1 Abstract representation theory

1.1 Basic functional analysis

1.1.1 Locally convex topological spaces

There are many texts on this subject. Two useful references are

Yoshida,K, *Functional Analysis*, Springer-Verlag, New York, 1974.

Treves, F, *Topological Vector Spaces, Distributions and Kernels,* Academic Press, New York,1967.

Let V be a vector space over $F = \mathbb{R}$ or $\mathbb{C}$. Then a subset U of V is said to be *convex* if whenever $x, y \in U$, $(1-t)x + ty \in U$ for all $0 \leq t \leq 1$. U is said to be *balanced* if whenever $x \in U$, $zx \in U$ for all $|z| \leq 1$. U is said to be *absorbing* if whenever $x \in V$ there exists $z \in F$ with $z \neq 0$ so that $zx \in U$.

Example 1 *$V = \mathbb{R}^n$ and $U = \{(x_1, ..., x_n) | \sum x_i^2 < 1\}$. Then U is convex,balanced and absorbing.*

Example 2 *V the space of all continuous F valued functions on $\mathbb{R}$ and $U = \{f | f(x) = 0$ if $|x| > r\}$. Then U is convex,balanced but not absorbing.*

If U is convex, balanced and absorbing then we define

$$p_U(v) = \inf\{t > 0 | t^{-1}v \in U\}.$$

Notice $p_U(v)$ is defined for all v since U is absorbing. Also $p_U(0) = 0.$. The function p_U is called the Minkowski gauge associated with U.

Lemma 3 *Let $p = p_U$. Then p satisfies*

1. $p(v) \geq 0$ for all $v \in V$.

2. $p(v + w) \leq p(v) + p(w)$ for all $v, w \in V$.

3. $p(zv) = |z|p(v)$ for all $z \in F$ and $v \in V$.

Proof. 1. is clear. If $z = 0$ or $v = 0$ then $p(zv) = p(0) = 0 = |z|p(v)$. So assume that $z \neq 0$. Then since U is balanced the set

$$\begin{aligned} &\{t > 0 | t^{-1}zv \in U\} = \{t > 0 | t^{-1}|z|v \in U\} \\ &= |z|\{t > 0 | t^{-1}v \in U\}. \end{aligned}$$

Now taking the infema of both sides of the equation proves 3. We now prove 2. Let $p(v) = s_o$ and $p(w) = t_o$. Let $\varepsilon > 0$ be arbitrary and let s, t be such that $s_o < s < s_o + \varepsilon$, $t_o < t < t_o + \varepsilon$ and $s^{-1}v, t^{-1}w \in U$. Then if $0 \leq u \leq 1$ the convexity of U implies

$$(1-u)s^{-1}v + ut^{-1}w \in U.$$

Choose u such that $ut^{-1} = (1-u)s^{-1}$ that is $u(t^{-1} + s^{-1}) = s^{-1}$. (Note that this is consistent with $0 < u < 1$.) This implies that

$$ut^{-1} = (1-u)s^{-1} = \frac{1}{t+s}.$$

Thus

$$\frac{1}{t+s}(v+w) \in U.$$

This implies that $p(v+w) \leq t+s$. Now take the limit as $\varepsilon \to 0$. $\blacksquare$

Definition 4 *A function, p, from V to $\mathbb{R}$ satisfying 1.,2.,3. in the previous lemma is called a* semi-norm *on V. If in addition it satisfies the condition $p(v) = 0$ implies $v = 0$ then p is said to be a* norm.

Exercise 5 *If p is a semi-norm on V then the sets $\{v \in V | p(v) < 1\}$, $\{v \in V | p(v) \leq 1\}$ are convex, balanced and absorbing. Further, U is a subset of the latter set.*

Definition 6 *V is said to be a topological vector space if it is a topological space such that the operations*

$$V \times V \longmapsto V, x, y \longmapsto x + y, F \times V \to V, x \longmapsto zx,$$

are continuous.

Definition 7 *We say that V is a locally convex topological vector space if V is a topological vector space that has a basis of neighborhoods of 0 consisting of open, convex, balanced sets.*

Example 8 *$V = F^n$ with the usual topology and the neighborhood basis consists of the open balls centered at 0.*

Example 9 *More generally if p is a norm on V a vector space over F then $d(v, w) = p(v - w)$ defines a metric on V. V is a locally convex, topological vector space with respect to the metric topology. If V is in addition metrically complete then V is said to be a* Banach space.

Example 10 *In this example $F = \mathbb{C}$. Let $\langle ..., ... \rangle$ be a Hermitian inner product on V (that is the function $v, w \longmapsto \langle v, w \rangle$ is bilinear over $\mathbb{R}$, linear in the first variable over $\mathbb{C}$ and $\langle w, v \rangle$ is the complex conjugate of $\langle v, w \rangle$ and further, $\langle v, v \rangle$ is a strictly positive real number if $v \neq 0$. If this is so we define $\|v\| = \langle v, v \rangle^{\frac{1}{2}}$. The* Schwarz inequality*,*

$$|\langle v, w \rangle| \leq \|v\| \|w\|,$$

implies the triangle inequality

$$\|v + w\| \leq \|v\| + \|w\|.$$

Thus $v \longmapsto \|v\|$ defines a norm on V. The pair $(V, \langle ..., ... \rangle)$ is called a pre-Hilbert space. *If V is a Banach space with respect to the norm $v \longmapsto \|v\|$ then it is called a* Hilbert space.

Example 11 *Let $\mathcal{S}$ be a set of semi-norms on V such that if $v \in V$ then there exists $p \in \mathcal{S}$ such that $p(v) > 0$. Set $U_{p,\varepsilon} = \{v \in V | p(v) < \varepsilon\}$ for $p \in \mathcal{S}$ and $\varepsilon > 0$. Then if we take the sets $U_{p,\varepsilon}$ as a basis of neighborhoods of 0 in V then this endows V with the structure of a locally convex topological vector space.*

Lemma 12 *Every structure of a locally convex vector space on V is obtained in this manner.*

Proof. Let $\mathcal{U}$ be a neighborhood basis for 0 in V consisting of open, convex, balanced, absorbing sets. If $U \in \mathcal{U}$ let p_U be the corresponding Minkowski gauge. Then we have

$$U = \{v \in V | p_U(v) < 1\}.$$

Indeed, since U is convex if $\frac{1}{t}v \in U$ and $0 < t < 1$ then $t\frac{1}{t}v + (1-t)0 \in U$. If $v \in U$ then $p_U(v) = t$ and $t \leq 1$. The set $\{s > 0 | s^{-1}v \in U\}$ is open in $\mathbb{R}$. Thus $t < 1$. ∎

Lemma 13 *Let V and W be locally convex topological spaces. Let $T : V \to W$ be a linear mapping. Then T is continuous on V if and only if for each continuous semi-norm, q, on W the map $v \longmapsto q(T(v))$ defines a continuous seminorm on V.*

Proof. The sufficiency is clear. Since T is linear it is enough to prove that if the condition is satisfied then T is continuous at 0. But the condition implies that for each q a semi-norm on W and each $\varepsilon > 0$ the set $T^{-1}(\{w \in W | q(w) < \varepsilon\})$ is an open neighborhood of 0. ∎

Example 14 *Let X be a locally compact (Hausdorff) topological vector space. Let $C(X, F)$ denote the space of all continuous functions on X with values in F. If K is a compact subset of X then to K we assign the semi-norm, p_K, defined by $p_K(f) = \max_{k \in K} |f(k)|$. The corresponding topology on V is called the* topology of uniform convergence on compacta. *Note that if X is compact we need only use the norm p_X.*

If V is a locally convex, topological vector space then a sequence $\{v_j\}$ in V is said to be *Cauchy* if for each continuous semi-norm p on V and each $\varepsilon > 0$ there exists N such that if $i, j \geq N$ then $p(v_i - v_j) < \varepsilon$. We say that V is *sequentially complete* if every Cauchy sequence in V converges. A sequentially complete topological vector space is said to be a *Fréchet space* if the topology is given by a countable set of semi-norms as in Example 11.

Lemma 15 *If V is a locally, convex, topological space with a countable basis of neighborhoods of 0 the V is metrizable. Or equivalently if the topology on V is given as in example 11 with the set $\mathcal{S}$ countable then V is metrizable.*

Proof. We will prove the equivalent second statement (see Lemma 12). Let $\mathcal{S} = \{p_j\}_{j<N}$ with $N \leq \infty$. Consider

$$d(v,w) = \sum_{j<N} \frac{p_j(v-w)}{2^j(1+p_j(v-w))}.$$

Since each term in the above series is at most equal to 2^{-j} the series converges uniformly as a series of functions on $V \times V$. If $d(v,w) = 0$ then all of the terms in the series are 0. Thus $v = w$. That d is a metric follows from the following exercise. That it gives the topology on V follows from the observation that

$$\lim_{j\to\infty} p_k(v_j - v) = 0$$

for all $k = 1, 2, ...$ if and only if

$$\lim_{j\to\infty} d(v_j, v) = 0.$$

■

Exercise 16 *Assume that $p(v+w) \leq p(v)+p(w)$ for all $v, w \in V$ and $p(v) \geq 0$ for all $v \in V$. Define $q(v) = \frac{p(v)}{1+p(v)}$. Then $q(v+w) \leq q(v)+q(w)$ for all $v, w \in V$. (Hint: Write out $q(v) + q(w) - q(v+w)$. Put the expression over the obvious common denominator*

$$(1+p(v))(1+p(w))(1+p(v+w))$$

and see that the numerator is non-negative.)

