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0. Introduction

This is a set of notes from a course given May–June 2000 at Hong Kong University on representation theory of real and p-adic groups. I would like to thank the mathematics department and particularly A. Borel and N. Mok for their hospitality.

The topics covered were:

1. Induced Modules.

2. Basic Results about $(\mathfrak{g}, K)$–modules.

3–6. Basics of representation theory of p–adic groups.

7. The Jacquet functor.

8. Cuspidal representations.

9–12. Iwahori–Hecke algebras.

13. Langlands parametrization of irreducible admissible modules.

14. Some remarks about Arthur's conjectures.

The notes are very informal. Most of the time I did not present full proofs and many results are not properly attributed. The reader consult the references which have complete proofs and ample historical comments. The note are only meant to be a guide to the various algebraic ideas used in representation theory. I have tried whenever possible to present results that are similar for both real groups and p–adic groups. More details are given in the p–adic case because the proofs are more elementary. The theme (to the extent that there is one) is the classification of irreducible representations and their parametrization following the ideas of Langlands.

Section 1 presents induction and its basic properties, in particular Harish–Chandra induction. Section 2 summarizes the basic properties of $(\mathfrak{g}, K)$ modules. The basic references are [Wallach] and [Knapp]. The focus is on the subrepresentation theorem which has a counterpart in the p–adic case. Sections 3–6 develop the basic properties of smooth representations. The basic references are [Bernstein] and [Casselman]. Most of the details follow them closely. An excellent summary can be found in [Cartier]. Sections 7–8 are along the same lines; The main theorem is that any irreducible representation is a submodule of an induced from a supercuspidal representation.

One major technique for studying representations of p–adic groups is to relate them to representations of Iwahori–Hecke type algebras associated to open compact groups. In sections 9–12 we present two special cases; maximal compact subgroups so that the group has an Iwasawa decomposition and Iwahori subgroups. Here we mostly illustrate the results by various examples. The main references are [Lusztig] and [Satake].

Section 13 introduces the Langlands parametrization for real groups. The basic reference is [ABV]; many of the results go back to work of Langlands, Shelstad–Langlands and Knapp–Zuckerman.

Section 14 illustrates Arthur's conjectures about the residual spectrum. The interested reader should consult [MW] and [M].

Inevitably there are plenty of errors. Please let me know what you find so I can fix them.

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1. Induced Representations

1.1. Let $H \subseteq G$ be a closed subgroup. A representation (ρ, V) is a group homomorphism

$$\rho : H \to Aut(V)$$

where V is a vector space. Typically we will assume that V is a Hilbert space and the representation is continuous.

Definition. *The induced representation $(\iota(\rho), V_{\iota(\rho)})$ is defined as*

$$V_{\iota(\rho)} := \{f : G \longrightarrow V \mid f(gh) = \rho(h^{-1})f(g)\}$$

with the action given by $\iota(g)f(x) := f(g^{-1}x)$.

An intertwining map $a : (\rho, V) \to (\pi, W)$ induces an intertwining map

$$\iota(a) : V_{\iota(\rho)} \to W_{\iota(\pi)}$$

by the formula $\iota(a)f := a \circ f$.

Proposition. *Induction is a functor*

$$Ind_H^G \; : \; \{Representations\ of\ H\} \to \{Representations\ of\ G\}.$$

We can add conditions to $V_{\iota(\rho)}$ such as asking that the functions f be continuous, C^∞, C^ω, or just measurable.
If $H \subseteq G$ is a subgroup, then there is a another functor

$$Res_H^G \; : \; \{\text{Representations of } G\} \;\to\; \{\text{Representations of } H\}.$$

This is simply the restriction of a representation of G to H.

Theorem. *(Frobenius Reciprocity) Let (π, W) be a representation of G and (ρ, V) be a representation of H. Then there is a natural isomorphism*

$$Hom_G\left(W, Ind_H^G(V)\right) \simeq Hom_H\left(Res_H^G\, W, V\right).$$

Proof. An element of the left hand side is an assignment $F : w \mapsto f_w$ such that

$$f_{\pi(g)w}(x) = [\iota(g)f_w](x) = f_w(g^{-1}x), \qquad f_w(gh) = \pi(h)^{-1}f_w(g).$$

Now consider the map $\Phi : F \mapsto \Phi_F$ where $F \in Hom_G[W, Ind_H^G(V)]$,

$$\Phi_F(w) := f_w(\mathbb{1}).$$

It satisfies

$$\Phi_F : W \to V, \quad \Phi_F(\pi(h)w) = f_{\pi(h)w}(\mathbb{1}) = f_w(h^{-1}) = \rho(b)[f_w(\mathbb{1})] = \rho(h)\Phi_F(w).$$

We show that Φ is an isomorphism:

(1) Φ *is* $1-1:$ Say $\Phi_F \equiv 0$. Then $f_w(\mathbb{1}) = 0$ for all $w \in W$, so

$$f_{\pi^{-1}(g)w}(\mathbb{1}) = 0 \Longrightarrow f_w(g) = 0 \quad \forall_{g,w} \quad \text{so} \quad f_w \equiv 0 \quad \forall w.$$

(2) Φ is onto: Suppose $\varphi \in \mathrm{Hom}_H(W, V)$. Define $F_\phi : W \to I_H^G(V)$ by the formula

$$F_\varphi(w)(g) = f_w(g) = \varphi\big(\pi(g^{-1})w\big),$$

(Recall W is a representation of G). Then

$$f_w(gh) = \varphi\big(\pi(h)^{-1}\pi(g)^{-1}w\big) = \pi(h)^{-1}\varphi\big(\pi(g)^{-1}w\big).$$

By definition,

$$f_{\pi(g)w}(x) = \varphi\big(\pi(x)^{-1}\pi(g)w\big) = f_w(g^{-1}x) = [\iota(\rho)(g)f_w]\,(x).$$

Exercise. Show that Φ_{F_φ} equals φ.

1.2. Compact Groups. Asssume the Haar measures dg, dh are normalized so that the volumes of the groups are 1. Then

(1) G/H carries a G-invariant measure. If $\varphi \in C_c^\infty(G/H)$, then let

$$f_\varphi(g) := \varphi(gH) \in C_c(G).$$

Then f_φ satisfies $f_\varphi(gh) = f_\varphi(g)$ and we define

$$\int_{G/H} \varphi(\dot{g})d\dot{g} := \int_G f_\varphi(g)dg.$$

(2) If $(\rho, \mathcal{H})$ is a representation on a Hilbert space $\mathcal{H}$, we can always assume it is unitary. This is called Weyl's unitary trick. If $(\ ,\ )$ is any inner product on $\mathcal{H}$, define

$$< v, w >= \int_G (\pi(g)v,\ \pi(g)w)dg.$$

We will often impose the condition that the functions in $V_{\iota(\rho)}$ be L^2. The induced representation is unitary in a *canonical* way:

$$< f_1, f_2 >:= \int_G f_1(g)\overline{f_2(g)}dg. \tag{1.2.1}$$

We call this L^2–induction.

Exercise. Show that (1.2.1) is an invariant inner product. $\square$

Exercise. Let $G := SU(2)$, and H be the diagonal subgroup. The irreducible representations of G are realized as

$$V_n := span\ \{x^k y^{n-k}\}$$

the space of homogeneous polynomials of degree n. The action is

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{C}) \quad \pi_n(g)x^k y^{n-k} := (ax+by)^k(cx+dy)^{n-k}$$

(1) Show that (π_n, V_n) is irreducible.
(2) Compute the restriction of π_n to H.
(3) Let $\chi_n : H \to \mathbb{C}$ be $\chi_n \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = z^n$. Compute $Ind_H^G(\chi_n)$. $\square$

Exercise. Let now

$$G = SL(2,\mathbb{R}) = \left\{ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad \det g = 1 \right\}.$$

Show that $SL(2,\mathbb{R}) \simeq SU(1,1)$. Let

$$H = P = \begin{pmatrix} a & x \\ 0 & 0^{-1} \end{pmatrix}$$

Show that P is <u>not</u> unimodular. Compute the modular function δ_P.
Hint: $\begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} \alpha & Y \\ 0 & -\alpha \end{pmatrix} \begin{pmatrix} a^{-1} & -x \\ 0 & a \end{pmatrix} = \begin{pmatrix} \alpha & a^2 Y \\ 0 & -\alpha \end{pmatrix}$ $\square$

1.3. Principal Series. Let $\chi : \mathbb{R}^\times \to \mathbb{C}$ be a character:

$$a \quad \mapsto \quad |a|^\nu (sgn\, a)^\varepsilon \quad \varepsilon = 0,1\ ,\ \nu \in \mathbb{C}$$

This extends to a character of P:

$$\chi_{(\nu,\varepsilon)} \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix} = |a|^\nu (sgn\, a)^\varepsilon$$

Consider the induced module $Ind_p^G(\chi_{(\nu,\varepsilon)})$. We want to analyze this as a module. Recall $K = SO(2) \subseteq SL(2,\mathbb{R})$ the maximal compact subgroup. Then

$$G = K \cdot P \quad \text{and} \quad K \cap P = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}$$

We will work with the K-finite part of Ind; call it $X(\nu,\varepsilon)$. Then

$$G/P \simeq K/K \cap P \simeq K/M \cap K$$

where $P = MAN$ with

$$M = \begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \quad A = \{a(t) \mid a(t) = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}\},$$
$$N = \{n(x) \mid n(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\}.$$

The principal series is

$$X(\nu,\varepsilon) = \{f : G \to \mathbb{C} \mid f(gp) = \chi_{\nu,\varepsilon}(ma)^{-1} f(g)\} \simeq$$
$$\{f : K \to \mathbb{C} \mid f(k\ m) = \chi_{\nu,\varepsilon}(m^{-1}) f(k)\} \simeq \bigoplus_{\mu \in \widehat{K}} m(\mu) V_\mu.$$

The representation V_μ is the character $\chi_\mu(r(\theta)) = e^{i\mu\theta}$ where μ is an integer. We write

$$r(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

Exercise. Show that if $\varepsilon = 0$ only the V_μ with μ even occur, and in this case $m(\mu) = 1$. If $\varepsilon = 1$, then only V_μ with μ odd occur, and $m(\mu) = 1$. $\square$

We describe this decomposition of the principal series explicitly. We start with the decomposition $G = KP$. Every g can be decomposed into $g = r(\theta)a(t)n(x)$ as follows. We use the (conformal) action of G on the upper half plane.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \alpha & x \\ 0 & \alpha^{-1} \end{pmatrix} \qquad \alpha > 0$$
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \infty = \frac{a\infty + b}{c\infty + d} = \frac{a}{c} = \frac{\cos\theta}{\sin\theta}$$
$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} i = \frac{di - b}{-ci + a} = \frac{\alpha^{-1} i - x}{\alpha} = \alpha^{-2} i - \alpha^{-2} x$$
$$\frac{(di - b)(ci + a)}{a^2 + c^2} = \frac{-(ab + dc) + i}{a^2 + c^2}$$

so

$$\alpha = \sqrt{a^2 + c^2}, \quad x = -(ab + dc), \quad \cos\theta = \frac{a}{\sqrt{a^2 + c^2}}, \quad \sin\theta = \frac{c}{\sqrt{a^2 + c^2}}.$$

Then

$$f_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \left(\frac{a + ic}{\sqrt{a^2 + c^2}} \right)^n (a^2 + c^2)^{\frac{\nu}{2}} = \frac{(a + ic)^{\frac{n}{2} + \frac{\nu}{2}}}{(a - ic)^{\frac{n}{2} - \frac{\mu}{2}}}$$

The Lie algebra $\mathfrak{g} = sl(2,\mathbb{R})$ is generated by

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

satisfying the relations

$$[H,E]=2E, \quad [H,F]=2F, \quad [E,F]=H.$$

If $X\in\mathfrak{g}$, and (π,V) is a representation, then the action on a vector $v\in C^\infty(V)$ is given by

$$X\cdot v=\left.\frac{d}{dt}\right|_{t=0}\pi(\exp(-tX))\cdot v$$

A basis for X is given by the functions $f_n(r(\theta)a(t)n(x))=e^{in\theta}\cdot e^{-t\nu}$. We can compute

$$\begin{aligned}
(H\cdot f_n)(\mathbb{1})&=\left.\frac{d}{dt}\right|_{t=0}f_n\begin{pmatrix}e^{-t}&0\\0&e^t\end{pmatrix}=\left.\frac{d}{dt}\right|_{t=0}e^{t\nu}=\nu\\
E\cdot f_n(\mathbb{1})&=\left.\frac{d}{dt}\right|_{t=0}f_n\begin{pmatrix}1&-t\\0&1\end{pmatrix}=0\\
F\cdot f_n(\mathbb{1})&=\left.\frac{d}{dy}\right|_{y=0}f_n\begin{pmatrix}1&0\\-y&0\end{pmatrix}=\left.\frac{d}{dy}\right|_{y=0}\left(\frac{1-iy}{\sqrt{1+y^2}}\right)^n\cdot(1+y^2)^{\frac{\nu}{2}}\\
&=\left.\frac{d}{dy}\right|_{y=0}(1-iy)^{\frac{n}{2}+\frac{\nu}{2}}(1+iy)^{-\frac{n}{2}+\frac{\nu}{2}}\\
&(-i)(\frac{n}{2}+\frac{\nu}{2})+i(-\frac{n}{2}+\frac{\nu}{2})=-in
\end{aligned}$$

because when $g=\begin{bmatrix}1&0\\y&1\end{bmatrix}$,

$$\cos\theta=\frac{1}{\sqrt{1+y^2}}, \quad \sin\theta=\frac{-y}{\sqrt{1+y^2}}, \quad t=\sqrt{1+y^2}.$$

Since the f_n are eigenfunctions for $E-F$, it is better to use the relations

$$\begin{aligned}
(iH+E+F)f_n&=(\nu-n)i, \qquad (E-F)f_n=-in,\\
(-iH+E+F)f_n&=(-\nu-n)i.
\end{aligned}$$

The point is that

$$[E-F,iH+E+F]=-2i(iH+E+F), \quad [E-F,-iH+E+F]=2i(-iH+E+F)$$

so that $(iH+E+F)f_n$ is a multiple of f_{n-2} and $(-iH+E+F)f_n$ is a multiple of F_{n+2}. Precisely,

$$(iH+E+F)f_n=i(\nu-n)f_{n-2}, \qquad (-iH+E+F)f_n=i(-\nu-n)f_{n+2}.$$

Consider the case $\varepsilon=0$. The module is only reducible at $\nu=2k$ an even integer. If $k\geq 0$, then $X(2k,0)$ has a quotient which is the finite dimensional representation and a submodule which is the direct sum of two discrete series. For negative integers we get a similar picture, but the composition series is reversed, the finite dimensional is a submodule and the sum of discrete series is a quotient.

We can make the picture more symmetric if we change ν to $\nu-1$; then $X(\nu,\varepsilon)$ and $X(-\nu,\varepsilon)$ are dual to each other and $X(-\overline{\nu},\varepsilon)$ is hermitian dual to $X(\nu,\varepsilon)$. Precisely we define $X(\nu,\varepsilon)$ to be formed of functions which satisfy $f(ga(t))=e^{-t(\nu-1)}f(g)$. This kind of shift has the effect that induction takes unitary modules to unitary modules.

1.4. The bundle corresponding to an induced representation. Let (ρ, V) be a representation of H. We define a vector bundle $\mathcal{E}$ as follows.

$$\mathcal{E} := G \times_H V = \{(g, v)\}/\left\{(g, v) \sim (gh, h^{-1}v)\right\},$$
$$\pi : \mathcal{E} \to G/H, \qquad \pi(g, v) = gH.$$

Proposition. *Sections of $\mathcal{E}$ are precisely the functions in $Ind_H^G(\rho)$.*

Proof. Exercise. □

1.5 Unitarity. We would like to say that induction takes unitary representations to unitary representations. It is not even clear that $I_H^G(\rho)$ forms a Hilbert space. As we saw for compact groups we need integration.

Recall the modular function δ_G defined by

$$\int_G f(xg)dx = \delta_G(g) \int_G f(x)dx.$$

Theorem. *$X = G/H$ has a G–invariant measure if and only if $\delta_H(h) = \delta_G(h)$ for all $h \in H$.*

Proof. Consider the map $f \mapsto \varphi_f$ from $C_c(G)$ to $C_c(X)$,

$$\varphi_f(gH) = \int_H f(gh)dh.$$

If μ is G-invariant on X, then

$$\widetilde{\mu}(f) := \mu(\varphi_f)$$

is a left invariant Haar measure, so

$$\mu(R_h f) = \delta_G(h)\mu(f).$$

But also

$$\varphi_{R_h f}(g) = \int_H (R_h f)(gh')dh' = \int_H f(gh'h)dh' = \delta_H(h)\varphi_f(g).$$

So $\delta_G(h) = \delta_H(h)$. To establish the converse, we need the following lemma.

Lemma. *The map $f \mapsto \varphi_f$ is onto.*

Sketch of Proof. Let $F \in C_c(X)$ with C=supp F Let $\widetilde{C}$ be open relatively compact so that $\pi(\widetilde{C}) \supset C$, and $\psi \in C_c(G)$ $\quad \psi \equiv 1$ on $\widetilde{C}$, $\psi \geq 0$. Then

$$f(g) = \begin{cases} F(gH)\frac{\psi(g)}{\varphi_\psi(g)} & g \in \widetilde{C} \\ 0 & g \notin \widetilde{C} \end{cases}$$

satisfies $\varphi_f = F$. $\square$

Let $\psi \subset C_c^\infty(G)$ be arbitrary. Then the formula

$$\int_G \psi(g) \int_H f(gh)dh\, dg = \int_G \varphi_\psi(g) f(g) dg$$

holds. We can choose $\psi's$ so that $\varphi_\psi \to \mathbb{1}$. Thus if $\varphi_f \equiv 0$ then $\int_G f(g)dg = 0$. This allows us to set

$$\mu(F) := \int_G f(g)dg$$

where f is any function such that $\varphi_f = F$. More details can be found in [Helgason]. $\square$

1.6 Harish-Chandra induction. We are interested in the following situation which we will call Harish–Chandra induction.

G is a reductive (linear) group over $\mathbb{R}$ with rational points $G_\mathbb{R}$ and Lie algebra $\mathfrak{g}$. The Cartan involution is denoted θ and the Cartan decomposition is $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$. The complexifications of any algebra $\mathfrak{h}$ will be denoted by $\mathfrak{h}_c$.

Let $\mathfrak{a} \subset \mathfrak{s}$ be a maximal abelian subalgebra. Let $P = MN$ be a minimal parabolic subgroup such that the split component of the centralizer of M is A with Lie algebra $\mathfrak{a}$. Since M is θ-stable, it decomposes as $M = M^\circ A$, where $M^\circ = M \cap K$. Let $\sigma \in \widehat{M_K}$ and $\nu \in \widehat{A}$. Then $\sigma \otimes \nu$ is a character of M. We can *inflate* it to P as follows. Since N is the unipotent radical of P, it is a normal subgroup. So there is a group homomorphism

$$P \longrightarrow P/N \cong M \longrightarrow 1,$$

and $\sigma \otimes \nu$ can be pulled back to a representation $\widetilde{\sigma \otimes \nu}$ on which N acts trivially.

Exercise. Show that the modular function of P is

$$\delta_P(man) = |det(\mathrm{Ad}\, a)|_\mathfrak{n}|, \qquad m \in M \cap K$$

where $\mathfrak{n} = Lie(N)$. If $\Delta(\mathfrak{n}, \mathfrak{a})$ are the roots of $\mathfrak{a}$ on $\mathfrak{n}$, and $2\rho = \sum_{\alpha \in \Delta(\mathfrak{n},\mathfrak{a})} \alpha$, then

$$\delta_P(a) = e^{2\rho(\log a)}. \quad \square$$

Definition. *The Harish-Chandra induced module from* (σ, ν) *is*

$$X(\sigma, \nu) := Ind_P^G[\sigma \otimes \widetilde{(\nu - \rho)}].$$

Exercise. Generalize this construction to the case of a standard parabolic subgroup $Q = LU$. $\square$

In this situation, $X = G/Q$ is compact. But since G *is* unimodular and Q *is not,* there is no way we can use the previous results; usual induction *does not* preserve unitarity.

1.7. Densities. We go back to the situation of of a closed subgroup $H \subset G$ and $X := G/H$. Let (σ, V) be a unitary representation. Then there is a bundle map

$$\mathcal{E} \times \mathcal{E} \longrightarrow \mathcal{O}$$

where $\mathcal{O}$ is the trivial bundle. This is simply the inner product map taking

$$(g, v) \times (g, w) \mapsto (g, < v, w > .$$

But while we cannot integrate functions, we can integrate densities. These are sections of the bundle $\mathcal{D}$ corresponding to $\delta = \delta_H/\delta_G$.

Definition. *We call the normalized induced module from σ, the induced module*

$$N - Ind_H^G(\sigma) := Ind_H^G[\sigma \otimes \delta^{1/2}].$$

We will mostly use this type of induction. Then we get a map

$$\mathcal{E} \cdot \mathcal{D}^{1/2} \times \mathcal{E} \cdot \mathcal{D}^{1/2} \longrightarrow \mathcal{D}.$$

If we have two sections $f_1,\ f_2$ of $\mathcal{E} \otimes \mathcal{D}^{1/2}$ we can define

$$< f_1, f_2 >:= \int_X < f_1(x), f_2(x) > dx.$$

The idea of why we can integrate densities is the same as in theorem 1.5. So in our case we find $\delta(a) = |\det(Ada|_{\mathfrak{n}})|^{-1}$ so we set

$$X(\sigma, \nu) := Ind_P^G\, [\sigma \otimes \nu - \rho]_{K-\text{finite}}$$

This accounts for shifting from ν to $\nu - 1$ in the example of $SL(2)$. For normalized induction, $\sigma \otimes \nu$ unitary implies $X(\sigma, \nu)$ unitary.

We now describe why densities can be integrated. Recall $\delta(h) := \frac{\delta_H(h)}{\delta_G(h)}$. If $F \in \Gamma_c(G/H, \mathcal{D})$ then integration makes sense. Indeed, define a map

$$\varphi : C_c(G) \to \Gamma(X, \mathcal{D}) \qquad f \mapsto \varphi_f(g) := \int_G f(gh)\delta(h)dh.$$

Theorem.

(1) φ *is* onto $\Gamma_c(X, \mathcal{D})$.

(2) *If* $f \in C_c(G)$ *is such that* $\varphi_f \equiv 0$ *then* $\int\limits_G f(x)dx = 0$

The theorem allows us to define

$$\int_X F := \int_G f \; dg$$

for any density F.

Proof of the theorem. The proof of (1) is like lemma 1.5 except the definition of φ_f is as in (1.7.1). For (2) we first observe that

$$\begin{aligned}\varphi_f(gh') &= \int\limits_G f(g\overbrace{h'h}^{u})\delta(h)dh = \int\limits_G f(gu)\delta({h'}^{-1}h)dh = \\ &= \delta(h')^{-1}\varphi_f(g).\end{aligned} \tag{1.7.1}$$

Let $\psi \in C_c(G)$ be arbitrary. Assuming $\varphi_f(g) = 0$ for all $g \in G$,

$$\begin{aligned}0 &= \int\limits_G \psi(g)\varphi_f(g)dg = \int\limits_G \psi(g) \int\limits_G f(gh)\delta(h)dh \; dg = \\ &= \int\limits_H \int\limits_G \psi(g)f(gh)\delta(h)dg \; dh = \int\limits_H \int\limits_G \psi(uh^{-1})f(u)\delta_G(h)\delta(h)du \; dh \\ &= \int\limits_G f(u) \int\limits_H \psi(uh^{-1})\delta_H(h)dh \; du = \int\limits_G f(u)(\int\limits_H \psi(uh)dh)du = \int_G f(u)\varphi_\psi(u)du.\end{aligned}$$

As before, choose ψ's so that $\phi_\psi \to \mathbb{1}$. Then

$$\int\limits_G f(u)du = 0. \quad \square$$

1.8. Frobenius Reciprocity. Let W be a $(\mathfrak{g}, K)$ module. Then

$$Hom_{(\mathfrak{g},K)}\left[W, X(\sigma,\nu)\right] \simeq Hom_{(\mathfrak{m},M\cap K)}\left[W/\mathfrak{n}W, \; \sigma\otimes\nu-\rho\right].$$

The $(\mathfrak{m}, M\cap K)$ module $W/\mathfrak{n}W$ is called the Jacquet module of W.

2. $(\mathfrak{g}, K)$ modules

We will summarize some of the basic properties. Details are in chapter 3 of [Wallach] volume 1, [Knapp] and [Vogan].

2.1. Definition. *a module for both $U(\mathfrak{g}_c)$ and K is called a $(\mathfrak{g}_c, K)$–module if the following conditions are satisfied.*

(1) $k \cdot X \cdot v = \mathrm{Ad}(k)X \cdot k \cdot v$,
(2) *dim span*$K \cdot v < \infty$,
(3) $\frac{d}{dt}|_{t=0} \exp(t, Y) \cdot v = Y \cdot v$ *for* $Y \in k$.

