A and Theorem 1, let us sketch the main idea of the proof of Theorem A (see Part B in [5] or Chapter 14 of [7]).

Let h be an entire function satisfying the assumption (1) in Theorem A. If $\Gamma e^h \in A(\mathcal{M}_0)$ and $a \in \mathbb{C}$, set $g = h - (az^j/j!)$ which satisfies the condition $\overline{N}\big(r, 1/g^{(j)}\big) = S(r, g^{(j)})$. Applying Lemma A below and using the fact that $T(r, \Gamma'/\Gamma) = r + o(r)$, one can conclude that $T(r, g^{(j)}) = O(r)$. On the other hand, g is an entire function with $\overline{N}\big(r, 1/g^{(j)}\big) = S(r, g^{(j)})$, thus $T(r, g^{(j+1)}/g^{(j)}) = o(r)$. Hence $g^{(j+1)}/g^{(j)}$ belongs to $\mathcal{M}_0$ which implies $g \in A(\mathcal{M}_0)$. Thus h and $h' \in A(\mathcal{M}_0)$. Since $h' = (e^h)'/(e^h)$, it follows that $e^h \in A(\mathcal{M}_0)$. Combining with the assumption that $\Gamma e^h \in A(\mathcal{M}_0)$, one can deduce a contradiction to the hypertranscendence of Γ over $\mathcal{M}_0$.

Actually, from the proof of Theorem A, it is not hard to see that the assumption $\Gamma e^h \in A(\mathcal{M}_0)$ and the condition (1) imply that $T(r, h^{(j)}) = O(T(r, \Gamma'/\Gamma))$. Our Theorem 1 considers a sort of complement assumption that $T(r, \Gamma'/\Gamma) = S(r, h^{(j)})$. Under this assumption, the condition (2) is less restrictive than the one on $\overline{N}(r, 1/(h^{(j)} - a))$ in Theorem A. In addition, a can also be nonconstant.

To produce more examples of hypertranscendental functions, Bank also investigated the hypertranscendency of the perturbation of hypertranscendental meromorphic functions by adding a small function.

Theorem B ([5])**.** *Let f be a meromorphic function on the complex plane which is hypertranscendental over a differential field $\mathcal{S} \subset \mathcal{S}_f$. Let g be a meromorphic function on the complex plane. Then, if $f + g$ satisfies an algebraic differential equation over $\mathcal{S}$, we have*

$$T(r, f) = O(\overline{N}(r, 1/f) + \overline{N}(r, f) + T(r, g))$$

as $r \to \infty$ outside of a possible exceptional set of finite measure.

In particular, if all $\overline{N}(r,1/f), \overline{N}(r,f)$ and $T(r,g)$ are $S(r,f)$, then $f+g$ must be hypertranscendental over $\mathcal{S}$.

The proofs of Theorem A and B in [4, 5] depend on the following Lemma first appeared in [8].

Lemma A ([8])**.** *Let $P(z,y,y',\ldots,y^{(n)})$ be a polynomial in $y,y',\ldots,y^{(n)}$ whose coefficients are meromorphic functions on $\mathbb{C}$. For each $r>0$, let $\Delta(r)$ be the maximum of the Nevanlinna characteristics of the coefficients of P. Let f be a nonzero meromorphic function on the complex plane satisfying the equation $P=0$, but for some nonnegative integer q, $P_q(f,f',\ldots,f^{(n)})\neq 0$, where P_q is the homogeneous part of P of total degree q in the indeterminates $y,y',\ldots,y^{(n)}$. Then*

$$T(r,f)=O(E(r)),$$

as $r\to\infty$, outside of a possible exceptional set of finite measure, where

$$E(r)=\overline{N}(r,1/f)+\overline{N}(r,f)+\Delta(r)+\log r.$$

In addition, for any $\alpha>1$, there exist positive constants c and r_0 such that

$$T(r,f)\leq cE(\alpha r), \text{ for all } r\geq r_0.$$

In 1991, Y. Z. He and C. C. Yang [9] proved that $\Gamma(g)$ is hypertranscendental over the field $\mathcal{M}^g$ of meromorphic functions y with $T(r,y)=O(T(r,g))$ by using Steinmetz's Reduction Theorem (Theorem C below). Their method can be applied to the general case (see Theorem 3). In 2007, Markus [10] applied the method of differential algebra to obtain the hypertranscendence of $\zeta(\sin z)$ and $\Gamma(\sin z)$ over $\mathbb{C}$, and he proved the differential independence between f_i and $f_j(\sin z)$ for $i,j=1,2$, where $f_1=\Gamma$ and $f_2=\zeta$.