Example 17 *If X, as in example 13, has a countable basis for its topology then $C(X;F)$ endowed with the topology of uniform convergence on compacta is a Fréchet space. To see this we note that there exists a countable collection $\{K_j\}$ of compact subsets having the property that if $x \in X$ and if U is open in X then there exists j such that $x \in interior(K_j) \subset K_j \subset U$. Let $p_j = p_{K_j}$. Then these seminorms define the topology on $C(X,F)$. We need only show the completeness. This follows from the fact that a uniform limit of continuous functions is continuous.*

We will need just one more general notion. Let V be a vector space over F and let I be a partially ordered set. We assume that for each $\alpha \in I$ we have $V_\alpha \subset V$ a subspace that is endowed with a locally convex topology such that

- If $\alpha < \beta$ then $V_\alpha \subset V_\beta$ and the subspace topology on V_α is the same as the given topology.
- $\bigcup_{\alpha \in I} V_\alpha = V$.

If these conditions are satisfied then we define a locally convex topology on V as follows we take a neighborhood basis for 0 consisting of the subsets U of V that are convex hulls of subsets of the form $\bigcup_{\alpha \in I} U_\alpha$ where U_α is an open, convex, balanced neighborhood of 0 in V_α. V with this topology is called the *(strict) direct limit* of $\{V_\alpha\}_{\alpha \in I}$. We write

$$V = \lim_{\rightarrow} V_\alpha.$$

Exercise 18 *Show that the sets U in the definition of direct limit are convex,balanced and absorbing and that these sets do form a basis for neighborhoods of* 0.

Exercise 19 *Show that if*

$$V = \lim_{\rightarrow} V_\alpha$$

then a linear map of V to F is continuous if and only if its restriction to each V_α is continuous.

Example 20 *X a locally compact topological space. If $f \in C(X;F)$ then define the support of f to be the closure of the set $\{x \in X | f(x) \neq 0\}$. Let $C_c(X;F)$ be the space of those elements of compact support. If K is a compact subset of X then we use the notation $C_K(X;F)$ for the continuous functions with support contained in K. Then each $C_K(X;F)$ is a Banach space relative to the norm p_K. We give $C_c(X;F)$ the direct limit topology (here the compact subsets are ordered by inclusion).*

Lemma 21 *Let $\{f_j\}$ be a sequence in $C_c(X;F)$ then $\lim_{j\to\infty} f_j = f_o$ in $C_c(X;F)$ if and only if there exists an N and a compact subset K of X such that if $j \geq N$ then the support of f_j is contained in K and $\lim_{j\to\infty} f_j = f_o$ uniformly on K.*

Proof. The sufficiency is clear. Replacing f_j by $f_j - f_o$ we may assume that $f_o = 0$. We assume that there is no such K. Then there is a subsequence $\{f_{j_l}\}$, an increasing sequence of compact subsets K_l and a sequence of points x_l in X such that $x_l \in K_l - K_{l-1}$ ($K_0 = \emptyset$), $f_{j_l}(x_l) \neq 0$ and the sequence is not contained in any compact subset of X. We define the semi-norm

$$p(f) = 2\sum_{l=1}^{\infty} \sup_{x \in K_l - K_{l-1}} \frac{|f(x)|}{|f_{k_l}(x_l)|}$$

(Notice that this is a finite sum.) Then the set $U = \{f \in C_c(X;F) | p(f) < 1\}$ is an open neighborhood of 0 in $C_c(X;F)$ but $f_{j_l} \notin U$ for any l. This proves the lemma. ■

Now assume that $X = X_1 \times X_2$ with X_1, X_2 locally compact spaces and assume that X has been endowed with the product topology. If $f \in C_c(X;F)$ then define for $x \in X_1$, $f(x,\cdot)$ to be the function $y \longmapsto f(x,y)$. We will need the following result in the next section.

Lemma 22 *The function* $x \longmapsto f(x,\cdot)$ *is a continuous map from* X_1 *into* $C_c(X_1;F)$.

Exercise 23 *Give a proof of this lemma. (Hint: Use the fact that a continuous function on a compact set is uniformly continuous.)*

1.1.2 Measures

Let V be a topological vector space over F. Then we use the notation, V^*, for the usual (algebraic) dual space of V. $\lambda \in V^*$ will be called a functional. We use the notation V' for the subspace of continuous elements of V^*.

If X is a locally compact topological space and if we endow $V = C_c(X;\mathbb{C})$ with the locally convex topology given in Example 20 in the previous section then an element, μ, of V' is called a *complex (Radon) measure.* If $\mu \neq 0$ and if $\mu(f)$ is real and non-negative for each f such that $f(x)$ is real and non-negative for all $x \in X$ then μ is called a measure.

The simplest examples of measures are given as follows: Let $x \in X$ then define $\delta_x(f) = f(x)$. Then δ_x is continuous on each of the subspaces $C_K(X;F)$ for K a compact subset of X and thus it defines a continuous functional on $C_c(X;F)$. Hence δ_x is a measure on X called the *Dirac delta function supported at* x.

Lemma 24 *Let* μ *be a measure on the locally compact space* X. *If* $f \in C_c(X)$ *then*

$$|\mu(f)| \leq \mu(|f|).$$

Here $|f|(x) = |f(x)|$. The result follows from $|f(x)| \pm f(x) \geq 0$ for all f. Thus $\mu(|f| \pm f) \geq 0$. Hence, $\mu(|f|) \geq \pm\mu(f)$.

Example 25 *Let* $X = \mathbb{R}$ *and define*

$$\mu(f) = \int_{-\infty}^{\infty} f(x)dx.$$

If the support of f *is contained in the interval* $[a,b]$ $(a < b$ *and both finite)*

$$|\mu(f)| \leq \int_{-\infty}^{\infty} |f(x)|dx = \int_a^b |f(x)|dx \leq \\ (b-a) \max_{a \leq x \leq b} |f(x)|.$$

The last formula defines a continuous semi-norm on $C_c(\mathbb{R},\mathbb{C})$. *Thus* μ *is a measure on* $\mathbb{R}$. *The usual Lebesgue measure.*

Example 26 *Let* X *and* Y *separable, locally compact topological spaces (i.e. they both have countable bases for their topologies). Let* $Z = X \times Y$ *with the product topology. Let* μ *and* ν *be respectively measures on* X *and* Y. *We have* *seen that if $f \in C_c(Z;\mathbb{C})$ then the map $x \longmapsto f(x,\cdot)$ is a continuous from X_1 to $C_c(X_2;\mathbb{C})$. This implies that $x \longmapsto \nu(f(x,\cdot))$ is a continuous function on X_1. It is not hard to see that this function has compact support. Thus we can define*

$$(\mu \times \nu)(f) = \mu(x \longmapsto \nu(f(x,\cdot))).$$

Exercise 27 *Show that $\mu \times \nu$ defines a measure on Z.*

Example 28 *Let X be a locally compact topological space and let Y be an open subspace of X. We may thus look upon $C_c(Y;\mathbb{C})$ as a topological subspace of $C_c(X;\mathbb{C})$. If μ is a complex measure on X then we define $\mu_{|Y} = \mu_{|C_c(Y;\mathbb{C})}$.*

Obviously the above method of construction of measures on product spaces extends to a product of arbitrary finite number of spaces. Thus we have the usual Lebesgue measure on $\mathbb{R}^n$. Obviously, this is not a good definition of Lebesgue measure since it seems to only allow integration of continuous functions of compact support. However, we will assume that the reader is aware of the actual Lebesgue integral and understands why it is determined by the corresponding functional.

If G is a topological group and if X is topological space then an action of G on X is a continuous mapping

$$F : G \times X \to X$$

with the following properties

- $F(1,x) = x$ for all $x \in X$.
- $F(g, F(h,x)) = F(gh,x)$ for all $g,h \in G, x \in X$.

Notice that the two conditions imply that the maps $x \longmapsto F(g,x)$ are homeomorphisms. We will denote this map by $F_g(x)$ or more simply as $g \cdot x$ (or even gx). If f is a function on X then we will use the notation $L_g(f) = f \circ F_{g^{-1}}$.