A $(\mathfrak{g}, K)$ module (π, V) is called finitely generated (f.g.) if it is so as an $U(\mathfrak{g}_c)$-module. It is called irreducible, if it has no nontrivial proper submodules.

Schur's Lemma. *If (π, V) is irreducible, then*

$$Hom_{\mathfrak{g}_c, K}[V, V] = \mathbb{C}Id.$$

Given $\mu \in \widehat{K}$, we can define a projection $P_\mu : V \to V$

$$P_\mu(v) := \int_K \chi_\mu(k)\pi(k)v dk.$$

The image $V(\mu)$ is the sum of all submodules K-isomorphic to V_μ.

Then $V = \bigoplus_{\mu \in \widehat{K}} V(\mu)$ (algebraic sum).

Definition. *We say V is admissible if* $\dim V(\mu) < \infty$. *This is the same as*

$$\dim Hom_K[V_\mu, V] < \infty.$$

In general, if $(\pi, \mathbb{H})$ is a Hilbert space representation, $\mathbb{H}_K$ is a $(\mathfrak{g}_c, K)$–module.

Theorem. *If V is finitely generated, then every $V(\mu)$ is finitely generated as a $Z(\mathfrak{g}_c)$–module.*

The main reason this is true is because of a result of Kostant–Rallis:

$$S(\mathcal{S}) = \mathcal{H} \otimes S(\mathcal{S})^K$$

where $\mathcal{S} = exp\ \mathfrak{s}$ and $\mathcal{H}$ is a subspace called the space of harmonics. Its main property that we need is that $\dim Hom_K[V_\mu : \mathcal{H}] < \infty$ for any $\mu \in \widehat{K}$.

Let V be a $(\mathfrak{g}_c, K)$ module generated by the finite dimensional K-invariant set W. Set $V_j := U(\mathfrak{g}_c)_j W$ and $Gr(V) := \oplus V_j / V_{j-1}$. This space inherits an $\mathcal{S}(\mathfrak{g}_c) = GrU(\mathfrak{g}_c)$ action. In this action, $S(\mathfrak{k})$ acts trivially. So $Gr(V)$ has a $\mathcal{S}(S)$–action only. The action of K does *not* become trivial, in fact

$$V \simeq Gr(V) \quad \text{as} \quad K - \text{modules}$$

and

$$k \cdot u \cdot v = (\mathrm{Ad}\, ku) \cdot kv.$$

So as a K–module, V is a quotient of

$$S(\mathcal{S})^K \otimes \mathcal{H} \otimes W.$$

But $Hom_K[V_\gamma, \mathcal{H} \otimes W] \simeq Hom_K[V_\gamma \otimes W^*, \mathcal{H}]$ and this is finite dimensional.

The action of $Z(\mathfrak{g}_c)$ *factors* through $S(\mathcal{S})^K$ and $S(\mathcal{S})^K$ is finitely generated over the image of $Z(\mathfrak{g}_c)$ in $S(\mathcal{S})$.

This completes the sketch of the proof of the theorem. $\square$

Exercise. Let $\mathfrak{g}_c = sl(2, \mathbb{C})$. Show that

$$S(\mathfrak{g}_c) \simeq \mathcal{H} \otimes S(\mathfrak{g}_c)^G$$

where $\mathcal{H} \cong \oplus V_\mu$ with V_μ the odd–dimensional irreducible representations each occuring with multiplicity one.

2.2. Consequences.

(1) Suppose $Z(\mathfrak{g}_c)$ acts on V by a character (V has an infinitesimal character). Then V is admissible.
(2) Suppose V is irreducible. Then V is admissible.
(3) A representation of G on a Hilbert space $\mathbb{H}$ is called admissible if $\mathbb{H}_K$ is admissible. $(\pi, \mathbb{H})$ unitary, is irreducible if and only if $(\pi, \mathbb{H}_K)$ is irreducible.
(4) Two unitary representations π and σ satisfy $(\pi, \mathbb{H}_K) \simeq (\sigma, \mathbb{H}'_K) \iff (\pi, \mathbb{H}) \sim (\sigma, \mathbb{H}')$

2.3. The Jacquet module. Let $P = MN$ be the minimal parabolic subgroup, and $\mathfrak{n} := Lie(N)$.

Theorem. *If V is admissible, then $V/\mathfrak{n}V \neq 0$ is finite dimensional.*

Corollary. *For any admissible irreducible module (π, V), there is a $\sigma \in \widehat{M}^\circ$ and $\nu \in \widehat{A}$ such that V embeds in $X(\sigma, \nu) := Ind_P^G(\widetilde{\sigma \otimes \nu})$.*

Proof. For any $\nu \in \widehat{A}$, let $\mathcal{J}(V)_\nu$ be the generalized ν–eigenspace. Since $\mathcal{J}(V)$ is finite dimensional, there is ν such that this eigenspace is nonzero. Consider the largest quotient on which A acts by ν. This is nonzero, finite dimensional and M° acts on it. Being reductive, M° acts semisimply, so there is $\sigma \in \widehat{M}^\circ$ such that $\sigma \otimes \nu$ is a quotient of $\mathcal{J}(V)$. Then Frobenius reciprocity implies

$$Hom_G[V, X(\sigma, \nu)] \simeq Hom_M[\mathcal{J}(V), \sigma \otimes \nu] \neq 0. \quad \square$$

Nedless to say, we can shift ν by ρ in the above formula.

The most difficult part of the proof is to show that the Jacquet module is nonzero. Here is a sketch assuming that V is the $(\mathfrak{g}_c, K)$ module of $(\pi, \mathbb{H})$ (which can be proved independently). Consider the functions $f_{v,w}(g) =< \pi(g)v, w >$ which are called matrix entries. They are eigenfunctions of the center of the enveloping algebra. Explicitly, there is a character $\chi : Z(\mathfrak{g}_c) \to \mathbb{C}$ such that

$$z \cdot f_{v,w} = \chi(z) f_{v,w} \text{ for all } z \in Z(\mathfrak{g}_c). \tag{2.3.1}$$

On the other hand,

$$G = K \cdot A^+ \cdot K.$$

So if we fix the K–types of v, w, then $f_{v,w}$ is determined by its values on A. So (2.3.1) becomes an equation on A. Let

$$\rho := \frac{1}{2} \sum_{\alpha \in \triangle(\mathfrak{n},\mathfrak{a})} \alpha.$$

Let A^{++} be the regular elements in A^+. Any solution of the differential equation (2.3.1) (in particular the function $f_{v,w}$) can be expressed as a finite sum

$$\sum_i p_i(\log a) F_i(\nu_i - \rho)(\log a) \tag{2.3.2}$$

where p_i are polynomials and $F_i(\lambda)$ has a convergent power series expansion on A^{++} of the form

$$F_i(\lambda)(\log a) = \sum_{Q \in \Lambda} c_{i,Q} a^{\lambda - Q} \tag{2.3.3}$$

where Λ is the lattice spanned by the positive roots. The F_i are normalized so that $c_{i,0} = 1$ and $\nu_i - \nu_j \notin \Lambda$ for $i \neq j$. The ν_i for which the p_i are nonzero are called *leading terms*. Let $\overline{P} = M\overline{N}$ be the opposite parabolic subgroup. The theorem is proved by showing that these leading terms occur as weights in $V/(\mathfrak{n}^- V)$ as an A–module. Indeed choose w so that p_i is nonzero and denote by $\ell_{v,w}$ the coefficient of one of its top degree terms. Then fix w and view it as a functional

$$\ell : V \to \mathbb{C}, \qquad \ell(v) := \ell_{v,w}.$$

It is nonzero, vanishes on $\mathfrak{n}^- V$ and satisfies $\ell(\pi(H)v) = (\nu_i - \rho)(H)\ell(v)$.

The change from $\mathfrak{n}^-$ to $\mathfrak{n}$ is trivial (interchange V and its K–finite dual $\widetilde{V}$.

More details can be found in [Knapp]. Full details and references for the proof of theorem 2.3 can be found in [Wallach].

2.4. The Harish–Chandra homomorphism. Let $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$ be a θ–stable Cartan subalgebra containing $\mathfrak{a}$. Then $\mathfrak{t} \subset \mathfrak{m}^\circ$ is a Cartan subalgebra. The Harish–Chandra homomorphism is an algebra isomorphism

$$\gamma : \mathcal{Z}(\mathfrak{g}) \longrightarrow S(\mathfrak{h}_c)^{W_c} \tag{2.4.1}$$

where $W_c := N(\mathfrak{h}_c)/C(\mathfrak{h}_c)$. It is defined as follows. Fix a positive system $\triangle \subset R(\mathfrak{g}_c, \mathfrak{h}_c)$. The Lie algebra decomposes

$$\mathfrak{g}_c = \mathfrak{u}_c + \mathfrak{h}_c + \mathfrak{u}_c^-$$

where $\mathfrak{u}_c$ is spanned by the root vectors E_α and $\mathfrak{u}_c^-$ is spanned by the root vectors $E_{-\alpha}$ with $\alpha \in \triangle$. Then

$$U(\mathfrak{g}_c) = U(\mathfrak{h}_c) + U(\mathfrak{g}_c)\mathfrak{u}_c.$$

The projection

$$p : \mathcal{Z}(\mathfrak{g}_c) \longrightarrow U(\mathfrak{h}_c)$$

is well defined. Since $\mathfrak{h}_c$ is abelian, $U(\mathfrak{h}_c) \cong S(\mathfrak{h}_c)$. Let $\rho := \frac{1}{2}\sum_{\alpha\in\triangle}\alpha$ and

$$\gamma' : S(\mathfrak{h}_c) \longrightarrow S(\mathfrak{h}_c), \qquad H \mapsto H - \rho(H)\mathbb{1}.$$

Then $\gamma = \gamma' \circ p$. $\square$

As a result, an *infinitesimal character* χ can be identified with a W_c–orbit of a $\lambda \in \mathfrak{h}_c^*$ by the formula

$$\chi(z) = \gamma(z)(\lambda).$$

For (2.3.1) we need a different decomposition. Let $a \in A^{++}$. Then

$$\mathfrak{g}_c = \operatorname{Ad} a^{-1}(\mathfrak{k}_c) + \mathfrak{a}_c + \mathfrak{k}_c.$$

As a consequence, Harish-Chandra proved the following theorem.

Theorem. *For any differential operator $u \in U(\mathfrak{g}_c)$ there is an operator $r(u) \in U(\mathfrak{a}_c)$ such that for any K–biinvariant C^∞ function f,*

$$(u \cdot f)|_A = r(u) \cdot f|_A.$$

Example/Exercise. Consider the spherical principal series $X(0,\nu)$ for $SL(2,\mathbb{R})$. Show that the center of $Z(\mathfrak{g}_c)$ is a polynomial algebra in $z = H^2 + c(EF + FE)$ (find the constant c). Let v_o be the spherical vector. Consider the matrix entry

$$< \pi(g)v_o, v_o >=< \pi(a)v_o, v_o >= f_\nu(a).$$

Find the equation on A that corresponds to (2.3.1). $\square$

For $u \in \mathcal{Z}(\mathfrak{g}_c)$, the system of equations in theorem 2.4 has regular singularities. This accounts for the expansion in (2.3.2)–(2.3.3). In addition the particular form of $r(u)$ implies the following corollary

Corollary. *If the infinitesimal character of (π, V) is λ, then the leading exponents satisfy $\nu_i = w\lambda|_{\mathfrak{a}_c}$.*

This accounts for the fact that there are finitely many leading exponents.

2.5. Tempered representations. Let

$$\Pi = \{\alpha_1, \dots, \alpha_l\}$$

be the simple restricted roots in $\triangle$ and $\omega_1, \dots, \omega_l$ the dual weights.

Theorem. *An irreducible representation is a* discrete series *if and only if for every leading exponent* ν,

$$< \nu, \omega_j >< 0.$$

Main idea of the proof. The condition means that there are $m_j < 0$ so that

$$\nu = \sum m_j \alpha_j.$$

The claim follows from the integration formula

$$\int_G f(g)\, dg = \int_{K\times\mathfrak{a}\times K} f(k_1 e^H k_2) \prod_{\alpha\in\triangle} |e^{\alpha(H)} - e^{-\alpha(H)}|\, dk_1\, dk_2\, dH.$$

Definition. *We say that an irreducible representation* (π, V) *is tempered if all the leading exponents satisfy*

$$< \nu, \omega_j >\leq 0.$$

2.6. Intertwining operators. Let $Q = LU$ and $Q' = LU'$ be two standard associate parabolic subgroups. Decompose $L = L^\circ A_Q$ where A_Q is the split component of the center and L^0. Let (σ, V) be an admissible representation of L° and ν a character of A_Q. Write

$$X_Q(\sigma,\nu) := Ind_Q^G[\sigma\otimes\widetilde{(\nu-\rho_Q)}].$$

Then we can define an intertwining operator

$$I_{Q,Q'} : X_Q(\sigma,\nu) \longrightarrow X_{Q'}(\sigma,\nu)$$

by the formula

$$If(g) := \int_{U'/U\cap U'} f(gu)du. \tag{2.6.1}$$

The relation between the leading terms and the growth of the matrix coefficients shows that if σ is a tempered representation, then the integral (2.6.1) converges for $Re\nu$ very dominant. A nontrivial argument shows that in fact this integral extends analytically in ν in the region where

$$(Re\nu, \alpha) \geq 0, \text{ for all } \alpha \in \triangle(\mathfrak{u}, \mathfrak{a}_Q). \tag{2.6.2}$$

A particular case is provided by $Q' = \overline{Q}$, the opposite parabolic subgroup.

2.7. Theorem. *(Langlands classification). Assume* σ *is tempered irreducible and* ν *satisfies (2.6.1). Then* $X_Q(\sigma,\nu)$ *has a unique irreducible quotient and* $X_{\overline{Q}}(\sigma,\nu)$ *has a unique irreducible submodule. This module which we call* $L(\sigma,\nu)$ *is characterized by the property that it is the image of* $I_{Q,\overline{Q}}$.
Every irreducible (π, V) *is of the form* $L(\sigma,\nu)$. *Two modules* $L(\sigma,\nu)$ *and* $L(\sigma',\nu')$ *are equivalent if and only if there is* $g\in G$ *so that*

$$(L', \sigma', \nu') = g\cdot(L,\sigma,\nu).$$

3. P–adic Groups

By a *local field* $\mathbb{F}$ we mean a a nonarchimedean local field with residue field $\mathbb{F}_q$. Then

$$\mathbb{F} \supset \mathcal{O}_{\mathbb{F}} \supset \mathcal{P}_{\mathbb{F}}$$

where $\mathcal{O}_{\mathbb{F}}$ is the ring of integers and $\mathcal{P}_{\mathbb{F}}$ is the unique nonzero prime ideal. Let ϖ (or π) be an element such that

$$\mathcal{P}_{\mathbb{F}} = \pi\mathcal{O}_{\mathbb{F}}.$$

Every element x in $\mathbb{F}$ is of the form (uniquely)

$$x = \pi^n r \text{ with } r \in \mathcal{O} \setminus \mathcal{P}_F.$$

We write

$$ord_F(x) = n, \qquad ord_F(0) = \infty, \qquad |x|_{\mathbb{F}} := q^{-n}.$$

According to the main classification theorem of such fields,

(A) $\mathbb{F}$ is a finite algebraic extension of $\mathbb{Q}_P$

(B) $\mathbb{F}$ is the quotient field of formal power series $\mathbb{F}_q\big[[t]\big]$.

We will mostly deal with (A).

3.1. Definition. *A locally compact topological group G will be called totally disconnected, (t.d.) if every neighborhood of $\mathbb{1}$ contains an open compact group.*

A space X is called t.d. if every point has a basis of open compact neighborhoods (also called an ℓ–space).

Lemma 1. *Let X be a t.d. space.*

1. *If $Y \subseteq X$ is locally closed and X is t.d. then so is Y*
2. *If $K \subseteq X$ t.d. is compact and $K \subseteq \bigcup_\alpha U_\alpha$ is an open covering, then there exist disjoint compact open $V_i \quad i = 1, \cdots, k$ such that $V_i \subseteq U_\alpha$ for some α and $K \subset \bigcup V_i$.*

Proof. Exercise.

Lemma 2. *Suppose G t.d. acts (continuously) on a t.d. space X with a finite number of orbits and G is countable at ∞. Then X has an open orbit $X_o \simeq G/H$ for some closed subgroup H.*

3.2. Definition. *We denote by $C^\infty(X)$ locally constant complex valued functions. Then $\mathcal{S}(X) \subseteq C^\infty(X)$ is the set of compactly supported functions locally constant functions. The linear functionals (distributions) on $\mathcal{S}(X)$ are denoted $\mathcal{S}^*(X)$.*

Proposition. *Suppose $U \subseteq X$ is open. Write $Z := X \setminus U$. Then*

$$0 \to \mathcal{S}(U) \to \mathcal{S}(X) \to \mathcal{S}(Z) \to 0.$$

Proof. We can extend any $f \in S(U)$ by zero because there is a compact open set K such that $f|_{X-K} \equiv 0$ already. On the other hand, if $f \in S(Z)$, we may as well assume Z compact and cover it by open U_α such that $f|_{U\alpha \cap Z} = c_\alpha$. Then use lemma 1. $\square$

Corollary.

$$0 \to \mathcal{S}^*(Z) \to \mathcal{S}^*(X) \to \mathcal{S}^*(U) \to 0.$$

3.3. Support of a distribution. *If $\mathcal{E} \in \mathcal{S}^*(X)$ define*

$$supp\mathcal{E} := \text{ the smallest closed set such that } \mathcal{E}|_{\mathcal{S}(X-Z)} = 0$$

Let $\mathcal{S}^*(X)_c$ be distributions with compact support. Then an element $f \in \mathcal{S}^*(X)_c$ defines a functional on all of $C^\infty(X)$ by the formula

$$< \mathcal{E}, f > := < \mathcal{E}, \delta_{supp\mathcal{E}} \cdot f > .$$

This makes sense when $supp\mathcal{E}$ is open compact. If not, you replace $supp\mathcal{E}$ by a larger compact open set and show that the definition makes sense.

3.4. We now look at the case of a *t.d.* group G. Let X_1, X_2 be compact open subsets of G. There exists a compact open subgroup K such that

$$X_1 : \bigcup_{i=1}^{n_1} x_{i_1} K, \quad X_2 = \bigcup_{j=1}^{n_2} x_{j_2} K$$

disjoint unions, Then define

$$(X_1 : X_2) := n_1/n_2$$

Then $(X_1 : X_2) = \mu(X_1)/\mu(X_2)$ where μ is any left invariant Haar measure. So if $\mu(X)$ is rational for one such set, it is rational for *all* such sets.

Examples.

(1) $(\mathbb{F}, +)$ and $(\mathbb{F}^\times, \cdot)$.
(2) $GL(n, \mathbb{F})$. $K = GL(n, \mathcal{O}_\mathbb{F})$ is open compact. Let

$$K_i : I + \varpi^i gl(n, \theta_\mathbb{F}).$$

These are open compact subgroups which form a system of open compact neighborhoods of $\mathbb{1}$.

3.5. If $K \subseteq G$ is open compact, let

$$\mathcal{H}_K = \mathcal{H}(G, K) := \{f \in S(G) \,|\, f(k_1 g k_2) = f(g)\}.$$

This is a convolution algebra under the operation

$$f_1 * f_2(g) := \int_G f_1(x) f_2(x^{-1}g) dx.$$

Exercise. Show that $* : \mathcal{H}(G,K) \times \mathcal{H}(G,K) \to \mathcal{H}(G,K)$. Also show that the algebra has a unit element

$$e_K(x) = \frac{1}{vol(K)}\delta_K. \quad \square$$

The union of all the $\mathcal{H}(G,K)$ is denoted $\mathcal{S}(G)$ or $\mathcal{H}(G)$. This is also an algebra under $*$. It is an idempotented algebra, *i.e.*

For any finite set $\{f_1, \cdots, f_k\}$ there is $e \in \mathcal{H}$ such that $ef_i = f_i e,\ e^2 = e$.

Definition. *A representation (π, V) of G is called* smooth *if for every $v \in V$, $Stab_G(v)$ is open compact.*

For such a representation we can define a representation of $\mathcal{H}(G)$ via

$$\pi(f)v := \int_G f(g)\pi(g)v\, dg.$$

3.6. Because our groups are reductive, we sometimes have to consider variants of the Hecke algebra. Recall G is unimodular, $Z \subseteq G$ the center, $\chi : Z \to \mathbb{C}$ a character. Then define $\mathcal{H}_\chi(G)$ to be the set of locally constant functions satisfying

(1) $f(gz) = \chi(z)^{-1} f(g)$
(2) Supp f is compact modulo Z

Let ν be a Haar measure on G/Z. Then define

$$f_1 *_\chi f_2(g) := \int_{G/Z} f_1(x) f_2(x^{-1}g) d\nu(x). \tag{3.6.1}$$

Note.
(a) $f_1(xz)f_2(z^{-1}x^{-1}g) = f_1(x)f_2(x^{-1}g)$, so (3.6.1) makes sense.
(b) $f_1 *_\chi f_2 \in \mathcal{H}_\chi(G)$.

3.7. Exercises/Corrections.

(1) Read/Review chapter 1 in A. Weil's Basic Number Theory.
(2) Suppose $Z \subseteq X$ is an open compact set, Z=supp $\mathcal{E}$. Show that if $f \in C^\infty(X)$

$$< \mathcal{E}, f > := < \mathcal{E}, \delta_Z f >$$

makes sense. Generalize to the case of an arbitrary compact set Z.
(3) Suppose (π, V) is a continuous representation on a Hilbert space. Define

$$V_{sm} := \{v \in V \mid \exists K \text{ open compact group such that } v \in V^K\}$$

Show that V_{sm} is dense in V.
(4) Let (π, V) be smooth, V *any* vector space. Show that $\int_G f(g)\pi(g)v dg$ makes sense for any $g \in \mathcal{H}(G)$.

(*Hint:* Since f is locally constant with compact support and V is smooth, there is K such that $v \in V^K$ and $f = \sum c_i \delta_{x_i K}$. Show that the integral is a finite sum.

3.8. Theorem. *There is a natural equivalence*

$$\{\textit{Smooth modules } V \textit{ for } G\} \longleftrightarrow \{\textit{modules } V \textit{ for } \mathcal{H}(G) \textit{ satisfying } \mathcal{H}V = V\}.$$

A module such that $\mathcal{H}V = V$ is called nondegenerate.

Proof. We need to show how to recover the G-module action from the $\mathcal{H}$–module. It is enough to show that the representation of $\mathcal{H}$ extends to $\mathcal{S}^*(G)_c$ with compact support (this is also an algebra with a $*$.)

Finst of all, note that $\mathcal{H}(G)$ embeds in $S^*(G)$; if $f \in \mathcal{H}(G)$, the inclusion is defined as

$$\mathcal{E}_f(h) := \int_G f(g)h(g)dg = (f * \widetilde{h})(\mathbb{1}) \quad \text{where} \quad \widetilde{h}(x) = h(x^{-1}).$$

Furthermore, recall that $G \times G$ acts on $\mathcal{H}$ via

$$L_g f(x) = f(g^{-1}x), \quad R_g f(x) = f(xg).$$

Then $G \times G$ also acts on $\mathcal{S}^*(G)$ via

$$(L_g\mathcal{E})(f) = \mathcal{E}(L_{g^{-1}f}), \quad (R_g\mathcal{E})(f) = \mathcal{E}(R_{g^{-1}}f).$$

Lemma 1. *The inclusion $\mathcal{H}(G) \hookrightarrow \mathcal{S}^*(G)$ maps into $\mathcal{S}^*(G)_c$ and is $G \times G$ equivariant.*

Proof. Exercise.

Lemma 2. *If $\mathcal{E} \in \mathcal{S}^*(G)_c$, then $\mathcal{E}$ defines an operator*

$$\widetilde{\mathcal{E}} : \mathcal{H} \to \mathcal{H}$$

with the property that

$$\widetilde{\mathcal{E}}(f_1 * f_2) = \widetilde{\mathcal{E}}(f_1) * f_2.$$

Proof. Define

$$\widetilde{\mathcal{E}}(f)(g) := \mathcal{E}(L_g f).$$

This has the desired properties because

$$\begin{aligned} L_g(f_1 * f_2)(x) &= (f_1 * f_2)(g^{-1}x) = \int_G f_1(y)f_2(y^{-1}g^{-1}x)dy \\ &(y^{-1}g^{-1} = u^{-1} \quad u = gu \quad g^{-1}u = y) \\ &= \int_G f_1(gu)f_2(u^{-1}x)du = L_g f_1 * f_2(x). \end{aligned}$$

Remark. More generally if (π, V) is a smooth representation and $\mathcal{E} \in \mathcal{S}^*(G)_c$, we can think of

$$(g, v) \longmapsto \pi(g)v$$

as an element of $C^\infty(G) \otimes V$. Then define

$$\pi(\mathcal{E})v := < \mathcal{E}, \pi(g)v > .$$

3.9. Projectives. Let U be an abelian category. We say $\rho \in Obj(U)$ is *projective* if the functor

$$X \longmapsto Hom(\rho, X)$$

is exact. This means that if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence, then

$$0 \to Hom\ (\rho, A) \to Hom\ (\rho, B) \to Hom\ (\rho, C) \to 0$$

is also exact. What is needed is that for any $\varphi : \rho \to C$ and $B \xrightarrow{\pi} C \to 0$, there is ψ such that $\pi \circ \psi = \phi$.

