# AVERAGE BOUND TOWARD THE GENERALIZED RAMANUJAN CONJECTURE AND ITS APPLICATIONS ON SATO-TATE LAWS FOR $GL(n)$

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Abstract. We give the first non-trivial estimate for the number of $GL(n)$ ($n \geq 3$) Hecke Maass forms whose Satake parameters at any given prime $p$ fail the Generalized Ramanujan Conjecture, and study some applications on the (vertical) Sato-Tate laws.

## 1. Introduction

There are great advances in the study of the Generalized Ramanujan Conjecture (GRC) and Sato-Tate laws for $GL(2)$. However, these two problems for $GL(n)$ ($n \geq 3$) are still very mysterious. Recently, Matz and Templier [12] made a breakthrough and proved the vertical Sato-Tate law conjectured by Sarnak [19] for $GL(n)$ where $n \geq 3$. An interesting consequence is a nontrivial estimate for the number of Hecke-Maass forms failing GRC *fairly*, which has been refined to those forms failing *marginally* in [8]. In this paper we improve further to give a nontrivial estimate for the Hecke-Maass forms failing GRC, and explore its applications on Sato-Tate laws for $GL(n)$ with $n \geq 3$.

Let $\mathcal{H}^\natural = \{\phi_j\}$ be an orthonormal basis of Hecke-Maass cusp forms for $SL_n(\mathbb{Z})$ with $n \geq 3$. For $T > 10^2$, define

$$\mathcal{H}_T = \left\{\phi \in \mathcal{H}^\natural : \mu_\phi \in i\mathbb{R}^n,\ \|\mu_\phi\|_2 \leq T\right\}$$

where $\mu_\phi$ is the Langlands parameters and $\|\cdot\|_2$ is the Euclidean norm. By Weyl's law (see [13]), $\#\mathcal{H}_T \ll T^d$, where $d = n(n+1)/2 - 1$. Matz and Templier [12, (1.1)] proved that $\#\mathcal{H}_T \gg T^d$.

Let $p$ be a fixed prime and $\phi \in \mathcal{H}_T$. The Satake parameters of $\phi$ consist of $n$ complex numbers $\pi_{\phi,1}(p), \pi_{\phi,2}(p), \ldots, \pi_{\phi,n}(p)$. It is well-known that

$$\pi_{\phi,1}(p)\pi_{\phi,2}(p)\cdots\pi_{\phi,n}(p) = 1$$

and (the unitary condition)

$$\{\pi_{\phi,1}(p), \ldots, \pi_{\phi,n}(p)\} = \left\{\overline{\pi_{\phi,1}(p)}^{-1}, \ldots, \overline{\pi_{\phi,n}(p)}^{-1}\right\}. \tag{1}$$

GRC asserts that

$$|\pi_{\phi,1}(p)| = |\pi_{\phi,2}(p)| = \cdots = |\pi_{\phi,n}(p)| = 1, \tag{2}$$

whose proof seems out of reach at present and the best bound towards GRC is (see [11])

$$|\pi_{\phi,\ell}(p)| \leq p^{1/2-1/(n^2+1)} \text{ for } \ell = 1, 2, \ldots, n. \tag{3}$$

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For applications, in addition to individual bounds, we may need an estimate of the forms $\phi\,(\in \mathcal{H}_T)$ that fail GRC at a given prime $p$. In this direction, following from Lau and Wang [8, Theorem 7.3] and Matz and Templier [12, Corollary 1.8], there are constants $c', T_0, \delta_0 > 0$ such that

$$\#\left\{\phi \in \mathcal{H}_T : \max_{1\le \ell\le n} \log|\pi_{\phi,\ell}(p)| > \theta\right\} \ll T^{d-c'\theta/\log p} \tag{4}$$

holds for all $T > T_0$ and $p \ll T^{\delta_0}$, where $c', T_0, \delta_0$ and the implied $\ll$-constant depend only on $n$. Results of the same fashion were earlier obtained by Sarnak [19] for $SL_2(\mathbb{Z})$ and Blomer, Buttcane and Raulf [2] for $SL_3(\mathbb{Z})$.

A defect of (4) is that, in view of Weyl's law, the upper bound becomes trivial when $\theta = 0$, which cannot give a sufficient control in some applications. In this paper we shall provide a non-trivial upper estimate for (4) in this boundary case of $\theta = 0$.

**Theorem 1.1.** *We have*

$$\#\left\{\phi \in \mathcal{H}_T : \max_{1\le \ell\le n} \log|\pi_{\phi,\ell}(p)| > 0\right\} \ll T^{d}\left(\frac{\log p}{\log T}\right)^3,$$

*where the implied constant depends on $n$.*

**Remark 1.1.** *In $GL(2)$ case, there are some similar results in* [9] *and* [23].

Next we explore applications of Theorem 1.1 on Sato-Tate laws for $GL(n)$. Write

$$\underline{\pi}_\phi(p) = (\pi_{\phi,1}(p), \cdots, \pi_{\phi,n}(p)) = (e^{i\theta_{\phi,1}(p)}, \cdots, e^{i\theta_{\phi,n}(p)})$$

where $\theta_{\phi,j}(p) \in \{a+bi : a \in [0,2\pi), b \in \mathbb{R}\}$ for $j = 1, \cdots, n$. This $\theta_{\phi,j}(p)$ is uniquely determined for each $\pi_{\phi,j}(p)$. Denote $\underline{\theta}_\phi(p) = (\theta_{\phi,1}(p), \ldots, \theta_{\phi,n}(p))$. Both $\underline{\theta}_\phi(p)$ and $\underline{\pi}_\phi(p)$ lie in $\mathbb{C}^n$. Since the order of $\underline{\pi}_\phi(p)$'s entries does not play role in GRC, we shall view $\underline{\pi}_\phi(p)$ in $\mathbb{C}^n/\mathfrak{S}_n$ (and $\underline{\theta}_\phi(p)$ in $\mathbb{C}^n/\mathfrak{S}_n$) where $\mathfrak{S}_n$ is the symmetric group of degree $n$.* GRC is equivalent to

$$\underline{\theta}_\phi(p) \in [0,2\pi)^n/\mathfrak{S}_n \quad \text{or} \quad \underline{\pi}_\phi(p) \in {S^1}^n/\mathfrak{S}_n$$

where $S^1$ is the unit circle in $\mathbb{C}$.

It is widely believed that the Satake parameters satisfy the (horizontal) generalized Sato-Tate conjecture: for any fixed $\phi \in \mathcal{H}^\natural$, the sets $\left\{\underline{\pi}_\phi(p) : p \le x\right\}$ become equidistributed with respect to the Sato-Tate measure $d\mu_{\mathrm{ST}}$ on ${S^1}^n/\mathfrak{S}_n$ as $x \to \infty$. This conjecture remains open. Parametrizing $S^1$ by $e^{i\theta}$, we have ${S^1}^n/\mathfrak{S}_n \cong [0,2\pi)^n/\mathfrak{S}_n$. We keep using $d\mu_{\mathrm{ST}}$ for the (push-forward) Sato-Tate measure on $[0,2\pi)^n/\mathfrak{S}_n$. Given any $I = \prod_{j=1}^n [a_j, b_j] \subset [0,2\pi)^n$. We denote by $\mathbf{\mathfrak{S}}I$ the image of $I$ under the canonical map $\rho : [0,2\pi)^n \to [0,2\pi)^n/\mathfrak{S}_n$. The generalized Sato-Tate conjecture can be formulated as

$$\lim_{x\to\infty} \frac{1}{\pi(x)} \#\left\{p \le x : \ \underline{\theta}_\phi(p) \in \mathbf{\mathfrak{S}}I\right\} = \int_{\mathbf{\mathfrak{S}}I} d\mu_{\mathrm{ST}}$$

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*Here and in the sequel we use the same notation for a vector $\underline{x} \in X$ and its image in $X/\mathfrak{S}_n$ whenever it is clear from the context.

where $\pi(x)$ counts the number of primes up to $x$.

Instead of varying primes, Sarnak [19, §4] considered the vertical perspective and conjectured that for any fixed prime $p$, the sets $\left\{\underline{\pi}_\phi(p) :\ \phi \in \mathcal{H}_T\right\}$ are equidistributed with respect to the Plancherel measure $d\mu_p$, as $T \to \infty$. This conjecture has been proved by Matz and Templier [12] recently, saying that for any $I = \prod_{j=1}^n [a_j, b_j] \subset [0, 2\pi)^n$,

$$\lim_{T\to\infty} \frac{1}{\#\mathcal{H}_T} \# \left\{\phi \in \mathcal{H}_T :\ \underline{\theta}_\phi(p) \in \mathfrak{S} I\right\} = \int\limits_{\mathfrak{S} I} d\mu_p.$$

There are various interesting earlier work than that of Matz and Templier. Zhou [24] proved essentially that the equidistribution theorem holds on average in the vertical sense under some orthogonality relation on the Fourier coefficients, and subsequently, he and Buttcane [3] confirmed the vertical version of equidistribution theorem on $GL(3)$. See Remark 1.2 for the relevant work in the $GL(2)$ case.

Our first application is to provide an explicit estimate on the rate of convergence, i.e. a quantitative version of the result of Matz and Templier.

**Theorem 1.2.** *For any fixed prime $p$ and any $I = \prod_{j=1}^n [a_j, b_j] \subset [0, 2\pi)^n$, we have*

$$\frac{1}{\#\mathcal{H}_T} \# \left\{\phi \in \mathcal{H}_T :\ \underline{\theta}_\phi(p) \in \mathfrak{S} I\right\} = \int\limits_{\mathfrak{S} I} d\mu_p + O\left(\frac{\log p}{\log T}\right),$$

*where the implied constant depends only on $n$.*

**Remark 1.2.** *(1) This generalizes the works in $GL(2)$ by Murty and Sinha* [14] *and Pujahari* [17] *for holomorphic cusp forms, Lau and Wang* [9] *for Maass cusp forms, and Lau, Li and Wang* [6] *for $GL(2)$ automorphic representations over totally real fields.*

*(2) Theorem 1.1 plays a key role in the proof since it helps to get around the difficulty in controlling the terms from the "exceptional" Satake parameters.*

*(3) The equidistribution (without an explicit rate of convergence) was firstly obtained by Sarnak* [19]*, and Knightly and Li* [5] *for $GL(2)$ Maass forms, and by Serre* [20] *and later independently by Conrey, Duke and Farmer* [4] *for $GL(2)$ holomorphic primitive forms. The counterpart for the Hilbert modular form is settled in the work of Li* [10]*.*

Theorem 1.2 yields immediately the following result towards an analogue of Lang-Trotter's problem.