Example 29 *$X = G$. $F(g,x) = gx$. The second property is just the associative law. This action is called the* left regular action. *There is also the* right regular action *given by $F(g,x) = xg^{-1}$.*

Example 30 *$X = \mathbb{R}^n$ and G is a closed subgroup of $GL(n,\mathbb{R})$ and $F(g,x) = gx$. That is the usual identification of a matrix with a linear map.*

Definition 31 *Let X be a separable, locally compact space and let G be a separable locally compact group acting on X. Let μ be a non-zero complex measure on X. Then μ is said to be* semi-invariant *if for each $g \in G$ there is a scalar $c(g)$ such that $\mu(L_g f) = c(g)\mu(f)$ for all $f \in C_c(X,\mathbb{C})$. It will be called* invariant *if $c(g) \equiv 1$.*

Exercise 32 *Show that, in the above context, $c(gh) = c(g)c(h)$ for $g,h \in G$.*

Lemma 33 *Let G, X be as in the previous definition. If $f \in C_c(X;\mathbb{C})$ then the map $G \to C_c(X;\mathbb{C})$ given by $g \longmapsto L_g f$ is continuous.*

This proved by observing that local compactness implies that if $a \in G$ then there is an open neighborhood of a, U, such that the closure, $\overline{U}$, of U is compact. There is thus a compact subset of X, K', such that the support of all of the functions $L_b f$, $b \in U$ is contained in $K^{'}$. Now use uniform continuity.

Lemma 34 *Let the notation be as in Definition 31. Then the function c is continuous on G.*

This follows by observing that $g \longmapsto \mu(L_g f)$ is continuous and that if $\mu(f) \neq 0$ then $c(g) = \frac{\mu(L_g f)}{\mu(f)}$.

Exercise 35 *Show that if G or X is compact then $c(g) = 1$ for all $g \in G$.*

Example 36 *Let the notation be as in Example 30. If μ is Lebesgue measure on $\mathbb{R}^n$ then the change of variables theorem implies that it is semi-invariant under the action of $GL(n,\mathbb{R})$ and $c(g) = |\det(g)|$.*

Example 37 *In this example $GL(n,\mathbb{R})$ acts on the $n \times n$ matrices, $M_n(\mathbb{R})$, by left multiplication. We use the matrix entries as coordinates and thereby identify $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$. We take μ to be Lebesgue measure. In this cas $c(g) = |\det(g)|^n$.*

Although we will be mainly concerned with real groups in these lectures we will include a few related remarks for other fields. Let F be a field. Then an absolute value on F is a function, $x \longmapsto |x|$ with values in the non-negative real numbers and satisfying.

- $|x| > 0$ if $x \neq 0$.
- $|xy| = |x||y|$.
- $|x + y| \leq |x| + |y|$.

The last property of a norm allows us to define a metric on F by $d(x, y) = |x - y|$. We will say that F is a local field if the metric is non-trivial (i.e. $|x| \neq 1$ for some $x \in F - \{0\}$) and if we endow F with the topology induced by the metric then the set $U = \{x \in F||x| \leq 1\}$ is compact. In Weil, A., *Basic Number Theory*,Second Edition, Springer, 1973 it is shown that these fields coincide (up to topological isomorphism) with the locally compact, non-discrete fields. Two absolute values are equivalent if they have the same sets U. In Weil's book it is shown there are two types up to equivalence which we will now describe. (The proof of this theorem by Weil uses some of the ideas below. However, we will just fix our attention on these examples and not worry about proving that they are the only ones.)

1. F isomorphic with $\mathbb{R}$ or $\mathbb{C}$ and $|...|$ is the usual absolute value.
2. F is non-Archimedian. That is to say $|x+y| \leq \max\{|x|,|y|\}$ for $x,y \in F$ and the set $\{\log|x| | x \in F^\times\}$ is a discrete subset of $\mathbb{R}$.

In case 1. we fix Lebesgue measure on $\mathbb{R}$ and we identify $\mathbb{C}$ with $\mathbb{R}^2$ and use Lebesgue measure. We will denote this measure by $\mu(f) = \int f(x)dx$ in both cases. We note that both measures are semi-invariant relative to the action of $GL(1,F) = F^\times = F - \{0\}$. In the case of $\mathbb{R}$, $c(x) = |x|$ in the case of $\mathbb{C}$, $c(x) = |x|^2$.

In case 2. we need the following theorem of Haar.

Theorem 38 *Let G be a locally compact, separable, topological group then there exists, up to positive scalar a unique measure μ on G invariant under the left regular action.*

With this in hand we can handle case 2. Let μ be a choice of left invariant measure on F thought of as a topological group under addition. Then this measure (or at least a positive multiple of it) will be denoted

$$\mu(f) = \int f(x)dx.$$

We note that if $a \in F^\times$ and we recall $L_a f(x) = f(a^{-1}x)$ and (so as not to confuse the two group actions) we set $t_y f(x) = f(x+y)$. Then by definition $\mu(t_y f) = \mu(f)$. We note that $L_a(t_y f)(x) = t_y f(a^{-1}x) = f(a^{-1}x+y) = t_{ay}(L_a f)(x)$. Thus if we set $\mu_a(f) = \mu(L_a f)$ then

$$\mu_a(t_y f) = \mu(t_{ay}(L_a f)) = \mu(L_a f) = \mu_a(f).$$

Thus μ_a is translation invariant so Haar's theorem implies that there is a real valued function that takes positive values on $F^\times$such that $\mu_a = c(a)\mu$.

Lemma 39 *In case 2. there is a positive real number, r, such that $c(a) = |a|^r$.*

Proof. Let $\nu(x) = -\log|x|$ for $x \in F^\times$. The subgroup $\nu(F^\times)$ is discrete in additive group $\mathbb{R}$. It is therefore infinite cyclic (since it is not just $\{0\}$). Let ξ be a positive generator of this group. Fix $\pi \in F$ such that $\nu(\pi) = \xi$. We note that $\nu(\pi) = e^{-\xi} < 1$. The condition 2. above implies that $R = \{x \in F | |x| \leq 1\}$ is a subring of F. This ring is compact and since the value set of $|...|$ is discrete we see that R is open. Also the set $\mathfrak{p} = \{x \in F | |x| < 1\}$ is an ideal in R. Also observe that $\pi \in \mathfrak{p}$. We note that if $x \in F^\times$ then $|x| = |\pi|^n$ for some integer n. Thus if $x \in \mathfrak{p}$, $n \geq 1$ thus $|\pi^{-1}x| \leq 1$. Hence $\pi^{-1}x \in R$. We have shown

1. $\mathfrak{p} = \pi R$.

2. If $U = \{x \in F | |x| = 1\}$ then U is a compact open subgroup of $F^\times$ and furthermore $F^\times = U\{\pi^n | n \in \mathbb{Z}\}$.

Finally, we are ready to prove the Lemma. Since U is a compact group and c is continuous and takes positive values we see that $c(u) = 1$ for all $u \in U$. Now $c(\pi^n) = c(\pi)^n$. We observe that we $c(\pi) < 1$. Indeed, if not then we have

$$\lim_{n\to\infty} c(\pi^n) \text{ is either } 1 \text{ or } \infty.$$

So if f has support in K then $L_{\pi^n}(f)$ has support in $\pi^n K$. As $\lim_{n\to\infty}\pi^n = 0$ this would imply that $0 = \lim_{n\to\infty}\mu(L_{\pi^n}f) = \lim_{n\to\infty} c(\pi^n)\mu(f)$ for all $f \in C_c(F;\mathbb{C})$. But then $\mu = 0$ which is a contradiction. Hence $c(\pi) < 1$. So if $c(\pi) = e^{-r\xi} = |\pi|^r$ then $r > 0$. We have if $x \in F^\times$ then $x = \pi^n u$ with $u \in U$. Hence $c(x) = c(\pi^n u) = c(\pi^n) = c(\pi)^n = |\pi|^{rn} = |x|^r$. ■

We will replace $|x|$ for a local field with $c(x)$. Thus in the case of $\mathbb{C}$ it is not an absolute value. We will call it a norm in all cases. Notice that in the non-Archimedian case it is still an absolute value.

We now consider the action of $GL(n,F)$ on F^n by the usual action of matrices on vectors. On F^n will use the product of the translation invariant measure on F discussed above. We denote this measure by μ. We will write

$$\mu(f) = \int_{F^n} f(x)dx.$$

Lemma 40 *If F is a local field and $g \in GL(n,F)$ then*

$$\int_{F^n} f(gx)dx = |\det(g)|^{-1}\int_{F^n} f(x)dx.$$

Proof. We observe that $GL(n,F)$ is generated by elementary matrices. That is matrices of the form

$$D_i(a) = \begin{bmatrix} 1 & & & & \\ & \ddots & & & \\ & & a & & \\ & & & \ddots & \\ & & & & 1 \end{bmatrix}$$

(with zeros off of the main diagonal, a nonzero a in the i-th diagonal position, all the rest of the diagonal entries 1) and $T_{ij}(y)$ which is the linear transformation of F^n such that if $T_{ij}(y)x = z$ then $z_k = x_k$ for $k \neq j$ and $z_j = x_j + yx_i$. The translation invariance of the measure and the previous lemma imply

$$\int_{F^n} f(T_{ij}(y)x)dx = \int_{F^n} f(x)dx$$

and

$$\int_{F^n} f(D_i(a)x)dx = |a|^{-1}\int_{F^n} f(x)dx.$$

Since $\det(T_{ij}(y)) = 1$ and $\det(D_i(a)) = a$ the lemma follows. ■

We note that if we look upon $GL(n,F)$ acting on $M_n(F)$ (the $n\times n$ matrices) by left multiplication then if we identify $M_n(F)$ with the n-fold product of F^n with itself then the matrix of left multiplication by g is the $n\times n$ block diagonal matrix with diagonal blocks equal to g. With the corresponding product measure on $M_n(F)$ we now have

Corollary 41 *Notation as above*

$$\int_{M_n(F)} f(gx)dx = |\det(g)|^{-n} \int_{M_n(F)} f(x)dx.$$

Lemma 42 *Let G be a locally compact, separable, topological group and let μ be a semi-invariant measure under the left regular representation. If*

$$\mu(L_g f) = c(g)\mu(f)$$

then the measure $f \longmapsto \mu(c^{-1}f)$ defines a Haar (left invariant) measure on G.