Theorem. *The category $\mathcal{M}(\mathcal{H})$ of nondegenerate modules has enough projectives.*

Proof. Let $e \in \mathcal{H}$ be idempotent ($e^2 = e$). Let $P_e = \mathcal{H}e$. Then $Hom_{\mathcal{H}}(P_e, X) = eX$. So P_e is projective. If $M \in \mathcal{M}(\mathcal{H})$, and $v \in M$, there is e_v such that $v = \pi(e)v$; $v = \sum \pi(f_i)w_i = \pi(e)(\sum \pi(f_i)w_i)$ for any e such that $ef_i = f_ie$. Then V is a quotient of $\bigoplus\limits_{v \in V} P_{e_v}$.

3.10. Injectives. Typically injective objects are dual to projective ones. But the dual of a smooth representation may not be smooth.

Definition. *If (π, V) is a smooth representation, define $(\widetilde{\pi}, \widetilde{V})$ as*

$$\widetilde{V} := (V^*)_{sm}, \qquad \widetilde{\pi}(g)\widetilde{v} := \widetilde{v} \circ \pi(g^{-1}).$$

Proposition. *Let $K \subseteq G$ be open compact. Then*

(1) $\widetilde{\pi}(e_k)\widetilde{V} \simeq (\pi(e_k)V)^*$,
(2) $Hom_G\ (V, \widetilde{W}) \simeq Hom_G\ (W, \widetilde{V})$,
(3) $V \hookrightarrow \widetilde{\widetilde{V}}$.

The proof is left as an exercise.

Lemma. *If P is projective, then $\widetilde{P}$ is injective.*

Definition. *(π, V) is called admissible if for every open compact $K \subseteq G$, $\dim V^K < \infty$.*

It is easy to check that (π, V) is admissible if and only if $V \simeq \widetilde{\widetilde{V}}$.

Theorem. *Every irreducible representation is admissible.*

We will sketch the ideas of the proof in the later sections.

Corollary. *$\mathcal{M}(\mathcal{H})$ has enough injectives.*

We will see important examples of projective and injective modules in section 4.5.

3.11. Induction. Let $H \subseteq G$ be a closed subgroup. There are two induction functors,

$$Ind_H^G := \{f : G \to V \text{ locally constant }\}$$

and

$$ind_H^G := \{f \in Ind_H^G \text{ with compact support modulo } H\}.$$

Proposition.

(1) *$ind \hookrightarrow Ind$*
(2) *both are exact functors*
(3) *If G/H is compact,* ind=Ind *maps admissible modules to admissible modules.*

3.12. The Jacquet functor. If (π, V) is a G-module, define $J_G(V) := V/\{\pi(g)v - v\}$ (also written V_G). This is the analogue of the functor defined in section 2 for real groups.

Proposition.

(1) *J_G is right exact.*
(2) *If G is compact, J_G is exact.*
(3) *If G is a union of compact subgroups $\cdots \subseteq H_i \subseteq H_{i+1} \cdots$ then J_G is exact.*

4. Irreducible Representations

4.1. A smooth representation (π, V) is called irreducible if it has no nontrivial invariant subspaces.

Schur's Lemma. *Suppose G is countable at infinity. If (ρ, V) is irreducible, then $End_G(V) \simeq \mathbb{C}$.*

Separation Lemma. *Suppose G is countable at infinity. Then for any $h \in \mathcal{H}(G)$, $h \neq 0$ there is an irreducible (ρ, V) such that $\rho(h) \neq 0$.*

Proof. Write $h^c(x) := \overline{h(x^{-1})}$. Then

$$(h * h^c)(g) = \int_G h(x) h^c(x^{-1}g)dx = \int_G h(x)\overline{h(g^{-1}x)}dx$$

Then $y = h * h^c \not\equiv 0$ (set $g = \mathbb{1}$). We need a representation so that $\rho(u) \neq 0$. Now $u^c = u$. So $0 \neq u \cdot u^c = u^2$ and so on, $u^n \neq 0$ for any n. The theorem follows from the following lemma. $\square$

Lemma. *Let $\mathcal{A}$ be an algebra of countable dimension over $\mathbb{C}$ with unit. Suppose $a \in \mathcal{A}$ is not nilpotent. Then there is a simple module so that $a \neq 0$.*

Proof of the lemma. There is $0 \neq \lambda \in \mathbb{C}$ such that $a - \lambda$ is not invertible. If all of them were invertible, the set $(a-\mu)^{-1}$ is uncountable, so linearly dependent. Then there are c_i and μ_i such that

$$\sum c_i (a-\mu_i)^{-1} = 0.$$

By multiplying with $(a - \mu_i)$ we find that a is the *zero* of a polynomial function. In other words there is $n_0 \geq 0$ and $\lambda_i \neq 0$ such that

$$a^{n_0} \prod_{\lambda_j \neq 0} (a - \lambda_j)^{nj} = 0.$$

Since $a^{n_0} \neq 0$, it follows that some $(a - \lambda_j)$ is *not* invertible.

Let M be an irreducible quotient of $\mathcal{A}/(a-\lambda)\mathcal{A}$. Then $M \neq 0$ and $(a-\lambda)\mathbb{1} = 0$ with $\lambda \neq 0$.

4.2. Here are some of the main basic results about irreducible representations.

Theorem. *Let (π, V) for any unitary irreducible representation. Then V_{sm} is dense in V and (π_{sm}, V_{sm}) is admissible.*

Definition. *Let χ be a character of $\mathcal{Z} = Z(G)$. We say (π, V) is a $\chi-$ representation if $\pi(z)v = \chi(z)v$ for all $v \in V$. We say (π, V) is absolutely cuspidal (supercuspidal, compact) if for any $\widetilde{v} \in \widetilde{V}$,*

$$g \longmapsto < \pi(g)v, \widetilde{v} >$$

has compact support modulo Z.

4.3. Recall $K \subset G$ open compact and $\mathcal{H}_K := \mathcal{H}(G//K) := \mathcal{H}(K\backslash G/K)$ the subalgebra of K biinvariant functions in $\mathcal{H}$.

Proposition. *Let (π, V) be a smooth representation.*

1. *If π is irreducible, then (π, V^K) is irreducible as an $\mathcal{H}_K$ module.*
2. *Suppose V satisfies (a) V is generated by V^K and (b) for any $0 \neq Y \subset V$, $Y^K \neq 0$*

Then V^K irreducible implies V irreducible.

Proof. For (1), suppose $0 \neq W \subset V^K$ is $\mathcal{H}_K$ invariant and $v_1,\ v_2 \in V^K$. If $v_1 \neq 0$, there is $h \in \mathcal{H}$ such that $v_2 = \pi(h)v_1$ because V is irreducible. Then $v_2 = \pi(e_K h e_K)v_1$ and $e_K h e_K \in \mathcal{H}_K$, proving that V^K is irreducible.

For (2), suppose $0 \neq U \subseteq V$. Then $U^K \neq 0$ so $U^K = V^K$. But then $U^K = V^K$ generates V so $U = V$.

Remark. We have a functor

$$V \longmapsto V^K$$

from $\mathcal{M}(G)$ to $\mathcal{M}(\mathcal{H}_K(G))$. There is also one from $\mathcal{M}(\mathcal{H}_K(G))$ to $\mathcal{M}(G)$

$$U \longrightarrow \mathcal{H}(G/K) \otimes_{\mathcal{H}_K} U$$

We would like to say that this is an equivalence of categories between $\mathcal{M}_K(G)$ and $\mathcal{M}(\mathcal{H}_K)$.

Exercise. Show that $\widetilde{\mathcal{H}(G)} \subset C^\infty(G)$.

4.4. Characters. Suppose (π, V) is admissible. If $f \in \mathcal{H}(G)$ then $\pi(f)$ has finite rank. This is because for an admissible module, for any open compact K,

$$V = \bigoplus V(\delta)$$

where $V(\delta) = span\left\{\int_K \chi_\delta(k)\pi(k)v dk\right\}$, and $\dim V(\delta) < \infty$.

Thus we define a distribution character

$$\Theta_\pi(f) := tr\ \pi(f).$$

($f = \sum c_i \delta_{Kx_iK}$ for some open compact K).

Proposition. *If $\{\pi_1, \cdots, \pi_n\}$ is a set of inequivalent admissible irreducible representations, then $\Theta_1, \cdots, \Theta_n$ are linearly independent.*

Proof. Choose a minimal nontrivial relation

$$\sum_{i=1}^k c_i \Theta_i(f) = 0, \text{ for all } f \in \mathcal{H}.$$

Let K be small enough so that $V_i^K \neq 0$ for all i. The $\mathcal{H}_K$–modules V_i^K are nonzero irreducible and the same character identity holds. By the Frobenius-Schur theorem, there is a function $h \in \mathcal{H}_K$ such that $\pi_1(h) = Id$ and $\pi_i(h) = 0$ for $i > 1$. Then

$$\sum_{i=1}^k c_i \Theta_i(h * f) = c_1 \Theta_i(f) = 0, \text{ for all } f \in \mathcal{H}.$$

Corollary. *Let (π_1, V_1), (π_2, V_2) be admissible finite length representations. Then*

$$\left\{\begin{array}{c} (\pi_1, V_1) \textit{ and } (\pi_2, V_2) \textit{ have} \\ \textit{the same composition factors} \\ \textit{with multiplicities} \end{array}\right\} \Longleftrightarrow \Theta_{\pi_1} = \Theta_{\pi_2}$$

5.1. Suppose (π, V) is a smooth representation and let $(\widetilde{\pi}, \widetilde{V})$ be the smooth dual. A matrix coefficient of V is a function

$$f_{v,\widetilde{v}}(g) := < \pi(g^{-1})v, \widetilde{v} > .$$

Definition 1. (π, V) *is called* compact *if* $f_{v,\widetilde{v}}$ *is compactly supported for any* $v \in V$, $\widetilde{v} \in \widetilde{V}$.

Remark. We can also make the following alternative definition.

Definition 2. π *is called compact if for every* $v \in V$ *and every open compact* K, *the map* $D_{v,K} : g \longmapsto \pi(e_K)\pi(g^{-1})v$ *has compact support.*

If $\widetilde{v} \in V^K$, it is easy to see that

$$\operatorname{supp} f_{v,\widetilde{v}} \subseteq \operatorname{supp} D_{v,K}.$$

In the other direction, it is enough to show that the image of $D_{v,K}$ is contained in a finite dimensional space. Indeed if this is the case, we can find $\widetilde{v}_i \in \widetilde{V}^K$ such that $\operatorname{supp} D_{v,K} \subseteq \bigcup_i \operatorname{supp} f_{v,\widetilde{v}_i}$.

So suppose this dimension is *not* finite. Let $\{g_i\}$ be such that $v_i = \pi(e_K g_i^{-1})v$ are linearly independent. This implies that the g_i cannot be contained in a fixed compact set. On the other hand let $\widetilde{v} \in \widetilde{V}^K$ be such that $< v_i, \widetilde{v} >= 1$ for all i. Then the $g_i \in \operatorname{supp}\ f_{v,\widetilde{v}}$, a contradiction.

5.2. Proposition. *A compact irreducible representation is unitarizable.*

Proof. Choose a nonzero vector $\widetilde{v} \in \widetilde{V}$. Then define

$$< v_1, v_2 > := \int_G < \pi(g^{-1})v_1, \widetilde{v} > \cdot \overline{< \pi(g)v_2, \widetilde{v} >} dg$$

Remark. In N. Wallach's course you saw the similar definition for square integrable representations. This is a special case.

5.3. For a compact representation (π, V) we can define a map

$$m : V \otimes \widetilde{V} \longrightarrow \mathcal{H}$$

by the formula $m : v \otimes \widetilde{v} \longmapsto f_{v,\widetilde{v}}$.

It is $G \times G$ equivariant, because

$$f_{\pi(g)v,\widetilde{v}}(x) =< \pi(x^{-1})\pi(g)v, \widetilde{v} >=< \pi(x^{-1}g)v, \widetilde{v} >= f_{v,\widetilde{v}}(g^{-1}x)$$

Similarly for the right action.

Exercise. Show that if (π_1, V_1) and (π_2, V_2) are irreducible modules, then $(\pi_1 \otimes \pi_2, V_1 \otimes V_2)$ is an irreducible $G \times G$ module.

Proposition. *Any finitely generated compact representation is admissible. In particular so is any irreducible compact representation.*

Proof. If $v_1, \cdots, v_n$ are a set of generators, $V^K = \pi(e_K)V$ is spanned by $\pi(e_K)\pi(g)v_i$. Since each of the functions D_{K,v_i} has compact support, span $\{\pi(e_K)\pi(g)v_i\}_{g,i}$ is finite dimensional.

5.4. Formal Dimension. Let (π, V) be a compact representation.

Proposition. *There exists a natural homomorphism*

$$\varphi : \mathcal{H} \to V \otimes \widetilde{V}.$$

This is unique (up to a scalar) and has the property that if $\varphi(h) = \sum v_i \otimes \widetilde{v}_i$, *then*

$$tr\pi(h) = d_\pi \sum < v_i, \widetilde{v}_i > \quad \textit{for some} \quad d_\pi \neq 0.$$

Proof. Note that $Hom_{\mathbb{C}}(V, V)$ is a $G \times G$ module with the action

$$(g_1, g_2) \cdot a := \pi(g_1) \circ a \circ \pi(g_2)^{-1}.$$

There is a natural map

$$\alpha : V \otimes \widetilde{V} \to Hom_{\mathbb{C}}(V, V),$$

namely $\alpha(v, \widetilde{v})(\xi) := \widetilde{v}(\xi)v$. This is an isomorphism onto $Hom_{\mathbb{C}}(V, V)_{sm}$. Clearly the map is injective into this latter module, we want to show α is onto. We estimate the dimension of the image. If K is compact open, $Hom_{\mathbb{C}}(V, V)^{K \times K}$ has dimension less than $Hom_{\mathbb{C}}(V^K, V^K)$ which is less than the dimension of $(V \otimes \widetilde{V})^{K \times K}$.

The map is $h \mapsto \pi(h) \mapsto \alpha^{-1}(\pi(h))$. The uniqueness of the map follows from

$$\begin{aligned} Hom_{G\times G}[\mathcal{H}, V \otimes \widetilde{V}] &= Hom_{G\times G}[\widetilde{V} \otimes V, \widetilde{\mathcal{H}}] \subset Hom_{G\times G}[\widetilde{V} \otimes V, C^\infty(G)] \\ &= Hom_{d(G)}[\widetilde{V} \otimes V, \mathbb{1}_G] \simeq Hom_G[V, V] = \mathbb{C}Id. \end{aligned}$$

The main point is that $C^\infty(G)$ is an induced module; it is

$$Ind_{d(G)}^{G\times G}(\mathbb{1}_G)_{\text{smooth}}.$$

where $d(G)$ is the diagonal subgroup of $G \times G$. Recall that

$$Ind_{d(G)}^{G\times G}(\mathbb{C}_G) = \{F : G \times G \to \mathbb{C} | F(g_1 x, g_2 x) = F(g_1, g_2)\}$$

Then there is a $G \times G$-equivariant isomorphism

$$F \longmapsto f(x) := F(x, 1)$$

Then

$$\begin{aligned} (L_{g_1,g_2}F)(x, 1) &= F(g_1^{-1}x, g_2^{-1}) = F(g_1^{-1}xg_2, \mathbb{1}) = \\ &= f(g_1^{-1}xg_2) = (L_{g_1} \times R_{g_2}(f)(x). \end{aligned}$$

Now compose the two maps $\varphi \circ m = d_\pi \cdot Id$.

Remark. d_π depends on the choice of Haar measure because $\pi(h)$ does.

Proposition. $d_\pi \neq 0$.

Proof. Suppose $x \in V \otimes \widetilde{V}$ is such that $h = m(x) \neq 0$. We want to show $\pi(h) \neq 0$. If (τ, W) is an irreducible representation not equivalent to π, we show $\tau(h) = 0$. Pick $w \in W$. There is a map $V \otimes \widetilde{V} \to W$ given by

$$v \otimes \widetilde{v} \longmapsto \tau(m(v \otimes \widetilde{v}))w$$

This is G-equivariant in the v–variable. So the image is a direct sum of modules isomorphic to V. Thus the image is zero for all w; so $\tau(h) = 0$. The proof of the theorem follows from the separation lemma. □

5.5. Consider the kernel $\mathcal{K}_\pi$ of ϕ in $\mathcal{H}$ and the image $\mathcal{U}_\pi$ of m. Then we get a decomposition

$$\mathcal{H} = \mathcal{K}_\pi \oplus \mathcal{U}_\pi$$

where $\mathcal{K}_\pi$ and $\mathcal{U}_\pi$ are ideals.

Theorem. *For any module (φ, W), we can decompose*

$$W = W_0 \oplus W_1 \qquad\qquad W_0 = \mathcal{U}_\pi W, \quad W_1 = \mathcal{K}_\pi W$$

Furthermore,

(1) *W_1 has no subquotients isomorphic to π*
(2) *W_0 is a direct sum of copies of π.*

Corollary. *A compact representation is projective in $\mathcal{M}(\mathcal{H})$.*

6. Structure of t.d. reductive groups

6.1. Most of the results will be stated without proof, but illustrated by the example of $GL(n)$. Let G be the $\mathbb{F}$–points of a connected reductive algebraic group.

Proposition. *Let G° be the subgroup generated by all compact subgroups.*

(1) *G° is an open normal subgroup with compact center.*
(2) *$G/G^\circ := \Lambda(G)$ is a lattice (in $\mathbb{R}^m$ for some $m > 0$)*
(3) *If Z is the center of G, then $G^\circ Z$ is finite index in G.*

Example. If $G = GL(n)$, then every matrix x in G° must have determinant in $\mathcal{O}^\times$ (otherwise $\{x^n\}$ is not compact). So $G^\circ \subset G^1 := \{x \in G | \det x \in \mathcal{O}^\times\}$. In fact we can easily check $G^\circ = G^1$. We can also check (2) and (3),

$$G/G^\circ \simeq \{diag(\pi^k z, \cdots, \pi^k z)\}/_{\{(z,\cdots,z)\}} \qquad z \in \mathcal{O}^\times$$

is isomprphic to $\mathbb{Z}$.

6.2 Definition. *A parabolic subgroup is a group of the form*

$$P = P_g = \{x \in G | \{Adg^n(x)\}_{n \geq 0} \quad \textit{is relatively compact}\}.$$

Then let $M := Pg \cap Pg^{-1}$ *and*

$$U = U_g = \{x \in G | Adg^n x \to 1 \quad as \quad n \to \infty\}. \qquad \square$$

Then U *is the unipotent radical of* P *and* $P = MU$.

Similarly write $\overline{P}$ for P_g^{-1}; then $\overline{P} = M\overline{U}$.

Let P_0 be a fixed, minimal parabolic subgroup. We say P is standard, if $P \supset P_0$.

All minimal parabolic subgroups are conjugate and there is a maximal compact subgroup K_0 such that $G = K_0 P_0$. In particular all minimal parabolic subgroups are conjugate to P_0 by elements of K_0.

Remark. K_0 is *not* unique even up to conjugacy. In $SL(2, \mathbb{F})$, there are two maximal compact conjugacy classes,

$$K_0 = SL(2, \mathcal{O}) \quad \text{and} \quad K_0' = \begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix} K_0 \begin{bmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{bmatrix}.$$

In $GL(2, \mathbb{F})$ all maximal compact subgroups are conjugate to $K_0 = GL(n, \mathcal{O})$. For P_0 we take the upper triangular group. Standard parabolic subgroups are the block upper triangular groups

$$\begin{array}{cc} & \begin{array}{ccc} n_1 & n_2 & \cdots \end{array} \\ \begin{array}{c} n_1 \\ n_2 \\ \vdots \end{array} & \begin{bmatrix} * & & \\ 0 & * & \\ 0 & 0 & \ddots \end{bmatrix} \end{array}$$

They correspond to partitions $(n_1, \cdots, n_k)$ of n. Let $\Lambda = \Lambda(M_0) := M_0/M_0^\circ$ and Λ_Z the image of $Z(M_0)$ in Λ. Then Λ can be lifted to M_0 as a set so that Λ_Z is an abelian subgroup. Let

$$\Lambda^+ := \{\lambda \in \Lambda | Ad\lambda|_{N_0} \text{ is contracting, but not necessarily strictly}\}.$$

In $GL(n)$ this is the diagonal subgroup $\Lambda \leftrightarrow \{(\pi^{m_1}, \cdots, \pi^{m_n})\}$ and Λ^+ is the subset satisfying $m_1 \geq \cdots \geq m_n$. We let Λ^{++} be the subset of strictly contracting elements, (corresponding to $m_1 > \cdots > m_n$ in $GL(n)$).

These notions make sense for any parabolic subgroup. If $\lambda \in \Lambda^+$ then $P_0 \subseteq P_\lambda$. If $\lambda \in \Lambda^{++}$ then $P_\lambda = P_0$.

6.3 Bruhat's Theorem. *There are arbitrarily small compact open subgroups K satisfying*

(1) *K is normalized by K_0.*
(2) *For any standard $P = MN$ there is a decomposition $K = (\overline{N} \cap K)(M \cap K)(N \cap K)$. Furthermore,*
   $M \cap K$ is $Ad\Lambda$ invariant,
   $N \cap K$ is $Ad\Lambda^+$ invariant
   $\overline{N} \cap K$ is $(Ad\Lambda^+)^{-1}$ invariant

For $GL(n)$ let $K_i = I + \pi^i gl(n, \mathcal{O})$. They satisfy the conclusion of the theorem.

We say K is *good with respect to P*, or *in good position with respect to P.*

Proposition. *Any standard unipotent subgroup U has a filtration by compact open subgroups.*

Proof. Choose K as in theorem 6.3. Then we say $\lambda \in \Lambda^+$ is strictly dominant with respect to P if $P = P_\lambda$. Such λ always exist. Then let

$$N_n = \lambda^{-n} N \cap K \lambda^n, \quad n \in \mathbb{Z}. \quad \square$$

For $GL(n)$ and a partition $(n_1, n_2, \cdots, n_k)$, take

$$\lambda = (\underbrace{\pi^k \cdots \pi^k}_{n_1}, \underbrace{\pi^{k-1} \cdots \pi^{k-1}}_{n_2}, \cdots, \underbrace{\pi \ldots \pi}_{n_k}).$$

6.4 Theorem. *(Cartan decomposition)*

$$G = K_0 \Lambda^+ K_0.$$

7. The Jacquet functor

7.1. Let $P = MN$ be a parabolic subgroup. If σ is a representation of M, we have defined normalized induction in the previous lectures. Recall that $\mathbb{F} \supseteq \mathcal{O} \supseteq \mathcal{P}$ and $K_0 := G(\mathcal{O})$ is a maximal compact subgroup. Then $G = K_0 \cdot P$. So $G/P \simeq K/K \cap P$ is compact and Ind_P^G coincides with ind_p^G.

Lemma 1. $Ind_P^K(\sigma)|_K \simeq Ind_{K \cap P}^K(\sigma|_{K \cap P})$.

Proof. Exercise.

Lemma 2. $Ind_P^G(\sigma) \simeq Ind_P^G(\widetilde{\sigma})$.

Proof. Exercie. $\sim$ refers to contragredient.

7.2. Suppose now N is any locally compact t.d. group possessing arbitrarily large open compact subgroups. This means that if $X \subseteq N$ is compact, there is an open compact subgroup $N_0 \subseteq N$ such that $X \subseteq N_0$.

Exercise. Let $N \subset GL(n)$ be the group of upper triangular matrices with 1's on the diagonal. Show that N has the aforementioned property. $\square$

Let (π, V) be a smooth representation of N. If $N_0 \subseteq N$ is an open compact subgroup, define

$$V(N_0) = \{v \in V \mid \int_{N_0} \pi(n)vdn = 0\}$$

Then $V = V^{N_0} \oplus V(N_0)$, in other words $V(N_0)$ is the complement of the largest subspace of V on which N_0 acts trivially.

Definition. *Let*

$$V(N) = \bigcup_{N_0 \subseteq N} V(N_0)$$

where the union is over all open compact subgroups N_0 of N.

Proposition.

$$V(N) = span\{\pi(n)v - v | v \in V\}.$$

Proof. Say $w = \pi(n)v - v$. Take N_0 small enough so that it fixes both $\pi(n)v$ and v. Then

$$\int_{N_0} \pi(n_0)(\pi(n)v - v)dv = 0.$$

Conversely, suppose $N_1 \subset N$ is open compact and $v \in V(N_1)$. Choose $N_0 \subseteq N_1$ such that $\pi(n_0)v = v$ for all $n_0 \in N_0$. Then

$$0 = \frac{1}{vol(N_1)} \int_{N_1} \pi(n)vdv = \frac{vol(N_0)}{vol(N_1)} \sum_{n \in N_1/N_0} \pi(n)v$$

Then

$$\frac{vol(N_0)}{vol(N_1)} \sum_{n \in N_1/N_0} (\pi(n)v - v) = -v,$$

so v is a linear combination of $\pi(n)v - v$ with $n \in N_1$. $\square$

7.3. Definition. *The Jacquet functor is*

$$J_N(V) := V/V(N) \quad \textit{(also denoted } V_N\textit{)}.$$

Proposition. *If U is any module on which N acts trivially, then*

$$Hom_N(V, U) \simeq Hom_{\mathbb{C}}(V_N, U).$$

Proof. Exercise, we've already mentioned this result earlier.