**Corollary 1.1.** *For any $(a_1, \ldots, a_n) \in \mathbb{C}^n$, we have*

$$\# \left\{\phi \in \mathcal{H}_T : (\pi_{\phi,1}(p), \ldots, \pi_{\phi,n}(p)) = (a_1, \ldots, a_n)\right\} \ll \#\mathcal{H}_T \frac{\log p}{\log T},$$

*where the implied constant depends only on $n$.*

Moreover we have an analogue of Theorem 1.1 in Shin and Templier's work [21] by (7).

**Corollary 1.2.** *Let $\{p_k\}$ be a strictly increasing sequence of primes. Suppose that $T = T(k)$ satisfies $\frac{\log p_k}{\log T} \to 0$ as $k \to \infty$. Then the Satake parameters*

$$\left\{ \underline{\pi}_\phi(p_k) : \ \phi \in \mathcal{H}_T \right\}_{k \geq 1}$$

*are equidistributed with respect to the Sato-Tate measure $d\mu_{\mathrm{ST}}$, as $k \to \infty$.*

Corollary 1.2 and the generalized Sato-Tate problem in horizontal or vertical sense are investigations for the statistics of $\left\{ \underline{\pi}_\phi(p) : \phi \in \mathcal{H}, \, p \text{ primes} \right\}$. With Theorem 1.1, we derive a central limit behaviour related to the horizontal Sato-Tate distribution of $\left\{ \underline{\pi}_\phi(p) : p \text{ primes} \right\}$ over $\phi \in \mathcal{H}$.

For any $\phi \in \mathcal{H}_T$ and any $I = \prod_{j=1}^n [a_j, b_j] \subset [0, 2\pi)^n$, we define

$$N_I(\phi; x) := \# \left\{ p \leq x : \ \underline{\theta}_\phi(p) \in \mathfrak{S} I \right\}.$$

The following theorem tells that the generalized Sato-Tate conjecture is true on average.

**Theorem 1.3.** *Suppose that $T = T(x)$ satisfies $\log T / \log x \to \infty$ as $x \to \infty$. For any $I = \prod_{j=1}^n [a_j, b_j] \subset [0, 2\pi)^n$, we have*

$$\frac{1}{\# \mathcal{H}_T \pi(x)} \sum_{\phi \in \mathcal{H}_T} N_I(\phi; x) = \mu_{ST}(\mathfrak{S} I) + O\left( \frac{\log x}{\log T} + \frac{\log \log x}{\pi(x)} \right),$$

*where $\mu_{ST}(\mathfrak{S} I) = \int_{\mathfrak{S} I} d\mu_{\mathrm{ST}}$ and the implied constant only depends on $n$.*

**Remark 1.3.** *The above theorem should be compared with Corollary 1.2,* [15, Theorem 1] *(holomorphic cusp forms),* [22, Theorem 1.3] *(Maass cusp forms) for $GL(2)$ and* [21, Theorem 1.1] *for a reductive group over a number field which has discrete series representations.*

Suppose we can model $\underline{\theta}_\phi(p)$ by independently and identically distributed random variable $X_p$ induced by the characteristic function $\mathbb{1}_{\mathfrak{S} I}$ on the probability space whose probability measure is the Sato-Tate measure $d\mu_{\mathrm{ST}}$. Then $\mathbb{E}[X_p] = \int \mathbb{1}_{\mathfrak{S} I} \, d\mu_{\mathrm{ST}}$ ($= \mu$, say) and thus the variance is

$$\sigma^2 := \mathbb{E}[(X_p - \mu)^2] = \int (\mathbb{1}_{\mathfrak{S} I} - \mu)^2 \, d\mu_{\mathrm{ST}} = \mu - \mu^2.$$

The central limit theorem asserts that

$$\mathrm{S}_x := \frac{\frac{1}{\pi(x)} \sum_{p \leq x} X_p - \mu}{\sigma / \sqrt{\pi(x)}} \xrightarrow{d} N(0,1) \qquad \text{as } x \to \infty,$$

i.e. the cumulative distribution function of $\mathrm{S}_x$ converges in distribution to the standard normal distribution. This heuristic argument can be worked out in the following sense.

**Theorem 1.4.** *Let $T = T(x)$ be a function satisfying $\frac{\log T}{\sqrt{x}\log\log x} \to \infty$ as $x \to \infty$. Then for any bounded continuous, real-valued function $h$ on $\mathbb{R}$, we have*

$$\lim_{x\to\infty} \frac{1}{\#\mathcal{H}_T} \sum_{\phi\in\mathcal{H}_T} h\left(\frac{N_I(\phi;x) - \pi(x)\mu_{\mathrm{ST}}(\mathfrak{S}I)}{\sqrt{\pi(x)(\mu_{\mathrm{ST}}(\mathfrak{S}I) - \mu_{\mathrm{ST}}(\mathfrak{S}I)^2)}}\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t)e^{-\frac{t^2}{2}}dt.$$

**Remark 1.4.** *This generalizes a result obtained by Prabhu and Sinha* [16] *for $GL(2)$.*

**Remark 1.5.** *In contrast to Theorem 1.4, we obtained in* [7] *the central limit behaviour for the smoothly weighted frequency. However, the result* [7, Theorem 1.4] *does not imply Theorem 1.4 here, for Theorem 1.1 plays a crucial role in the proof of Theorem 1.4.*

**Remark 1.6.** *There are two main technical differences from the work by Prabhu and Sinha* [16]. *First, we need a tool in several variables instead of the Beurling-Selberg polynomials of one variable to approximate $N_I(\phi; x)$. Second, the control on the "exceptional" Satake parameters – using Theorem 1.1 – is more complicated than the case of $GL(2)$.*

## 2. Preparations

2.1. **An Arthur-Selberg Trace formula.** One of our main tools is the following trace formula of Matz and Templier [12, Theorem 1.4] with modification to fit our situation.

**Theorem 2.1** (Matz-Templier)**.** *Let $n \geq 3$. Given any $m \in \mathbb{N}$, any distinct primes $p_1, \dots, p_m$ and any $g_1, \dots, g_m \in \mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]^{\mathfrak{S}_n}$. We have*

$$\left| \sum_{\phi\in\mathcal{H}_T} \prod_{i=1}^{m} g_i(\alpha_\phi(p_i)) - \#\mathcal{H}_T \prod_{i=1}^{m} \int_{S^{1^n}/\mathfrak{S}_n} g_i d\mu_{p_i} \right| \leq c_1 T^{d-1/2} \prod_{i=1}^{m} p_i^{A\deg'(g_i)} \|g_i\|_{\max}$$

*where $c_1, A$ are constants only depending on $n$. Here $\|g\|_{\max}$ denote the maximum of the absolute values of its coefficients and constant term. The degree function $\deg'(g)$ denotes the degree when $g$ is expressed in terms of the elementary symmetric polynomials $e_0, \cdots, e_m$ ($e_0 := 1$ and $e_n = x_1 \cdots x_n$) with $\deg'(e_0) = \deg'(e_n) = 0$ and $\deg(e_i) = 1$ for $1 \leq i \leq n-1$.*

**Remark 2.1.** *(i) Here and in what follows, $f \in \mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]^{\mathfrak{S}_n}$ means a polynomial $f$ in $x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1}$ over $\mathbb{C}$ and*

$$f(x_{\sigma(1)}, \cdots, x_{\sigma(n)}, x_{\sigma(1)}^{-1}, \cdots, x_{\sigma(n)}^{-1}) = f(x_1, \cdots, x_n, x_1^{-1}, \cdots, x_n^{-1})$$

*for any $\sigma \in \mathfrak{S}_n$. (ii) Note that $\#\mathcal{H}_T \asymp T^d$.*

*Proof.* Put

$$S = \{p_1, \dots, p_m\}.$$

Let $\tau_{p_i}$ correspond to $g_i$ under the Satake correspondence for $i = 1, \dots, m$. Let

$$\tau = \prod_{p\in S} \tau_p \prod_{p\notin S} \mathbf{1}_{G(\mathbb{Z}_p)}.$$

The first term in the theorem agrees with that of [12, Theorem 1.1]. For the second term,

$$\begin{aligned}\prod_{i=1}^{m}\int_{S^{1^n}/\mathfrak{S}_n} g_i d\mu_{p_i} &= \prod_{p\in S}\mathrm{vol}(\mathbb{Z}_p)^{-1}\int_{Z(\mathbb{Q}_p)}\tau_p(z)dz\\ &= \prod_{p\in S}\sum_{z\in Z(\mathbb{Q}_p)/Z(\mathbb{Z}_p)}\tau_p(z)\\ &= \sum_{\gamma\in Z(\mathbb{Q})/\{\pm1\}}\prod_{p\in S}\tau_p(\gamma)\prod_{p\notin S}\mathbf{1}_{G(\mathbb{Z}_p)}(\gamma),\end{aligned}$$

which agrees with that of [12, Theorem 1.1]. Then, by [12, Theorem 1.1], we have to estimate

$$\|\prod_{p\in S}\tau_p\prod_{p\notin S}\mathbf{1}_{G(\mathbb{Z}_p)}\|_{L^1(G(\mathbb{A}_f))} = \prod_{p\in S}\|\tau_p\|_{L^1(G(\mathbb{Q}_p))}.$$