Exercise 43 *Prove this by the obvious calculation.*

Corollary 44 *Haar measure on $GL(n, F)$ is given by*

$$\int_{GL(n,F)} f(g) \frac{dg}{|\det(g)|^n}.$$

Here $\int_{GL(n,F)} f(g)dg$ means the restriction of the translation invariant measure on $M_n(F)$ (see Example 28).

The following lemma is a special case of a much more general result but it will be sufficient for our purposes.

Lemma 45 *Let X be a locally compact, separable topological space and let G be a locally compact, separable group acting on X. We assume that the action is transitive ($Gx_o = X$ for some, hence all, $x_o \in X$). We also assume that if $x_o \in X$ then*

$$G_{x_o} = \{g \in G | gx_o = x_o)$$

is compact. Then up to positive multiple there is at most one G-invariant measure on X.

Proof. For this we need the following fact (which is not completely trivial). Let $x_o \in X$ then X is homeomorphic with G/G_{x_o}under the map $gG_{x_o} \longmapsto gx_o$ (for a proof see Helgason,S., *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, 1978). Set $K = G_{x_o}$. Fix a Haar measure on K which will be denoted with the usual integral notation. We define for $f \in C_c(G; \mathbb{C})$,

$$T(f)(gx_o) = \int_K f(gk)dk.$$

We leave it to the reader to check that $T : C_c(G; \mathbb{C}) \to C_c(X; \mathbb{C})$ is linear and continuous. Let μ be a G-invariant measure on X then we define

$$\nu(f) = \mu(T(f))$$

for $f \in C_c(G;\mathbb{C})$. This is easily checked to be left invariant on G. We leave it to the reader to see that it is a measure on G. The lemma now follows from Haar's theorem. ■

We conclude this section with a result that will be used often in the rest of these lectures. First we need a bit more notation. We note that if G is a group then we can define the right regular action of G on G by $r_g x = xg^{-1}$. We set $R_g f(x) = f(xg)$. We note that if μ is a left invariant measure on G then the measure $f \longmapsto \mu(R_g f)$ is also a left invariant measure. Thus Haar's theorem implies that

$$\mu \circ R_g = \delta(g)\mu.$$

Definition 46 *The function δ is called the* modular function *of G. It is continuous by Lemma 34. If $\delta \equiv 1$ then we say that G is* unimodular.

Exercise 47 *Show that the measure $f \longmapsto \mu(\delta^{-1} f)$ is invariant under the right regular action. Such a measure will be called right invariant.*

Proposition 48 *Let G be a locally compact, separable, unimodular topological group and suppose that A and B are two closed subgroups of G such that*

$$AB = \{ab | a \in A, b \in B\} = G$$

and $A \cap B$ is compact. Then if dg denotes invariant measure on G, da denotes left invariant measure on A and db right invariant measure on B then up to constants of normalization

$$\int_G f(g)dg = \int_A \int_B f(ab)dadb.$$

In other words if $T : C_c(G) \to C_c(A \times B)$ is given by $T(f)(a,b) = f(ab)$, if μ is left invariant on A, ν is right invariant on B then $(\mu \times \nu) \circ T$ is invariant on G.

Proof. Let $A \times B$ act on G by $(a,b) \cdot x = axb^{-1}$. Then the stability group of 1 is $\{(k,k)|k \in A \cap B\}$ which is compact. The measure $(\mu \times \nu) \circ T$ and Haar measure are both invariant under this action of $A \times B$. Lemma 45 now implies the result. ■

Addendum

In this addendum we will discuss integration on locally compact, separable topological spaces in more detail. Fix such an X.

We say that an open covering $\mathcal{U}$ of X is locally finite if for each $p \in X$ there is an open neighborhood, U, of p in X such that the set $\{V \in \mathcal{U} | V \cap U \neq \emptyset\}$ is finite. A sequence $\{\phi_j\}$ in $C_c(X)$ is called a partition of unity if there is a locally finite open covering $\mathcal{U} = \{U_j\}$ of X such that for each j, the support of ϕ_j is contained in $\{U_j\}$ and

- $\phi_j(x) \geq 0$ for all $x \in X$.
- $\sum_j \phi_j(x) = 1$ for all $x \in X$.

Notice that the local finiteness assumption implies that for each $x \in X$ there is a neighborhood of x such that the sum in the second part of the definition is uniformly finite. We say that the partition of unity is subordinate to an open covering $\mathcal{V}$ of X if the $\mathcal{U}$ as above is a refinement of $\mathcal{V}$.

Theorem A.1 *Let V be an open covering of X then there exists a partition of unity subordinate to V.*

This is a classic theorem of Dieudonne. It is proved in the same way as the standard differential geometric result using Urysohn's lemma to construct "patch functions". We note that if $\phi \in C_c(X)$ and $f \in C(X)$ then $\phi f \in C_c(X)$.

Lemma A.2 *Let μ be a measure on X. Let $f \in C(X)$ then if for some partition of unity $\{\phi_j\}$*

$$\sum_j \mu(\phi_j |f|) < \infty$$

then this is so for every partition of unity. Furthermore, under the hypothesis of the above convergence the series

$$\sum_j \mu(\phi_j f)$$

converges absolutely and its value is independent of the choice of partition of unity.

Proof. Let $\{\psi_j\}$ be another partition of unity. Then we have for each j and for each $g \in C(X)$

$$\sum_i \psi_i \phi_j g = \phi_j g$$

and the sum is finite. Thus we have (using both locally finite conditions)

$$\sum_j \mu(\phi_j |f|) = \sum_j \sum_i \mu(\psi_i \phi_j |f|) = \sum_i \sum_j \mu(\psi_i \phi_j |f|) = \sum_i \mu(\psi_i |f|).$$

Now $|\mu(\phi_j f)| \leq \mu(|\phi_j f|) = \mu(\phi_j |f|)$. Thus the sum in the second formula is absolutely convergent. The last assertion is now proved by the same interchange of orders of integration. ■

We say that $f \in C(G)$ is *μ-summable* if

$$\sum_j \mu(\phi_j |f|) < \infty$$

for some, hence every, partition of unity for X. If f is summable and if $\{\phi_j\}$ is a partition of unity for X then we set

$$\mu(f) = \sum_j \mu(\phi_j f).$$

The content of the previous result is that this expression is independent of the choice of partition of unity.

Lemma A.3 *If $\phi \in C(X)$ is summable and if $f \in C(X)$ satisfies $|f(x)| \leq |\phi(x)|$ for all $x \in X$ then f is summable.*

Proof. We have $\mu(\phi_j|f|) \leq \mu(\phi_j|\phi|)$. Thus

$$\sum_j \mu(\phi_j|f|) \leq \sum_j \mu(\phi_j|\phi|).$$

■

A measure, μ, on X is said to be normal if whenever $\phi \in C_c(X)$ is such that $\phi(x) \geq 0$ for all $x \in X$ and $\phi \neq 0$ then $\mu(\phi) > 0$.

Lemma A.4 *If G is a group acting transitively on X and if μ is a G-invariant measure on X then μ is normal.*

Proof. Suppose that $f \in C_c(X)$ and $f(x) \geq 0$ for all $x \in X$ and that $f \neq 0$. Assume that $\mu(f) = 0$. Then $\mu(L_g f) = 0$ for all $g \in G$. Let $U = \{x \in X | f(x) > 0\}$. Then U is open and non-empty. Let $\{\phi_j\}$ be a partition of unity subordinate to the covering $\{gU\}_{g \in G}$. Then for each j there exists $g_j \in G$ such that the support of ϕ_j is contained in $\{g_j U\}$. Thus $L_{g_j} f(x) > 0$ if x is in the support of ϕ_j, K_j. Let $c_j = \min_{k \in K_j} f(k)$. Then $c_j > 0$. We have

$$L_{g_j} f(x)\phi_j(x) \geq c_j \phi_j(x).$$

Also

$$L_{g_j} f(x)\phi_j(x) \leq L_{g_j} f(x).$$

Putting this together we have

$$c_j \mu(\phi_j) \leq \mu((L_{g_j} f)\phi_j) \leq \mu(L_{g_j} f) = 0.$$

Hence $\mu(\phi_j) = 0$ for all j. If $g \in C_c(X)$ and if $C = \max_{x \in X} |g(x)|$. Then

$$|\mu(\phi_j g)| \leq \mu(\phi_j |g|) \leq C\mu(\phi_j) = 0.$$

Thus $\mu(\phi_j g) = 0$. Thus $0 = \sum_j \mu(\phi_j g) = \mu(g)$. Since g is arbitrary we have a contradiction to the convention that $\mu \neq 0$. ■

1.1.3 Operator algebras

For the rest of these lectures $|z| = (z\bar{z})^{\frac{1}{2}}$ for a complex number z.