Applying the same idea of He and Yang in [9], we obtain the following general result which covers the results of He and Yang [9].

Theorem 3. *Let f be hypertranscendental over the rational function field $\mathbb{C}(z)$ and g be a nonconstant entire function. Then $f\circ g$ is hypertranscendental over the field $\mathcal{S}^g$.*

As a consequence, we can generalize a result of L. Markus (see Lemma 1 in [10]) by using a different method.

Corollary 1. *Let a be a nonzero complex number. Then both $\Gamma(\sin az)$ and $\zeta(\sin az)$ are hypertranscendental over the field of meromorphic functions y with $T(r,y)=O(r)$ as $r\to\infty$ outside some set of finite measure.*

It is natural to consider the hypertranscendency of $g \circ f$ over some fields for entire hypertranscendental f and meromorphic g. This seems to be a more difficult problem as Steinmetz's Reduction Theorem cannot be applied directly here (see Remark 2 in Section 3). However, we do obtain one related result in Theorem 4(1).

Inspired by the results of Bank, He-Yang and Markus, in this paper, we will first prove a result similar to Lemma A, that is $T(r, f)$ can be controlled by one counting function $N(r, 1/f)$ (see Lemma 2). Using Lemma 2, we then obtain the following results on the hypertranscendency of perturbations of hypertranscendental functions, including that of Γ and Γe^h.

Theorem 4. *Let g and f be meromorphic functions and $\mathcal{S}$ be the field of meromorphic functions y with $T(r, y) = S(r, f'/f)$, i.e. $\mathcal{S} = \mathcal{S}_{f'/f}$. Let $\mathcal{O}$ be the set of entire functions on $\mathbb{C}$. Suppose f is hypertranscendental over $\mathcal{S}$.*

(1) If $f \in \mathcal{O}$, and $g - R$ has finitely many zeros, where R is a non-constant rational function, then $g \circ f$ is hypertranscendental over $\mathcal{S}$.

(2) Assume that $f \in \mathcal{S}_g$ and $\delta(a, g) > 0$ for some $a \in \mathcal{S} \setminus \{0\}$, then fg is hypertranscendental over $\mathcal{S}$.

(3) If there exists a non-negative integer k such that $T(r, f) = S(r, g^{(k)})$ and $\delta(a, g^{(k)}) > 0$ for some $a \in \mathcal{S}$, then $f + g$ is hypertranscendental over $\mathcal{S}$.

(4) Assume that $g \in \mathcal{O}$, and if there exists a nonnegative integer k such that $T(r, f'/f) = S(r, g^{(k)})$ and

$$\delta(a, g^{(k)}) > 0 \tag{3}$$

for some $a \in \mathcal{S}$, then fe^g is hypertranscendental over $\mathcal{S}$.

(5) If $g \in \mathcal{O}$ and $f \in \mathcal{S}_{\exp(g)}$, then $P(z, fe^g, (fe^g)', \ldots, (fe^g)^{(n)}) \not\equiv 0$ for any nontrivial distinguished polynomial $P(z, u_0, \ldots, u_n)$ over $\mathcal{S}$.

In Section 5, we will use Theorem 4 to prove Theorem 1 and 2. Section 2 introduces the basics of Nevanlinna Theory. Theorem 3 and 4 will be proven in Section 3 and 4, respectively.

2. Nevanlinna Theory

We recall the basic notations and results of Nevanlinna theory [7] which are main tools for proving our results.