Proposition. *If $U \to V \to W$ is exact, then so is*

$$U_N \to V_N \to W_N.$$

Proof. Exercise, This was also mentioned earlier under more general conditions.

8. Cuspidal Representations

8.1 Definition. *A smooth module (π, V) is called quasi–cuspidal if $V_N = 0$ for any standard parabolic subgroup.*

Recall that a representation is called *compact modulo the center* if for every open compact K,

$$D_{v,K}(g) = \pi(e_K)\pi(g^{-1})v \quad \text{has compact support modulo the center.}$$

8.2. Theorem. *(Harish–Chandra) (π, V) is quasi–cuspidal if and only if it is compact modulo the center.*

Proof. This is a sketch of the main idea. We may as well assume v is fixed by $K \subset K_0$ and that K is good with respect to P. We use the Cartan decomposition. Consider

$$\pi(e_K)\pi(\lambda^m)v = \int\limits_{K\cap N} \int\limits_{K\cap M} \int\limits_{K\cap N^-} \pi(n_1 n_0 n_1^-)\pi(\lambda^m) v\, dn_1^- dn_0 dn_1. \tag{8.2.1}$$

Choose $\lambda \in \Lambda^{++}(M)$. This integral becomes

$$\int\limits_{K\cap N} \pi(n)\pi(\lambda)v\,dn = \pi(\lambda)\left[\int\limits_{\lambda^{-m}(K\cap N)\lambda^m} \pi(n')v\,dn'\right]\pi(\lambda)^{-m}\cdot\delta(\lambda)^m. \tag{8.2.2}$$

If $D_{v,K}$ has compact support, the integral in (8.2.2) must be zero for m large enough. Since

$$\bigcup \lambda^{-m}(K\cap N)\lambda^m = N,$$

it follows from proposition 7.2 that the image of v in V_N is zero.

For the converse consider $g = k_1 a k_2$ with $k_i \in K_0$ and $a \in \Lambda^+$. Then

$$f_{v,\widetilde{v}}(g) =< \pi(a)\pi(k_2)v, \widetilde{\pi}(k_1^{-1})\widetilde{v} > .$$

Since $\pi(k)v$ as well as $\widetilde{\pi}(k)\widetilde{v}$ spans a finite dimensional space for $k \in K_0$ it is enough to show that $f_{v,\widetilde{v}}(a) = 0$ except for finitely many $a \in \Lambda^+$. Choose a positive simple root α of A and let $P = MN$ be the standard parabolic corresponding to it. We want to show that if a has too large a component along $\Lambda^+(M)$, then $f(a)$ must be zero. Since $V = V(N)$, we can find a compact open subgroup $N_1 \subset N$ so that $v = \sum \pi(n_k)w_k - w_k$ with $n_k \in N_1$. On the other hand there is a compact open subgroup $N_2 \subset N$ that fixes $\widetilde{v}$. So

$$f_{v,\widetilde{v}}(g) = \int_{N_2} < \pi(a)v, \pi(n_2^{-1})\widetilde{v} >= \int_{N_2} < \pi(a)\pi(a^{-1}n_2 a)v, \widetilde{v} > .$$

If the component of a along $\Lambda(M)^+$ is large enough, then $a^{-1})N_2 a \supset N_1$, so the integral is zero. □

8.3. Here is a summary of the basic results so far.

Theorem 1. *Any cuspidal irreducible representation is admissible.*

Theorem 2. *Any irreducible representation is admissible.*

Theorem 3. *Any irreducible module is a subrepresentation of a Harish–Chandra induced module from a supercuspidal module.*

Theorem 4. *Let $P = MN$ be an arbitrary parabolic subgroup.*

$$Hom_G\left(V, Ind_P^G(\widetilde{\sigma\delta_P^{-1/2}})\right) \simeq Hom_M\left(V_N, \sigma\delta_P^{-1/2}\right).$$

This is of course Frobenius reciprocity.

8.4 Theorem. *If (π, V) is finitely generated and smooth, then so is (π_N, V_N). If (π, V) is admissible, so is (π_N, V_N).*

Proof. Since N is normal in P, it is not hard to check that $V(N)$ is M–stable. So M acts on V_N and we get an action on V_N which is smooth. The resulting module is (π_N, V_N).

If X is a finite set generating V, let $K \subseteq K_0$ be open compact so that $X \subseteq V^{K_0}$. Then $\Gamma = K/K_0$ is finite and $G = P\Gamma K$. So V_N is generated by $\pi(\Gamma)X$ as an M-module.

For the admissibility we need a lemma. Let $K \subseteq K_0$ be open compact in good position with respect to P.

Lemma. *If* $\dim V^K < \infty$ *then* $\dim V_N^{M\cap K} < \infty$. *When this is the case,* V^K *maps* onto $V_N^{M\cap K}$.

The proof of the theorem follows. □

Proof of the lemma. Let $v \in V^{(M\cap K)(N^-\cap K)}$. Assume the volumes of all the compact groups are 1. Then

$$V_0 = \pi(e_k)V = \int_K \pi(k)vdk = \int_{N\cap K} \pi(n)\left(\int_{(M\cap K)(N^-\cap K)} \pi(mn^-)Vdmdn^-\right)dn$$
$$= \int_{N\cap K} \pi(n)vdn.$$

Thus V^K and $V^{(M\cap K)(N^-\cap K)}$ have the same image in V_N. Indeed $V^K \subseteq V^{(M\cap K)(N^-\cap K)}$ and if $v \in V^{(M\cap K)(N^-\cap K)}$, then the above formula shows that $v - v_0 \in V(N\cap K)$ (expand the last integral into a sum).

Let now $\overline{U} \subseteq V^{M\cap K}$ be any finite dimensional subspace and $U \subseteq V^{M\cap K}$ mapping N to it. We can find $N_1^- \subseteq N^-$ open compact so that $U \subseteq V^{(M\cap K)N_1^-}$. There is $a \in \Lambda^+$ such that $a^{-1}(N^-\cap K)a \subseteq N_1^-$. Then $\pi(a)U \subseteq V^{(M\cap K)(N^-\cap K)}$. So $\pi_N(a)\overline{U} \subseteq V_N^{(M\cap K)(N^-\cap K)}$ comes from V^K by the previous argument. So $\dim \overline{U} \leq \dim V^K$.

To prove the last claim, note that if $\overline{U}$ contains a basis of $V_N^{M\cap K}$ then so does $\pi_N(a)\overline{U}$. □

Exercise.

(1) Let P be the upper triangular subgroup of $GL(n)$. Show that $Ki = I + \pi^i\mathcal{H}(n)$ has an Iwahori factorization.
(2) If $r = (n_1, \cdots, n_k)$ is a partition of n, let P_r be the block upper triangular group with blocks of size $n_1, \ldots, n_k)$. Find a basis of neighborhoods of $\mathbb{1}$ of open compact subgroups in good position with respect to P_r. Find λ so that $P_\lambda = P_r$ and compute $\Lambda(M_r)$, $\Lambda^+(M_r)$ and $\Lambda^{++}(M_r)$.

8.5 Relations between G and G°. We say that a character $\psi : G \to \mathbb{C}^*$ is *unramified* if $\psi|_{G^\circ} = Id$. Call the set of unramified character $\Psi(G)$. Note that

$$\psi(G) \simeq \mathrm{Hom}(\Lambda(G), \mathbb{C}^*) \simeq (\mathbb{C}^*)^m.$$

Proposition. *Let (ρ, V) and (ρ', V') be irreducible representations of G.*

(1) *$\rho|_{G^\circ}$ is finite length.*
(2) *$\rho|_{G^\circ} \simeq \rho'|_{G^\circ}$ if and only if there is $\psi \in \Psi(G)$ such that $\rho \simeq \psi\rho'$.*

Proof. (1) is clear because Z acts by scalars and $G^\circ Z$ has finite index in G. For (2), suppose $\rho|_{G^\circ} \simeq \rho'|_{G^\circ}$. Let $\mathcal{A} = \mathrm{Hom}_{G^\circ}(V, V')$. This has an action of G, $g \cdot \ell := \rho'(g) \circ \ell \circ \rho(g')$. Then G° acts trivially. So we get an action of $\Lambda(G)$. Let $h \in \mathcal{H}$ be an eigenvector with eigenvalue ψ, *i.e.* $\tau(g)h = \psi(g)h$. Clearly this is an unramified character. We get a nonzero h satisfying

$$\rho'(g) \circ h \circ \rho(g^{-1}) = \psi(g)h \qquad \text{for all } g \in G.$$

This means $\rho' \simeq \psi\rho$. □

9. Hecke Algebras

9.1. We would like to find groups K so that lemma 8.6 gives an isomorphism. This would relate representations of $\mathcal{H}_K$ to the modules V_N. In particular composition series of induced modules would be related to corresponding problems for $\mathcal{H}_K$.

We can rephrase one of the earlier results as follows. Let $\mathcal{M}_K(\mathcal{H})$ be the category of finitely generated admissible modules such that all subquotients are generated by their K–fixed vectors.

Theorem. *The functor $V \longmapsto V^K$ is an equivalence of categories between $\mathcal{M}_K(\mathcal{H})$ and $\mathcal{F}(\mathcal{H}_K)$ where $\mathcal{F}(\mathcal{H}_K)$:= finite dimensional modules of $\mathcal{H}_K$.*

9.2. Recall $K_0 = G(\mathcal{O})$ (in the case of a split group) or just a maximal subgroup such that G has an Iwasawa decomposition $G = K_0P_0$. We will first study the algebra $\mathcal{H}(G//K_0)$. A representation (π, V) is called spherical if $V^{K_0} \neq 0$.

Proposition. *ASsume G has an automorphism σ such that $\sigma(K_0gK_0) = K_0g^{-1}K_0$ for all $g \in G$. Then $\mathcal{H}_{K_0}$ is abelian.*

Proof. We have to show that $f_1 * f_2 = f_2 * f_1$. Write $\breve{f}(x) := f(x^{-1})$. Then $\breve{f}_1 * \breve{f}_2 = (f_2 * f_1)\breve{}$. On the other hand, consider $(\sigma f)(x) := f(\sigma x)$. This is an automorphism of $\mathcal{H}_{K_0}$ *i.e.* $\sigma(f_1 * f_2) = \sigma f_1 * \sigma f_2$. The assumption implies $\sigma f = \breve{f}$. Then

$$\sigma(f_1 * f_2) = \sigma f_1 * \sigma f_2 = \breve{f}_1 * \breve{f}_2 : (f_2 * f_1)^v = \sigma(f_2 * f_1)$$

and so the claim follows. $\square$

This is the case when G is a real group with a Cartan involution θ and K is the corresponding maximal compact subgroup. If $g = ks$, then $KgK = KsK$ and $\theta(KsK) = Ks^{-1}K = Kg^{-1}K$. Similarly for $G = GL(n, \mathbb{F})$ and $K_0 = GL(n, \mathcal{O})$ use $\sigma(g) := {}^tg^{-1}$. If $g = k_1ak_2^{-1}$ with a diagonal, then $\sigma(K_0gK_0) = K_0g^{-1}K_0$.

9.3. For a p–adic group the following refinement holds.

Theorem. *$\mathcal{H}_{K_0} \simeq \mathbb{C}[\Lambda]^W$ as algebras.*

We will sketch a proof later.

9.4 Example. Let $G = GL(2)$ and $K_0 = GL(2, \mathcal{O})$. Then $\Lambda = \mathbb{Z}^2$ can be lifted as

$$(n_1, n_2) \longleftrightarrow a(n_1, n_2) := \begin{bmatrix} \pi^{n_1} & 0 \\ 0 & \pi^{n_2} \end{bmatrix}$$

To understand the multiplication structure of $\mathcal{H}_{K_0}$ we need to decompose

$$K_0\lambda K_0 \cdot K_0\mu K_0$$

into double cosets. It is not true that this is equal to $K_0\lambda\mu K_0$. For example if

$$\lambda = \mu = \begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix}$$

then

$$K_0\lambda K_0 \cdot K_0\lambda K_0 = K_0\lambda^2 K_0 \cup K_0 \begin{bmatrix} 1 & 0 \\ 0 & \pi \end{bmatrix} K_0. \tag{9.4.1}$$

Indeed,

$$K_0 = \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{bmatrix} \cup \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \mathcal{O} & \pi\mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}.$$

Then

$$\begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{O} & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{bmatrix} \in K_0, \begin{bmatrix} \pi^{-1} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \mathcal{O} & \pi\mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix} \begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix} \in K_0.$$

Thus $K_0\lambda K_0\lambda K_0 = K_0\lambda^2 K_0 \cup K_0\lambda \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \lambda K_0$. But

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & \pi \end{bmatrix}$$

and (9.4.1) follows. □

9.5 Example. Let $K = K_i$ be a congruence subgroup in $GL(n)$ with $i \geq 1$. Denote by $\mathcal{H}_0$ the subalgebra of K biinvariant functions supported on K_0. Then

$$\mathcal{H}_K \simeq \mathcal{H}_0 \cdot C \cdot \mathcal{H}_0$$

where C is the union of $K\lambda K$. This follows because K is normal in K_0; if $x, y \in K_0$, then $y^{-1}Ky \subset K$ so

$$KxKyK = Kxy(y^{-1}K) \cdot K = KxyK.$$

So any

$$K_0\lambda K_0 = \bigcup_{i,j} Kx_iK\lambda KxjK \subseteq \mathcal{H}_0 C \mathcal{H}_0$$

for $x_1 \ldots$ a (finite) set of representatives of cosets of K in K_0. C is in fact a finitely generated commutative algebra. Because $K = K_N \cdot K_{M_0} \cdot K_{N^-}$ and λ, μ can be taken in Λ^+, we get

$$K\lambda K\mu K = K\lambda K_N K_{M_0} K_{N^-}\mu K = K\lambda\mu K.$$

In particular C is generated by n elements.

Theorem. *(Bernstein) Let (π, V) be irreducible . There is a constant $c(G, K)$ such that*

$$\dim V^K \leq c(G, K).$$

Proof. (Sketch) Recall that V^K is an irreducible $\mathcal{H}_K$ module (or zero). Let M be a (nonzero finite dimensional) irreducible module for $\mathcal{H}_K$. We know (by Burnside's theorem) that

$$\pi : \mathcal{H}_K \to \mathrm{End}(M)$$

is surjective. So if $m = \dim M$,

$$m^2 \leq (\dim \mathcal{H}_0)^2 \dim \pi(C).$$

Lemma. *If $C \subseteq End(M)$ is abelian generated by n elements, then*

$$\dim C \leq m^{2-\frac{1}{2^{n-1}}}.$$

Then

$$m^2 \leq (\dim \mathcal{H}_0)^2 m^{2-\frac{1}{2^{n-1}}}$$
$$m \leq (\dim \mathcal{H}_0)^{2^n}.$$

See [Bernstein] for a reference to the proof.

Corollary 1. *There are finitely many isomorphism classes of irreducible cuspidal representations (π, V) satisfying $V^K \neq 0$.*

Proof. (Sketch) There is a fixed (independent of V) open compact set C such that the matrix entries

$$D_{v,K} : g \longmapsto, \pi(e_K)\pi(g^{-1})v$$

all have support in C. Then use the character theory.

Corollary 2. *If (π, V) is irreducible there is a quasi-cuspidal representation σ of M such that*

$$Hom_G(V, Ind_P^G(\sigma)) \neq 0.$$

If there is another pair (M', σ') with this property, then there is $g \in G$ and an unramified ψ' so that

$$g \cdot (M, \sigma) \simeq (M', \sigma'\psi')$$

This is one of the basic theorems of the theory of representations of p–adic groups. For the second part see ???.

10. Sketch of proof of the Satake isomorphism

10.1. Theorem. $\mathcal{H}(G, K) \simeq \mathbb{C}[\Lambda]^W$

For complete details see [Satake].

We recall the notation. We denote by K a maximal compact subgroup and $P = MN$ a minimal parabolic subgroup so that $G = K \cdot P$ has an Iwasawa decomposition. Let A be a maximal split torus such that $M = C_G(A)$, and let M° be the subgroup generated by compact subgroups. It also equals $M \cap K$. Then $M/M^\circ = \Lambda$ is the lattice that appears in theorem 10.1. Let $W := N(A)/_{C_G(A)} \cong N(A) \cap K/_{M^\circ}$ be the Weyl group.

We break down the proof into several steps.

10.2. $\mathcal{H}(M, M^\circ) \simeq \mathbb{C}[\Lambda]$.

This follows from the fact the $M/M^\circ \simeq \Lambda$ is a lattice.

10.3. Integration formulas. The subgroup P is *not* unimodular, so it has a left and a right invariant measure.

$$\int\limits_P f(P)d_lP = \int\limits_M\int\limits_N f(mn)dmdn = \int\limits_N\int\limits_M f(nm)\delta(m)^{-1}dmdn$$

$$\int\limits_P f(P)d_nP = \int\limits_M\int\limits_N f(mn)\delta(m)dndm = \int\limits_N\int\limits_M f(nm)\delta(m)dmdn$$

$$\int\limits_N f(n)dn = \triangle(m)\int\limits_N f(nmnm^{-1})dm$$

where $\delta(m) = |\det Adm|_{\underline{n}}|$ and $\triangle(m) := |\det(Adm - 1\!\!1)|_{\mathfrak{n}}|$. For m regular $\triangle(m)\neq 0$.

Finally let

$$F_f(m) := \triangle(m)\int\limits_{G/A} f(gmg^{-1})dg.$$

Then $F_f(m) = \int\limits_N f(nm)dn$.

10.4. Definition of S. The isomorphism in theorem 10.1 is implemented by

$$Sf(m) := \delta(m)^{\frac{1}{2}}\int\limits_N f(mn)dn.$$

It is clear that $S : \mathcal{H}(G,K)\to\mathcal{H}(M,M^\circ)$.

The fact that the image is in $\mathbb{C}[\Lambda]^W$ follows from

$$Sf(m) = \delta(m)^{-\frac{1}{2}}\triangle(m)\int\limits_{G/A} f(gmg^{-1})dg$$

and the observation that

$$\begin{aligned}\delta(m)^{-1}\triangle(m)^2 = |\det \mathrm{Ad}\, m|_{\mathfrak{n}}|^{-1}\cdot|\det(\mathrm{Ad}\, m-1)|_{\mathfrak{n}}|\cdot|(\det \mathrm{Ad}\, m-1)|_{\mathfrak{n}}| =\\ |\det(\mathrm{Ad}\, m-1)|_{\mathfrak{n}}|\cdot|\det(Adm-1)|_{n^-}|.\end{aligned}$$

which is W–invariant.

10.5. S is an algebra homomorphism. This follows from

$$\int\limits_N f_1 * f_2(mn)dn = \int\limits_N\int\limits_G f_1(g)f_2(g^{-1}mn)dgdn =$$

$$= \int\limits_N\int\limits_{MN} f_1(m'n')f_2\left(n'^{-1}m'^{-1}mn\right)dn'dm'dn =$$

$$= \int\limits_M\left(\int\limits_N f_1(m'n')dn'\right)\left(\int\limits_N f_2\left(m'^{-1}mn\right)dn\right)dm'$$

Then observe that $\delta^{\frac{1}{2}}(m) = \delta^{\frac{1}{2}}(m')\cdot\delta^{\frac{1}{2}}\left(m'^{-1}m\right)$.

10.6. S is onto. Let $\varphi_\lambda := \sum_{w \in W} \delta_{M^\circ w \lambda M^\circ}$. These form a basis of $\mathcal{H}(M, M^\circ)$. We need to show that φ_λ is in the image of S. We will illustrate this by an example only.

Example. Let $G = GL(2)$, $M = \text{diag}\,(m_1, m_2)$, $P = \begin{bmatrix} * & * \\ 0 & * \end{bmatrix}$ and so on. Say $m = \begin{bmatrix} \pi^{\mu_1} & 0 \\ 0 & \pi^{\mu_2} \end{bmatrix}$ $\mu_i \in \mathbb{Z}$. Let $\lambda = (\lambda_1, \lambda_2)$. Then $\delta(m)^{(\mu_1 - \mu_2)} = q$ where $q = |\pi|$. Then

$$\int_N f(mn)dn = \#\left\{\alpha \in \mathbb{Z} | \begin{bmatrix} \pi^{\mu_1} & 0 \\ 0 & \pi^{\mu_2} \end{bmatrix} \begin{bmatrix} 1 & \pi^\alpha \\ 0 & 1 \end{bmatrix} \in K_0 \begin{bmatrix} \pi^{\lambda_1} & 0 \\ 0 & \pi^{\lambda_1} \end{bmatrix} K_0 \right\}$$

times the volume of $\mathcal{O}$.

Assume $\lambda_1 \le \lambda_2$ and $\mu_1 \le \mu_2$ as we may because of the W–symmetry. Then

$$\begin{bmatrix} \pi^{\mu_1} & \pi^{\mu_1 + \alpha} \\ 0 & \pi^{\mu_2} \end{bmatrix} \sim \begin{cases} \begin{bmatrix} \pi^{\mu_1} & 0 \\ 0 & \pi^{\mu_2} \end{bmatrix} & \text{for} \quad 0 \le \alpha \\ \begin{bmatrix} \pi^{\mu_1 + \alpha} & 0 \\ 0 & \pi^{\mu_2 - \alpha} \end{bmatrix} & \text{for} \quad \alpha < 0 \end{cases}$$

So $Sf(m) = 0$ unless $(\mu_1, \mu_2) + m(1, -1) = (\lambda_1, \lambda_2)$ for some nonpositive integer m. In addition $Sf(m) \neq 0$ if $(\mu_1, \mu_2) = (\lambda_1, \lambda_2)$. Write

$$S(\delta_{K_0 \lambda K_0}) = \sum c(\lambda, \mu) \varphi_\mu$$

Then $c(\lambda, \lambda) \neq 0$ and

$$c(\lambda, \mu) \neq 0 \;\text{ only if } (\mu_1, \mu_2) = (\lambda_1 + m, \lambda_2 - m) \text{ with } m \ge 0.$$

Note that $\mu_1 \le \mu_2$ so there are only finitely many (μ_1, μ_2) for which $c(\lambda, \mu) \neq 0$.

This is true in general,

$$S\delta_{K_0 \lambda K_0} = \sum c(\lambda, \mu) \varphi_\mu$$

with $\lambda < \mu$ in a certain lexicographic ordering and $c(\lambda, \lambda) \neq 0$. The $c(\lambda, \mu)$ form a nondegenerate upper triangular matrix which is therefore invertible. $\square$

11. Spherical functions for real and p–adic groups

11.1. We follow [Helgason]. Suppose W is a finite group acting on a complex vector space $\mathfrak{X}$. Then is also acts on the polynomial algebra $\mathbb{C}[\mathfrak{X}]$.

Lemma 1. $\mathbb{C}[\mathfrak{X}]$ *is an integral extension of* $\mathbb{C}[\mathfrak{X}]^W$. $\mathbb{C}[\mathfrak{X}]^W$ *is a finitely generated algebra.*

Proof. If $p \in \mathbb{C}[\mathfrak{X}]$, then it is a root of

$$\prod_{w \in W} (t - wp) = 0$$

which has coefficients in $\mathbb{C}[\mathfrak{X}]^W$. Let $\mathcal{A}$ be the ideal generated by the polynomials with zero constant term in $\mathbb{C}[\mathfrak{X}]^W$. Then let $p_1, \cdots, p_s$ be a set of homogeneous, generators of $\mathcal{A}$. Any $p \in \mathbb{C}[\mathfrak{X}]^W$ can be written as

$$p = h_1 p_1 + \cdots + h_s p_s$$

We do an induction on $\deg p$. Average over W;

$$p = \widetilde{h}_1 \widetilde{p}_1 + \cdots + \widetilde{h}_s \widetilde{p}_s$$

But the degrees of $\widetilde{h}_i$ are strictly smaller so are in the algebra generated by the $\widetilde{p}_i$.

Remark. Part of this argument works for any compact group action. Furthermore if W is a Weyl group, Chevalley's theorem says that $\mathbb{C}[\mathfrak{X}]^W$ is a polynomial algebra.

Lemma 2. *Any algebra homomorphism*

$$\chi : \mathbb{C}[\mathfrak{X}]^W \to \mathbb{C}$$

extends an algebra homomorphism φ of $\mathbb{C}[\mathfrak{X}]$. An algebra homomorphism $\varphi : \mathbb{C}[\mathfrak{X}] \to \mathbb{C}$ is given by evaluation at a $\lambda \in \mathfrak{X}$:

$$(p) = \varphi_\lambda(p) = p(\lambda).$$

Two such homomorphisms φ_λ, $\varphi_{\lambda'}$ coincide on $\mathbb{C}[\mathfrak{X}]^W$ if and only if $\lambda' \in W \cdot \lambda$.