Let V and W be vector spaces over $\mathbb{C}$. We will use the notation $Hom_{\mathbb{C}}(V, W)$ for the space of all linear maps from V to W. If V and W are topological spaces then we denote by $L(V, W)$ the subspace of continuous linear maps.

Let V, W be locally convex topological spaces with the topology of V given by the semi-norms in the set $\mathcal{S}$ and that of W given by the semi-norms in the set $\mathcal{T}$ both as in Example 11 We note

Lemma 49 *If p is a continuous semi-norm on V then there exists $q \in \mathcal{S}$ and $A > 0$ such that $p(v) \leq Aq(v)$ for all $v \in V$.*

Proof. The sets $U_{q,\varepsilon}$ for $q \in \mathcal{S}$ and $\varepsilon > 0$ form a basis of neighborhoods of 0 in V. Since p is continuous, the set $U = \{v \in V | p(v) < 1\}$ is open in V. This implies that there exist $q \in \mathcal{S}$ and $\varepsilon > 0$ such that $U_{q,\varepsilon} \subset U$. Fix $0 < \delta < \varepsilon$. If $v \in V$ and $q(v) > 0$ then $q(\frac{\delta}{q(v)} v) = \delta < \varepsilon$. Thus $p(\frac{\delta}{q(v)} v) < 1$. So if $q(v) > 0$ we have $p(v) < \left(\frac{1}{\delta}\right) q(v)$. If $q(v) = 0$ then $p(tv) < 1$ all $t > 0$. Thus $p(v) = 0$. Take $A = \frac{1}{\delta}$. ■

Corollary 50 *$T \in Hom_{\mathbb{C}}(V, W)$ is continuous if and only if for every $p \in \mathcal{T}$ there exists $q \in \mathcal{S}$ and $C > 0$ such that $p(Tv) \leq Cq(v)$ for all $v \in V$.*

This follows from the above lemma and Lemma 13.

Definition 51 *Let V and W be locally convex spaces as above. Then the strong (operator) topology on $L(V, W)$ is the locally convex topology defined by the semi-norms $q(Tv)$ for $q \in \mathcal{T}$, $v \in V$.*

If V, W are normed spaces with norms by $|...|_V$ and $|...|_W$ respectively then the condition for continuity is just that there exists $A > 0$ such that $|T(v)|_W \leq A|v|_V$. This is the reason why a continuous linear map is called a *bounded linear operator.*

Definition 52 *In the context above, if $T \in L(V, W)$ we define*

$$\|T\| = \inf\{A > 0 | \, |T(v)|_W \leq A|v|_V, forallv \in V\}.$$

Exercise 53 *Show that this defines a norm on $L(V, W)$ (*called the operator norm*). Furthermore, if $S \in L(W, Z)$ with Z a normed space then $\|S \circ T\| \leq \|S\| \, \|T\|$.*

We note that if $V = W$ then $L(V, V)$ is an algebra under addition and composition. This is an example of a normed algebra. We now give the general definition.

Definition 54 *A pair $(R, \|...\|)$ of an algebra R over $\mathbb{C}$ and a norm $\|...\|$ on R is said to be a* normed algebra *if $\|ab\| \leq \|a\| \, \|b\|$ for all $a, b \in R$. It is called a* Banach algebra *if R is a Banach space with respect to $\|...\|$.*

Example 55 *Let X a compact topological space then $C(X)$ is a Banach algebra with respect to the uniform norm p_X (see Example 14).*

Lemma 56 *If V, W are Banach spaces then $L(V,W)$ is a Banach space with respect to the operator norm. In particular, $L(V,V)$ is a Banach algebra.*

Proof. Suppose that $\{T_j\}$ is a Cauchy sequence in $L(V,W)$. Then if $v \in V$ we have

$$|T_i v - T_j v|_W = |(T_i - T_j)v|_W \leq \|T_i - T_j\| \, |v|_V.$$

Thus $\{T_j v\}$ is a Cauchy sequence in W. Since W is a Banach space the sequence has a limit. Which is easily seen to define a linear function of v. This we have $S \in Hom_{\mathbb{C}}(V,W)$ such that

$$\lim_{j\to\infty} T_j v = Sv, v \in V.$$

By its very definition

$$\lim_{j\to\infty} |T_j v|_W = |Sv|_W, v \in V.$$

Thus since $| \, \|T_i\| - \|T_j\| \, | \leq \|T_i - T_j\|$ we see that the $\|T_j\|$ form a Cauchy sequence. Hence this set is bounded which implies that S is bounded and

$$\lim_{j\to\infty} \|T_j\| = \|S\|.$$

Finally, if $\varepsilon > 0$ is given there exists N such that $\|T_i - T_j\| < \varepsilon$ for $i, j \geq N$. Also, if $v \in V$ then there exists $N_1 \geq N$ such that if $j \geq N_1$ then $\|T_j v - Sv\| \leq \varepsilon |v|_V$. Thus if $i \geq N$

$$\begin{aligned} \|(T_i - S)v\| = \|(T_i - T_j)v + (T_j - S)v\| \leq \\ \|T_i - T_j\| \, |v|_V + \|(T_j - S)v\|. \end{aligned}$$

Taking $j \geq N_1$ we have the inequality

$$\|(T_i - S)v\| \leq 2\varepsilon |v|_V.$$

Hence $\|T_i - S\| \leq 2\varepsilon$ if $i \geq N$. ∎

If $(R, \|...\|)$ is a normed algebra then if we complete the underlying Banach space the continuity of the algebra structure implies that the complete Banach space has a structure of a Banach algebra. Which we will call the completion of that normed algebra R.

We now look at a second important example of a Banach algebra. Let H be a Hilbert space.

Definition 57 *An element $T \in L(H,H)$ is said to be* completely continuous *(or* compact*) if whenever $S \subset H$ is a bounded set then the closure of $T(S)$ is compact. T is said to be of* finite rank *if* $\dim T(H) < \infty$.

Obviously finite rank implies completely continuous. The converse of this is also almost true.

Lemma 58 *$T \in L(H,H)$ is completely continuous if and only if it is a limit in the norm topology of a sequence of operators of finite rank.*

The proof uses the following lemma and in the course of the proof we will also develop a great deal of the Hilbert space theory that we will be using. We will use the notation $CC(H)$ for the set of completely continuous operators on H.

Lemma 59 *$CC(H)$ is a closed ideal (right and left) in the Banach algebra $L(H,H)$.*

Proof. It is clear that $CC(H)$ is a subspace. Let $T \in L(H,H)$ and $S \in CC(H)$. If U is a bounded subset of H then so is $T(U)$. Hence $S(T(U))$ has compact closure. Let W be the closure of $S(U)$. Then $T(S(U))$ is compact since T is continuous and it contains $TS(U)$. Hence, $T(S(U))$ has compact closure. Thus $CC(H)$ is an ideal. We now prove that it is closed. So assume that $\{T_j\}$ is a sequence in $CC(H)$ and that $\lim_{j\to\infty} T_j = T \in L(H,H)$. We prove that if $\{u_n\}$ is an infinite bounded sequence in H then $\{Tu_n\}$ contains a convergent subsequence. We choose $\{u_{n_{1,j}}\}$ an infinite subsequence of $\{u_n\}$ such that $\{T_1 u_{n_{1,j}}\}$ is convergent. We choose $\{u_{n_{2,j}}\}$ an infinite subsequence of $\{u_{n_{1,j}}\}$ such that $\{Tu_{n_{2,j}}\}$ is convergent, etc. If we set $v_i = u_{n_{i,i}}$ then $\{v_i\}$ is an infinite subsequence of $\{u_n\}$ such that $\{T_j v_i\}$ converges (in i) for all j.

Let $\varepsilon > 0$ be given and let N be such that if $j \geq N$ then$\|T - T_j\| < \varepsilon$. Let N_1 be such that if $n, m \geq N_1$ then $\|T_N(v_n - v_m)\| < \varepsilon$. If $n, m \geq N_1$ then

$$\|T(v_m - v_n)\| = \|(T - T_N)(u_m - u_n) + T_N(u_m - u_n\| \leq$$
$$\|T - T_N\| 2C + \varepsilon \leq (2C+1)\varepsilon.$$

The result now follows. ■

We will also use some aspects of Hilbert space theory that are completely standard and I will just quote many of them. For example all can be found in Yoshida's book.

So for the time being H is a Hilbert space with inner product $\langle ..., ... \rangle$. The most important single result is the Riesz representation theorem (p.82 in Yoshida).

Theorem 60 *Let V be a closed subspace of H and set $V^{\perp} = \{z \in H | \langle z, v \rangle = 0, v \in V\}$. Then $H = V \bigoplus V^{\perp}$.*

This result implies

Corollary 61 *If $\lambda \in H'$ then there exists a unique element $f \in H$ such that $\lambda(x) = \langle x, f \rangle$ for all $x \in H$.*

We will do the easy argument that shows how the theorem does imply the corollary. We may assume that $\lambda \neq 0$. Set $V = \ker \lambda$ which is closed and codimension 1. Then $V^{\perp}$ is one dimensional. Since $\lambda \neq 0$ there is a unique element $u \in V^{\perp}$ such that $\lambda(u) = 1$. Take $f = \frac{u}{\|u\|^2}$.