By the proof of [12, Theorem 1.4], we obtain

$$\|\tau_p\|_{L^1(G(\mathbb{Q}_p))} \leq c' p^{A\deg(\phi)}$$

and the theorem follows plainly. □

2.2. **Integration formulas for the Sato-Tate measure and Plancherel measure.** As $S^{1^n}/\mathfrak{S}_n$ is identified with $[0,2\pi)^n/\mathfrak{S}_n$ via parametrizing $S^1$ by $e^{i\theta}$, we may view the Sato-Tate measure $d\mu_{\mathrm{ST}}$, which is supported on $\{\underline{x}\in S^{1^n}/\mathfrak{S}_n : \prod_i x_i = 1\}$, as a measure supported on $T_0/\mathfrak{S}_n \subset [0,2\pi)^n/\mathfrak{S}_n$, where $T_0 = \{\underline{\theta}\in[0,2\pi)^n : \sum_i\theta_i = 0\}$ (here for $a,b\in[0,2\pi)$, $a=b$ means $a\equiv b \bmod 2\pi$). The integration formula for $d\mu_{\mathrm{ST}}$ is then given by

$$d\mu_{\mathrm{ST}}(\underline{\theta}) = \frac{1}{n!(2\pi)^{n-1}}\prod_{1\leq\ell<m\leq n}|e^{i\theta_\ell}-e^{i\theta_m}|^2 d\theta_1\cdots d\theta_{n-1} \tag{5}$$

where $\sum_{\ell=1}^n\theta_\ell = 0$, $\theta_j\in[0,2\pi)$ for $j=1,\cdots,n-1$. Let $\rho:[0,2\pi)^n\to[0,2\pi)^n/\mathfrak{S}_n$. Thus if $I=\prod_{j=1}^n[a_j,b_j]\subset[0,2\pi)^n$, then $\rho(I)=\mathfrak{S}I$ in Section 1. For any measurable $f$ on $T_0/\mathfrak{S}_n$,

$$\int f\,d\mu_{\mathrm{ST}} = \int_{[0,2\pi)^{n-1}} f\circ\rho(\underline{\theta})\,d\mu_{\mathrm{ST}}(\underline{\theta}).$$

The Plancherel measure $d\mu_p$ is given by the formula

$$d\mu_p(\underline{\theta}) = \prod_{j=2}^{n}\frac{1-p^{-j}}{1-p^{-1}}\prod_{1\leq\ell<m\leq n}\left|e^{i\theta_\ell}-p^{-1}e^{i\theta_m}\right|^{-2} d\mu_{\mathrm{ST}}.\ ^{\dagger} \tag{6}$$

Plainly, we have

$$d\mu_p = (1+O(1/p))\,d\mu_{\mathrm{ST}} \to d\mu_{\mathrm{ST}} \qquad \text{as } p\to\infty. \tag{7}$$

† We may suppress $\underline{\theta}$ from $d\mu_{\mathrm{ST}}(\underline{\theta})$ or $d\mu_p(\underline{\theta})$ once no confusion arises.

## 3. Proof of Theorem 1.1

We need the following analogue of [12, Lemma 3.1] with a very similar proof.

**Lemma 3.1.** *Let $s$ be any positive integer. Then there exist symmetric polynomials $f_1, \ldots, f_s \in \mathbb{C}[x_1, \ldots, x_s]^{\mathfrak{S}_s}$ such that for all $\alpha \in \mathbb{C}^s$,*

$$\max\big(|f_1(\alpha)|, \ldots, |f_s(\alpha)|\big) > |\alpha|_\infty^2 \quad \text{if} \quad |\alpha|_\infty > 1,$$

*and*

$$\max\big(|f_1(\alpha)|, \ldots, |f_s(\alpha)|\big) \leq 3^s \cdot |\alpha|_\infty^2 \quad \text{if} \quad |\alpha|_\infty \leq 1.$$

*Proof.* For any integer $m > 0$, and $x = (x_1, \ldots, x_s) \in \mathbb{C}^s$ with $|x|_\infty > 1$, let

$$e_m(x) := \frac{2^m}{m!(s-m)!} \sum_{\sigma \in \mathfrak{S}_s} x_{\sigma(1)}^2 \cdots x_{\sigma(m)}^2 = 2^m \sum_{\substack{A \subset \{1, \cdots, s\} \\ \#(A) = m}} \prod_{j \in A} x_j^2.$$

In particular $e_0 = 1$. Note that by convention $x_{s+1} = x_{s+2} = 0$, and so on; in other words we view $x$ in $\mathbb{C}^s \subset \mathbb{C}^{s+1} \subset \mathbb{C}^{s+2}$ and so on, by adding zero coordinates at the end. Thus by convention $e_m = 0$ if $m > s$. Let $x_{\max} \in \{x_1, \ldots, x_s\}$ be such that $|x_{\max}| = |x|_\infty = \max_{1 \leq j \leq s} |x_j| > 1$. Let $x^- \in \mathbb{C}^{s-1}$ be the vector obtained from $x$ by omitting the coordinate $x_{\max}$. Then for every $0 < m \leq s$,

$$e_m(x) = 2x_{\max}^2 e_{m-1}(x^-) + e_m(x^-).\ ^{\ddagger}$$

Hence we have

$$|e_m(x)| \geq |x_{\max}^2 e_{m-1}(x^-)| \tag{8}$$

or

$$|e_m(x^-)| \geq |x_{\max}^2 e_{m-1}(x^-)|. \tag{9}$$

Note that there exists $m \in \{1, \ldots, s\}$ such that (8) holds (namely, it holds as least for $m = s$). Let $m_0$ be the smallest $m$ such that (8) holds. Then for every $0 < m \leq m_0 - 1$ the inequality (9) holds so that

$$|e_{m_0-1}(x^-)| \geq |x_{\max}|^2 |e_{m_0-2}(x^-)| \geq \cdots \geq |x_{\max}|^{2(m_0-1)}.$$

Therefore,

$$|e_{m_0}(x)| \geq |x_{\max}|^2 |e_{m_0-1}(x^-)| \geq |x_{\max}|^{2m_0} > 1.$$

The lemma follows with $f_i := e_i$ for $i = 1, \ldots, s$, as $\max_{1 \leq m \leq s} |e_m(x)| \geq |e_{m_0}(x)|$ and $|e_m(x)| \leq 2^m s! |x|_\infty^{2m} / (m!(s-m)!) \leq 3^s |x|_\infty^{2m}$. □

[‡]This can be seen as follows: Suppose, without loss of generality, $x_s = x_{\max}$. Using the second expression for $e_m(x)$, we split the sum into two sums according as $s \in A$ or not. The latter case (i.e. $s \notin A$) gives $e_m(x^-)$; the former case leads to

$$\sum_{\substack{s \in A \subset \{1, \cdots, s\} \\ \#(A) = m}} \prod_{j \in A} x_j^2 = x_s^2 \sum_{\substack{A \subset \{1, \cdots, s-1\} \\ \#(A) = m-1}} \prod_{j \in A} x_j^2.$$

Now we are ready to prove Theorem 1.1. We only consider the case that $\log T/\log p$ is sufficiently large compared to $n$. Otherwise, Theorem 1.1 is trivial. Let $L \in \mathbb{N}_0$ and define for any $x \neq y \in \mathbb{C}$,

$$U_L(x,y) := \frac{1}{L+1}\frac{x^{L+1}-y^{L+1}}{x-y}.\text{[§]} \tag{10}$$

Let $s = n(n-1)/2$. For $\underline{x} = (x_1, \cdots, x_n) \in \mathbb{C}^n$ with $x_1 \cdots x_n \neq 0$, we consider the unordered tuple

$$\mathrm{U}_L(\underline{x}) = \{U_L(x_\ell, x_m)\}_{1\leq \ell<m\leq n} \tag{11}$$

which lies in $\mathbb{C}^s/\mathfrak{S}_s$. Write $\underline{x}_\sigma = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$, $\sigma \in \mathfrak{S}_n$, and $\underline{x}^{-1} = (x_1^{-1}, \cdots, x_n^{-1})$. As $U_L(x,y) = U_L(y,x)$, it follows that $\mathrm{U}_L(\underline{x}_\sigma) = \mathrm{U}_L(\underline{x})$ for any $\sigma \in \mathfrak{S}_n$. We choose $f_1, \dots, f_s$ as in Lemma 3.1 with $s = n(n-1)/2$, and define

$$F_L(x_1, \dots, x_n) := \sum_{i=1}^{s} f_i(\mathrm{U}_L(\underline{x}))f_i(\mathrm{U}_L(\underline{x}^{-1})).$$

Then $F_L$ is in $\mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]^{\mathfrak{S}_n}$. We will take $L$ to be sufficiently large (compared to $n$) such that the coefficients of $F_L$ are less than one.

Abbreviate $\underline{\alpha}_\phi = (\pi_{\phi,1}(p), \pi_{\phi,2}(p), \dots, \pi_{\phi,n}(p))$. We have $\mathrm{U}_L(\underline{\alpha}_\phi^{-1}) = \overline{\mathrm{U}_L(\underline{\alpha}_\phi)}$ by the unitary condition (1), and therefore for all $\phi \in \mathcal{H}_T$,

$$F_L(\underline{\alpha}_\phi) = \sum_{i=1}^{s} |f_i(\mathrm{U}_L(\underline{\alpha}_\phi))|^2 \geq 0. \tag{12}$$

Suppose $\phi$ does not satisfy (2). The unitary condition (1) yields (at least) one pair of $\pi_{\phi,\ell}(p)$, $1 \leq \ell \leq n$, whose absolute value are not equal to 1. Without loss of generality, we assume that $\pi_{\phi,1}(p)$ and $\pi_{\phi,2}(p)$ are such a pair and write

$$\pi_{\phi,1}(p) = \rho_\phi(p)e^{i\theta'_\phi(p)} \quad \text{and} \quad \pi_{\phi,2}(p) = \rho_\phi(p)^{-1}e^{i\theta'_\phi(p)},$$

for some $\rho_\phi(p) > 1$ and real $\theta'_\phi(p)$. Then for these $\phi$, we have

$$\left|U_L(\pi_{\phi,1}(p), \pi_{\phi,2}(p))\right| = U_L(\rho_\phi(p), \rho_\phi(p)^{-1}) > 1,$$

by (10). Hence $\left|\mathrm{U}_L(\underline{\alpha}_\phi)\right|_\infty > 1$ if $\phi$ does not satisfy (2), and together with Lemma 3.1,

$$F_L(\underline{\alpha}_\phi) = \sum_{i=1}^{s} |f_i(\mathrm{U}_L(\underline{\alpha}_\phi))|^2 \geq 1 \tag{13}$$

if $\phi$ does not satisfy (2).