Proof. The first part is a general fact about integral extensions. The second part is standard; φ is determined by the values of φ on the coordinate functions x_i.

Finally suppose $W \cdot \lambda \cap W \cdot \lambda' = \phi$. There is p such that

$$|p - 1| < \frac{1}{3} \quad \text{on} \quad W \cdot \lambda, \qquad |p - 0| < \frac{1}{3} \quad \text{on} \quad W \cdot \lambda'.$$

Let

$$\widetilde{p} := \frac{1}{|W|} \sum_{w \in W} w \cdot p.$$

Then

$$\widetilde{p}(\lambda) \neq \widetilde{p}(\lambda'),$$

a contradiction. This part of the argument also works for an action of a compact group. $\square$

11.2. Recall M, M° and $\Lambda = M/M^\circ \simeq \mathbb{Z}^\ell$. Then

$$\mathfrak{X} = \Lambda \otimes_{\mathbb{Z}} \mathbb{C}.$$

Suppose now

$$\varphi : \mathcal{H}(M, M^\circ) \to \mathbb{C}$$

is an algebra homomorphism. It defines a character

$$\chi : M \to \mathbb{C}$$

by the formula $\chi(m) := \frac{\varphi(\delta_m M^\circ)}{vol(M^\circ)}$. Its relation to φ is

$$\varphi(f) = \int_M \chi(m) f(m) dm.$$

This is an unramified character ($\chi|_{M^\circ}$ = trivial). Two such characters define the same algebra homomorphism on $\mathcal{H}(M, M^\circ)^W$ if and only if they are conjugate under W.

11.3. Let φ be a homomorphism of $\mathcal{H}_K$. As in 11.2,

$$\varphi(f) = \int_G f(g)\Gamma(g) dg \quad \text{with} \quad \Gamma(g) = \frac{\varphi(\sigma_{KgK})}{vol(KgK)}.$$

for a K–biinvariant function satisfying $\Gamma(\mathbb{1}) = 1$. Γ is called a *zonal spherical function.*

11.4. Suppose φ corresponds to the character χ. Then

$$\varphi(f) = \int_M \chi(m)\delta(m)^{\frac{1}{2}} \int_N f(mn) dn dm$$

Let $\Phi_\chi(mnk) := \chi(m)\delta(m)^{\frac{1}{2}}$. Then

$$\varphi(f) = \int_K \int_M \int_N \Phi_\chi(mnk) f(mnk) dm dn dk$$

It follows that

$$\Gamma_\chi(g) = \int_K \Phi_\chi(kg) dk.$$

11.5. Let Γ be a K-biinvariant function. It defines a linear functional on $\mathcal{H}_K$.

Theorem. *The following are equivalent*

(1) Γ *defines an algebra homomorphism* $\varphi : \mathcal{H}_K \to \mathbb{C}$

(2) $\Gamma(g_1)\cdot\Gamma(g_2) = \int\limits_K \Gamma(g_1 k g_2)dk$

(3) *For each* $f \in \mathcal{H}_K$ *there is a constant* $c(f)$ *so that*

$$\Gamma * f = f * \Gamma = c(f)\Gamma.$$

Proof.

$$\varphi(f_1)\cdot\varphi(f_2) = \int\limits_{G\times G} f_1(g_1)f_2(g_2)\Gamma(g_1)\Gamma(g_2)dg_1dg_2$$

$$\begin{aligned}\varphi(f_1 * f_2) &= \int\limits_G \Gamma(g)\int\limits_G f_1(x)f_2(x^{-1}g)dxdg = \\ &= \int\limits_G f_1(g_1)\int\limits_G \Gamma(g)f_2(\overbrace{g_1^{-1}g}^{g_2})dgdg_1 \\ &= \int\limits_G f_1(g_1)\int\limits_G \Gamma(g_1g_2)f_2(g_2)dg_2 = \\ &\int\limits_{G\times G} f_1(g_1)f_2(g_2)\left(\int\limits_K \Gamma(g_1kg_2)dk\right)dg_1dg_2.\end{aligned}$$

This shows the equivalence of (1) and (2). We need to integrate Γ over K because the functions $f_1,\ f_2$ are arbitrary but K–biinvariant.

Next write

$$\begin{aligned}\varphi(f_1 * f_2) &= \int\limits_G f_2(g_2)\int\limits_G f_1(g_1)\Gamma(g_1g_2)dg_1dg_2 = \\ &= \int\limits_G f_2(g)(\check{f}_1 * \Gamma)(g)dg = \int\limits_G f_1(g)\Gamma * \check{f}_2(g)dg \\ &\text{where}\quad \check{f}(g) = f(g^{-1}).\end{aligned}$$

This gives the equivalence between (1) and (3). It also follows that $c(f) = \varphi(\check{f})$. $\square$

11.6. Let (π, V) be an irreducible spherical representation. Then V^K has dimension 1 and $\mathcal{H}_K$ acts by a homomorphism corresponding to the unramified character χ. Let v be a spherical vector and $\widetilde{v} \in \widetilde{V}$ be a spherical vector so that $(v, \widetilde{v}) = 1$. Consider the matrix entry

$$\Gamma(g) := (\pi(g)v, \widetilde{v})$$

This is a spherical function satisfying $\Gamma(1\!\!1) = 1$. Furthermore if $f \in \mathcal{H}_K$,

$$\int\limits_G f(g)(\pi(g)v, \widetilde{v})dg = (\pi(f)v, \widetilde{v}) = \varphi_\chi(f)(v, \widetilde{v}) = \varphi_\chi(f),$$

so

$$\Gamma_\chi(g) = (\pi(g)v, \widetilde{v}).$$

In addition consider the (possibly reducible) induced module

$$\mathrm{Ind}_P^G(\chi\delta^{-\frac{1}{2}}) \qquad \text{(unnormalized induction)}.$$

The spherical vector is $v(kmn) = \chi^{-1}, \delta^{+\frac{1}{2}}(m)$. The smooth dual is $\mathrm{Ind}_P^G(\chi^{-1}\delta^{\frac{1}{2}})$ with corresponding spherical vector $\widetilde{v}(kmn) = \chi(m)\delta^{\frac{1}{2}}(m)$. Then

$$(\pi(g)v, \widetilde{v}) = \int\limits_K \Phi_{K,\chi}(kg)dk = \int\limits_K \Phi_{K,\chi}(k^{-1}g)dk.$$

If $g \in G$, write $g = k(g)m(g)n(g)$ for the decompositon $G = KMN$. Then

$$\Phi_{K,\chi}(kg) = (\delta^{\frac{1}{2}}\chi)(m(g^{-1}k)).$$

On the other hand,

$$v(g) = (\delta^{\frac{1}{2}}\chi)(m(g)^{-1}) \quad \widetilde{v}(g) = (\delta^{\frac{1}{2}}\chi^{-1})(m(g)^{-1})$$

and

$$(v, \widetilde{v}) = \int\limits_K v(k)\widetilde{v}(k)dk.$$

Then

$$(\pi(g)v, \widetilde{v}) = \int\limits_K (\delta^{\frac{1}{2}}\chi)(m(g^{-1}k)^{-1})\underbrace{(\delta^{\frac{1}{2}}\chi^{-1})(m(k))}_{=1}dk$$

So one can compute the spherical function Γ_χ using the full induced module from $\chi\delta^{-\frac{1}{2}}$. These ideas have the following corollary.

Corollary. *Every spherical irreducible module is a subquotient of an induced from an unramified character.*

We will present a different proof later.

The real case

11.7. Everything mentioned so for is the same for the real case. But there is another characterization of spherical functions besides the ones in theorem 11.5. Let $\mathbb{D}(G/K)$ be left invariant differential operators invariant under the adjoint action of K.

Theorem. *A (continuous) biinvariant function is spherical if and only if it is a joint eigenfunction for $\mathbb{D}(G/K)$.*

Proof.

(1) Eigenfunctions for $\mathbb{D}(G|K)$ are analytic.

Recall $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$. Choose an orthonormal hasis X_i for $\mathfrak{s}$ under any K–invariant inner product. Then

$$C := \sum X_i^2 \in U(\mathfrak{g}) \text{ is K–invariant elliptic.}$$

By the elliptic regularity theorem, eigenfunctions are analytic.

(2) Spherical functions are eigenfunctions.

If $D \in \mathbb{D}(G|K)$, let $D_0 := \int\limits_K Adk \cdot Ddk$. If $F \in C^\infty(G)$ is G–invariant, then $DF(\mathbb{1}) = D_0F(\mathbb{1})$. Suppose $f \neq 0$ satisfies the relation $f(x) \cdot f(y) = \int\limits_K f(xky)dk$. There is x_0 such that $f(x_0) \neq 0$. Then $f(x_0)f(y) = \int\limits_K f(x_0ky)dk$. Translate y on the right by an element in K; the invariance of the Haar measure implies that f is right K–invariant. The same argument setting $y = x_0$ shows f is left K–invariant. Also $f(\mathbb{1}) = 1$.

If $\rho \in C_c^\infty(G)$ is such that $\int\limits_G \rho(g)f(g)dy \neq 0$, then

$$f(x)\int\limits_G f(y)\rho(y)dy = \int\limits_G\int\limits_K \rho(xk^{-1}y)dkf(y)dy.$$

This shows that f is C^∞ in x. Furthermore, for any $D_0 \in \mathbb{D}(G|K)$,

$$f(x)[D_0f](y) = \int\limits_K (D_0f)(xky)dk$$

Setting $y = \mathbb{1}$,

$$f(x)[D_0f](\mathbb{1}) = D_0f(x)$$

i.e. f is an eigenfunction.

(3) A joint eigenfunction is spherical.

Consider

$$F(y) := \int\limits_K f(xky)dk.$$

Then

$$(D_0F)(y) = \int\limits_K (D_0f)(xky)dk = \chi(D_0)F(y).$$

We conclude that $D_0[f(\mathbb{1})F - F(\mathbb{1})f](\mathbb{1}) = 0$ for all $D_0 \in \mathbb{D}(G/K)$. So $D[f(\mathbb{1})F - F(\mathbb{1})f](\mathbb{1}) = 0$, or equivalently

$$f(\mathbb{1})F = F(\mathbb{1})f.$$

But $f(\mathbb{1}) = 1$ and $F(\mathbb{1}) = f(x)$ so the claim follows. $\square$

11.7.1. Remains to determine the homomorphisms of $\mathbb{D}(G/K)$. In this case, $M = M^\circ \cdot A$ where A is a maximal torus in $S := exp(\mathfrak{s})$ with Lie algebra $\mathfrak{a}$.

Theorem. *$\mathbb{D}(G/K)$ is isomorphic to $S(\mathfrak{a})^W$ where $\mathfrak{g} := Lie\ (A)$ and $W := N(A)/M$.*

So the p–adic and real case are controlled by the lattice Λ which gives rise to the space on or $\mathfrak{X}$. We will try to formalize this in the next section.

11.8. Example. Let $G = SL(2,\mathbb{R})$, $K = SO(2)$. The Cartan decomposition $G = KA^+K$ is

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} e^{\frac{r}{2}} & 0 \\ 0 & e^{-\frac{r}{2}} \end{pmatrix} \begin{pmatrix} \cos\theta_2 & \sin\theta_2 \\ -\sin\theta_2 & \cos\theta_2 \end{pmatrix}$$

with $r \geq 0$. Then $\mathbb{D}(G/K)$ is generated by one operator $\triangle$. It is the Laplace-Beltrami operator on G/K. In coordinates (θ, r),

$$\triangle = \frac{\partial^2}{\partial r^2} + \frac{\cosh r}{\sinh r}\frac{\partial}{\partial r} + \frac{1}{(\sinh r)^2}\frac{\partial^2}{\partial \theta^2}.$$

Then an eigenfunction with eigenvalue $\alpha = \rho(\rho+1)$ is

$$\psi(r) = P_\rho(\cosh r) = \frac{1}{2\pi}\int_0^{2\pi} (\cosh r + \sinh r \cos u)^\rho du.$$

See [Helgason] for more details.

Exercise. Compute Γ_χ in the case $G = GL(2,\mathbb{F})$ $K = GL(2,\mathcal{O})$.
Hint: Use $\Gamma_\chi(a) = \int_K \Phi_\chi(ka)dk \quad a \in \Lambda^+$ or Λ^-.
A reference for the general case is [MacDonald].

11.9. Suppose G is a connected linear algebraic group $/\mathbb{F}$ where $\mathbb{F}$ is an infinite field (of characteristic zero). To such a group one associates

$$X^*(G) := \mathrm{Hom}_{\mathbb{F}-alg}(G, \mathbb{G}_m), \quad X_*(G) = \mathrm{Hom}_{\mathbb{Z}}(X^*(G), \mathbb{Z})$$

There is a pairing $<\varphi, \lambda>$ between these two groups. If $\varphi \in X_*$, $\lambda \in X^*$ with values in $\mathbb{Z}$,

$$<\varphi, \lambda> := \varphi(\lambda) \in \mathbb{Z}$$

Here $\mathbb{G}_m$ is the multiplicative group.

Now suppose A is a (split) torus of rank ℓ. Then

$$A \simeq \underbrace{\mathbb{G}_m \times \cdots \times \mathbb{G}_m}_{\ell-\text{times}}. \tag{11.9.1}$$

$(\lambda_1, \cdots, \lambda_\ell) \in X^*(A)$ is a basis for this $\mathbb{Z}$-module if and only if $s \to (\lambda_1(s), \cdots, \lambda_\ell(s))$ implements the isomorphism (11.9.1). Now $X^*(A)$ form a basis for the algebra $\mathcal{A}$ of regular (polynomial) functions on A. In algebraic geometry one says that A is the spectrum of $\mathcal{A} = \mathbb{F}[X^*(A)]$. If $\mathbb{L}$ is an extension of $\mathbb{F}$, then the $\mathbb{L}$–points of A can be recovered by the formula

$$A(\mathbb{L}) = \mathrm{Hom}(X^*(A), \mathbb{L}^*) \simeq X_*(A) \otimes_{\mathbb{Z}} \mathbb{L}^*.$$

11.10. One can attach the following data called a *root datum* to a linear connected algebraic group $\mathbb{G}$. Let $\mathbb{T} \subseteq \mathbb{G}$ be a maximal torus. Define

$$X^*(\mathbb{T}) := \mathrm{Hom}(\mathbb{T}, \mathbb{G}_m), \quad X_* := \mathrm{Hom}(\mathbb{G}_m, \mathbb{T})$$

These are in duality via

$$< \varphi, \lambda >:= \quad \varphi(\lambda(t)) = t^{<\varphi,\lambda>} \quad < \varphi, \lambda > \in \mathbb{Z}$$

Let $R := \Phi(\mathbb{G}, \mathbb{T})$ be the roots and $R^\vee := \check{\Phi}(\mathbb{G}, \mathbb{T})$ be the coroots. Choose a Borel subgroup $\mathbb{B}$ containing $\mathbb{T}$. This defines a positive root system $\triangle$ and corresponding coroots $\check{\triangle}$. We call

$$\Phi = (X^*, R, X_* \check{R}) \tag{11.10.1}$$

(or $(X^*, \triangle, X_*, \check{\triangle})$ if we want to keep track of $\mathbb{B}$ as well) the *root datum* attached to $\mathbb{G}$. Abstractly a root datum is a quadruplet as in (11.10.1) along with a correspondence $\alpha \longleftrightarrow \check{\alpha}$ and a perfect pairing $< \, , \, >: X^* \times X_* \longleftrightarrow \mathbb{Z}$. In addition

(1) $< \alpha, \check{\alpha} >= 2$ for all $\alpha \in \Phi$.
(2) If $s_\alpha(x) := x- < x, \check{\alpha} > \alpha$ then $s_\alpha(\Phi) = \Phi$.
(3) If $s_{\check{\alpha}}(u) = u- < u, \alpha > \check{\alpha}$ then $s_{\check{\alpha}}(\check{\Phi}) = \check{\Phi}$.

Theorem. *Over an algebraically closed (characteristic zero) field, there is a 1–1 correspondence between linear connected reductive groups and root datums.*

11.11. The root datum is "independent of the field." Given $\Psi = (X^*, R, X_*, \check{R})$ define the *dual root datum*

$$\check{\Psi} := (X_*, \check{R}, X^*, R).$$

We call $\check{G}$ the complex group corresponding to $\check{\Psi}$. Then $\check{G}$ has a torus $\check{T}$ satisfying

$$X^*(\check{T}) = X_*(T), \quad X_*(\check{T}) = X^*(T)$$

11.12. Now suppose $\mathbb{F} \subseteq \overline{\mathbb{F}}$. Then $\Gamma := Gal(\overline{\mathbb{F}}/\mathbb{F})$ acts on $\overline{\mathbb{F}}$ so also on $gl(n, \overline{\mathbb{F}})$ and $GL(n, \mathbb{F})$. A rational form is a map

$$\sigma : \Gamma \longrightarrow Aut(\mathbb{G})$$

where $\Gamma := Gal(\overline{\mathbb{F}}/\mathbb{F})$ with the property that for any regular function f on $\mathbb{G}$,

$$\gamma(f(\sigma(\gamma^{-1})x))$$

is also a regular function. A cocycle is a map

$$a : \Gamma \longrightarrow Aut(\mathbb{G})$$

satisfying $a(st) = a(s) \circ s(a(t))$. The homomorphism σ is related to a cocycle by the formula

$$\sigma(\gamma) = a(\gamma) \circ \gamma.$$

Exercise. Show that if a satisfies $a(st) = a(s) \circ s(a(t))$, then $\sigma(st) = \sigma(s) \circ \sigma(t)$.

Example. Let $\mathbb{F} := \mathbb{R}$. The action of $\Gamma := Gal(\overline{\mathbb{F}}/\mathbb{F}) = \mathbb{Z}_2$ on $gl(n, \mathbb{C})$ is complex conjugation. A rational form of $GL(n)$ is an antiholmorphic involutive automorphism σ of $GL(n, \mathbb{C})$ such that if f is an algebraic (regular) function of $GL(n, \mathbb{C})$, then so is $g \longmapsto \overline{f(\sigma g)}$. $\square$

We only need to classify the forms up to equivalence.

Definition.

(1) *σ is called* equivalent *to σ' if there is $g \in \mathbb{G}$ so that*

$$\sigma'(\gamma) = Adg \circ \sigma(\gamma) \circ Adg^{-1} = Ad(g \circ \sigma(\gamma) g^{-1}) \circ \sigma(\gamma).$$

(2) *σ and σ' are called* inner *if for each $\gamma \in \Gamma$,*

$$\sigma'(\gamma) \circ \sigma(\gamma)^{-1} \text{ is inner.}$$

Equivalent forms are inner because

$$Adg \circ \sigma(\gamma) \circ Adg^{-1} \cdot \sigma(\gamma)^{-1} = Adg \cdot Ad\sigma(\gamma^{-1})g = Ad(g\sigma(\gamma)^{-1}(g)).$$

Exercise. Classify the equivalence classes of real forms of $GL(n)$. Show that you get

$$U(p, q),\ p + q = n, \qquad GL(n, \mathbb{R}), \quad \text{and if } n \text{ is even} \quad U^*(n).$$

Which ones are inner of each other? $\square$

11.13. If $\gamma \in \Gamma$, then there is $g_\gamma \in \mathbb{G}$ such that

$$(\sigma(\gamma)\mathbb{B}, \sigma(\gamma)\mathbb{T}) = (g_\gamma \mathbb{B}, g_\gamma \mathbb{T}).$$

There is a choice of g_γ such that $\sigma'(\gamma) = Adg_\gamma^{-1} \circ \sigma(\gamma)$ is again a rational form. A form that admits a Γ–stable pair $(\mathbb{B}, \mathbb{T})$ is called quasisplit. The form is called split if $\mathbb{T}^{\sigma'(\Gamma)}$ is split. Every reductive group has a split form.

The action of Γ induces an action on the root datum; inner forms induce the same action. This also induces an action of Γ on $\check{\Psi}$, which in turn gives a homomorphism

$$\Gamma \longrightarrow Aut(\check{G}).$$

(These are complex linear homomorphisms). The (Langlands), dual group ${}^L G$ is defined as the semidirect product

$${}^L\mathbb{G}^\circ := \check{\mathbb{G}} \rtimes \Gamma$$

This group has a canonical map to Γ. If G is split, this action is trivial and the dual group is just $\check{G} \times \Gamma$. We will focus on this case because it is easier.

11.14. Let A be maximal split torus, $M = C_G(A)$ and so on as before. Then we have inclusions $X_*(A) \subseteq \Lambda \subseteq X_*(M)$. When G is split, $A = M$ so the above sets are equal. Let $\check{T}$ be the dual torus. Then unramified characters of A correspond to elements in $\check{T}$. The group W acts on all the objects, so W–conjugacy classes of unramified characters correspond to W–conjugacy classes of elements in $\check{T}$. Since $\check{T}$ embeds in $\check{G}$, we get

$$\left\{\begin{matrix}\text{Irreducible spherical}\\ \text{representations of } G\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{Conjugacy classes}\\ \text{of semisimple elements}\\ \text{in } \check{G}\end{matrix}\right\}. \tag{11.14.1}$$

More generally, the local LAnglands conjectures predict that $\check{G}$–conjugacy classes of maps

$$\begin{tikzcd} \mathcal{W}_{\mathbb{F}} \arrow[rr] \arrow[dr] & & {}^LG \arrow[dl] \\ & \Gamma & \end{tikzcd}$$

parametrize irreducible representations. $\mathcal{W}_{\mathbb{F}}$ is called the *Weil group* and comes equipped with a canonical map $\mathcal{W} \to \Gamma$. In the case of spherical representations, the homomorphisms factor through the inertia subgroup $\mathcal{W}_{\mathbb{I}}$. This has the property that $\mathcal{W}_{\mathbb{F}}/\mathcal{W}_{\mathbb{I}} \cong \mathbb{Z}$. Thus (11.14.1) becomes $\check{G}$–conjugacy classes of maps

$$\mathbb{Z} \longrightarrow {}^LG$$

with semisimple image compatible with the maps to Γ (when G is split the compatibility is trivial).

12. Iwahori-Hecke Algebra

12.1. Assume G is split. An Iwahori subgroup $\mathbb{I}$ is a group conjugate to the inverse image of a Borel subgroup $B(\mathbb{F}_q) \subseteq G(\mathbb{F}_q)$ in $\mathbb{G}(\mathcal{O})$. Recall that $G = K \cdot B$ where $K = G(\mathcal{O})$ and $B = AN$. Then consider $\mathbb{I}_\circ = K \cap A$, $\mathbb{I}_+ = K \cap N$ and $\mathbb{I}_- = K \cap N^-$. Then

$$\mathbb{I} = \mathbb{I}_+ \cdot \mathbb{I}_\circ \cdot \mathbb{I}_-$$

is an Iwahori subgroup which is good with respect to B.

12.2. In a previous section we proved that $V^{\mathbb{I}}$ maps onto V_N. We want to show that the map is an isomorphism. We will sketch two proofs in the case of $GL(2)$. The first one is from [MP]. Recall

$$G = GL(2),\ P = \left\{\begin{bmatrix} * & * \\ 0 & * \end{bmatrix}\right\},\ A = \left\{\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}\right\},\ N = \left\{\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix}\right\},\ \mathbb{I} = \left\{\begin{bmatrix} a & b \\ \pi c & d \end{bmatrix}\right\}$$

where $a, b, c, d \in \mathcal{O}$.

Write $a = \begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix}$. We want to show that if $0 \neq v \in V^{\mathbb{I}}$ then its image in V_N is $\neq 0$. To this end we construct a sequence $v_0 = v,\ v_1, \ldots$ such that all $v_i \neq 0$ map to the same element in V_N. We know that v_0 is invariant under $\mathbb{I}_+$. Consider

$$a^{-1}\mathbb{I}_+ a = \left\{\begin{bmatrix} 1 & \pi^{-1}b \\ 0 & 1 \end{bmatrix}\right\}, a^{-1}\mathbb{I}_- a = \left\{\begin{bmatrix} 1 & 0 \\ \pi^2 c & 1 \end{bmatrix}\right\}, a^{-1}\mathbb{I}_0 a = \mathbb{I}_0$$

Then $a^{-1}\mathbb{I}a$ is another Iwahori subgroup. Let

$$v_1 := \int\limits_{a^{-1}\mathbb{I}a} \pi(i)v_0 di = \int\limits_{a^{-1}\mathbb{I}_+a} \pi(i)v_0 di$$

Since $a^{-1}\mathbb{I}a \subset N$, $v_1 = \sum\limits_{n_k \in N} \pi(n_k)v_0, \pi_N(v_1) = \pi_N(v_0)$. Continue in the same manner by conjugating by a^{-1}. Recall that $V_N = V/V(N)$ where

$$V(N) = \left\{ v \in V \,\middle|\, \int\limits_{N'} \pi(n')v dn' = 0 \quad \text{for some open compact} \quad N' \subset N \right\}.$$

So if the image of v_0 is zero, one of the v_i has to be zero. This is because the union of $a^{-n}\mathbb{I}a^n$ is N, and if $v = \sum \pi(n_k)w_k - w_k$ eventually all the n_i are in $a^{-n}\mathbb{I}a^n$ and the integral is zero. So we need only show

$$v_k \neq 0 \quad \text{then} \quad v_{k+1} \neq 0.$$

We only need to do one step from v_0 to v_1; it's the same situation for the other k.