Definition 62 *If V is a closed subspace of H then we define $P_V : H \to V \subset H$ by setting $P_V(v) = v$ if $v \in V$ and $P_V(u) = 0$ if $u \in V^{\perp}$ and extending by linearity. P_V is called the* orthogonal projection *onto V.*

If $T \in L(H,H)$ then we note that if $w \in H$ the map $v \longmapsto \langle Tv, w \rangle$ defines an element of H'. Hence, there exists a unique element in H, denoted $T^*(w)$, such that

$$\langle Tv, w \rangle = \langle v, T^*(w) \rangle .$$

We note that it is clear that T^* defines a linear map of H to H. We also note that if $T^*(v) \neq 0$ then

$$\|T^*v\|^2 = \langle T^*v, T^*v \rangle = \langle TT^*v, v \rangle \leq \|T\| \, \|T^*v\| \, \|v\| .$$

So if we divide $\|T^*v\|$ from both sides we have $\|T^*v\| \leq \|T\| \, \|v\|$. This T^* is bounded and has norm at most $\|T\|$. Noting that $(T^*)^* = T$ we see that $\|T^*\| = \|T\|$.

Definition 63 *The operator T^* will be called the adjoint of T.*

Lemma 64 *Let $T \in CC(H)$ be such that $T = T^*$. Then there exists an orthonormal basis, $\{v_n\}$, of $\ker T^{\perp}$ and $\lambda_j \in \mathbb{R}$ such that $Tv_n = \lambda_n v_n$ and the dimension of $T^{\perp}$ is infinite then $\lim_{n\to\infty} \lambda_n = 0$.*

This result is completely standard. The simplest proof of it that we know is in N.Wallach, Real Reductive GroupsI, Academic Press, 1988, p.326, 8,A.1.2.

We can now complete the proof of Lemma 58. If $T \in CC(H)$ then $T = \frac{T+T^*}{2} + i\frac{T-T^*}{2}$. Each term is self adjoint and compact. We may therefore assume that $T = T^*$. Let v_j be as in Lemma 64. If there are only a finite number then T has finite rank. So assume that the number is infinite. Let for each m, V_m be the span of $v_1, ..., v_m$. Let P_m be the orthogonal projection onto V_m. Then $\|P_m T - T\| = \sup\{|\lambda_n| |n > m\} \to 0$ as $m \to \infty$. The necessity is a consequence of Lemma 59.

The upshot of all of this is that $CC(H)$ is an example of a Banach algebra.

We now come to our third main example of a Banach algebra. Let G be a locally compact separable (i.e countable basis for the topology) group and let μ be a choice of left invariant measure. if $f, g \in C_c(G)$ then we define $\check{g}(u) = g(u^{-1})$ and

$$f * g(x) = \mu(f L_x \check{g}).$$

In integral notation this says

$$f * g(x) = \int_G f(u)g(u^{-1}x)du.$$

Let A, B denote the supports of f and g respectively. Then the above expression is 0 if $x \notin AB$. Also one sees from the first formula that $f * g$ is continuous and hence it is an element of $C_c(G)$. This element is called the *convolution* of f and g. We observe that this bilinear operation is an algebra structure. Indeed,

$$\begin{aligned}
f_1 * (f_2 * f_3)(x) &= \int_G f_1(g_1)(f_2 * f_3)(g_1^{-1}x)dg_1 \\
&= \int_G f_1(g_1)(\int_G f_2(g_2)f_3(g_2^{-1}g_1^{-1}x)dg_2)dg_1 \\
&= \int_G \int_G f_1(g_1)f_2(g_2)f_3((g_1g_2)^{-1}x)dg_2dg_1 \\
&= \int_G \int_G f_1(g_1)f_2(g_1^{-1}g_2)f_3(g_2{}^{-1}x)dg_2dg_1
\end{aligned}$$

by the left invariance of the integral in g_2. The last expression can be written

$$\int_G (\int_G f_1(g_1)f_2(g_1^{-1}g_2)dg_1)f_3(g_2{}^{-1}x)dg_2 = (f_1 * f_2) * f_3(x).$$

We have therefore endowed $C_c(G)$ with the structure of an algebra.

Exercise 65 *Show that convolution defines a continuous map of $C_c(G) \times C_c(G)$ to itself. Thus $C_c(G)$ is a topological algebra under convolution.*

Exercise 66 *Show that if $f, g \in C_c(G)$ then $\int_G (f*g)(x)dx = \int_G f(x)dx \int_G g(x)dx$.*

If $f \in C_c(G)$ then we set $\|f\|_1 = \mu(|f|)$. This is the usual L^1-norm.

Lemma 67 *$C_c(G)$ endowed with the L^1-norm is a normed algebra under convolution.*

Proof. We have

$$\begin{aligned}
\|f_1 * f_2\|_1 &= \int_G |f_1 * f_2(x)|dx = \int_G |\int_G f_1(g)f_2(g^{-1}x)dg|dx \\
&\leq \int_G \int_G |f_1(g)f_2(g^{-1}x)|dgdx = \int_G |f_1(g)|(\int_G |f_2(g^{-1}x)|dx)dg \\
&= \int_G |f_1(g)| \int_G |f_2(x)|dxdg = \|f_1\|_1 \|f_2\|_1 .
\end{aligned}$$

■

This implies that under convolution $C_c(G)$ is a normed algebra relative to the L^1-norm. We denote the corresponding competed algebra by $L^1(G)$. One can show that elements of $L^1(G)$ are represented by functions on G (not necessarily continuous, nor of compact support).

If $f_1, f_2 \in C_c(G)$ then we define

$$\langle f_1, f_2 \rangle = \mu(f_1 \overline{f_2}) = \int_G f_1(g)\overline{f_2(g)}dg.$$

Then this defines a Hermitian inner product on $C_c(G)$ and thereby endows it with the structure of a pre-Hilbert space. The Hilbert space completion of $C_c(G)$ will be denoted $L^2(G)$. As is customary, we write $\|f\|_2 = \langle f, f \rangle^{\frac{1}{2}}$.

Definition 68 *If $(H, \langle ..., ... \rangle)$ is a Hilbert space then an element, T, of $Hom_{\mathbb{C}}(H, H)$ is said to be* unitary *if it is surjective and $\langle Tf_1, Tf_2 \rangle = \langle f_1, f_2 \rangle$ for all f_1, f_2 in H.*

We note that if T is unitary on H and if $\|...\|$ is the corresponding norm on H then $\|Tv\| = \|v\|$ for all $v \in H$. In particular, T is bounded and injective. Since it is surjective it is bijective. The inverse is given by T^*.

Lemma 69 *If $g \in G$ then L_g defines a unitary operator on $L^2(G)$. Furthermore, the map $G \to L(L^2(G), L^2(G))$, $g \longmapsto L_g$ is continuous if we endow $L(L^2(G), L^2(G))$ with the strong topology.*

We will use some more general concepts to prove this result.

Definition 70 *Let G be a locally compact topological group. Then a representation of G is a pair (π, V) of a locally convex space V and a homomorphism, π of G into the continuous invertible operators on V such that the map $\pi : G \to L(V, V)$ is continuous in the strong, operator topology. If V is a Banach (resp. Hilbert, rep. Fréchet) space then (π, V) is called a Banach (respect Hilbert, Fréchet) representation. If V is a Hilbert space and $\pi(g)$ is a unitary operator for each $g \in G$ then (π, V) is called a unitary representation.*

Thus the content of the above lemma is that if we set $\pi(g) = L_g$ then $(\pi, L^2(G))$ is a unitary representation called the left regular representation. Notice that in the Banach case we do not use the norm topology on $L(V, V)$.

Remark 71 *If we replace left invariant measure with right invariant measure then R_g defines a unitary representation of G on the corresponding $L^2(G)$.*

We will be mainly dealing with Hilbert representations in these lectures. When we need more general spaces it will be because we have imposed stronger conditions than continuity on our representations. We need a general method of checking the strong continuity. For this we will use the principle of uniform boundedness.

Theorem 72 *Let V, W be Banach spaces and let $\mathcal{U}$ be a subset of $L(V, W)$. Suppose that for each $v \in V$ there exists $C_v < \infty$ such that $\|Tv\| \leq C_v \|v\|$ for all $T \in \mathcal{U}$. Then there exists $C < \infty$ such that $\|T\| \leq C$ for all $T \in \mathcal{U}$.*

This surprising theorem is a consequence of the Baire category theorem. A proof can be found in Yoshida (cited at the beginning of 1.1.1).

Lemma 73 *Let H be a Hilbert space, let G a locally compact, separable topological group and let π be a homomorphism of G into bounded, invertible operators on H. Then π defines a representation of G (that is, it is continuous in the strong topology) if and only if the following two conditions are satisfied*

1. There is a dense subspace, $V \subset H$ such that if $v \in V, w \in H$ the function $c_{v,w}(g) = \langle \pi(g)v, w \rangle$ is continuous.

2. If K is a compact subset of G then there exists a positive constant C_K such that $\|\pi(g)\| \leq C_K$ for all $k \in K$.