It follows from (12) and (13) that

$$\sum_{\substack{\phi\in\mathcal{H}_T \\ \text{(2) holds}}} 1 \geq \sum_{\phi\in\mathcal{H}_T} \Big(1 - F_L(\underline{\alpha}_\phi)\Big).$$

[§] $U_L(e^{i\theta}, e^{-i\theta}) = \frac{1}{L+1}U_L(\cos\theta)$ where $U_L(x)$ is a Chebyshev polynomial of the second kind.

We infer that

$$\#\mathcal{H}_T \geq \sum_{\substack{\phi\in\mathcal{H}_T \\ (2)\text{ holds}}} 1 \geq \sum_{\phi\in\mathcal{H}_T} \big(1 - F_L(\underline{\alpha}_\phi)\big)$$

$$= \#\mathcal{H}_T \int_{[0,2\pi)^{n-1}} (1 - F_L(e^{i\theta_1}, \dots, e^{i\theta_n}))d\mu_p + O(p^{LA'}T^{d-1/2}). \tag{14}$$

Here we have applied Theorem 2.1 in the last step where $A'$ is a constant depending only on $n$ and $\theta_1 + \cdots + \theta_n = 0$.

By (10), we have $|U_L(x, y)| \leq 1$ for any $x, y \in S^1$, and by Lemma 3.1 with (11),

$$\begin{aligned} F_L(e^{i\theta_1}, \dots, e^{i\theta_n}) &= \sum_{i=1}^{s} f_i(\mathrm{U}_L(e^{i\theta_1}, \dots, e^{i\theta_n}))f_i(\mathrm{U}_L(e^{-i\theta_1}, \dots, e^{-i\theta_n})) \\ &= \sum_{i=1}^{s} |f_i(\mathrm{U}_L(e^{i\theta_1}, \dots, e^{i\theta_n}))|^2 \\ &\leq 3^s s \cdot \left|\mathrm{U}_L(e^{i\theta_1}, \dots, e^{i\theta_n})\right|_\infty^4. \end{aligned}$$

Consequently,

$$F_L(e^{i\theta_1}, \dots, e^{i\theta_n}) \ll_n \left|\mathrm{U}_L(e^{i\theta_1}, \dots, e^{i\theta_n})\right|_\infty^4 \leq \max_{1\leq \ell < m \leq n} |U_L(e^{i\theta_\ell}, e^{i\theta_m})|^4.$$

Together with (6) (which implies $d\mu_p \ll d\mu_{ST}$), we get

$$\int\limits_{[0,2\pi)^{n-1}} F_L(e^{i\theta_1}, \dots, e^{i\theta_n})d\mu_p \ll_n \max_{1\leq \ell < m \leq n} \int\limits_{[0,2\pi)^{n-1}} |U_L(e^{i\theta_\ell}, e^{i\theta_m})|^4 d\mu_{\mathrm{ST}}. \tag{15}$$

In view of (10) and (5), two factors of the denominator of $|U_L(e^{i\theta_\ell}, e^{i\theta_m})|^4$ are cancelled out by factors in the product inside the integration formula of $d\mu_{\mathrm{ST}}$. Bounding the other factors trivially, we see that

$$\int\limits_{[0,2\pi)^{n-1}} |U_L(e^{i\theta_\ell}, e^{i\theta_m})|^4 d\mu_{\mathrm{ST}} \;\ll_n\; \frac{1}{L^2} \int\limits_{[0,2\pi)^{n-1}} |U_L(e^{i\theta_\ell}, e^{i\theta_m})|^2 d\theta_1 \cdots d\theta_{n-1} \ll \frac{1}{L^3}$$

after an integration with the expansion

$$U_L(e^{i\theta_\ell}, e^{i\theta_m}) = \frac{1}{L+1} \sum_{\substack{0\leq \alpha,\beta\leq L \\ \alpha+\beta=L}} e^{i\alpha\theta_\ell}e^{i\beta\theta_m}.$$

Putting the estimate into (15), we deduce from (14) that

$$0 \leq \#\mathcal{H}_T - \sum_{\substack{\phi\in\mathcal{H}_T \\ (2)\text{ holds}}} 1 \ll \#\mathcal{H}_T L^{-3} + p^{LA'}T^{d-1/2}.$$

Since $\#\mathcal{H}_T \asymp T^d$ and $\log T/\log p$ is sufficiently large compared to $n$, Theorem 1.1 follows by choosing

$$L = \left[\frac{\log T}{4A' \log p}\right].$$

## 4. Further preparations

Let $n \in \mathbb{N}$ and $\Omega$ be some set. Suppose $D \subset \Omega^n$ and $F : \Omega^n \to \mathbb{C}$ is any function supported on $D$. For every $\sigma \in \mathfrak{S}_n$, we define

$$D_\sigma := \{\underline{x}_\sigma : \underline{x} \in D\} \qquad \text{and} \qquad F \circ \sigma(\underline{x}) := F(\underline{x}_\sigma).$$

(Recall $\underline{x}_\sigma = (x_{\sigma(1)}, \cdots, x_{\sigma(n)})$ if $\underline{x} = (x_1, \cdots, x_n)$.)

Set

$$\mathfrak{S}D := \bigcup_{\sigma \in \mathfrak{S}_n} D_\sigma. \tag{16}$$

Then $\mathfrak{S}D$ is the preimage for the image of $D$ in $\Omega^n/\mathfrak{S}_n$ (denoted by $\boldsymbol{\mathfrak{S}}D$) under the canonical projection from $\Omega^n$ to $\Omega^n/\mathfrak{S}_n$.

Define $\mathfrak{S}F$ and $|\mathfrak{S}|F$ by the equations

$$\prod_{\sigma \in \mathfrak{S}_n} (\mathbb{1}_{\mathfrak{S}D} - F \circ \sigma) = \mathbb{1}_{\mathfrak{S}D} - \mathfrak{S}F \quad \text{and} \quad \prod_{\sigma \in \mathfrak{S}_n} (\mathbb{1}_{\mathfrak{S}D} + F \circ \sigma) = \mathbb{1}_{\mathfrak{S}D} + |\mathfrak{S}|F \tag{17}$$

where $\mathbb{1}_{\mathfrak{S}D} : \Omega^n \to \{0, 1\}$ is the characteristic function on $\mathfrak{S}D$ and $\prod$ denotes the product (not composite) of functions. Alternatively,

$$\mathfrak{S}F = \sum_{1 \le r \le n!} (-1)^{r-1} \sum_{\substack{A \subset \mathfrak{S}_n \\ \#A = r}} \prod_{\sigma \in A} F \circ \sigma \quad \text{and} \quad |\mathfrak{S}|F = \sum_{1 \le r \le n!} \sum_{\substack{A \subset \mathfrak{S}_n \\ \#A = r}} \prod_{\sigma \in A} F \circ \sigma. \tag{18}$$

**Remark 4.1.** *(1) For $\mathfrak{S}D \subset D' \subset \Omega^n$, we also have*

$$\prod_{\sigma \in \mathfrak{S}_n} (\mathbb{1}_{D'} - F \circ \sigma) = \mathbb{1}_{D'} - \mathfrak{S}F \quad \textit{and} \quad \prod_{\sigma \in \mathfrak{S}_n} (\mathbb{1}_{D'} + F \circ \sigma) = \mathbb{1}_{D'} + |\mathfrak{S}|F.$$

*(2) For $\mathfrak{F} = \mathfrak{S}F$ or $|\mathfrak{S}|F$, $\mathfrak{F}(\underline{x}_\sigma) = \mathfrak{F}(\underline{x})$ for all $\sigma \in \mathfrak{S}_n$ by* (17)*; $|\mathfrak{S}F| \le |\mathfrak{S}||F|$ by* (18)*.*
*(3) Suppose $0 \le F \le G \le 1$ on $\Omega^n$ (which implies $\mathrm{supp}(F) \subset \mathrm{supp}(G)$). Then*

$$0 \le \mathfrak{S}F \le \mathfrak{S}G \le 1. \tag{19}$$

*This is seen as follows: We have $0 \le \prod_\sigma (\mathbb{1}_{\mathfrak{S}D} - G \circ \sigma) \le \prod_\sigma (\mathbb{1}_{\mathfrak{S}D} - F \circ \sigma) \le 1$ where $D = \mathrm{supp}(G)$ and the product runs over all $\sigma \in \mathfrak{S}_n$, as $0 \le F \circ \sigma \le G \circ \sigma \le 1$ for all $\sigma \in \mathfrak{S}_n$. Then* (19) *follows readily from* (17) *and Part (1) of this remark.*

### 4.1. A few inequalities.

We develop some tools for approximation using the work of Barton, Montgomery and Vaaler [1].