Let f and a be meromorphic functions in the complex plane $\mathbb{C}$ and $\mathbb{D}_r = \{|z| < r\}$. Denote the number of poles of f in $\mathbb{D}_r$ by $n(r, f)$, and let $n(r, a) = n(r, a, f) =$

$n(r,1/(f-a))$. When the number of distinct poles of f in $\mathbb{D}_r$ is denoted by $\overline{n}(r,f)$, we then let $\overline{n}(r,a)=\overline{n}(r,1/(f-a))$. Correspondingly we define the counting function and truncated counting function in Nevanlinna theory as follows:

$$N(r,a,f):=\int_0^r \frac{n(t,a)-n(0,a)}{t}dt+n(0,a)\log r;$$

$$\overline{N}(r,a,f):=\int_0^r \frac{\overline{n}(t,a)-\overline{n}(0,a)}{t}dt+\overline{n}(0,a)\log r.$$

The proximity function is defined as

$$m(r,f):=\frac{1}{2\pi}\int_0^{2\pi}\log^+\left|f(re^{i\theta})\right|d\theta$$

and

$$m(r,a,f):=m(r,1/(f-a)),$$

where $\log^+ x=\max\{0,\log x\}$. The Nevanlinna characteristic function of f is defined by

$$T(r,f)=m(r,f)+N(r,f).$$

The First Main Theorem of Nevanlinna theory for small functions [11] says that for any meromorphic function a with $T(r,a)=S(r,f)$,

$$T(r,f)=T(r,a,f)+S(r,f)$$

where $T(r,a,f):=m(r,a,f)+N(r,a,f)$. Finally, we denote the Nevanlinna order of f by

$$\rho(f):=\limsup_{r\to\infty}\frac{\log T(r,f)}{\log r},$$

and the deficiency of a for f by

$$\delta(a,f):=\liminf_{r\to\infty}\frac{m(r,a,f)}{T(r,f)}=1-\limsup_{r\to\infty}\frac{N(r,a,f)}{T(r,f)}.$$

If $\delta(a,f)>0$, then we say a is a deficient function of f.

The logarithmic derivative lemma states that

Lemma 1 ([7]). *Let f be a transcendental meromorphic function and $k\geq 1$ be an integer. Then*

$$m\left(r,\frac{f^{(k)}}{f}\right)=S(r,f),$$

and if f is of finite order of growth, then

$$m\left(r,\frac{f^{(k)}}{f}\right)=O(\log r).$$

3. Proof of Theorem 3

To prove Theorem 3, we first introduce the Steinmetz's Reduction Theorem.

Theorem C (Steinmetz's Reduction Theorem [12, 13])**.** *Let $F_j, 1 \leq j \leq N$ be meromorphic functions on $\mathbb{C}$. Let $h_j, 1 \leq j \leq N$ be meromorphic and g be entire on $\mathbb{C}$ such that for each j,*

$$T(r, h_j) = O(T(r, g))$$

as $r \to \infty$ outside some set of finite measures. Given a functional equation of the form

$$F_1(g(z))h_1(z) + \cdots + F_N(g(z))h_N(z) = 0,$$

then there exist polynomials p_j, not all zeros, such that

$$p_1(g(z))h_1(z) + \cdots + p_N(g(z))h_N(z) = 0.$$

Furthermore, if $h_j \not\equiv 0$ for some j, then there exist polynomials Q_j, not all zeros, such that

$$F_1(z)Q_1(z) + \cdots + F_N(z)Q_N(z) = 0.$$

Proof of Theorem 3. We will follow the idea of the proof of Theorem 4 in [9].

Suppose that $f \circ g$ satisfies a nontrivial algebraic differential equation with coefficients in $\mathcal{S}^g$, i.e., there exists a nontrivial differential polynomial $P(z, w, w', \ldots, w^{(n)})$ with coefficients in $\mathcal{S}^g$ such that

$$P(z, f \circ g, (f \circ g)', \ldots, (f \circ g)^{(n)}) = \sum_j (M_j(f) \circ g)(H_j(g)(z)) = 0$$

where $M_j(f)$ is a differential monomial of f with constant coefficients and $H_j(g)(z)$ is a differential polynomial of $g(z)$ whose coefficients are some linear combinations of the coefficients of the original differential polynomial $P(z, w, w', \ldots, w^{(n)})$.

Now, set $F_j(z) = M_j(f)(z)$ and $h_j(z) = H_j(g)(z)$, it follows from the second result of Theorem C that there exist polynomials Q_i, not all zeros, such that

$$F_1(z)Q_1(z) + \cdots + F_N(z)Q_N(z) = 0.$$

which implies that f satisfies a nontrivial algebraic differential equation with coefficients in $\mathbb{C}(z)$. This is a contradiction to our assumption that f is hypertranscendental over $\mathbb{C}(z)$.