Now both $\mathbb{I}$ and $a^{-1}\mathbb{I}a$ are contained in

$$a^{-1}Ka = \left\{ \begin{bmatrix} a & \pi^{-1}b \\ \pi c & d \end{bmatrix} \right\}$$

and both contain

$$a^{-1}K_1a = \left\{ \begin{bmatrix} a & c \\ \pi^2 c & d \end{bmatrix} \right\}$$

Their quotients in $a^{-1}Ka/a^{-1}K_1a \simeq GL(2, \mathbb{F}_q)$ are

$$\mathbb{I} \leftrightarrow \left\{ \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \right\} \quad \text{and} \quad a^{-1}\mathbb{I}a \leftrightarrow \left\{ \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right\}$$

The claim will follow from the following lemma.

12.3. Lemma. *If $0 \neq v$ is invariant under $\left\{ \begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} \right\}$, then*

$$\int\limits_{\mathbb{F}} \pi\left(\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right) v \; dx \neq 0.$$

Proof. Write $G = GL(2, \mathbb{F}_q)$ and use $N, \overline{N}$ to denote the images of $\mathbb{I}$, $a^{-1}\mathbb{I}a$ and so on. Let $\mathcal{F}(G)$ be the space of functions on G with the left regular action. It is enough to show that if $v \in \mathcal{F}(G)$ is fixed by N, then

$$\sum_{\overline{n} \in \overline{N}} \pi(\overline{n}) \cdot v \neq 0.$$

An irreducible representation with an N–fixed vector embeds in an $\mathrm{Ind}_B^G \xi$ for some character ξ of B trivial on N. It is enough to show that

$$A : \mathrm{Ind}_B^G \xi \longrightarrow \mathrm{Ind}_B^G \xi A f(x) = \frac{1}{|\overline{N}|} \sum_{\overline{n}} f(\overline{n} x)$$

is injective on $(\mathrm{Ind}_B^G \xi)^N$. Recall that if $w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then $G = B \cup NwB$. So the N–invariant vectors have a basis f_0, f_1 where

$$\begin{aligned} f_0(n_1 w d n_2) &= 0, & f_0(dn) &= \xi^{-1}(d) \\ f_1(n_1 w d n_2) &= \xi^{-1}(d), & f_1(dn) &= 0 \end{aligned}$$

On the other hand $G = \overline{N} B \cup wB$. Then

$$\begin{aligned} Af(\overline{n}_0 dn) &= \xi(d)^{-1} \frac{1}{|\overline{N}|} \sum_{\overline{n}} f(\overline{n}) \\ Af(\overline{n}_0 w dn) &= \xi(d)^{-1} \frac{1}{|\overline{N}|} \sum_{\overline{n}} f(\overline{n} w d) = \xi(d)^{-1} \frac{1}{|\overline{N}|} \sum_{n} f(wn) \\ &= \xi(d)^{-1} f(w) \end{aligned}$$

So $Af_0(\overline{n} dn) = \xi(d)^{-1} \sum_{\overline{n}} f_0(\overline{n}) = \xi(d)^{-1}$ because

$$\begin{bmatrix} 1 & 0 \\ y & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \quad \text{only if} \quad x = y = 0.$$

On the other hand

$$Af_0(\overline{n}_0 w dn) = \xi(d)^{-1} f_0(w) = 0.$$

Furthermore,

$$Af_1(wdn) = \xi(d)^{-1} f_1(w) = \xi(d)^{-1}, \quad Af_1(\overline{n}_0 w dn) \xi(d)^{-1} \frac{1}{|\overline{N}|} \sum_{\overline{n}} f_1(\overline{n}).$$

Now $\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = n_1 w d n_2$ means

$$\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix} = \begin{pmatrix} 1 & x_1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & x \\ 0 & a^{-1} \end{pmatrix},$$

so $a = y$. Therefore,

$$Af_1(\overline{n}_0 w dn) = \xi(d)^{-1} \frac{1}{|\overline{N}|} \left(f_1(\mathbb{1}) + \sum_{y \neq 0} \xi \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \right)$$

Recall that $f_1(\mathbb{1}) = 0$. The other sum is zero unless ξ restricted to $\{\begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix}\}$ is trivial. Either way A is invertible.

12.3.1. Another version. This is the original proof of Borel–Casselman. Let $v \in V^{\mathbb{I}}$ be such that $\pi_N(v) = 0$. Then v is in the span of $\pi(n_i)w - w$, for n_i in a large open compact subgroup. Let

$$a = \begin{bmatrix} \pi & 0 \\ 0 & \pi^{-1} \end{bmatrix}.$$

Then

$$\pi(a)^n \pi(n_i)w - \pi(a)^n w = \pi(a^n n_i a^{-n})\pi(a)^n w - \pi(a)^n w$$

and $a^n n_i a^{-n} \in \mathbb{I}$ for large n (and all i). Thus $\pi(e_{\mathbb{I}} a^n e_{\mathbb{I}})v = 0$. It is enough to check that $e_{\mathbb{I}} a^n e_{\mathbb{I}}$ is invertible in $\mathcal{H}_{\mathbb{I}}$. We will describe the structure of the algebra $\mathcal{H}_{\mathbb{I}}$ and leave it as an exercise to check that this element is invertible.

12.4. We determine the structure of $\mathcal{H}$ in several examples, $GL(2)$, $SL(2)$ and the adjoint group of type A_1. Consider the elements $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}$. They generate a group

$$W = \left\{ \begin{pmatrix} \pi^{a_1} & 0 \\ 0 & \pi^{a_2} \end{pmatrix} \cup \begin{pmatrix} 0 & \pi^{a_1} \\ \pi^{a_2} & 0 \end{pmatrix} \right\}$$

Proposition. $G = \bigcup_{w \in W} \mathbb{I} w \mathbb{I}$.

Proof. Exercise.

Note that $W = W_0 \rtimes \Lambda$ where W_0 is generated by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\Lambda = \left\{ \begin{pmatrix} \pi^{a_1} & 0 \\ 0 & \pi^{a_2} \end{pmatrix} \right\}$
Another realization of W is as follows. Consider the elements

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & \pi \\ -\pi^{-1} & 0 \end{pmatrix}.$$

They generate a group W_{aff}. Let

$$\Omega := \left\{ \begin{pmatrix} \pi^{a_1} & 0 \\ 0 & \pi^{a_2} \end{pmatrix} \right\} \Big/ \left\{ \begin{pmatrix} \pi^{a} & 0 \\ 0 & \pi^{-a} \end{pmatrix} \right\}$$

This is a product

$$\Omega = \mathbb{Z} \times \left\{ \begin{pmatrix} \pi^{a} & 0 \\ 0 & \pi^{a} \end{pmatrix} \right\}$$

W_{aff} acts on this, trivially on the center and by affine transformations on $\mathbb{Z}$; $\mathbb{Z}$ is generated by $\begin{pmatrix} \pi & 0 \\ 0 & 1 \end{pmatrix}$ $\left(\text{modulo } \begin{pmatrix} \pi^{a} & 0 \\ 0 & \pi^{a} \end{pmatrix}\right)$.

We consider the W_{aff} part. For $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ we can just work inside K_0. We compute in the finite group.

$$G = B \cup BwB, \ |B| = (q-1)^2 q, \ |NwB| = q^2(q-1)^2 = q|B|.$$

Let

$$\frac{1}{|B|}\delta_B = \mathbb{1}, \qquad \delta_{BwB} = \frac{1}{|B|}T_w.$$

Then

$$T_w^2(x) = \frac{1}{|B|^2}\int\limits_{BwN} \delta_{NwB}(g^{-1}x)dg = \frac{1}{|B|}\int\limits_{WN} \delta_{BWN}(g^{-1}x)dg.$$

Then if $x = \mathbb{1}$ we get $\frac{q}{|B|}$, if $x = w$ we get $\frac{q-1}{q|B|}$. So $T_w^2 = (q-1)T_w + q\mathbb{1}$. Write T_1 for T_w and T_0 for the element corresponding to $\begin{pmatrix} 0 & -\pi \\ \pi^{-1} & 0 \end{pmatrix}$. Then $T_0^2 = (q-1)T_0 + q\mathbb{1}$ as well by considering the other maximal compact subgroup K_0'.

Now W_{aff} has a length function. It is generated by two reflections s_1 and s_0. If $w \in W_{aff}$ let $w = s_1 \cdots s_k$ be a minimal decomposition. We write $\ell(w) = k$; it is independent of the reduced decomposition. If $w \in W_{aff}$ and T_w is the corresponding coset, then $T_w = T_1 \cdots T_s$ independent of the reduced decomposition.

12.5. In order to determine the irreducible representations of $\mathcal{H}$ we need to determine the center. Set $\theta = \frac{1}{q}T_0T_1$. Then

$$\begin{aligned}
\theta T_1 \theta &= \frac{1}{q^2}T_0\underbrace{T_1T_1}T_0T_1 = \frac{1}{q^2}T_0\big((q-1)T_1 + q\big)T_0T_1 = \\
&= \frac{(q-1)}{q^2}T_0T_1T_0T_1 + \frac{1}{q}, T_0^2T_1 = (q-1)\theta^2 + \frac{(q-1)}{q}T_0T_1 + T_1 \\
&= T_1 + (q-1)\left[\theta^2 + \theta\right]
\end{aligned}$$

We can rewrite this as

$$\theta T_1 = T_1\theta^{-1} + (q-1)(\theta+1) = T_1\theta^{-1} + (q-1)\frac{\theta - \theta^{-1}}{1-\theta^{-1}} \tag{12.5.1}$$

Proposition.

(1) *Every element of $\mathcal{H}$ can be written uniquely as $TP(\theta,\theta^{-1}) + Q(\theta,\theta^{-1})$. Similarly as $P^1(\theta,\theta^{-1})T + Q^1(\theta,\theta^{-1})$*

(2) *The center of $\mathcal{H}$ is $\mathbb{C}\left[\theta + \theta^{-1}\right]$.*

Proof. (1) is clear from (12.5.1); uniqueness is an exercise. For (2)

$$\begin{aligned}
&T(\theta+\theta^{-1}) = \theta^{-1}T + (q-1)(\theta+\theta^{-1}) + \theta T - (q-1)(\theta+\theta^{-1}) = (\theta+\theta^{-1})T \\
&\theta TP(\theta,\theta^{-1}) + \theta Q(\theta,\theta^{-1}) = T\theta^{-1}P(\theta,\theta^{-1}) + (q-1)(\theta+1) + \theta Q(\theta,\theta^{-1}) \\
&= T\theta P(\theta,\theta^{-1}) + \theta Q(\theta,\theta^{-1})
\end{aligned}$$

So $(\theta - \theta^{-1})P(\theta,\theta^{-1}) = 0$, therefore $P(\theta,\theta^{-1}) \equiv 0$.

Suppose $TQ(\theta,\theta^{-1}) = Q(\theta,\theta^{-1})T$. Then note that

$$T\theta^n = \theta^{-n}T + (q-1)(\text{ terms in } \theta,\theta^{-1})$$

The proof follows by an induction in the degree of Q. $\square$

12.6. Representations. Since $\mathcal{A} := \operatorname{span}\{\theta, \theta^{-1}\}$ is abelian, any (finite dimensional) representation (π, V) of $\mathcal{H}$ must have a nonzero vector v such that

$$\theta v = \nu v \quad \text{for some} \quad \nu \in \mathbb{C}^{\times}.$$

Let

$$X(v) := \mathcal{H} \otimes_{\mathcal{A}} \mathbb{C}_{\nu}$$

Then there is a nonzero map

$$X(\nu) \longrightarrow V. \tag{10.6.1}$$

If V is irreducible, then this map is onto.

Remark. $X(\nu)$ is really the set of $\mathbb{I}$–fixed vectors of the module induced from the unramified character ν.

12.7. We analyze $X(\nu)$, in particular when it has a nontrivial quotient. First observe that $X(\nu)$ has a basis

$$1 \otimes 1\!\!1_{\nu}, \quad T \otimes 1\!\!1_{\nu}.$$

Let $\mathcal{H}_K \subseteq \mathcal{H}$ be the subalgebra generated by $T := T_1$. Then

$$T(T+1) = q(T+1) \quad \text{and} \quad T(T-q) = -(T-q) \tag{9.7.1}$$

This implies that as an $\mathcal{H}_K$ module, $X(\nu)$ decomposes into a direct sum.

$$X(\nu) \simeq (1+T) \otimes 1\!\!1_{\nu} \oplus (T-q) \otimes 1\!\!1_{\nu}$$

12.8. The finite Hecke algebra. Let (D) by any Dynkin diagram. The Weyl group W is the group generated by $s_1, \cdots, s_n$ (n the rank of (D)) subject to the relations:

(1) $s_i^2 = 1$,
(2) $(s_i s_j)^{m(i,j)} = 1$ ($m(i,j)$ the (i,j) entry in the Cartan matrix).

The Hecke algebra $\mathcal{H}_K$ attached to (D) with indeterminate q is the algebra generated over $\mathbb{C}[q, q^{-1}]$ by $T_1 \cdots, T_n$ subject to the relations

(1) $T_i^2 = (q-1)T_i + q$
(2) $\underbrace{T_i T_j \cdots T_i T_j}_{m(i,j)} = \underbrace{T_j T_i \cdots T_j T_i}_{m(i,j)}$

Theorem. *When q is specialized to a positive real number,*

$$\mathcal{H}_K \simeq \mathbb{C}[W].$$

An account of the basic properties of the finite Hecke algebra can be found in [Carter].

12.9. We return to the case of $SL(2)$. For $X(\nu)$ to have a proper nontrivial quotient, either $(1+T)\otimes \mathbb{1}_\nu$ or $(T-q)\otimes \mathbb{1}_\nu$ must be invariant under θ. We get

$$\begin{aligned} \theta(1+T) &= (1+T)\theta^{-1} + q\theta - \theta^{-1} + (q-1), \\ \theta(T-q) &= (T-q)^{-1} - \theta + q\theta^{-1} + (q-1). \end{aligned} \tag{12.9.1}$$

Tensoring with $\mathbb{1}_\nu$ sets $\theta = \nu$ in these equations. Then $(1+T)\otimes \mathbb{1}_\nu$ is a submodule when $\nu = -1$ or q^{-1} and $(T-q)\otimes \mathbb{1}_\nu$ is a submodule when $\nu = -1$ or q.

At $\nu = -1$, $X(\nu)$ becomes completely reducible and has two factors. At $\nu = q$, $(1+T)\otimes \mathbb{1}_\nu$ given rise to a quotient where T and θ both act by q. This is called the trivial module.

At $\nu = q^{-1}$, $X(\nu)$ has a quotient where T acts by -1 and θ acts by q^{-1}. This is called the Steinberg module.

12.10. We describe the Iwahori–Hecke algebra for $G = GL(2)$ in more detail. Write

$$T \longleftrightarrow \mathbb{I}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\mathbb{I} \quad \tau \longleftrightarrow \mathbb{I}\begin{pmatrix} 0 & 1 \\ \pi & 0 \end{pmatrix}\mathbb{I}$$

and let

$$\theta_1 = \frac{1}{q}(\tau T), \quad \theta_2 = \frac{1}{q}(T^{-1}\tau)$$

Then

$$T\theta_1 = \theta_2 T + (q-1)\theta_1, \quad T\theta_2 = \theta_1 T - (q-1)\theta_1, \quad \theta_1\theta_2 = \theta_2\theta_1.$$

In fact both $\theta_1\theta_2$ and $\theta_1+\theta_2$ are in the center of the algebra,

$$T(\theta_1+\theta_2) = \theta_2 T + (q-1)\theta_1 + \theta_1 T - (q-1)\theta_1 = (\theta_1+\theta_2)T.$$

12.11 The general case. Let $\Phi = (X, R, Y, \check{R})$ be a root datum where R is assumed reduced. We associate a Hecke algebra to this datum.

Let W_0 be the (finite) Weyl group associated to R, and fix $\Pi \subseteq R$ a set of simple roots. Then let $\mathcal{H}_K$ be the Hecke algebra associated to this group, generated by $T_1, \cdots, T_n$ subject to

(1) $T_i^2 = (q-1)T_i + q$
(2) $\underbrace{T_iT_j\cdots}_{m(i,j)} = \underbrace{T_jT_i\cdots}_{m(i,j)}$
(3) if $\ell(xy) = \ell(x)+\ell(y)$, then $T_xT_y = T_{xy}$.

Let $\mathcal{A} := \mathbb{C}[X]$ be the group algebra of X generated by $\{\theta_x\}_{x\in X}$ subject to

$$\theta_x\cdot\theta_y = \theta_{x+y}, \quad \theta_0 = \mathbb{1}.$$

The Iwahori–Hecke algebra $\mathcal{H}$ is the algebra generated by $\mathcal{H}_K$ and $\mathcal{A}$ subject to

$$T_s\theta_x = \theta_{s(x)}T_s + (q-1)\frac{\theta_x - \theta_{s(x)}}{1-\theta_{-\alpha}^2},$$

where $s = s_\alpha$ is the reflection corresponding to the simple root $\alpha \in \Pi$ (recall that W_0 acts on X and Y).

12.12. If G is a split p-adic group, its algebra $\mathcal{H}_{\mathbb{I}}$ is isomorphic to the above for the dual root system.

Example 1. $G = Sp(2)$ the symplectic group of rank 2. Then $\check{G}$ is $SO(5)$.

$$X = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2,\ R = \pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\},$$
$$Y = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2\ \check{R} = \{\pm\varepsilon_1, \pm\varepsilon_2, \pm\varepsilon_1 \pm \varepsilon_2\}.$$

Let $\theta_1 \longleftrightarrow \varepsilon_1, \quad \theta_2 \longleftrightarrow \varepsilon_2$. The simple roots are $\alpha_1 = \varepsilon_1 - \varepsilon_2,\ \alpha_2 = \varepsilon_2$. Let T_1 and T_2 be the corresponding elements in $\mathcal{H}_K$.

$$T_1\theta_1 = \theta_2 T_1 + (q-1)\frac{\theta_1 - \theta_2}{1 - \theta_2\theta_1^{-1}} = \theta_2 T_1 + (q-1)\theta_1$$

$$T_2\theta_1 = \theta_1 T_2 \qquad T_2\theta_2 = \theta_2^{-1}T_2 + (q-1)\frac{\theta_2 - \theta_2^{-1}}{1 - \theta_2^{-1}} = \theta_2^{-1}T_2 + (q-1)(\theta_2 + 1)$$

$$T_1\theta_2 = \theta_1 T_1 + (q-1)\frac{\theta_2 - \theta_1}{1 - \theta_2\theta_1^{-1}} = \theta_1 T_1 - (q-1)\theta_1$$

Example 2. Let $G = SO(5)$. Then $\check{G}$ is $Sp(2, \mathbb{C})$.

$$X = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2, R = \{\pm 2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j\}$$
$$Y = \mathbb{Z}\varepsilon_1 + \mathbb{Z}\varepsilon_2, \check{R} = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j\}.$$

Again let $\theta_1 \longleftrightarrow \varepsilon_1 \quad \theta_2 \longleftrightarrow \varepsilon_2$

$$T_1\theta_1 = \theta_2 T_1 + (q-1)\frac{\theta_1 - \theta_2}{1 - \theta_2\theta_1^{-1}} = \theta_2 T_1 + (q-1)\theta_1.$$

The other relations are the same, except

$$T_2\theta_2 = \theta_2^{-1}T_2 + (q-1)\frac{\theta_2 - \theta_2^{-1}}{1 - \theta_2^{-2}} = \theta_2^{-1}T_2 + (q-1)\theta_2.$$

Even though $sp(2) \simeq so(5)$, the corresponding Hecke algebras are not.

12.13 Theorem. *(Kazhdan–Lusztig) There is a 1–1 correspondence*

$$\left\{\begin{matrix}\textit{Irreducible Representations}\\ \textit{of } G \textit{ with } \mathbb{I} \textit{ - fixed vectors}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}(s, X, \sigma) : s \in \check{G} \textit{ semisimple},\\ X \in \check{\mathfrak{g}}, sXs^{-1} = qX, \sigma \in \widehat{A}(s, X)\end{matrix}\right\}\Big/ \check{G}$$

where $A(s, X) := C_{\check{G}}(s, X) \,/ [\mathcal{Z}(\check{G})C_{\check{G}}(s, X)_0]$.

Not all σ occur; one can tell precisely which ones do.

12.14. We now describe the parameter space in more detail and give some application.

Definition. *An element $X \in \mathfrak{g}$ is called nilpotent if* $\operatorname{ad} X$ *is nilpotent. An element $u \in G$ is called unipotent if $Adu - I$ is nilpotent.*

Proposition. *The exponential map gives a 1–1 correspondence between nilpotent elements in $\mathfrak{g}$ and unipotent elements in G.*

A Lie triple $\{X,\ H,\ Y\}$ are three elements in the Lie algebra satisfying

$$[H, X] = 2X, [H, Y] = 2Y, [X, Y] = H.$$

They span an $sl(2, \mathbb{C})$, in fact Lie triples are in 1–1 correspondence with homomorphisms

$$\phi :\ sl(2, \mathbb{C}) \longrightarrow \mathfrak{g}.$$

Theorem. *(Jacobson–Morozov) Let G be a (complex) connected semisimple Lie group with Lie algebra $\mathcal{G}$. Then There is a $1-1$ correspondence*

$$\{e \in \mathfrak{g} : \operatorname{ad} X \quad \textit{is nilpotent} \quad \}/_G \quad \longleftrightarrow \quad \{\varphi : s\ell(2, \mathbb{C}) \longrightarrow G\}/_G \ .$$

Two nilpotent elements $X,\ X'$ are conjugate under G if and only if the corresponding elements $H,\ H'$ are conjugate.

12.15. We want to analyze conjugacy classes under G of pairs (s, X), where $s \in G$ is semisimple and $X \in \mathfrak{g}$ is such that $Ads \cdot X = qX$.

By theorem 12.14 we can embed X in a Lie triple $\{X, H, Y\}$ Then let

$$s_X := \exp\left(\frac{1}{2} \log qH\right),$$

It satisfies

$$Ads_X \cdot X = qX.$$

Thus $s \cdot s_X^{-1}$ centralizes X.

Lemma 1. *$C_G(X) = L \cdot U$ where U is the unipotent radical of $C_G(X)$ and $L = C_G(H, X) = C_G(H, X, Y)$ is reductive.*

Thus we can write $s = s_0 \cdot s_X \cdot u$ where $s_0 \in L$ and $u \in U$.

Lemma 2. *Assume $s \in L$ is semisimple. If $su \in LU$ is semisimple, then su is conjugate (under U) to s.*

Proof. Decompose $\mathfrak{u} = \mathfrak{u}_1 + \dots \mathfrak{u}_k$ so that each $\mathfrak{u}_i$ is the sum of eigenspaces for $\operatorname{Ad} s$ with eigenvalues having the same real part. Let $U_i = exp(\mathfrak{u}_i)$. Then

$$U = U_1 \cdot U_2 \cdot \dots \cdot U_k.$$

Assume that the real parts of the eigenvalues are in strictly increasing order. Let $\alpha_1 \in U_1$. Then

$$\operatorname{Ad} \alpha_1(su_1u_2 \dots u_k) = s \cdot (s^{-1}(\alpha_1)\alpha u_1 \alpha_1^{-1}) \operatorname{Ad}(\alpha_1)(u_2 \dots u_k).$$

Now $\mathrm{Ad}(\alpha_1)(u_2 \ldots u_k) \in U_2 \cdot \ldots \cdot U_k)$. If none of the eigenvalues in $\mathfrak{u}_1$ are zero, then

$$\alpha_1 \mapsto s^{-1}(\alpha_1)\alpha u_1 \alpha_1^{-1}$$

is onto U_1 (modulo terms in U_2 and higher). So we can conjugate su into an element of the form $su_2 \cdot \ldots \cdot u_k$. We could continue this way to get rid of u_2 and so on to show that su is conjugate to s unless $\mathrm{Ad}\, s$ has an eignevalue of 0 (on $\mathfrak{u}$.) At this stage the element is of the form $su^0 u_j \cdot \ldots \cdot u_k$ with none of the u_i coming from an eigenspace with eigenvalue 0. We can continue the procedure using su^0 instead of s to get rid of the rest of the u_i. We end up with an element su^0 with s and u^0 commuting, s semisimple and u unipotent. But this is the Jordan decomposition. Since the original su was assumed semisimple, $u^0 = 1$. $\square$

Thus we may conjugate the data in theorem 12.13 to $s_0 s_X$ with $s_0 \in C_{\check{G}}(H, X)$ semisimple.

Lemma 3. *$(s_0 s_X, X)$ is conjugate to $(s_0' s_X', X')$ if and only if there is $g \in G$ such that*

$$Adg'(s_0, s_X, X) = (s_0', s_X', X').$$

So in theorem 12.13 conjugacy classes of (s, X) are in 1–1 correspondence with $C_{\check{G}}(s_X, X)$ conjugacy classes of semisimple elements s_0.