Let $\varphi_{u,v} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ be the normalized characteristic functions defined as,

$$\varphi_{u,v}(\theta) = \begin{cases} 1 & \text{if } u < \theta - n < v \text{ for some } n \in \mathbb{Z}, \\ \frac{1}{2} & \text{if } u - \theta \in \mathbb{Z} \text{ or if } v - \theta \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

where $u < v < u + 1$. We may also view $\varphi_{u,v}$ as a periodic function on $\mathbb{R}$. Define two functions $\alpha_{u,v}, \beta_{u,v}$ on $\mathbb{R}/\mathbb{Z}$ (viewed as periodic function on $\mathbb{R}$ as well) by

$$\alpha_{u,v}(\theta) = \widehat{\alpha}_{u,v}(0) + \sum_{1\le|\ell|\le M} \widehat{\alpha}_{u,v}(\ell)e(\ell\theta) \tag{20}$$

and

$$\beta_{u,v}(\theta) = (2M+2)^{-1} \sum_{|\ell|\le M} \widehat{\beta}_{u,v}(\ell)e(\ell\theta), \tag{21}$$

where $e(x) = e^{2\pi i x}$, $\widehat{\alpha}_{u,v}(0) = v - u$, $\widehat{\beta}_{u,v}(0) = 2$ and for $\ell \neq 0$,

$$\widehat{\alpha}_{u,v}(\ell) = \frac{1}{2\pi i\ell}\widehat{J}\Big(\frac{\ell}{M+1}\Big)\big(e(-\ell u) - e(-\ell v)\big)\big) \tag{22}$$

with $\widehat{J}(t) = \pi t(1 - |t|)\cot \pi t + |t|$ for $0 < |t| < 1$, and

$$\widehat{\beta}_{u,v}(\ell) = \left(1 - \frac{|\ell|}{M+1}\right)(e(-\ell u) + e(-\ell v)). \tag{23}$$

The functions $\alpha_{u,v}$ and $\beta_{u,v}$ define polynomials on $\mathbb{C}$. Write

$$\widetilde{\alpha}_{u,v}(z) = \widehat{\alpha}_{u,v}(0) + \sum_{1\le|\ell|\le M} \widehat{\alpha}_{u,v}(\ell)z^{\ell} \quad \text{and} \quad \widetilde{\beta}_{u,v}(z) = (2M+2)^{-1} \sum_{|\ell|\le M} \widehat{\beta}_{u,v}(\ell)z^{\ell}.$$

**Proposition 4.1.** *We have (i)* $\overline{\widetilde{\alpha}_{u,v}(z)} = \widetilde{\alpha}_{u,v}(\overline{z}^{-1})$, $\overline{\widetilde{\beta}_{u,v}(z)} = \widetilde{\beta}_{u,v}(\overline{z}^{-1})$; *(ii) moreover, for any* $\theta \in \mathbb{C}$, $\widetilde{\alpha}_{u,v}(z) = \alpha_{u,v}(\theta)$ *and* $\widetilde{\beta}_{u,v}(z) = \beta_{u,v}(\theta)$ *if* $z = e(\theta)$.

*(iii) Suppose* $u, v \in \frac{1}{M+1}\mathbb{Z}$. *We have*

- $|\varphi_{u,v}(\theta) - \alpha_{u,v}(\theta)| \le \beta_{u,v}(\theta)$ *for all* $\theta \in \mathbb{R}$,
- $0 \le \alpha_{u,v}(\theta), \beta_{u,v}(\theta) \le 1$ *for all* $\theta \in \mathbb{R}$, *hence* $0 \le \widetilde{\alpha}_{u,v}(z), \widetilde{\beta}_{u,v}(z) \le 1$ *for* $z \in S^1$.

*Proof.* Write $f(\ell) = \widehat{\alpha}_{u,v}(\ell)$ or $\widehat{\beta}_{u,v}(\ell)$ with $\ell \in \mathbb{Z}$. By (22) and (23), $f(0) \in \mathbb{R}$ and $\overline{f(\ell)} = f(-\ell)$, leading to (i). The assertion (ii) is obvious. The remaining assertions in (iii) come from [1, (2.6) and (2.10)], for $\alpha_{u,v}(\theta), \beta_{u,v}(\theta)$ are the functions $\varphi_{u,v} * j_M(\theta)$ and $(2M+2)^{-1}\{k_M(u-\theta) + k_M(\theta - v)\}$ in [1]. □

4.2. **Approximation of the characteristic function of $\mathfrak{S}I$.** We take $\Omega = \mathbb{R}/(2\pi\mathbb{Z})$ which is identified with $[0, 2\pi)$ and $\emptyset \neq I_j = [a_j, b_j] \subset (0, 2\pi)$ for $j = 1, \cdots, n$. Let $M$ be any sufficiently large number. Choose $u_j^{\pm}, v_j^{\pm} \in \frac{1}{M+1}(2\pi\mathbb{Z})$ $(1 \le j \le n)$ so that $|u_j^- - u_j^+|$ and $|v_j^- - v_j^+|$ are $\ll M^{-1}$ and

$$0 \le u_j^+ < a_j < u_j^- < v_j^- < b_j < v_j^+ < 2\pi.$$

Set $I = \prod_{j=1}^n I_j$, $I^{\pm} = \prod_{j=1}^n [u_j^{\pm}, v_j^{\pm}]$ and define for $\underline{x} \in \Omega^n$,

$$\Phi^{\pm}(\underline{\theta}) = \prod_{j=1}^{n} \varphi_j^{\pm}(\theta_j) \quad \text{where} \quad \varphi_j^{\pm}(\theta) = \varphi_{\frac{1}{2\pi}u_j^{\pm}, \frac{1}{2\pi}v_j^{\pm}}(\tfrac{1}{2\pi}\theta). \tag{24}$$

Clearly $0 \le \Phi^- \le \mathbb{1}_I \le \Phi^+ \le 1$ and $\Phi^\pm = \mathbb{1}_{I^\pm}$ a.e. with respect to the Lebesgue measure on $\mathbb{R}^n$. Let $\mathfrak{S}I$ be defined as in (16). With $\mathbb{1}_{\mathfrak{S}I} = \mathfrak{S}\mathbb{1}_I$ and (19), we have

$$0 \le \mathfrak{S}\Phi^- \le \mathbb{1}_{\mathfrak{S}I} \le \mathfrak{S}\Phi^+ \le 1 \quad \text{and} \quad \mathfrak{S}\Phi^\pm = \mathbb{1}_{\mathfrak{S}I^\pm} \text{ a.e..} \tag{25}$$

Next we extend the domains of $\alpha_{u,v}(\theta)$ and $\beta_{u,v}(\theta)$ to $\mathbb{C}$, and set (for $\underline{\theta} \in \mathbb{C}^n$)

$$\alpha_j^\pm(\theta) = \alpha_{\frac{1}{2\pi}u_j^\pm, \frac{1}{2\pi}v_j^\pm}(\tfrac{1}{2\pi}\theta) \quad \text{and} \quad \beta_j^\pm(\theta) = \beta_{\frac{1}{2\pi}u_j^\pm, \frac{1}{2\pi}v_j^\pm}(\tfrac{1}{2\pi}\theta),$$

and define two functions on $\mathbb{C}^n$:

$$\boldsymbol{\alpha}^\pm(\underline{\theta}) = \prod_{j=1}^n \alpha_j^\pm(\theta_j) \quad \text{and} \quad B^\pm(\underline{\theta}) = \sum_{j=1}^n \beta_j^\pm(\theta_j). \tag{26}$$

Recall that $\boldsymbol{\mathfrak{S}}I$ denotes the image of $I$ under the projection $\rho : [0, 2\pi)^n \to [0, 2\pi)^n/\mathfrak{S}_n$, i.e. $\boldsymbol{\mathfrak{S}}I = \rho(I)$, then $\mathfrak{S}I$ defined as in (16) is the preimage of $\rho(I)$, i.e. $\mathfrak{S}I = \rho^{-1}(\rho(I))$, or $\mathfrak{S}I = \rho^{-1}(\boldsymbol{\mathfrak{S}}I)$. Analogously for a function $f$ on $[0, 2\pi)^n$, the functions $\mathfrak{S}f$ and $|\mathfrak{S}|f$ on $[0, 2\pi)^n$ descend to functions on $[0, 2\pi)^n/\mathfrak{S}_n$, denoted by $\boldsymbol{\mathfrak{S}}f$ and $|\boldsymbol{\mathfrak{S}}|f$, i.e. $\mathfrak{S}f = \boldsymbol{\mathfrak{S}}f \circ \rho$ and $|\mathfrak{S}|f = |\boldsymbol{\mathfrak{S}}|f \circ \rho$.**

**Lemma 4.1.** *We have the following:*

1. *On $\mathbb{R}^n$, $0 \le \mathfrak{S}\boldsymbol{\alpha}^\pm \le 1$ and $0 \le |\mathfrak{S}|B^\pm < (n+1)^{n!}$.*
2. *On $\mathbb{R}^n$, $|\mathfrak{S}\Phi^\pm - \mathfrak{S}\boldsymbol{\alpha}^\pm| \le |\mathfrak{S}|B^\pm$.*
3. *On $\{\underline{\theta} \in \mathbb{C}^n : \underline{\theta}_\sigma = \overline{-\underline{\theta}}$ for some $\sigma \in \mathfrak{S}_n\}$, both $\mathfrak{S}\boldsymbol{\alpha}^\pm$ and $|\mathfrak{S}|B^\pm$ are $\mathbb{R}$-valued.*

*Moreover, for $\ell = 1$ or $2$,*

$$\int \left|(\boldsymbol{\mathfrak{S}}\boldsymbol{\alpha}^\pm)^\ell - \mathbb{1}_{\boldsymbol{\mathfrak{S}}I}\right| d\mu_p \ll M^{-1} \quad \text{and} \quad \int |\boldsymbol{\mathfrak{S}}|B^\pm \, d\mu_p \ll M^{-1}. \tag{27}$$

*Proof.* In view of (26) and Proposition 4.1 (iii), we get

$$0 \le \boldsymbol{\alpha}^\pm(\underline{\theta}) \le 1 \text{ and } 0 \le B^\pm(\underline{\theta}) \le n \text{ for } \underline{\theta} \in \mathbb{R}^n.$$

Thus in (1), the first and second inequalities follow respectively from (19) and trivially bounding (18).