$\square$

Remark 2. Here, we will explain the reason why the Steinmetz's Reduction Theorem does not work for the hypertranscendency of $g \circ f$. We use the same idea of proof of Theorem 3. Suppose that $g \circ f$ satisfies a nontrivial algebraic differential equation over a suitable field such that we can apply the Steinmetz's Reduction Theorem, thus we have

$$p_1(f(z))h_1(z) + \cdots + p_N(f(z))h_N(z) = 0$$

or

$$F_1(z)Q_1(z) + \cdots + F_N(z)Q_N(z) = 0$$

where $h_j(z)$ is a differential polynomial of $f(z)$ whose coefficients are some linear combinations of the coefficients of the algebraic differential equation $g \circ f$ satisfied, and $F_j(z)$ is a differential monomial of g with constant coefficients. From these two equalities, we cannot deduce any contradictions even through we have known the hypertranscendency of f.

4. Proof of Theorem 4

In this section, we are now going to prove our main result (Theorem 4). To prove Theorem 4, we need the following lemmata.

Lemma 2. *Let f be a nonzero meromorphic function on the complex plane and $P(z, y, y', \ldots, y^{(n)})$ be a polynomial in $y, y', \ldots, y^{(n)}$ whose coefficients are in the field $\mathcal{S}_f$. Suppose f satisfies the equation $P = 0$. Rewrite $P = 0$ as $P_q = \sum_{j=k}^{m} P_j$, for some nonnegative integers q and $k(> q)$ such that $P_q \neq 0$ for each $j \geq k$, where P_j is the homogeneous part of P of total degree j in the indeterminates $y, y', \ldots, y^{(n)}$. Then for any integer N with $q \leq N \leq k$,*

$$m(r, P_q/f^N) = S(r, f).$$

In addition if $q = 0$, then

$$T(r, f) = N(r, 0, f) + S(r, f).$$

Remark 3. Lemma 2 is essentially B.Q. Li's Lemma 4.1 in [14]

Proof of Lemma 2. Let $P(z, u_0, \ldots, u_n)$ be a polynomial in $u_0, \ldots, u_n$ with coefficients in $\mathcal{S}_f$. Assume that

$$I = \{i := (i_0, i_1, \ldots, i_n) |\ i_j \text{ is a nonnegative integer and } 0 \leq j \leq n\}$$

is an index set with finite cardinal numbers. Define

$$|i| = \sum_{j=0}^{n} i_j \quad \text{and} \quad I_p = \{i \in I : |i| = p\}.$$

For each $l \geq q$, let

$$P_l = \sum_{i \in I_l} a_i(z) u_0^{i_0} \dots u_n^{i_n}$$

where $a_i \in \mathcal{S}_f$.

Take any point $z \in \mathbb{C}$, we consider several cases.

Case (i) $|f(z)| \geq 1$. Since $P_q = \sum_{i \in I_q} a_i(z) u_0^{i_0} \dots u_n^{i_n}$,

$$\left| \frac{P_q(f, f', \dots, f^{(n)})}{f^N}(z) \right| \leq \left| \frac{P_q(f, f' \dots, f^{(n)})}{f^q}(z) \right|$$
$$\leq \sum_{i \in I_q} \left| a_i(z) \frac{f^{i_0}(f')^{i_1} \cdots (f^{(n)})^{i_m}}{f^q} \right| := G_1(z).$$

Case (ii) $|f(z)| \leq 1$. Then by $P_q = \sum_{j=k}^{m} P_j, q \leq N \leq k$, we have

$$\left| \frac{P_q(f, f', \dots, f^{(n)})}{f^N}(z) \right| = \left| \sum_{j=k}^{m} \frac{P_j(f, f' \dots, f^{(n)})}{f^j}(z) f^{j-N} \right|$$
$$\leq \sum_{j=k}^{m} \left| \frac{P_j(f, f' \dots, f^{(n)})}{f^j}(z) \right| |f|^{j-N}$$
$$\leq \sum_{j=k}^{m} \sum_{i \in I_j} \left| a_i(z) \frac{f^{i_0}(f')^{i_1} \cdots (f^{(n)})^{i_m}}{f^j} \right| := G_2(z).$$