12.16 Example. Let $\check{G} = SL(2, \mathbb{C})$. There are two conjugacy classes of nilpotent elements,

$$X_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

For X_0, the Lie triple is $\{X_0 = 0, H_0 = 0, Y_0 = 0\}$. For X_1 the Lie triple is

$$X_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad H_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Y_1 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The centralizer of H_1 is the torus

$$T := \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} : z \neq 0 \right\}.$$

The only elements centralizing X_0 are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For X_0 any conjugacy class of a semisimple element in $SL(2, \mathbb{C})$ will do.

12.17 Example. Let $\check{G} = SO(3, \mathbb{C})$. Realize it as 3×3 matrices of determinant 1 stabilizing the form $x_1 x_3 + x_2^2$. The Lie algebra is formed of 3×3 matrices M such that

$$M \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot^t M = 0.$$

There are again two conjugacy classes of nilpotent elements,

$$X_0 = 0, \qquad X_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For X_0 we just get conjugacy classes of semisimple element in $SO(3)$. They are all conjugate to elements in the maximal torus

$$T = \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \right\}.$$

For X_1 we get

$$H_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \quad , \quad Y_1 = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 0 & -2 & 0 \end{pmatrix} \quad .$$

The centralizer $C_G(H_1, X_1)$ is just T. So there is only one representation attached to X_1.

On the other hand, for X_0 and

$$s = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

the centralizer is disconnected,

$$\begin{pmatrix} z & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & z^{-1} \end{pmatrix} \cup \begin{pmatrix} 0 & 0 & z \\ 0 & -1 & 0 \\ z^{-1} & 0 & 0 \end{pmatrix}$$

so the component group is a $\mathbb{Z}_2$. This has two characters. $\square$

Example 12.17 corresponds to

$$\Psi = (X = \mathbb{Z}\varepsilon, R = \{\pm 2\varepsilon\}, Y = \mathbb{Z}\varepsilon, \check{R} = \{\pm\varepsilon\}).$$

The relation in the Hecke algebra is

$$T\theta = \theta^{-1}T + (q-1)\frac{\theta - \theta^{-1}}{1 - \theta^{-2}} = \theta^{-1}T + (q-1)\theta.$$

For this algebra we get nontrivial submodules at $\nu = \pm q^{\frac{1}{2}}$ or $\nu = \pm q^{-\frac{1}{2}}$.

12.18. Fix an infinitesimal character, *i.e.* an orbit of a semisimple element $s \in \check{G}$. Let

$$H := C_{\check{G}}(s), \quad \mathcal{Y} = \{x \in \mathfrak{g} | \operatorname{Ad} s \cdot X = qX\}$$

Then H is reductive and acts on $\mathfrak{g}$.

Proposition. *H acts with finitely many orbits on $\mathcal{Y}$. There is a 1–1 correspondence*

$$\left\{ \begin{matrix} \textit{Irreducible representations} \\ \textit{of } \mathcal{H} \textit{ with infinitesimal characters} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \textit{irreducible } H\textit{–equivariant local} \\ \textit{systems on } \mathcal{Y} \end{matrix} \right\}.$$

More precisely a local system is a pair $\mathcal{O}_{\mathcal{Y}}, \xi)$ where $\mathcal{O}_{\mathcal{Y}} \subseteq \mathcal{Y}$ is an orbit of H and ξ is an irreducible representation of $C_H(\mathcal{Y}) \big/ C_H(\mathcal{Y})^0$ (Not all ξ appear in this correspondence).

12.19. In this correspondence, $\mathcal{O}_0 = \{0\}$ and $\xi = triv$ corresponds to the (unique) spherical irreducible representation with infinitesimal character s.

12.20. A parameter (s, X, ξ) is called distinguished if it is not contained in any Levi component $\check{M}$ of a proper parabolic subgroup of $\check{G}$.

Recall that an irreducible representation (π, V) of G is called a discrete series if any matrix entry $f_{\nu,\widetilde{\nu}} \in L^2(G)$.

Proposition. *(Kazhdan–Lusztig) An irreducible module for $\mathcal{H}$ is the $\mathbb{I}$-fixed vectors of a discrete series if and only if the corresponding (s, X) is distinguished.*

12.21. An irreducible module (π, V) is called tempered if it is a constituent of a (unitarily Harish–Chandra) induced module from a discrete series tensored with a unitary character on a Levi component.

Proposition. *(Kazhdan-Lusztig) A representation is tempered if and only if s_0 is elliptic (the closure of the group generated by s_0 is compact).*

Remark. General results about tempered representations and discrete series for p–adic groups can be found in chapter 4 of [Casselman]. They explain how one can detect whether a representation is tempered from $V^{\mathbb{I}}$. Another reference is [BW] where a proof of the Langlands classification is given.

12.22. $\mathcal{H}$ has an involution called the Iwahori–Matsumoto involution ι with the property that

$$\iota(T_i) = -qT_i \qquad i = 0, \ldots n.$$

It is also a $*$ algebra, i.e. it has a complex conjugate autiautomorphism

$$f^*(x) := \overline{f(x^{-1})}$$

So we can talk about hermitian and unitary modules for $\mathcal{H}$.

Theorem. *(Barbasch–Moy)*

(1) (π, V) *is unitary if and only if* $V^{\mathbb{I}}$ *is unitary.*
(2) (π, V) *is unitary if and only if* $(\iota(\pi), \iota(V))$ *is unitary.*
(3) *The irreducible module* $L = L(s_X, X, triv)$ *is tempered, therefore unitary. Then* $\iota(L) = L(s_X, Triv, triv)$ *is also unitary.*

(1) is an old conjecture of Borel–Casselman. Since $V^{\mathbb{I}}$ is finite dimensional, it replaces the problem of checking that a form is positive definite on an infinite dimensional space with the same problem on a finite dimensional space. In (3) the second module is spherical and is typically isolated in the unitary dual.

13. Langlands parametrization

13.1. Dual group. Recall that to each inner class of rational forms we have associated the (complex) *L–group* ${}^L G$ as follows. Let $\check{G}$ be the dual group. This is the usual object; if $G \leftrightarrow \Psi(G) := (X^*, R, X_*, \check{R})$, then $\check{G} \leftrightarrow \Psi(\check{G}) := (X_*, \check{R}, X^*, R)$. Then the automorphism σ induces a homomorphism

$$\check{\sigma} : \Gamma \longrightarrow Aut(\Psi(\check{G})), \tag{13.1.1}$$

but unlike for the rational forms the automorphisms are *algebraic.*

Definition. *Fix an inner class of rational forms corresponding to* $\sigma : \Gamma \to Aut(\Psi(G))$ *with dual* $\check{\sigma}$*. Then* ${}^L G$ *is defined as*

$${}^L G := \check{G} \ltimes \Gamma.$$

This is a complex reductive algebraic group having the following structure.

(1) There is a short exact sequence

$$1 \longrightarrow \check{G} \longrightarrow {}^L G \longrightarrow \Gamma \longrightarrow 1. \tag{13.1.2}$$

(2) If $\gamma \in \Gamma$, then any preimage g_γ acts by $\check{\sigma}(\gamma)$.

(3) There is an open subgroup $\Gamma_0 \subset \Gamma$ such that $\check{G} \times \Gamma_0$ is isomorphic to an open subgroup of ${}^L G$.

Typically, a Langlands parameter is a continuous homomorphism

$$\phi : W_F \longrightarrow {}^L G \tag{13.1.3}$$

which is compatible with the canonical maps into Γ, *i.e.* the diagram

$$\begin{array}{ccc} W_F & \xrightarrow{\phi} & {}^L G \\ \downarrow & & \downarrow \\ \Gamma & \xrightarrow{id} & \Gamma \end{array} \tag{13.1.4}$$

is commutative.

13.2. Real case. In this case $\Gamma = \mathbb{Z}_2$. We write σ for the automorphism corresponding to the nontrivial element of Γ and $\check{\sigma}$ for the automorphism defining ${}^L G$.

The *Weil group* $W_{\mathbb{R}}$ is the group

$$W_{\mathbb{R}} = \mathbb{C}^{\times} \times \{1, j\}, \qquad jzj^{-1} = \overline{z},\ j^2 = -1. \tag{13.2.1}$$

Definition. *A* Langlands parameter *is a (real) Lie group homomorphism*

$$\phi \ :\ W_{\mathbb{R}} \longrightarrow {}^L G \tag{13.2.2}$$

compatible with the maps to Γ *and such that* $\phi(\mathbb{C}^{\times})$ *is formed of semisimple elements.*

Write this set as $\Pi({}^L G)$ and let $\check{G}$ act in the obvious way.

Definition. *The* pure Langlands component group *is the component group*

$$A^{loc}_{\phi} = \check{G}(\phi)/\check{G}(\phi)_0, \tag{13.2.3}$$

where $\check{G}(\phi)$ *is the centralizer of the image of* ϕ.

The pairs (ϕ, τ), where $\tau \in \widehat{A^{loc}_{\phi}}$, are called *complete Langlands parameters* and the set of equivalence classes under the action of $\check{G}$ will be denoted $\Xi(G/\mathbb{R})$.

Here is the main theorem.

Theorem. *There is a natural bijection*

$$\Pi(G/\mathbb{R}) \leftrightarrow \Xi(G/\mathbb{R}),$$

where $\Pi(G/\mathbb{R})$ *denotes the set of equivalence classes of irreducible admissible modules for all the inner forms of* G.

Remark. This is one of the main results in [ABV]. Langlands and Shelstad have proved such statements much earlier. The difference is perhaps in the details. For example the set $\widehat{A}(\phi)$ always has a distinguished element, namely the trivial local system. To make the correspondence as canonical as possible, one has to choose a similar irreducible representation in each packet. For this one can fix a Whittaker model. Another difference is that the definition of a real form is modified so as to make the statement true. I will ignore most of these finer points and concentrate on examples.

We describe the parametrization of $\Pi(G/\mathbb{R})$ in the case of regular infinitesimal character without recourse to the Weil group homomorphisms. Suppose that $G(\mathbb{R})$ is the set of fixed points of the conjugation σ. Let θ be a Cartan involution commuting with σ. Then a parameter is a triplet $\gamma = (H(\mathbb{R}), \Lambda, \lambda)$ as follows:

- $H = T \times A$ is a θ–stable σ–stable Cartan subgroup with Cartan subalgebra $\mathfrak{h}$,
- Λ is a character of $H(\mathbb{R})$, and λ is a character of $\mathfrak{h}(\mathbb{R})$; we are assuming that λ is regular.
- Let $R_{im}(\lambda) = R_{im,c} \cup R_{im,nc}$ be the positive system of imaginary roots determined by λ. Then
$$d\Lambda = \lambda + \rho(R_{im}) - 2\rho(R_{im,cpct}).$$

By conjugating by the (real) Weyl group, we can also choose the parameter so that λ is dominant with respect to some fixed positive system of real roots. Then to such a parameter one can associate a *Standard module* $M(\gamma)$ which is induced from a Discrete series on a Levi component and has a unique irreducible quotient $L(\gamma)$. This is a variant of the Langlands calssification stated in section 2.7.

For singular infinitesimal character the situation is even more complicated , but of the same type.

13.3. Langlands Parameters. We now describe the Langlands parameters more explicitly. First an elementary lemma.

Lemma. *Let H be a complex group with Lie algebra $\mathcal{L}(H) = \mathfrak{h}$. Then there is a bijection between the set of* real *Lie homomorphisms $\phi : \mathbb{C}^\times \longrightarrow H$ and the set of pairs $(\lambda, \mu) \in \mathfrak{h} \times \mathfrak{h}$ such that $[\lambda, \mu] = 0$ and $\exp(2i\pi(\lambda - \mu)) = 1$.*

We recall the formula

$$\phi(e^t) = \exp(t\lambda + \bar{t}\mu).$$

Proposition. *The set of Langlands parameters can be put in 1–1 correspondence with the set of pairs*

$$\{\ (y, \lambda) \in {}^L G \times \check{\mathfrak{g}} \mid \lambda \textit{ semisimple } \}$$

subject to the conditions

(1) $[\lambda, Ad(y)\lambda] = 0$,
(2) $y^2 = \exp(2i\pi\lambda)$,
(3) $y \in \check{G}^\Gamma - \check{G}$ *is semisimple.*

We write $e(\lambda) = \exp(2i\pi\lambda)$. We also recall that the μ in lemma 13.3 is $\mu = Ad(\phi(j))\lambda$ and $y = \exp(i\pi\lambda)\phi(j)$.

Example I. Suppose G a complex group viewed as a real group. Choose a conjugation which we denote by $\overline{}$ so that we can identify G_c with $G \times G$ and take

$$\sigma(x, y) = (\overline{y}, \overline{x}) \tag{13.3.1}$$

for such a real form. Then $\check{G} = G \times G$ and $\check{\sigma}(x, y) = (y, x)$. Write $y = [(a, b), \tau]$ and $\lambda = (\lambda_L, \lambda_R)$. By conjugating everything by $\check{G}$, we may assume that $y = [(1, \beta), \tau]$. Because $y^2 = [\beta, \beta, 1]$, conditions (1)–(3) become $\beta = \exp(2i\pi\lambda_L) = \exp(2i\pi\lambda_R)$ and $[\lambda_L, \lambda_R] = 0$. The component group is trivial.

Suppose $\beta = 1$; then these data give the usual conditions for a parameter of an irreducible Harish–Chandra module. There is a single conjugacy class of Cartan subgroups with representative $H = TA$. If $\mathfrak{h}$ is the Cartan subalgebra, it decomposes into $\sqrt{-1}\mathfrak{h}_\mathbb{R} + \mathfrak{h}_\mathbb{R}$, the first one is $\mathfrak{t}$, the second one $\mathfrak{a}$. Since $\mu = \lambda_L - \lambda_R$ is integral, it defines a character $\mathbb{C}_\mu$ of T and $\nu = \lambda_L + \lambda_R$ defines a character $\mathbb{C}_\nu$ of A. Let $B \supset H$ be a Borel subgroup. Then we can take

$$M(\gamma) = Ind_B^G(\mathbb{C}_\mu \otimes \mathbb{C}_\nu)_{K-finite}.$$

This has a unique irreducible subquotient $L(\gamma)$ determined by the fact that it contains the K–type $V(\mu)$ with extremal weight μ. This is a variant of section 2.7 for complex groups proved by Parthasarathy-Rao-Varadarajan. Furthermore, we note that the λ in Proposition 13.3 corresponds to the infinitesimal character of the representation. $\square$

Example II. Suppose $G = SL(2,\mathbb{C})$. Then

$$\sigma(g) = \overline{g}, \qquad \text{the usual conjugation on each entry} \tag{13.3.2}$$

gives the real form $SL(2,\mathbb{R})$. The dual group is $PSL(2,\mathbb{C}) \times \Gamma$. Let $y = [a,\tau]$. Then $a^2 = e(\lambda)$ and $[\lambda, Ad(a)\lambda] = 0$. Conjugate the data so that $\lambda,\ Ad(a)\lambda$ are in the diagonal Cartan subalgebra. Assume $e(\lambda) = 1$ and λ is regular for simplicity. We can conjugate the data so that $\lambda = \begin{bmatrix} m & 0 \\ 0 & -m \end{bmatrix}$ for some positive integer m. Then a is in the normalizer of the diagonal Cartan subalgebra,

$$\begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \cup \begin{bmatrix} 0 & z \\ -z^{-1} & 0 \end{bmatrix}.$$

There are three orbits of the centralizer of λ, $a_1 = 1$, $a_2 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$ and $a_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The component group is trivial in the first two cases and $\mathbb{Z}_2$ in the third case,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}.$$

(this is $PSL(2)$!).

On the other hand, in $SL(2,\mathbb{R})$ there are two conjugacy classes of Cartan subgroups:

$$H_c = \{\begin{bmatrix} \cos r & \sin r \\ -\sin r & \cos r \end{bmatrix}\} \qquad \mathfrak{h}_c = \{\begin{bmatrix} 0 & r \\ -r & 0 \end{bmatrix}\},$$
$$H_s = \{\pm\begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix}\} \qquad \mathfrak{h}_s = \{\begin{bmatrix} t & 0 \\ 0 & -t \end{bmatrix}\}.$$

Given a fixed integral infinitesimal character λ, there are two parameters attached to H_c and two attached to H_s. These are the two discrete series and the two principal series. Then a_1 corresponds to the principal series $X(\varepsilon,\nu)$ which is irreducible. Next a_2 corresponds to the (other) principal series which contains a finite dimensional representation. Finally the two local systems for a_3 correspond to the two discrete series.

Example III. Consider rational forms of the adjoint group $PSL(2,\mathbb{C})$. These can be identified with $SO(3)$ and $SO(2,1)$. Note that $SO(2,1)$ is disconnected. You can see this in the realization of $PSL(2)$ as $SL(2)/\pm Id$ by observing that $\begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$ is fixed by σ.

The dual group is ${}^LG = SL(2,\mathbb{C}) \times \Gamma$. In this case we assume again that λ is regular integral. We may again conjugate the data so that λ is in the diagonal Cartan subalgebra, but there are two cases $e(\lambda) = Id$ and $e(\lambda) = -Id$. They correspond to λ being diagonal with integer or half–integer entries,

$$\lambda = \begin{bmatrix} m & 0 \\ 0 & -m \end{bmatrix}, \qquad m > 0.$$

When $e(\lambda) = Id$, the choices for a are $a_{1,3} = \pm\, Id$. These correspond to irreducible principal series of $SO(2,1)$ (M° is disconnected).

In the case $e(\lambda) = -Id$, there are three orbits of a's with representatives

$$a_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad a_2 = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \qquad a_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

The local component groups are trivial for the first two a's, but a $\mathbb{Z}_2$ for the third case. The local systems on the third case correspond to discrete series on $SO(2,1)$. The first two orbits correspond one to the finite dimensional representations, one for $SO(2,1)$ the other for $SO(3)$.

The rest of the section is devoted to describing a pair $(H, \mathcal{Y})$ where $\mathcal{Y}$ is a space on which H acts with finitely many orbits. The H–equivariant local systems on the orbits are in 1–1 correspondence with the Langlands parameters. The geometry of the singularities has representation theoretic significance. The serious reader should consult [ABV], particularly the introduction.

An *infinitesimal character* is an orbit $\mathcal{O}_{\mathbb{R}} = (\mathcal{Y}, \mathcal{O})$ where

$$\begin{aligned} &\mathcal{O} \text{ is a semisimple } \check{G}\text{–conjugacy class of an element } \lambda \in \check{\mathfrak{g}} \\ &\mathcal{Y} \text{ is a } \check{G}\text{–conjugacy class of an element } y \in \check{G}^{\Gamma} - \check{G}, \\ &y^2 \text{ is conjugate to } e(\lambda) = exp(2i\pi\lambda) \text{ under } \check{G}. \end{aligned} \tag{13.3.3}$$

Denote by $\mathcal{C}(\mathcal{O}_{\mathbb{R}})$ the $\check{G}$–conjugacy class $\mathcal{Y}^2 = e(\mathcal{O})$.

Definition. *The set of Langlands parameters with infinitesimal character $\mathcal{O}_{\mathbb{R}}$ is*

$$P(\mathcal{O}_{\mathbb{R}}, \check{G}^{\Gamma}) = \{\ \phi = (y, \lambda) \in P({}^L G)\ :\ y \in \mathcal{Y},\ \lambda \in \mathcal{O}\ \}. \tag{13.3.4}$$

We make the simplifying assumption that $\mathcal{C}(\mathcal{O}_{\mathbb{R}})$ is trivial.

This is the same as requiring that the infinitesimal character be integral.

Then the $\mathcal{Y}$ in the orbit $\mathcal{O}_{\mathbb{R}}$ can be identified with just $\{y \mid y^2 = 1\}$. For $y \in \mathcal{Y}$, set

$$K(y) := \text{ centralizer of } y \text{ in } \check{G}. \tag{13.3.5}$$

Then y defines an involutive automorphism $\theta(y)$ of $\check{G}$ with fixed point group $K(y)$. Then set

$$\mathcal{O}^{comm}(y) := \{\lambda \in \mathcal{O} \mid [\lambda, \theta(y)\lambda] = 0\}. \tag{13.3.6}$$

Proposition. *There is a $\check{G}$–equivariant map between $P(\mathcal{O}_{\mathbb{R}}, {}^L G)$ and the* fiber product

$$\check{G} \times_{K(y)} \mathcal{O}^{comm}(y).$$

In particular, the orbits of $\check{G}$ on $P(\mathcal{O}_{\mathbb{R}}, {}^L G)$ are in one–to–one correspondence with orbits of $K(y)$ on $\mathcal{O}^{comm}(y)$; and there are finitely many orbits. Furthermore, these orbits are closed.

This is the analog of the maps $\lambda : W_F \longrightarrow {}^L G$ in the p–adic case. But because the orbits are closed, this is not good for doing geometry.

13.4. Geometric Parameter Space. We now give the description of the *geometric parameters* which will have singular orbits, even though the group acting and the parametrization of orbits will be the same.

Given λ, let $\check{\mathfrak{g}}_i = \{ \ x \ : \ [\lambda, x] = ix \ \}$, and recall the map $e(\lambda) = \exp(2i\pi\lambda)$. The *canonical flat through* λ is defined to be

$$\mathcal{F}(\lambda) := \lambda + \sum_{i \in \mathbb{N}^+} \check{\mathfrak{g}}_i. \tag{13.4.1}$$

Write $\mathcal{F}(\check{\mathfrak{g}})$ for the set of all canonical flats. For a fixed λ, we consider

$$\begin{aligned}
\check{\mathfrak{g}}(\lambda) &:= \sum_{n \in \mathbb{Z}} \check{\mathfrak{g}}(\lambda)_n, & H(\lambda) &:= C_G(e(\lambda)) \\
\check{\mathfrak{l}}(\lambda) &:= \check{\mathfrak{g}}(\lambda)_0 = C_{\check{\mathfrak{g}}}(\lambda)_n, & L(\lambda) &:= C_H(\lambda) \\
\check{\mathfrak{n}}(\lambda) &:= \sum_{n \in \mathbb{N}} \check{\mathfrak{g}}(\lambda), & N(\lambda) &\text{ the corresponding Lie group,} \\
\check{\mathfrak{p}}(\lambda) &:= \check{\mathfrak{l}}(\lambda) + \check{\mathfrak{n}}(\lambda), & P(\lambda) &:= L(\lambda)N(\lambda).
\end{aligned} \tag{13.4.2}$$

Then $H(\lambda)$ is a reductive group, $P(\lambda)$ is a parabolic subgroup and most important,

$$N(\lambda) \cong \lambda + \check{\mathfrak{n}}(\lambda), \qquad \text{via} \qquad n \mapsto Ad(n)\lambda.$$

so that

$$\mathcal{F}(\lambda) = Ad(P(\lambda)) \cdot \lambda = Ad(N(\lambda)) \cdot \lambda = \lambda + \check{\mathfrak{n}}(\lambda). \tag{13.4.3}$$

Thus $\mathcal{F}(\lambda)$ is an affine space for $N(\lambda)$ and a homogeneous space for $P(\lambda)$.

We now specialize to the case when $\lambda \in \mathcal{O}$ for a fixed orbit $\mathcal{O}$. Fix $\Lambda \in \mathcal{F}(\mathcal{O})$. Write $P(\Lambda)$ for the stabilizer of Λ in $\check{G}$. Then $P(\Lambda) = L(\Lambda)N(\Lambda)$ is a parabolic subgroup, and if $\lambda \in \Lambda$, then $\mathcal{F}(\Lambda) = Ad(N)\lambda$. Thus e is constant on Λ and $\mathcal{F}(\mathcal{O})$ can be identified with the variety $\mathcal{P}$ of parabolic subgroups of type $P(\Lambda)$.

Definition. *The set of Langlands parameters with infinitesimal character $\mathcal{O}_{\mathbb{R}}$ is*

$$P(\mathcal{O}_{\mathbb{R}}, \check{G}^{\Gamma}) = \{ \ \phi = (y, \lambda) \ : \ y \in \mathcal{Y}, \ \lambda \in \mathcal{O} \ \}.$$

The geometric parameter space of infinitesimal character $\mathcal{O}_{\mathbb{R}}$ is the fiber product

$$X(\mathcal{O}_{\mathbb{R}}, \check{G}^{\Gamma}) = \mathcal{Y} \times_{\mathcal{C}(\mathcal{O}_{\mathbb{R}})} \mathcal{F}(\mathcal{O}),$$

obtained by using the map e and the squaring map from $\mathcal{Y}$ to $\mathcal{C}(\mathcal{O}_{\mathbb{R}})$.

Remark. In general (even when $e(\lambda) \neq 1$), the space $\mathcal{F}(\mathcal{O})$ inherits an algebraic structure and so does $\mathcal{C}(\mathcal{O}) \cong \check{G}/C_{\check{G}}(e(\lambda))$. The map $e : \mathcal{F}(\mathcal{O}) \longrightarrow \mathcal{C}(\mathcal{O})$ is then algebraic.

Fix $y \in \mathcal{Y}$ such that $y^2 = 1$ (recall $\mathcal{C}(\mathcal{O}_{\mathbb{R}}) = 1$). Write $K(y)$ for the centralizer of y, a symmetric subgroup of $\check{G}$.