Next we observe

$$(*) : \left|\Phi^\pm - \boldsymbol{\alpha}^\pm\right| \le B^\pm \text{ (on } \mathbb{R}^n),$$

**Let us compare the known integration formulas for $d\mu_{\mathrm{ST}}$ ($n = 2, 3$) with (5) under this identification. When $n = 2$, the Sato-Tate measure of $I$ is often given as $\frac{2}{\pi}\int_I \sin^2\theta \, d\theta$ for $I \subset [0, \pi]$, which equals $\frac{1}{2! \cdot 2\pi}\int_{\mathfrak{S}I} |e^{i\theta} - e^{-i\theta}|^2 \, d\theta$. When $n = 3$, the Sato-Tate measure is given as in [2, p.899-900]: for any measurable function $f$ on the support $\mathcal{R}$ of the Sato-Tate measure $d\mu_{\mathrm{ST}}$,

$$\int_{\mathcal{R}} f(z) \, d\mu_{\mathrm{ST}}(z) = \int_{S^1 \times S^1} f \circ \Phi(\theta_1, \theta_2) \, d\alpha(\theta_1, \theta_2)$$

where $\Phi : (S^1 \times S^1)/W \to \mathcal{R}$ is bijective, and $W$ is the group of 6 maps $S^1 \times S^1 \to S^1 \times S^1$ generated by $(e^{i\theta_1}, e^{i\theta_2}) \mapsto (e^{i\theta_2}, e^{i\theta_1})$ and $(e^{i\theta_1}, e^{i\theta_2}) \mapsto (e^{i\theta_1}, e^{-i(\theta_1+\theta_2)})$. Observe that $(S^1 \times S^1)/W \cong T_0/\mathfrak{S}_3$ and $d\alpha(\theta_1, \theta_2) = d\mu_{\mathrm{ST}}(\underline{\theta})$. This verifies the case for $n = 3$.

following plainly from the inequality

$$(\star): \left|\prod_{j=1}^{n} x_j - \prod_{j=1}^{n} y_j\right| \leq \sum_{j=1}^{n} |x_j - y_j|$$

for $0 \leq x_j, y_j \leq 1$. (Note that $|\varphi_j^{\pm} - \alpha_j^{\pm}| \leq \beta_j^{\pm}$ and $0 \leq \varphi_j^{\pm}, \alpha_j^{\pm} \leq 1$ due to Proposition 4.1 (iii).) Then (2) is proved with (18), and the inequalities $(\star)$ and $(*)$.

We lift $\alpha_j^{\pm}, \beta_j^{\pm}$ to $\widetilde{\alpha}_j^{\pm}, \widetilde{\beta}_j^{\pm}$ as in Proposition 4.1 (ii). Correspondingly for $\mathfrak{F} = \mathfrak{S}\boldsymbol{\alpha}^{\pm}$ or $|\mathfrak{S}|B^{\pm}$, we have a lift $\widetilde{\mathfrak{F}}(\underline{x})$ of $\mathfrak{F}(\underline{\theta})$. By construction, $\widetilde{\mathfrak{F}}(x_1, \cdots, x_n) \in \mathbb{C}[x_1^{\pm}, \cdots, x_n^{\pm}]^{\mathfrak{S}_n}$ (cf. Remark 4.1 (2)) and by Proposition 4.1 (1), for any $\underline{x} = (x_1, \cdots, x_n) \in \mathbb{C}^n$,

$$\overline{\widetilde{\mathfrak{F}}(x_1, \cdots, x_n)} = \widetilde{\mathfrak{F}}(\overline{x_1}^{-1}, \cdots, \overline{x_n}^{-1}).$$

Thus $\widetilde{\mathfrak{F}}(\underline{x}) \in \mathbb{R}$ if $\underline{x}_\sigma = \overline{\underline{x}^{-1}}$ (the unitary condition), which is equivalent to $\underline{\theta}_\sigma = \overline{-\underline{\theta}}$ when $\underline{\theta} = (\theta_1, \cdots, \theta_n)$ and $\underline{x} = (e^{i\theta_1}, \cdots, e^{i\theta_n})$, for some $\sigma \in \mathfrak{S}_n$. This confirms (3).

Recall the integral formulas (6) and (5) for $d\mu_p$ and $d\mu_{\mathrm{ST}}$. From (25), we get

$$\int |\mathbb{1}_{\mathfrak{S}I} - \mathfrak{S}\Phi^{\pm}|\, d\mu_p \ll \int |\mathbb{1}_{\mathfrak{S}I} - \mathfrak{S}\Phi^{\pm}| = \int |\mathbb{1}_{\mathfrak{S}I} - \mathbb{1}_{\mathfrak{S}I^{\pm}}| \ll 1/M$$

where the last two integrals are against the Lebesgue measure on $[0, 2\pi)^n$. With the just shown (1) and (2), we see that

$$\int \left|\left(\mathfrak{S}\boldsymbol{\alpha}^{\pm}\right)^2 - \mathbb{1}_{\mathfrak{S}I}\right| d\mu_p \leq 2\int \left|\mathfrak{S}\boldsymbol{\alpha}^{\pm} - \mathbb{1}_{\mathfrak{S}I}\right| d\mu_p \ll M^{-1} + \int |\mathfrak{S}|B^{\pm}\, d\mu_p.$$

From $0 \leq B^{\pm} \leq n$ on $\mathbb{R}^n$ and (18), we infer

$$0 \leq |\mathfrak{S}|B^{\pm} \ll_n \sum_{\sigma \in \mathfrak{S}_n} B^{\pm} \circ \sigma.$$

Consequently,

$$\int |\mathfrak{S}|B^{\pm}\, d\mu_p \ll_n \int_{[0,2\pi)^n} B^{\pm}(\underline{\theta}) \ll 1/M,$$

where the last estimate comes from $\int_0^{2\pi} \beta_j(\theta)\, d\theta = 1/(M+1)$ (by (21)) and (26). $\square$

## 5. Contributions of Exceptional Satake parameters

Let $\phi \in \mathcal{H}_T$. All the four functions $\boldsymbol{\alpha}^{\pm}(\underline{\theta}_\phi(p))$ and $B^{\pm}(\underline{\theta}_\phi(p))$ are defined, no matter whether $\phi$ satisfies (GRC) or not. But when $\phi$ satisfies (GRC), the vector $\underline{\theta}_\phi(p)$ lies in $(\mathbb{R}/(2\pi\mathbb{Z}))^n$ and Lemma 4.1 (1) and (2) hold. Otherwise, for the "exceptional" Satake parameters, we shall give a control via the lemma below. Define

$$\mathcal{H}_{T,p}^{*} = \left\{\phi \in \mathcal{H}_T : \max_{1 \leq \ell \leq n} \log|\pi_{\phi,\ell}(p)| > 0\right\}.$$

**Lemma 5.1.** *Let $h, l \geq 1$ be any integers. For $\mathfrak{F} = \mathfrak{S}\boldsymbol{\alpha}^{\pm}$ or $|\mathfrak{S}|B^{\pm}$, we have $\mathfrak{F}(\underline{\theta}_\phi(p)) \in \mathbb{R}$ and*

$$\frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}^*_{T,p}} \left|\mathfrak{F}\left(\underline{\theta}_\phi(p)\right)\right|^h \ll_l \left(\frac{\log p}{\log T}\right)^{3(1-\frac{1}{2l})} \left(c^h + T^{-\frac{1}{4l}} p^{A'Mh}\right) \tag{28}$$

*where $c = 1$ or $(n+1)^{n!}$ according as $\mathfrak{F} = \mathfrak{S}\boldsymbol{\alpha}^{\pm}$ or $|\mathfrak{S}|B^{\pm}$, the positive constant $A'$ depends at most on $n$ and the implied constant depends at most on $n$ and $l$.*

*Proof.* Lemma 4.1 yields $\mathfrak{F}(\underline{\theta}_\phi(p)) \in \mathbb{R}$. We apply Hölder's inequality with the pair $(2l, 2l/(2l-1))$ to the left-hand side of (28) and get

$$\frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}^*_{T,p}} \left|\mathfrak{F}\left(\underline{\theta}_\phi(p)\right)\right|^h \leq \left(\frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}^*_{T,p}} 1\right)^{1-\frac{1}{2l}} \left(\frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}_T} \mathfrak{F}\left(\underline{\theta}_\phi(p)\right)^{2hl}\right)^{\frac{1}{2l}}.$$

We apply Theorem 2.1 to get, as $\deg'(\mathfrak{F}) \ll_n 1$,

$$\frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}_T} \mathfrak{F}\left(\underline{\theta}_\phi(p)\right)^{2hl} = \int_{T_0/\mathfrak{S}_n} \boldsymbol{\mathfrak{F}}^{2hl} d\mu_p + O\left(T^{-1/2} p^{2A'Mhl}\right)$$

for some constant $A' > 0$ depending at most on $n$, where the function $\boldsymbol{\mathfrak{F}}$ is defined by $\mathfrak{F} = \boldsymbol{\mathfrak{F}} \circ \rho$. The right-hand side is $\ll_n c^{2hl} + T^{-1/2} p^{2A'Mhl}$ since $0 \leq \mathfrak{F} \leq c$, cf. Lemma 4.1 (1). The assertion follows from Theorem 1.1. ∎