Combining the above results, we see that in any case

$$\left| \frac{P_q(f, f', \dots, f^{(n)})}{f^N}(z) \right| \leq G_1(z) + G_2(z)$$

for any $z \in \mathbb{C}$. By the well-known Logarithmic Derivative Lemma and $a_i \in \mathcal{S}_f$, we deduce that

$$m(r, P_q/f^N) \leq m(r, G_1 + G_2) = S(r, f).$$

Now if $q = 0$, then by taking $N = 1$, we have

$$m(r, 1/f) \leq m(r, P_0/f) + m(r, 1/P_0) + O(1) = S(r, f)$$

as $T(r, P_0) = S(r, f)$. Hence the result follows from the First Main Theorem of Nevanlinna theory. □

As a consequence, one can also obtain the following lemma first proved by A. Mohon'ko in 1982.

Lemma 3 ([11])**.** *Let f be a transcendental meromorphic solution of an algebraic differential equation $P(y) = P(z, y, y', \dots, y^{(k)}) = 0$ with coefficients in $\mathcal{S}_f$. If a meromorphic function ϕ with $T(r, \phi) = S(r, f)$ does not solve $P(z, y, y', \dots, y^{(k)}) = 0$ i.e. $P(z, \phi, \phi', \dots, \phi^{(k)}) \not\equiv 0$, then*

$$m\left(r, \frac{1}{f - \phi}\right) = S(r, f)$$

Proof. Let $g = f - \phi$, then $T(r, g) = T(r, f) + S(r, f)$. Since $P(f) \equiv 0$, we have

$$P(f) = P(g + \phi) = Q(g) + P(\phi) \equiv 0$$

where Q is a differential polynomial over $\mathcal{S}_f$ with lowest degree at least one, as $T(r, \phi) = S(r, f)$. The result follows immediately from Lemma 2 as $P(\phi) \not\equiv 0$. □

Lemma 4 ([15])**.** *Let f be a transcendental entire function and let g be a transcendental meromorphic function in the complex plane, then $T(r, f) = o(T(r, g \circ f))$ as $r \to \infty$.*

Proof of Theorem 4. 1). Without loss of generality, we can assume $R(z) = z$, since if f is hypertranscendental over $\mathcal{S}$, it is easy to show that $R \circ f$ is also hypertranscendental over $\mathcal{S}$.

Suppose $g(z) - z = 0$ has d roots, then $g(z) - z = Q(z)A(z)$ where Q is a polynomial with degree d, and A is a transcendental meromorphic function which is nowhere zero. Hence if f is an entire function, we have

$$N(r, 0, g \circ f - f) = N(r, 0, Q(f)A(f)) = N(r, 0, Q(f)) \leq dT(r, f) + S(r, f).$$

By Lemma 4, we have $T(r, f) = o(T(r, g \circ f))$. Suppose $g \circ f$ is not hypertranscendental over $\mathcal{S}$, that is, $g \circ f$ is a solution of an algebraic differential equation $P(z, y, y', \dots, y^{(k)}) = 0$ with coefficients in $\mathcal{S}$ (hence in $\mathcal{S}_{g \circ f}$ as well). By Lemma 3 and the assumption that f is hypertranscendental over $\mathcal{S}$, we have

$$m\left(r, \frac{1}{g \circ f - f}\right) = S(r, g \circ f).$$

By the First Main Theorem of Nevanlinna Theory for small functions [11],

$$\begin{aligned} T(r, g \circ f) &= T(r, g \circ f - f) + S(r, g \circ f) \\ &= m(r, 0, g \circ f - f) + N(r, 0, g \circ f - f) + S(r, g \circ f) \\ &\leq S(r, g \circ f) + dT(r, f) = S(r, g \circ f) \end{aligned}$$

which is a contradiction. This completes the proof of the first part.

2). If $a \not\equiv 0$, since f is hypertranscendental over $\mathcal{S}$, it is easy to show that af is also hypertranscendental over $\mathcal{S}$, as $a \in \mathcal{S}$.

Since $T(r,f) = S(r,g), T(r,a) = S(r,f'/f) = S(r,f)$, one can obtain that $T(r,af) = T(r,f) + S(r,f) = S(r,fg)$.