Proof. If (π, H) is a representation then condition 1. is clearly satisfied. The strong continuity implies that the functions

$$g \longmapsto \|\pi(g)v\|$$

are continuous on G. Condition 2 now follows from the principle of uniform boundedness. We now consider the converse. So we assume the two conditions. We observe that they imply

$1'$. $c_{v,w}$ is continuous for all $v, w \in H$.

This is proved by a "3ε argument". Let $v \in H$, $g \in G$, $\{v_j\}$ a sequence in V such that $\lim v_j = v$, $\{g_n\}$ a sequence in G such that $\lim g_n = g$. Then there exists a compact subset K of G containing g and each of the g_n. We must show that $\lim_{n\to\infty} c_{v,w}(g_n) = c_{v,w}(g)$. We note that Condition 2 implies that $\|\pi(k)\| \leq C_K < \infty$ for all $k \in K$. Set $C = C_K$. We have

$$\begin{aligned}
|c_{v,w}(g_n) - c_{v,w}(g)| &= |c_{v,w}(g_n) - c_{v_j,w}(g_n) + c_{v_j,w}(g_n) - c_{v_j,w}(g) + c_{v_j,w}(g) - c_{v,w}(g)| \\
&\leq |c_{v,w}(g_n) - c_{v_j,w}(g_n)| + |c_{v_j,w}(g_n) - c_{v_j,w}(g)| + |c_{v_j,w}(g) - c_{v,w}(g)| \\
&= |\,\langle \pi(g_n)(v - v_j), w \rangle\,| + |c_{v_j,w}(g_n) - c_{v_j,w}(g)| + |\,\langle \pi(g)(v_j - v), w \rangle\,| \\
&\leq 2C\,\|v - v_j\|\,\|w\| + |c_{v_j,w}(g_n) - c_{v_j,w}(g)|.
\end{aligned}$$

Now let $\varepsilon > 0$ be given then there exists N such that if $j \geq N$ then $\|v - v_j\| < \varepsilon$. Fix one such j. There exists N_1 such that if $n \geq N_1$ then $|c_{v_j,w}(g_n) - c_{v_j,w}(g)| < \varepsilon$. Putting all of this together we have that if $n \geq N_1$ then $|c_{v,w}(g_n) - c_{v,w}(g)| < (2C+!)\varepsilon$. This proves $1'$.

We will now begin the proof of the lemma. Ideas in this proof will be used in the next section. Let H_o be the subspace of all $v \in H$ such that the map $g \longmapsto \pi(g)v$ is continuous from G to H. Then using an argument as in the proof of $1'$. one can show that condition 2. implies that H_o is closed. Also, it is not hard to see that if we can show that $H_o = H$ then the result is proved.

If $f \in C_c(G)$ then we set

$$\mu_f(v, w) = \int_G f(g)\,\langle \pi(g)v, w \rangle\,dg.$$

If the support of f is contained in the compact set K and if $\phi \in C_c(G)$ is such that $\phi(k) = 1$ for all $k \in K$ (such a ϕ exists by Urysohn's theorem) then we have

$$\left|\mu_f(v,w)\right| \leq C_K \left\|\phi\right\|_1 p_K(f) \left\|v\right\| \left\|w\right\|.$$

Thus the Riesz representation theorem implies that for each $v \in H$ there exists an element $T_f(v) \in H$ such that $\langle T_f(v), w\rangle = \mu_f(v,w)$. It is easy to see that T_f is a linear map of H to H. The estimate above now shows that $\|T_f(v)\| \leq C_K \|\phi\|_1 p_K(f) \|v\|$. Thus the map $f \longmapsto T_f$ of the completion of $C_c(G)$ into $L(H,H)$ is strongly continuous. We note that

$$T_{L_g f} = \pi(g) T_f.$$

Hence $T_f(H) \in H_o$ for all $f \in C_c(G)$. Now, since G is separable and locally compact we can find a sequence of open subsets $U_j \subset G$ such that $\overline{U_j}$ is compact, $U_j \supset \overline{U}_{j+1}$ and $\cap_j U_j = \{1\}$. Urysohn's lemma implies that there exists $\phi_j \in C_c(G)$ such that the support of ϕ_j is contained in U_j, $\phi_j(x) \geq 0$ for all $x \in G$ and $\phi_j(x) = 1$ for all $x \in \overline{U}_{j+1}$. Set $u_j(x) = \frac{\phi_j(x)}{\|\phi_j\|_1}$. Then if $v, w \in H$

$$\lim_{j\to\infty} \left\langle T_{u_j}(v), w\right\rangle = \langle v, w\rangle.$$

Before we prove this we will show how it completes the proof.

We need to show that $H_o^\perp = 0$. But if $w \in H_o^\perp$ then $\left\langle T_{u_j}(w), w\right\rangle = 0$ for all j. Hence the limit formula implies that $\langle w, w\rangle = 0$.

To prove the limit formula we note that

$$\left\langle T_{u_j}(v), w\right\rangle - \langle v, w\rangle = \int_G u_j(g)(c_{v,w}(g) - c_{vw}(1))dg.$$

Let $\varepsilon > 0$ be given then there exists N such that if $j \geq N$ then

$$|c_{v,w}(g) - c_{vw}(1)| < \varepsilon \text{ for } g \in U_j.$$

Thus if $j \geq N$ then

$$\int_G u_j(g)(c_{v,w}(g) - c_{vw}(1))dg \leq \varepsilon \int_G u_j(g)dg = \varepsilon.$$

This completes the proof. ■

Definition 74 *If (π, H) is a Hilbert representation of G then the operator T_f as defined in the above proof will be denoted $\pi(f)$.*

Definition 75 *If G is a locally compact separable topological group then a sequence $\{u_j\}$ of non-negative functions in $C_c(G)$ such that $\|u_j\|_1 = 1$ for all j and there exist open subsets U_j of G such that $\overline{U_j}$ is compact, $U_j \supset \overline{U}_{j+1}$ and $\cap_j U_j = \{1\}$ and the support of u_j is contained in U_j then $\{u_j\}$ will be called a* delta sequence *or* approximate identity *on G.*

Exercise 76 *Prove Lemma 69.*

1.2 Basic representation theory

1.2.1 Schur's lemma.

Throughout this section G will denote a locally compact, separable topological group.

Perhaps Schur's lemma (in its many guises) the most fundamental single result of representation theory. In this section we will give several variants of this result. The first is the most standard and is a direct consequence of the spectral theorem. We first need some definitions.

Definition 77 *Let (π, V) be a representation of G on a locally convex space V. A subspace W of V is said to be* invariant *if $\pi(g)W \subset W$. The representation is said to be irreducible if the only closed, invariant subspaces are $\{0\}$ and V.*

Definition 78 *If (π_i, V_i), $i = 1, 2$ are respectively representations of G on locally convex spaces V_1, V_2 then a continuous map $T : V_1 \to V_2$ will be called a* G-homomorphism *or* G-intertwining operator *if $T \circ \pi(g) = \pi(g) \circ T$ for all $g \in G$. We will use the notation $L_G(V_1, V_2)$ for the space of all G-homomorphisms from V_1 to V_2. The representations are said to be equivalent if there exists an element then is bijective with continuous inverse in $L(V_1, V_2)$.*

Remark 79 *In the literature the notation $Hom_G(V_1, V_2)$ is often used for what we are calling $L_G(V_1, V_2)$.*

Here is the first version of Schur's Lemma which is a direct consequence of the spectral theorem.

Proposition 80 *Let (π, H) be a unitary representation of G. Then it is irreducible if and only if $L_G(H, H) = \mathbb{C}I$.*

Proof. Suppose $L_G(H, H) = \mathbb{C}I$. Let V be a closed invariant subspace of H. Let P denote the orthogonal projection of H onto V. If $v \in H$ then $v = v_1 + v_2$ with $v_1 \in V$ and $v_2 \in V^\perp$. If $w \in V^\perp$ and if $v \in V$ then for each $g \in G$ we have $0 = \langle \pi(g)v, w \rangle = \langle v, \pi(g^{-1})w \rangle$ by the assumption of unitarity. But then $V^\perp$ is an invariant space. Thus $\pi(g)v_1 \in V$ and $\pi(g)v_2 \in V^\perp$. Hence $P\pi(g)v = \pi(g)v_1 = \pi(g)Pv$. Thus $P \in L_G(H, H)$. Thus $P = zI$, $z \in \mathbb{C}$. Since P is a projection $z = 0$ or I. Thus $V = \{0\}$ or $V = H$.