## 6. Proof of Theorem 1.2

In view of (25) and Lemma 4.1 (2), we obtain

$$\mathfrak{S}\boldsymbol{\alpha}^- - |\mathfrak{S}|B^- \leq \mathbb{1}_{\mathfrak{S}I} \leq \mathfrak{S}\boldsymbol{\alpha}^+ + |\mathfrak{S}|B^+ \tag{29}$$

on $\mathbb{R}^n$, which is thus applicable to $\phi \in \mathcal{H}_T \setminus \mathcal{H}^*_{T,p}$. The remnants due to $\phi \in \mathcal{H}^*_{T,p}$ are then controlled by Lemma 5.1 with $h = l = 1$, so

$$\frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}_T} \mathfrak{S}\boldsymbol{\alpha}^-(\underline{\theta}_\phi(p)) - \mathcal{E} \leq \frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}_T} \mathbb{1}_{\mathfrak{S}I}(\underline{\theta}_\phi(p)) \leq \frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}_T} \mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p)) + \mathcal{E}$$

where

$$\mathcal{E} \ll_n \frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}_T} \sum_{\epsilon = \pm} |\mathfrak{S}|B^\epsilon(\underline{\theta}_\phi(p)) + \left(\frac{\log p}{\log T}\right)^{3/2} \left(1 + p^{A'M} T^{-1/4}\right).$$

Note that $\mathbb{1}_{\boldsymbol{\mathfrak{S}}I} \circ \rho = \mathbb{1}_{\mathfrak{S}I}$ (and $\underline{\theta}_\phi(p)$ is identified with $\rho(\underline{\theta}_\phi(p))$). Invoking (27) gives that

$$\frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}_T} \mathbb{1}_{\mathfrak{S}I}(\underline{\theta}_\phi(p)) - \int \mathbb{1}_{\boldsymbol{\mathfrak{S}}I}\, d\mu_p \ll \frac{1}{M} + \left(\frac{\log p}{\log T}\right)^{3/2} \left(1 + p^{A'M} T^{-1/4}\right). \tag{30}$$

The proof is completed with

$$M = \left[\frac{\log T}{8A' \log p}\right].$$

## 7. Proof of Theorem 1.3

By (7), we see that $\int \mathbb{1}_{\mathbf{S}I}\, d\mu_p = \mu_{\mathrm{ST}}(\mathbf{S}I) + O(p^{-1})$. Summing (30) over $p \le x$, we get

$$
\begin{aligned}
&\frac{1}{\#\mathcal{H}_T \pi(x)} \sum_{\phi\in\mathcal{H}_T} N_I(\phi; x) - \mu_{\mathrm{ST}}(\mathbf{S}I) \\
\ll\ & \frac{\log\log x}{\pi(x)} + \frac{1}{M} + \frac{1}{\pi(x)} \sum_{p\le x} \left(\frac{\log p}{\log T}\right)^{3/2} (1 + p^{A'M} T^{-1/4}) \\
\ll\ & \frac{\log\log x}{\pi(x)} + \frac{1}{M} + \left(\frac{\log x}{\log T}\right)^{3/2} + T^{-1/4} x^{A'M} \log x. \qquad (31)
\end{aligned}
$$

The proof is completed by taking

$$M = \left[\frac{\log T}{8A' \log x}\right].$$

## 8. Central Limit Behaviour

In [7, Section 4], we formulate a set-up to yield the central limit behaviour for the statistics of a family of objects. This section is devoted to recall this so as to show Theorem 1.4. Here we only need the case for one-dimensional Gaussian distribution.

Let $\{\mathcal{X}_x\}_{x\in(0,\infty)}$ and $\{\mathcal{T}_t\}_{t\in(0,\infty)}$ be two collections of finite sets such that (i) $\mathcal{X}_i \subseteq \mathcal{X}_j$ (resp. $\mathcal{T}_i \subseteq \mathcal{T}_j$) for $i \le j$, and (ii) both $\mathcal{X} = \bigcup_x \mathcal{X}_x$ and $\mathcal{T} = \bigcup_t \mathcal{T}_t$ are infinite sets. Besides we are given a family of objects $\{a_\phi(p) : \phi \in \mathcal{T}, p \in \mathcal{X}\}$ and a family of independent real random variables $\{\mathrm{A}_p : p \in \mathcal{X}\}$ over possibly different probability spaces. Suppose for some real constants $\mu$ and $\nu$,

(I) $\displaystyle \frac{1}{\sqrt{|\mathcal{X}_x|}} \sum_{p\in\mathcal{X}_x} \left|\mathbb{E}[\mathrm{A}_p] - \mu\right| \to 0$ as $x \to \infty$,

(II) $\displaystyle \frac{1}{|\mathcal{X}_x|} \sum_{p\in\mathcal{X}_x} \mathbb{E}[\mathrm{A}_p^2] \to \nu$ as $x \to \infty$,

(III) $\mathbb{E}[|\mathrm{A}_p|^r] \le c_0^r$ for all $r \ge 0$ and all $p \in \mathcal{X}$, for some constant $c_0 \ge 1$.

**Proposition 8.1.** *Let $a_\phi(p)$ and $\mathrm{A}_p$ be defined as above. Suppose the above conditions (I)-(III) for $\{\mathrm{A}_p\}$ hold, and there exists a function $T_{\mathrm{A}}(x)$ satisfying $T_{\mathrm{A}}(x) \to \infty$ as $x \to \infty$ such that for any $x > 0$ and any $t \ge T_{\mathrm{A}}(x)$,*

$$\frac{1}{|\mathcal{T}_t|} \sum_{\phi\in\mathcal{T}_t} \prod_{p\in\mathcal{X}_x} a_\phi(p)^{u_p} = \prod_{p\in\mathcal{X}_x} \mathbb{E}[\mathrm{A}_p^{u_p}] + O_a(|\mathcal{X}_x|^{-a/2-1}) \qquad (32)$$

*for any $u_p \in \mathbb{N}_0$ ($p \in \mathcal{X}$), where $a = \sum_p u_p$ and the implied constant depends at most on $a$. Define*

$$Z_x(\phi) = \frac{1}{\sqrt{|\mathcal{X}_x|}} \sum_{p\in\mathcal{X}_x} (a_\phi(p) - \mu). \qquad (33)$$

*If* $t = t(x) \geq T_{\mathrm{A}}(x)$*, then*

$$\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} h(Z_x(\phi)) \xrightarrow[x\to\infty]{} \frac{1}{\sqrt{2\pi(\nu-\mu^2)}}\int h(x)e^{-x^2/(2(\nu-\mu^2))}\,dx^{\dagger\dagger}$$

*for any bounded continuous function* $h:\mathbb{R}\to\mathbb{R}$*.*

## 9. Proof of Theorem 1.4

We start with two lemmas so as to prove Proposition 9.1, which leads to Theorem 1.4.

**Lemma 9.1.** *Suppose the number* $M$ *in Subsection 4.2 satisfies* $M \leq \pi(x)$*. We have*

$$(34)\quad \frac{1}{\#\mathcal{H}_T}\sum_{\phi\in\mathcal{H}_T}\Big(\sum_{p\leq x}|\mathfrak{S}|B^{\pm}\left(\underline{\theta}_\phi(p)\right)\Big)^2 \ll (M^{-2}+T^{-1/2}x^{A'M})\pi(x)^2$$

$$(35)\quad \frac{1}{\#\mathcal{H}_T}\sum_{\phi\in\mathcal{H}_T}\Big|\sum_{p\leq x}\left(\mathfrak{S}\boldsymbol{\alpha}^{+}(\underline{\theta}_\phi(p))-\mathfrak{S}\boldsymbol{\alpha}^{-}(\underline{\theta}_\phi(p))\right)\Big|^2 \ll (M^{-2}+T^{-1/2}x^{A'M})\pi(x)^2$$

*where the two implied constants and the positive constant* $A'$ *depend at most on* $n$*.*

*Proof.* By Theorem 2.1, for $\mathfrak{F},\mathfrak{G}\in\{\mathfrak{S}\boldsymbol{\alpha}^{\pm},|\mathfrak{S}|B^{\pm}\}$, and for any two primes $p\neq q$,

$$\frac{1}{\#\mathcal{H}_T}\sum_{\phi\in\mathcal{H}_T}\mathfrak{F}(\underline{\theta}_\phi(p))^2-\int\mathfrak{F}^2\,d\mu_p \ll T^{-1/2}p^{2AM},$$

$$\frac{1}{\#\mathcal{H}_T}\sum_{\phi\in\mathcal{H}_T}\mathfrak{F}(\underline{\theta}_\phi(p))\mathfrak{G}(\underline{\theta}_\phi(q))-\int\mathfrak{F}\,d\mu_p\int\mathfrak{G}\,d\mu_q \ll T^{-1/2}(pq)^{AM}.$$

Squaring out the sum over $p\leq x$, the left-side of (34) equals

$$\Sigma_1(x)+\mathcal{E}_1(x)$$

where

$$\begin{aligned}\Sigma_1(x) &= \sum_{p\leq x}\int\left(|\mathfrak{S}|B^{\pm}\right)^2 d\mu_p+\sum_{p\neq q\leq x}\int|\mathfrak{S}|B^{\pm}\,d\mu_p\int|\mathfrak{S}|B^{\pm}\,d\mu_q\\ \mathcal{E}_1(x) &\ll T^{-1/2}\sum_{p\leq x}p^{2AM}+T^{-1/2}\sum_{p\neq q\leq x}(pq)^{AM}.\end{aligned}$$

Clearly $\mathcal{E}_1(x)\ll T^{-1/2}x^{2AM}\pi(x)^2$ and, by Lemma 4.1 (1) and (27),

$$\Sigma_1(x)\ll M^{-1}\pi(x)+M^{-2}\pi(x)^2.$$

The assertion (34) is complete.

††We would take this opportunity to correct some typos in [7]: The factor $\frac{1}{2\pi}$ on the right-side of the equations in Theorem 4.1 (ii), Remark 4 (c) and (d) of p.177, and Theorem 4.2 of p.178 should be $\frac{1}{\sqrt{2\pi}}$.