Suppose fg is not hypertranscendental over $\mathcal{S}$, that is, fg is a solution of an algebraic differential equation $P(z,y,y',\ldots,y^{(k)}) = 0$ with coefficients in $\mathcal{S}$ (hence in $\mathcal{S}_{fg}$ also). By Lemma 3 and the hypertranscendence of af over $\mathcal{S}$, we have

$$m\left(r, \frac{1}{fg - af}\right) = S(r,fg) = S(r,g).$$

On the other hand, by the First Main Theorem of Nevanlinna Theory for small functions, as $T(r,af) = S(r,fg)$,

$$\begin{aligned}
T(r,fg) &= T(r,fg - af) + S(r,fg) \\
&= m(r,0,fg - af) + N(r,0,fg - af) + S(r,fg) \\
&\le N(r,0,g-a) + N(r,0,f) + S(r,g) \\
&= N(r,0,g-a) + S(r,g).
\end{aligned}$$

Since $T(r,fg) = T(r,g) + S(r,g)$, it follows that $T(r,g) = N(r,a,g) + S(r,g)$ which is a contradiction to the assumption that $\delta(a,g) > 0$.

3). If $f + g \in A(\mathcal{S})$, so does $f^{(k)} + g^{(k)}$, that is, there exists a nontrivial algebraic differential equation $P(z,y,y',\ldots,y^{(n)}) = 0$ over $\mathcal{S}$ such that

$$P(z, f^{(k)} + g^{(k)}, f^{(k+1)} + g^{(k+1)}, \ldots, f^{(k+n)} + g^{(k+n)}) \equiv 0.$$

Set

$$Q(z, g^{(k)}, g^{(k+1)}, \ldots, g^{(n+k)}) := P(z, f^{(k)} + g^{(k)}, f^{(k+1)} + g^{(k+1)}, \ldots, f^{(k+n)} + g^{(k+n)}),$$

then $Q(z, g^{(k)}, g^{(k+1)}, \ldots, g^{(n+k)}) \equiv 0$. It is easy to check that all the Nevanlinna characteristic functions of the coefficients of $Q(z, g^{(k)}, g^{(k+1)}, \ldots, g^{(n+k)})$ are $S(r, g^{(k)})$, as $T(r,f) = S(r, g^{(k)})$ and $T(r, f^{(k)}) \le (k+1)T(r,f) + S(r,f)$

On the other hand, since f is hypertranscendental over $\mathcal{S}$, so is $f^{(k)} + a$ for any $a \in \mathcal{S}$, hence

$$Q(z, a, a', \ldots, a^{(n)}) = P(z, f^{(k)} + a, f^{(k+1)} + a', \ldots, f^{(k+n)} + a^{(n)}) \not\equiv 0.$$

By Lemma 3, we have

$$m\left(r, \frac{1}{g^{(k)} - a}\right) = S(r, g^{(k)})$$

which is a contradiction to the assumption that $\delta(a, g^{(k)}) > 0$ for some $a \in \mathcal{S}$.

4). If $fe^g \in A(\mathcal{S})$, then clearly, $\dfrac{f'}{f} + g' = \dfrac{(fe^g)'}{fe^g} \in A(\mathcal{S})$, and hence so does $\left(\dfrac{f'}{f}\right)^{(k)} + g^{(k+1)}$ for any nonnegative integer k, that is, there exists an algebraic differential equation $P(z, y, y', \ldots, y^{(n)}) = 0$ over $\mathcal{S}$ such that

$$P\left(z, \left(\frac{f'}{f}\right)^{(k)} + g^{(k+1)}, \left(\frac{f'}{f}\right)^{(k+1)} + g^{(k+2)}, \ldots, \left(\frac{f'}{f}\right)^{(k+n)} + g^{(k+n+1)}\right) \equiv 0.$$

Set

$$Q(z, g^{(k)}, g^{(k+1)}, \ldots, g^{(n+k+1)}) := P\left(z, \left(\frac{f'}{f}\right)^{(k)} + g^{(k+1)}, \ldots, \left(\frac{f'}{f}\right)^{(k+n)} + g^{(k+n+1)}\right),$$

then

$$Q(z, g^{(k)}, g^{(k+1)}, \ldots, g^{(n+k+1)}) \equiv 0$$

and all the Nevanlinna characteristic functions of the coefficients of $Q(z, g^{(k)}, g^{(k+1)}, \ldots, g^{(n+k+1)})$ are $S(r, g^{(k)})$ from

$$T(r, f'/f) = S(r, g^{(k)})$$

and

$$T(r, (f'/f)^{(j)}) \leq (j+1)T(r, f'/f) + S(r, f'/f)$$

for any nonnegative integer j.