Proposition. *The group $\check{G}$ acts on $P(\mathcal{O}_{\mathbb{R}}, \check{G}^{\Gamma})$ and $X(\mathcal{O}_{\mathbb{R}}, \check{G}^{\Gamma})$ in a natural fashion. There is a $\check{G}$–equivariant isomorphism*

$$X(\mathcal{O}_{\mathbb{R}}, \check{G}^{\Gamma}) \cong \check{G} \times_{K(y)} \mathcal{P}.$$

The orbits of $\check{G}$ on X are in 1–1 correspondence with the orbits of $K(y)$ on $\mathcal{P}$; in particular there are only finitely many of them. The isotropy group $\check{G}^x$ at a point $x = (y, \Lambda) \in X$ is $K(y) \cap P(\Lambda)$.
The map

$$p : P(\mathcal{O}_{\mathbb{R}}, \check{G}^{\Gamma}) \longrightarrow X(\mathcal{O}_{\mathbb{R}}, \check{G}^{\Gamma})$$

induces a bijection between $\check{G}$–conjugacy classes on these two spaces. The component groups of the centralizers correspond.

13.5. Example. Let $G = SL(2, \mathbb{R})$. Then $\check{G}^{\Gamma} = PSL(2) \times \{1, \tau\}$, direct product, where $\tau^2 = 1$. Take λ to be regular integral, in fact such that $e(\lambda) = 1$. Then $\mathcal{P} = \mathcal{B}$, the variety of Borel subgroups. There are two conjugacy classes of y's,

$$\begin{aligned} &y = \tau, && K(y) = PSL(2, \mathbb{C}), \\ &y = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \tau, && K(y) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \cup \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \end{aligned} \tag{13.5.1}$$

Consider the second case. Identifying $\mathcal{B}$ with a sphere, there are two orbits, the union of the North and South pole, and the rest. The component group for the open orbit is $\mathbb{Z}_2$. The two characters correspond to $DS(\pm)$. The component group for the poles is trivial. The corresponding representation is the trivial representation occuring in the spherical principal $PS(sph)$. The other (irreducible) principal series $PS(sgn)$ of $SL(2)$, matches the orbit of the *other* y.

14. Arthur's Conjectures

14.1. In the previous sections we have seen statements describing the equivalence classes of irreducible admissible modules in terms of homomorphisms

$$\varphi \ : \ \mathcal{W}_{\mathbb{F}} \longrightarrow {}^{L}G. \tag{14.1.1}$$

These are special cases of more general conjectures of Langlands. Let k be a global (number) field. Then homomorphisms

$$\varphi \ : \ \mathcal{W}_{\mathbb{A}} \longrightarrow {}^{L}G \tag{14.1.2}$$

are conjectured to describe the discrete spectrum of $L^2(G(\mathbb{A})/G(k))$. I refer to the lectures of A. Borel and refereneces therein for more details about the spectrum of $L^2(G(\mathbb{A})/G(k))$. If $\mathbb{F} = k_v$ for some place v of k, then $\mathcal{W}_{\mathbb{F}}$ maps to $\mathcal{W}_{\mathbb{A}}$ and (14.1.1) is obtained from (14.1.2) by composition.

Arthur refined these conjectures in [Arthur]. He replaced (14.1.2) by maps

$$\Psi \ : \ \mathcal{W}_{\mathbb{A}} \times SL(2, \mathbb{C}) \longrightarrow {}^{L}G. \tag{14.1.3}$$

In particular, the discrete spectrum is supposed to correspond to maps Ψ such that the image is not contained in any Levi component of a proper parabolic subgroup. For $GL(n)$, the residual spectrum is given in [MW], confirming many of these conjectures for this case.

14.2. Further work along these lines can be found in [M]. Suppose $\mathbb{G}$ is split. Let $\mathbb{K} = \prod_v K_v \subset G(\mathbb{A})$. Then consider the special case of automorphic forms with $\mathbb{K}$ fixed vectors. For this special case, $\Psi|_{\mathcal{W}_{\mathbb{A}}}$ is trivial.

14.3. The part of the continuous spectrum attached to the case in 14.2 comes from Eisenstein series attached to the minimal parabolic subgroup. One starts with Paley–Wiener functions ϕ on $\mathfrak{a} \cong \mathbb{C}^\ell$ where ℓ is the rank. These correspond to Eisenstein series. The inner product is

$$\langle \phi, \psi \rangle = \int_{Re\lambda = \Lambda_0} \sum_{\sigma \in W} r(\sigma, \lambda) \phi(\lambda) \overline{\psi(-\sigma\overline{\lambda})}$$

for Λ_0 very dominant. The $r(\sigma, \lambda)$ are meromorphic functions which we will define later. We deform the integration from Λ_0 to 0.

14.4. Consider the case of $\mathbb{G} = SO(5)$. The dual group is $Sp(4)$, the symplectic group of rank 2. We write $\lambda = (\nu_1, \nu_2)$. Note that we are working in the dual group $\check{G} = Sp(4)$ of rank two. The roots are all *coroots* already. The set of positive roots is

$$\Delta^+ = \{\ \varepsilon_1 - \varepsilon_2,\ \varepsilon_1 + \varepsilon_2,\ 2\varepsilon_1,\ 2\varepsilon_2\ \}.$$

Write

$$\mathcal{L}(s) = \frac{L(s)}{\varepsilon(s) L(1+s)}.$$

for the functions that enter in the formula for the intertwining operator $r(\sigma, \lambda)$

$$r(\sigma, \lambda) = \prod_{\substack{\alpha \in \Delta^+ \\ w\alpha \notin \Delta^+}} \mathcal{L}(< \lambda, \alpha >).$$

Then the $\mathcal{L}$ have the following properties:

$$\begin{aligned} \mathcal{L}(s) &= \frac{a_{-1}}{s-1} + a_0 + a_1(s-1) + \ldots, \qquad &\text{near } s = 1, \\ \mathcal{L}(s) &= -1 + c_1 s + \ldots, \qquad &\text{near } s = 0. \end{aligned}$$

We also have the relation

$$\mathcal{L}(s) \cdot \mathcal{L}(-s) = 1.$$

We get the following table.

σ^{-1}	σ	Roots	Intertwining Operator
$(3,1)$	$(3,1)$	$-$	1
$(3,-1)$	$(3,-1)$	$(0,2)$	$\mathcal{L}(2\nu_2)$
$(1,3)$	$(1,3)$	$(1,-1)$	$\mathcal{L}(\nu_1-\nu_2)$
$(1,-3)$	$(-1,3)$	$(2,0),\ (1,-1)$	$\mathcal{L}(2\nu_1)\cdot\mathcal{L}(\nu_1-\nu_2)$
$(-1,3)$	$(1,-3)$	$(0,2),\ (1,1)$	$\mathcal{L}(2\nu_2)\cdot\mathcal{L}(\nu_1+\nu_2)$
$(-3,1)$	$(-3,1)$	$(2,0),\ (1,1),\ (1,-1)$	$\mathcal{L}(2\nu_1)\cdot\mathcal{L}(\nu_1-\nu_2)\cdot\mathcal{L}(\nu_1+\nu_2)$
$(-1,-3)$	$(-1,-3)$	$(2,0),\ (0,2),\ (1,1)$	$\mathcal{L}(2\nu_1)\cdot\mathcal{L}(2\nu_2)\cdot\mathcal{L}(\nu_1+\nu_2)$
$(-3,-1)$	$(-3,-1)$	$(2,0),\ (0,2),\ (1,1),\ (1,-1)$	$\mathcal{L}(2\nu_1)\cdot\mathcal{L}(2\nu_2)\cdot\mathcal{L}(\nu_1+\nu_2)\cdot\mathcal{L}(\nu_1-\nu_2)$

14.5. We continue $\Lambda_0=(\nu_1,\nu_2)$ from the dominant chamber to $(0,0)$. There are poles on the lines

$$\nu_1-\nu_2=1,\qquad 2\nu_2=1,\qquad \nu_1+\nu_2=1,\qquad 2\nu_1=1. \tag{14.5.0}$$

Choose a path that crosses exactly one line at a time in the order specified in (14.5.0). We summarize the terms that we get. On each line we pick up a residue that we analyze further.

- On $\nu_1-\nu_2$ introduce coordinates $(\frac{1}{2}+\nu,-\frac{1}{2}+\nu)$. Possible residues occur at $(3/2,1/2)$, $(1,0)$ and $(1/2,-1/2)$. The point $(3/2,1/2)$ gives a discrete contribution, $(1,0)$ gives 0 and $(1/2,-1/2)$ gives a 1–dimensional integral along $Re\nu=0$. The calculations are in (14.5.1).
- On $2\nu_2=1$ introduce coordinates $(\nu,\frac{1}{2})$. Possible poles are at $(1/2,1/2)$ and $(0,1/2)$ The term at $(1/2,1/2)$ cancels with the term on $\nu_1+\nu_2=1$. The term for $(0,1/2)$ gives a 1–dimensional integral along $Re\nu=0$. The calculations are in (14.5.2).
- On $\nu_1+\nu_2=1$ introduce coordinates $(1/2+\nu,1/2+\nu)$. The only possible pole occurs at $\nu=0$. The residue cancels the one in (14.5.2). This calculation is in (14.5.3).
- On the line $2\nu_1=1$ the coordinates are $(1/2,\nu)$ and the residue gives an integral along $Re\nu=0$. This combines with the integral in (14.5.1) on $(\nu,1/2)$. The calculations are in (14.5.4).
- Finally at $(0,0)$ we get a double integral.

14.5.1. On $\nu_1-\nu_2=1$, we write

$$\lambda=(1/2+\nu,-1/2+\nu),\qquad -\overline{\lambda}=(-\frac{1}{2}-\nu,\frac{1}{2}-\nu). \tag{14.5.1.1}$$

The residue turns out at $\sigma = (1,3),\ (-1,3),\ (-3,1),\ (-3,-1)$:

$$a_{-1}\phi(\frac{1}{2}+\overline{\nu},-\frac{1}{2}+\overline{\nu})\cdot\{\overline{\psi}(\frac{1}{2}-\overline{\nu},-\frac{1}{2}-\overline{\nu})+\overline{\psi}(\frac{1}{2}-\overline{\nu},\frac{1}{2}+\overline{\nu})\mathcal{L}(1+2\nu)+$$
$$\overline{\psi}(\frac{1}{2}+\overline{\nu},\frac{1}{2}-\overline{\nu})\mathcal{L}(1+2\nu)\mathcal{L}(2\nu)+\overline{\psi}(\frac{1}{2}+\overline{\nu},-\frac{1}{2}+\overline{\nu})\mathcal{L}(1+2\nu)\mathcal{L}(2\nu)\mathcal{L}(-1+2\nu)\} \tag{14.5.1.2}$$

The term can have a pole at $\nu = 1,\ \frac{1}{2}$ and $\nu = 0$.

$\nu = 1$. In this case, $Re(\lambda) = (\frac{3}{2},\frac{1}{2})$ and all the terms are holomorphic except for $\sigma = (-3,-1)$,

$$a_{-1}^2\phi(3/2,1/2)\overline{\psi}(3/2,1/2). \tag{14.5.1.3}$$

$\nu = \frac{1}{2}$. In this case $Re(\lambda) = (1,0)$. The first two terms do not have poles, the last two give residues

$$a_{-1}^2\phi(1,0)\overline{\psi}(1,0)\cdot(\mathcal{L}(2)+\mathcal{L}(2)\mathcal{L}(0)) = 0. \tag{14.5.1.4}$$

$\nu = 0$. The term coincides with (I.2.3.8).

14.5.2. On $2\nu_2 = 1$, we write

$$\lambda = (\nu,1/2), \qquad -\overline{\lambda} = (-\overline{\nu},-\frac{1}{2}). \tag{14.5.2.1}$$

The residue turns out at $\sigma = (3,-1),\ (1,-3),\ (-1,-3),\ (-3,-1)$:

$$a_{-1}\phi(\nu,\frac{1}{2})\cdot\{\overline{\psi}(-\overline{\nu},\frac{1}{2})+\overline{\psi}(\frac{1}{2},-\overline{\nu})\mathcal{L}(\frac{1}{2}+\nu)+$$
$$\overline{\psi}(\frac{1}{2},\overline{\nu})\mathcal{L}(2\nu)\mathcal{L}(\frac{1}{2}+\nu)+\overline{\psi}(\overline{\nu},\frac{1}{2})\mathcal{L}(2\nu)\mathcal{L}(\frac{1}{2}+\nu)\mathcal{L}(-\frac{1}{2}+\nu)\} \tag{14.5.2.2}$$

The term can have a pole at $\nu = \frac{1}{2}$ and $\nu = 0$.

$\nu = \frac{1}{2}$. In this case $Re(\lambda) = (1/2,1/2)$. The first term does not have a pole, the second has a simple pole with residue

$$a_{-1}a_{-1}\phi(1/2,1/2)\overline{\psi}(1/2,-1/2), \tag{14.5.2.4}$$

Expand in powers of $\nu - \frac{1}{2}$:

$$\begin{aligned}
\mathcal{L}(\frac{1}{2}+\nu) &= \frac{a_{-1}}{\nu-\frac{1}{2}}+a_0+a_1(\nu-\frac{1}{2}),\\
\mathcal{L}(2\nu) &= \frac{a_{-1}}{2(\nu-1/2)}+a_0+a_1(2(\nu-1/2)),\\
\mathcal{L}(\nu-\frac{1}{2}) &= -1+c_1(\nu-\frac{1}{2}),\\
\phi(\nu,\frac{1}{2})\overline{\psi}(\frac{1}{2},\overline{\nu}) &= \phi\overline{\psi}+[\partial_1\phi\overline{\psi}+\phi\partial_2\overline{\psi}](\nu-\frac{1}{2}),\\
\phi(\nu,\frac{1}{2})\overline{\psi}(\overline{\nu},\frac{1}{2}) &= \phi\overline{\psi}+[\partial_1\phi\overline{\psi}+\phi\partial_1\overline{\psi}](\nu-\frac{1}{2}).
\end{aligned} \tag{14.5.2.5}$$

Then the residue of the third term (corresponding to $\sigma = (-1, -3)$) is

$$\frac{a_{-1}^2}{2}[\partial_1\phi\overline{\psi} + \phi\partial_2\overline{\psi}] + \frac{a_{-1}a_0}{2}\phi\overline{\psi} + a_0a_{-1}\phi\overline{\psi}. \tag{14.5.2.6}$$

The residue of the fourth term (corresponding to $\sigma = (-3, -1)$) is

$$\frac{c_1a_{-1}^2}{2}\phi\overline{\psi} - \frac{a_{-1}^2}{2}[\phi\partial_2\overline{\psi} - \phi\partial_1\overline{\psi}] - \frac{a_{-1}a_0}{2}\phi\overline{\psi} - a_0a_{-1}\phi\overline{\psi}. \tag{14.5.2.7}$$

The total comes out

$$\frac{a_{-1}^3}{2}[\phi\partial_2\overline{\psi} - \phi\partial_1\overline{\psi}] + \frac{c_1a_{-1}^3}{2}\phi\overline{\psi} \tag{14.5.2.8}$$

evaluated at $(1/2, 1/2)$.

$\nu = 0$**.** The term gives an integral over the imaginary part of a line.

14.5.3. On $\nu_1 + \nu_2 = 1$, we write

$$\lambda = (1/2 + \nu, 1/2 - \nu), \qquad -\overline{\lambda} = (-\frac{1}{2} - \overline{\nu}, -\frac{1}{2} + \overline{\nu}). \tag{14.5.3.1}$$

The residue turns out at $\sigma = (1, -3),\ (-3, 1),\ (-1, -3),\ (-3, -1):$

$$\begin{aligned} &a_{-1}\phi(\frac{1}{2} + \nu, \frac{1}{2} - \nu) \cdot \{\overline{\psi}(\frac{1}{2} + \overline{\nu}, -\frac{1}{2} - \overline{\nu})\mathcal{L}(1 - 2\nu) + \overline{\psi}(\frac{1}{2} + \overline{\nu}, -\frac{1}{2} + \overline{\nu})\mathcal{L}(1 + 2\nu)\mathcal{L}(2\nu) \\ &+\overline{\psi}(\frac{1}{2} - \overline{\nu}, \frac{1}{2} + \overline{\nu})\mathcal{L}(1 + 2\nu)\mathcal{L}(1 - 2\nu) + \overline{\psi}(\frac{1}{2} + \overline{\nu}, \frac{1}{2} - \overline{\nu})\mathcal{L}(1 + 2\nu)\mathcal{L}(2\nu)\mathcal{L}(1 - 2\nu)\} \end{aligned} \tag{14.5.3.2}$$

The only point of concern is $\nu = 0$. The first two terms have a simple pole; the residue is

$$a_{-1}\phi(\frac{1}{2}, \frac{1}{2})\overline{\psi}(\frac{1}{2}, -\frac{1}{2})\{\frac{a_{-1}}{-2} + \frac{a_{-1}\mathcal{L}(0)}{2}\} = -a_{-1}^2\phi(\frac{1}{2}, \frac{1}{2})\overline{\psi}(\frac{1}{2}, -\frac{1}{2}). \tag{14.5.3.3}$$

For the other terms we have the expansions

$$\begin{aligned} \mathcal{L}(1 + 2\nu) &= \frac{a_{-1}}{2\nu} + a_0 + a_1(2\nu), \\ \mathcal{L}(1 - 2\nu) &= \frac{a_{-1}}{-2\nu} + a_0 + a_1(-2\nu), \\ \mathcal{L}(2\nu) &= -1 + c_1(2\nu), \\ \phi(\frac{1}{2} + \nu, \frac{1}{2} - \nu)\overline{\psi}(\frac{1}{2} + \overline{\nu}, \frac{1}{2} - \overline{\nu}) &= \phi\overline{\psi} + [(\partial_1 - \partial_2)\phi\overline{\psi} - \phi(\partial_1 - \partial_2)\overline{\psi}]\nu, \\ \phi(\frac{1}{2} + \nu, \frac{1}{2} - \nu)\overline{\psi}(\frac{1}{2} + \overline{\nu}, \frac{1}{2} - \overline{\nu}) &= \phi\overline{\psi} + [(\partial_1 - \partial_2)\phi\overline{\psi} + \phi(\partial_1 - \partial_2)\overline{\psi}]\nu \end{aligned} \tag{14.5.3.3}$$

The residue of the third term (corresponding to $\sigma = (-1, -3)$ is

$$\frac{a_{-1}^2}{-4}[(\partial_1 - \partial_2)\phi\overline{\psi} - \phi(\partial_1 - \partial_2)\overline{\psi}]. \tag{14.5.3.4}$$

The residue of the fourth term (corresponding to $\sigma = (-3, -1)$) is

$$\frac{c_1 a_{-1}^2}{-2}\phi\overline{\psi} + \frac{a_{-1}^2}{4}[(\partial_1 - \partial_2)\phi\overline{\psi} + \phi(\partial_1 - \partial_2)\overline{\psi}]. \tag{14.5.3.5}$$

The total contribution is

$$-\frac{c_1 a_{-1}^3}{2}\phi\overline{\psi} + \frac{a_{-1}^3}{2}\phi[\partial_1\overline{\psi} - \partial_2\overline{\psi}] \tag{14.5.3.6}$$

evaluated at $(1/2, 1/2)$.

$\nu = 0$. The term gives an integral over a line with imaginary coordinate.

14.5.4. On $2\nu_1 = 1$, we write

$$\lambda = (1/2, \nu), \qquad -\overline{\lambda} = (-\frac{1}{2}, -\overline{\nu}). \tag{14.5.4.1}$$

The residue turns out at $\sigma = (-1, 3),\ (-3, 1),\ (-1, -3),\ (-3, -1):$

$$\begin{aligned}\alpha_1\phi(\frac{1}{2}, \nu)\cdot\{&\overline{\psi}(-\overline{\nu}, \frac{1}{2})\mathcal{L}(\frac{1}{2} - \nu) + \overline{\psi}(\frac{1}{2}, -\overline{\nu})\mathcal{L}(\frac{1}{2} + \nu)\mathcal{L}(\frac{1}{2} + \nu) + \\ &\overline{\psi}(\overline{\nu}, \frac{1}{2})\mathcal{L}(2\nu)\mathcal{L}(\frac{1}{2} + \nu) + \overline{\psi}(\frac{1}{2}, \overline{\nu})\mathcal{L}(2\nu)\mathcal{L}(\frac{1}{2} + \nu)\mathcal{L}(\frac{1}{2} - \nu)\}\end{aligned} \tag{14.5.4.2}$$

$\nu = 0$. The term gives a 1–dimensional integral over a line with imaginary coordinate.

14.5.5. At $(0, 0)$ we get a double integral over $Re\lambda = (0, 0)$.

Discrete Spectrum

We get a single term

$$a_{-1}a_{-1}\phi(3/2, 1/2)\overline{\psi}(3/2, 1/2).$$

Conjugates of (1/2,0)

$$\begin{aligned}&< \phi, \psi >= \\ &2\phi(\nu, \frac{1}{2})\cdot\{\overline{\psi}(-\overline{\nu}, \frac{1}{2}) + \overline{\psi}(\frac{1}{2}, -\overline{\nu})\cdot\mathcal{L}(\nu + \frac{1}{2}) + \\ &\qquad \overline{\psi}(\frac{1}{2}, \overline{\nu})\mathcal{L}(2\nu)\mathcal{L}(\nu + \frac{1}{2}) + \overline{\psi}(\overline{\nu}, \frac{1}{2})\mathcal{L}(2\nu)\mathcal{L}(\nu + \frac{1}{2})\mathcal{L}(\nu - \frac{1}{2})\} + \\ &2\phi(\frac{1}{2}, \nu)\cdot\{\overline{\psi}(-\overline{\nu}, \frac{1}{2})\mathcal{L}(-\nu + \frac{1}{2}) + \overline{\psi}(\frac{1}{2}, -\overline{\nu})\cdot\mathcal{L}(\nu + \frac{1}{2})\mathcal{L}(-\nu + \frac{1}{2}) + \\ &\qquad \overline{\psi}(\frac{1}{2}, \overline{\nu})\mathcal{L}(2\nu)\mathcal{L}(\nu + \frac{1}{2})\mathcal{L}(-\nu + \frac{1}{2}) + \overline{\psi}(\overline{\nu}, \frac{1}{2})\mathcal{L}(2\nu)\mathcal{L}(\nu + \frac{1}{2})\}.\end{aligned}$$

$$
\begin{aligned}
&< \phi, \psi >= \\
&2\phi(\frac{1}{2}+\nu, -\frac{1}{2}+\nu)\cdot\{\overline{\psi}(\frac{1}{2}-\overline{\nu}, -\frac{1}{2}-\overline{\nu}) + \overline{\psi}(\frac{1}{2}-\overline{\nu}, \frac{1}{2}+\overline{\nu})\cdot\mathcal{L}(1+2\nu)+ \\
&\overline{\psi}(\frac{1}{2}+\overline{\nu}, \frac{1}{2}-\overline{\nu})\mathcal{L}(1+2\nu)\mathcal{L}(2\nu) + \overline{\psi}(\frac{1}{2}+\overline{\nu}, -\frac{1}{2}+\overline{\nu})\mathcal{L}(-1+2\nu)\mathcal{L}(1+2\nu)\mathcal{L}(2\nu)\}+ \\
&2\phi(\frac{1}{2}+\nu, \frac{1}{2}-\nu)\cdot \\
&\{\overline{\psi}(\frac{1}{2}-\overline{\nu}, -\frac{1}{2}-\overline{\nu})\mathcal{L}(1-2\nu) + \overline{\psi}(\frac{1}{2}-\overline{\nu}, \frac{1}{2}+\overline{\nu})\cdot\mathcal{L}(1-2\nu)\mathcal{L}(1+2\nu)+ \\
&\overline{\psi}(\frac{1}{2}+\overline{\nu}, \frac{1}{2}-\overline{\nu})\mathcal{L}(1-2\nu)\mathcal{L}(1+2\nu)\mathcal{L}(2\nu) + \overline{\psi}(\frac{1}{2}+\overline{\nu}, -\frac{1}{2}+\overline{\nu})\mathcal{L}(1+2\nu)\mathcal{L}(2\nu)\}.
\end{aligned}
$$

It is interesting to see that these expressions have no singularities on the imaginary axis.

14.6. Finally we mention how this matches the homomorphisms in 14.1.3. There are four nilpotent orbits parametrized by their Jordan canonical forms which in turn corerspond to partitions of 4. They are

$$(4), \qquad (22), \qquad (211), \qquad (1111).$$

Only (4) is distinguished (not meeting any Levi component of a proper parabolic subalgebra). According to Arthur we associate to each nilpotent $\frac{1}{2}H$, where H is the semisimple element in the associated Lie triple. These are

$$(3/2, 1/2), \qquad (1/2, 1/2), \qquad (1/2, 0), \qquad (0, 0).$$

These correspond exactly to the integrals appearing in the inner product. $\square$

Remark. The original calculation of Langlands for G_2 has another term in the discrete spectrum. This is because there are two distinguished nilpotent orbits in this algebra.

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