By the unitary condition of $\underline{\pi}_\phi(p)$, Lemma 4.1 (3) implies $\mathfrak{F}(\underline{\theta}_\phi(p)) \in \mathbb{R}$. We obtain

$$\left|\sum_{p\le x}\left(\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p)) - \mathfrak{S}\boldsymbol{\alpha}^-(\underline{\theta}_\phi(p))\right)\right|^2 = S_{\phi,1}(x) - 2S_{\phi,2}(x)$$

where

$$\begin{aligned} S_{\phi,1}(x) &= \sum_{p\le x}\Big(\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p))^2 - 2\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p))\mathfrak{S}\boldsymbol{\alpha}^-(\underline{\theta}_\phi(p)) + \mathfrak{S}\boldsymbol{\alpha}^-(\underline{\theta}_\phi(p))^2\Big), \\ S_{\phi,2}(x) &= \sum_{p\ne q\le x}\Big(\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p))\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(q)) + \mathfrak{S}\boldsymbol{\alpha}^-(\underline{\theta}_\phi(p))\mathfrak{S}\boldsymbol{\alpha}^-(\underline{\theta}_\phi(q))\Big) \\ &\quad -2\sum_{p\ne q\le x}\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p))\mathfrak{S}\boldsymbol{\alpha}^-(\underline{\theta}_\phi(q)). \end{aligned}$$

After averaging over $\phi \in \mathcal{H}_T$ and assembling, the left-side of (35) equals

$$S_1(x) + S_2(x) + T^{-1/2}\mathcal{E}_2(x)$$

where

$$\begin{aligned} S_1(x) &= \sum_{p\le x}\int\left(\mathfrak{S}\boldsymbol{\alpha}^+ - \mathfrak{S}\boldsymbol{\alpha}^-\right)^2 d\mu_p \\ S_2(x) &= \sum_{p\ne q\le x}\int\left(\mathfrak{S}\boldsymbol{\alpha}^+ - \mathfrak{S}\boldsymbol{\alpha}^-\right) d\mu_p \int\left(\mathfrak{S}\boldsymbol{\alpha}^+ - \mathfrak{S}\boldsymbol{\alpha}^-\right) d\mu_q \\ \mathcal{E}_2(x) &\ll \sum_{p\le x} p^{2AM} + \sum_{p\ne q\le x}(pq)^{AM} \ll x^{2AM}\pi(x)^2. \end{aligned}$$

With Lemma 4.1 (1) and (27),

$$\begin{aligned} S_1(x) &\ll \sum_{p\le x}\sum_{\epsilon=\pm}\int\left|\mathfrak{S}\boldsymbol{\alpha}^\epsilon - \mathbb{1}_{\mathfrak{S}I}\right| d\mu_p \ll M^{-1}\pi(x) \\ S_2(x) &\ll M^{-2}\sum_{p\le x} 1 \ll M^{-2}\pi(x)^2. \end{aligned}$$

Then (35) follows. □

**Lemma 9.2.** *Let $M = M(x) := \lfloor\sqrt{\pi(x)}\log\log x\rfloor$. Suppose that $T = T(x)$ satisfies $\frac{\log T}{M\log x} \to \infty$ as $x \to \infty$. Then*

$$\lim_{x\to\infty}\frac{1}{\#\mathcal{H}_T}\sum_{\phi\in\mathcal{H}_T}\frac{1}{\pi(x)}\left|N_I(\phi;x) - \sum_{p\le x}\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p))\right|^2 = 0.$$

*Proof.* On $\mathbb{R}^n$, we have $\mathfrak{S}\boldsymbol{\alpha}^- - |\mathfrak{S}|B^- \le \mathbb{1}_{\mathfrak{S}I} \le \mathfrak{S}\boldsymbol{\alpha}^+ + |\mathfrak{S}|B^+$ (from (29)). With $N_I(\phi,x) = \sum_{p\le x}\mathbb{1}_{\mathfrak{S}I}(\underline{\theta}_\phi(p))$, we obtain that for $\phi \in \mathcal{H}_T \setminus \mathcal{H}^*_{T,p}$,

$$\left|N_I(\phi,x) - \sum_{p\le x}\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p))\right| \le \mathcal{E}_\phi(x)$$

where

$$\mathcal{E}_\phi(x) := \Big|\sum_{p\le x}\big(\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p)) - \mathfrak{S}\boldsymbol{\alpha}^-(\underline{\theta}_\phi(p))\big)\Big| + \sum_{\epsilon=\pm}\sum_{p\le x}|\mathfrak{S}|B^\epsilon(\underline{\theta}_\phi(p)).$$

For $\phi \in \mathcal{H}^*_{T,p}$, $N_I(\phi, x) = 0$, and thus

$$\sum_{\phi\in\mathcal{H}^*_{T,p}}\Big|N_I(\phi;x) - \sum_{p\le x}\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p))\Big|^2 \le \pi(x)\sum_{p\le x}\sum_{\phi\in\mathcal{H}^*_{T,p}}\Big|\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p))\Big|^2.$$

Hence we infer by Lemma 5.1 with $h = l = 2$ that

$$\begin{aligned} &\frac{1}{\#\mathcal{H}_T}\sum_{\phi\in\mathcal{H}_T}\frac{1}{\pi(x)}\Big|N_I(\phi;x) - \sum_{p\le x}\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p))\Big|^2 \\ (36)\qquad \le\ &\frac{1}{\#\mathcal{H}_T}\sum_{\phi\in\mathcal{H}_T}\frac{1}{\pi(x)}\mathcal{E}_\phi(x)^2 + \Big(\frac{\log x}{\log T}\Big)^{\frac{9}{4}}\Big(\pi(x) + T^{-\frac18}x^{2A'M}\pi(x)\Big).\end{aligned}$$

The second term tends to 0 as $x \to \infty$ because $\pi(x) = o((\log T)^2)$ and $M\log x = o(\log T)$. Using $(a+b)^2 \ll a^2 + b^2$, Cauchy-Schwarz's inequality and Lemma 9.1, the first term of (36) is

$$\ll M^{-2}\pi(x) + T^{-1/2}x^{A'M}\pi(x) \to 0 \qquad \text{as } x\to\infty.$$

The proof is thus complete. □

For $\phi \in \mathcal{H}_T$, we set

$$\mathfrak{Z}_x(\phi) = \frac{1}{\sqrt{\pi(x)}}\sum_{p\le x}\big(\mathfrak{S}\boldsymbol{\alpha}^+(\underline{\theta}_\phi(p)) - \mu_{\mathrm{ST}}(\mathfrak{S}I)\big).$$

**Proposition 9.1.** *Let $M = M(x) := \lfloor\sqrt{\pi(x)}\log\log x\rfloor$. Suppose that $T = T(x)$ satisfies $\frac{\log T}{M\log x} \to \infty$ as $x \to \infty$. Then*

$$\lim_{x\to\infty}\frac{1}{\#\mathcal{H}_T}\sum_{\phi\in\mathcal{H}_T}h(\mathfrak{Z}_x(\phi)) = \frac{1}{\sqrt{2\pi\upsilon}}\int h(x)e^{-x^2/(2\upsilon)}\,dx$$

*where $\upsilon = \mu_{\mathrm{ST}}(\mathfrak{S}I) - \mu_{\mathrm{ST}}(\mathfrak{S}I)^2$, for any bounded continuous function $h : \mathbb{R} \to \mathbb{R}$.*

*Proof.* We shall apply Proposition 8.1. Set $\mathcal{X}_x = \{p \le x : p \text{ primes}\}$, $\mathcal{T}_t = \mathcal{H}_t$ and $a_\phi(p) = \mathfrak{S}\boldsymbol{\alpha}(\underline{\theta}_\phi(p))$. Here and in the sequel, we suppress the superscript + in $\mathfrak{S}\boldsymbol{\alpha}^+$ for simplicity. For every prime $p$, we consider $T_0/\mathfrak{S}_n$ as a probability space with the probability measure $d\mu_p$. The function $\mathfrak{S}\boldsymbol{\alpha}(\underline{\theta})$ on $T_0/\mathfrak{S}_n$ induces a random variable, playing the role of $\mathrm{A}_p$. Put $\mu = \mu_{\mathrm{ST}}(\mathfrak{S}I)$. Then for $\ell = 1$ or 2, by (27) we have

$$\begin{aligned}\mathbb{E}[\mathrm{A}_p^\ell] &= \int\mathfrak{S}\boldsymbol{\alpha}^\ell\,d\mu_p = \int\mathbb{1}_{\mathfrak{S}I}\,d\mu_p + O(M^{-1}) \\ &= \int\mathbb{1}_{\mathfrak{S}I}\,d\mu_{\mathrm{ST}} + O(M^{-1} + p^{-1}).\end{aligned}$$

Hence Conditions (I) and (II) in Proposition 8.1 are fulfilled. As $0 \le \mathfrak{S}\boldsymbol{\alpha}(\underline{\theta}) \le 1$ for $\underline{\theta} \in \mathbb{R}^n$, it follows that $\mathbb{E}[|\mathrm{A_p}|^r] \ll 1$, yielding Condition (III).

It remains to check (32). For any $u_p \in \mathbb{N}_0$, we have

$$\frac{1}{\#\mathcal{H}_T} \sum_{\phi \in \mathcal{H}_T} \prod_{p \le x} \mathfrak{S}\boldsymbol{\alpha}(\underline{\theta}_\phi(p))^{u_p} \;=\; \prod_{p \le x} \int \mathfrak{S}\boldsymbol{\alpha}^{u_p}\, d\mu_p + O(T^{-1/2} \prod_{p \le x} p^{AMu_p}).$$

The main term equals $\prod_{p \le x} \mathbb{E}[\mathrm{A}_p]$. The $O$-term is $\ll T^{-1/2} \exp(aAM \log x)$ where $a = \sum_{p \le x} u_p$. Thus the $O$-term is $\ll_a x^{-a/2-1}$ for $T \ge T_{\mathrm{A}}(x)$ if the function $T_{\mathrm{A}}(x)$ is chosen so that $T_{\mathrm{A}}(x)/(M \log x) \to \infty$ as $x \to \infty$. □

Renormalizing the integral in Proposition 9.1 to the standard Gaussian density function $\frac{1}{\sqrt{2\pi}} e^{-t^2/2}$ completes the proof of Theorem 1.4.

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