On the other hand, since f is hypertranscendental over $\mathcal{S}$, so is $(f'/f)^{(k)}$ for any nonnegative integer k, and hence so is $(f'/f)^{(k)} + a'$ for any $a \in \mathcal{S}$. Therefore,

$$Q(z, a, a', \ldots, a^{(n+1)}) = P\left(z, \left(\frac{f'}{f}\right)^{(k)} + a', \left(\frac{f'}{f}\right)^{(k+1)} + a'', \ldots, \left(\frac{f'}{f}\right)^{(k+n)} + a^{(n+1)}\right) \not\equiv 0.$$

By Lemma 3, we have

$$m\left(r, \frac{1}{g^{(k)} - a}\right) = S(r, g^{(k)})$$

which is a contradiction to the inequality (3).

5). Let

$$P(z, u_0, u_1, \ldots, u_n) = \sum_{i=0}^{m} P_i(z, u_0, u_1, \ldots, u_n)$$

be a distinguished polynomial over $\mathcal{S}$, where $P_i(z, u_0, u_1, \ldots, u_n)$ contains only one term $a_i(z)u_0^{i_0}u_1^{i_1} \cdots u_n^{i_n}$ with coefficient $a_i \in \mathcal{S}$ and $i = i_0 + i_1 + \cdots + i_n$.

We first notice that the assumption $f \in \mathcal{S}_{\exp(g)}$ and Lemma 4 imply that

$$\begin{aligned} T\left(r, \frac{(fe^g)'}{fe^g}\right) &= T\left(r, \frac{f'}{f} + g'\right) \\ &\leq 2T(r,f) + S(r,f) + 2T(r,g) + S(r,g) = S(r, fe^g) \end{aligned}$$

Assume to the contrary that $P(z, fe^g, (fe^g)', \ldots, (fe^g)^{(n)}) \equiv 0$. Let q be a non-negative integer such that $a_q \neq 0$ and $a_j \equiv 0, j = 0, 1, \ldots, q-1$. Applying Lemma 2 to $N = q+1$, one can conclude that

$$m(r, P_q/(fe^g)^{q+1}) = S(r, fe^g)$$

hence $m(r, 1/(fe^g)) = S(r, fe^g)$ as $T(r, P_q/(fe^g)^q) = S(r, fe^g)$. However, $m(r, 0, fe^g) + N(r, 0, fe^g) = S(r, fe^g) + N(r, 0, f) \leq T(r, f) = S(r, fe^g)$, which is impossible, thus $a_q \equiv 0$. Repeating the above argument, one can obtain that $a_i \equiv 0$ for all $i = 0, 1, \ldots, m$. Hence the result follows.

This completes the proof of Theorem 4. □

5. Proof of Theorem 1 and 2

In this section, we will prove Theorem 1 and 2 by using Theorem 4.

Proof of Theorem 1. This follows immediately from part (4) of Theorem 4 and the fact that $T(r, \Gamma'/\Gamma) = r + o(r)$ in [3]. □

Proof of Theorem 2. We consider the following two cases.
Case 1. If $\rho(e^h) < \infty$, then Γe^h is hypertranscendental over $\mathcal{M}_0$ (see p.271 of [5]). Actually, in this case, h is a polynomial, hence it is not hard to see that $e^h \in A(\mathcal{M}_0)$ as $h' = (e^h)'/e^h$. If $\Gamma e^h \in A(\mathcal{M}_0)$, one can conclude that $\Gamma \in A(\mathcal{M}_0)$ which is a contradiction to the hypertranscendence of Γ over $\mathcal{M}_0$.
Case 2. If $\rho(e^h) = \infty$, then $\Gamma \in \mathcal{S}_{\exp(h)}$, hence the result follows immediately from Theorem 4(5). □

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