We now prove the converse. We first note that if $T \in L_G(H, H)$ then so is T^*. Since $T = \frac{T+T^*}{2} + i\frac{T-T^*}{2i}$ we must only show that if T is a self adjoint intertwining operator then T is a multiple of the identity. So we assume $T \in L_G(H, H)$ and $T^* = T$. To such an operator there is an associated family of spectral projections, P_S, for $S \subset \mathbb{R}$ a Borel set. (See Reed,M. and Simon,B., *Functional Analysis I,*Academic Press,1972., p.234.) The uniqueness of the spectral resolution and the fact that $\pi(g)T\pi(g)^{-1} = T$ implies that $\pi(g)P_S\pi(g)^{-1} = P_S$ for all S. Then $T = pI$ if and only if $P_{\{p\}} = I$. If the real interval $[a, b]$ contains the spectrum of T then $P_{[a,b]} = I$. Let $J_1 = [a, b]$. If we bisect J_1, then $J_1 = A \cup B$ and one of P_A or P_B is non-zero. Thus $P_A = I$ or $P_B = I$. Let J_2 be one of

A, B such that $P_{J_2} = I$. We can bisect again and get J_3 one of the halves such that $P_{J_2} = I$. We this have a nested sequence of intervals $J_1 \supset J_2 \supset J_3 \supset ...$ such that J_i has length $2^{-i}(b-a)$ and $P_{J_i} = I$. We note that $\cap_i J_i = \{p\}$ for some $p \in \mathbb{R}$. The definition of spectral projections implies that the limit of the P_{J_i} in the strong operator topology is $P_{\{p\}}$. Thus $P_{\{p\}} = I$. Hence $T = pI$. ■

We will rephrase this result in the context of operator algebras. Let $A \subset L(H,H)$ be a subalgebra. Then it is called a **algebra* if whenever $T \in A$, $T^* \in A$. We say that A is an *irreducible subalgebra* if whenever $V \subset H$ is a closed subspace invariant under all the elements of A, $V = \{0\}$ or $V = H$.

Definition 81 *If A is a subset of $L(H,H)$ then we denote by A' the set $\{T \in L(H,H) | Ta = aT, a \in A\}$. A' is called the commutant of A.*

The above is a standard notation. It unfortunately conflicts with our notation for dual space. It will be able to keep track of this ambiguity through the context of its usage.

We will now restate Schur's lemma.

Corollary 82 *(to the proof)A *algebra $A \subset L(H,H)$ is irreducible if and only if $A' = \mathbb{C}I$.*

Proof. We note that if $V \subset H$ is a closed subspace invariant under every element of A then so is $V^{\perp}$. Thus as above $P_V \in A'$. Thus if $A' = \mathbb{C}I$. Then A is irreducible (as above). To prove the converse, we note that if $T \in A'$ then so is T^*. If $a \in A$ is such that $a^* = a$ then the element

$$e^{ia} = \sum_{n=0}^{\infty} \frac{(ia)^n}{n!}$$

defines a unitary operator on H. Since

$$\frac{d}{dt}_{|t=0} e^{ita} T e^{-ita} = i(aT - Ta).$$

We see that $T \in A'$ if and only if $e^{ia} T e^{-ia} = T$ for all $a \in A$ such that $a = a^*$. We can now argue in exactly the same way as we did in the proof of the previous proposition to prove that $A' = \mathbb{C}I$. ■

We now come to the *Von Neumann density theorem.*

Theorem 83 *Let $A \subset L(H,H)$ be a *subalgebra containing I. Let $T \in (A')'$ and let $\{x_n\}_{n=1}^{\infty}$ be a sequence in H such that $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$. Then given $\varepsilon > 0$ there exists $a \in A$ such that $\sum_{n=1}^{\infty} \|(T-a)x_n\|^2 < \varepsilon$.*

Proof. Let V be a Hilbert space and let B be a *subalgebra of $L(V,V)$ containing I. Then

(1) If $v \in V$ then $(B')'v \subset \overline{Bv}$.

Indeed, $\overline{Bv}^{\perp}$ is B invariant since B is invariant under *. This implies that if P is the orthogonal projection of V onto $\overline{Bv}$ then $P \in B'$. Thus if $T \in (B')'$

then $TP = PT$. Hence $T(\overline{Bv}) \subset \overline{Bv}$. We therefore see that $(B')'(\overline{Bv}) \subset \overline{Bv}$. This proves the result since $v \in \overline{Bv}$.

We will apply this result to the Hilbert space V that consists of all sequences $\{x_n\}$ with $\sum_{n=1}^{\infty} \|x_n\|^2 < \infty$ and inner product $\langle \{x_n\}, \{y_n\} \rangle = \sum_n \langle x_n, y_n \rangle$. Let B be the algebra of operators $\sigma(a)$, $a \in A$ given by $\sigma(a)\{x_n\} = \{ax_n\}$. Let $P_m\{x_n\} = x_m$ and let $Q_m x = \{\delta_{n,m}x\}_{n=1}^{\infty}$. Then $P_m^* = Q_m$.

Suppose that $T \in B'$ and $a \in A$. Then

$$P_m T\sigma(a)\{x_n\} = P_m\sigma(a)T\{x_n\} = aP_mT\{x_n\}.$$

Also

$$TQ_m ax = T\sigma(a)Q_m x = \sigma(a)TQ_m x.$$

This implies

(2) If $T \in B'$ then $P_n T Q_m \in A'$ for all $n, m \geq 1$.

This implies that if $S \in (A')'$ then the operator $\{x_n\} \longmapsto \{Sx_n\}$ is in $(B')'$. Hence if $T \in (A')'$ then

$$\{Tx_n\} \in \overline{B\{x_n\}}.$$

This implies that given $\varepsilon > 0$ there exists $a \in A$ such that $\sum_{n=1}^{\infty} \|(T-a)x_n\|^2 < \varepsilon$. ■

The following result is also referred to as the Von Neumann density theorem in the literature.

Corollary 84 *Let A be a *subalgebra of $L(H,H)$ then if I is in the closure of A with respect to the strong operator topology then the algebra $(A')'$ is the closure of A in the strong operator topology.*

Proof. Let $C = A + \mathbb{C}I$. Then the above result implies that $(C')'$ is contained in the closure of C in the strong topology. Now the closure of C is the same as the closure of A by our hypothesis. Also it is clear that $A' = C'$. Thus $(A')'$ is contain in the closure of A. Since the reverse inclusion is clear, the result follows. ■

This result yields an analog of Burnside's theorem.

Corollary 85 *Let A be a *subalgebra of $L(H,H)$ containing the identity in its closure in the strong operator topology and acting irreducibly on H then the closure of A in the strong topology is $L(H,H)$.*

Proof. $A' = \mathbb{C}I$. ■

At this point we can introduce an important class of algebras for abstract representation theory.

Definition 86 *A *subalgebra of $L(H,H)$ is called a* Von Neumann algebra *of it is closed in the strong operator topology and contains the identity.*

The above results imply

Proposition 87 *A* **subalgebra,* A*, of* $L(H,H)$ *is a Von Neumann algebra if and only if* $(A')' = A$*.*

We will use this result to give a useful variant of Schur's lemma.

Proposition 88 *Let* (π, H) *be an irreducible unitary representation of* G*. Let* D *be a dense subspace of* H *such that* $\pi(g)D \subset D$ *for all* $g \in G$ *and let* T *be a linear map of* D *to* H *such that* $T\pi(g)v = \pi(g)Tv$ *for all* $g \in G, v \in D$*. Assume that there exists a dense subspace* D' *in* H *and a linear map* S *from* D' *to* H *such that*

$$\langle Tv, w\rangle = \langle v, Sw\rangle$$

for all $v \in D$*,* $w \in D'$*. Then* $T = \lambda I$ *for some* $\lambda \in \mathbb{C}$*.*

Remark 89 *Notice that there is no topology assumed on* D *or* D' *and* T, S *are general subject to the assumptions in the proposition.*

Proof. Assume that $v \in D$ and v and Tv are linearly independent. Then there exists $B \in L(H,H)$ with $Bv = v$, $BTv = v$. Let A be the subalgebra of $L(H,H)$ spanned by $\{\pi(g)|g \in G\}$. Then A satisfies the hypothesis of Theorem 83. Schur's lemma implies that $(A')' = L(H,H)$. Hence there exists a sequence $a_j \in A$ such that

$$\lim_{j\to\infty} a_j v = v, \lim_{j\to\infty} a_j Tv = v.$$

On the other hand $a_j D \subset D$ and $Ta_j = a_j T$. Thus if $w \in D'$ then

$$\langle v, w\rangle = \lim_{j\to\infty} \langle a_j Tv, w\rangle = \lim_{j\to\infty} \langle Ta_j v, w\rangle = \lim_{j\to\infty} \langle a_j v, Sw\rangle = \langle v, Sw\rangle = \langle Tv, w\rangle .$$

Since D' is dense this yields the absurd conclusion that of v, Tv are linearly independent then $Tv = v$. Thus v, Tv are linearly dependent for all $v \in D$. This implies that T is a scalar multiple of the identity. ∎

Exercise 90 *Show that if* V *is a vector space and* T *is a linear operator on* V *(no topology) and if for every* $v \in V$*,* v *and* Tv *are linearly dependent then* T *is a multiple of the identity.*

We will now give the main application of the preceding generalization of Schur's lemma. This application involves the development of the machinery of C^∞ vectors for representations of Lie groups. We will be assuming the standard, basic Lie group theory.

Let G be an n-dimensional Lie group with a finite number of connected components. Then G is a locally compact, separable topological group. Let $Lie(G)$ denote, as usual, the Lie algebra of G. Let ω be a differential form of degree n on G such that $L_g^*\omega = \omega$ for all $g \in G$. Here we recall that ω is an assignment $x \longmapsto \omega_x$ with $\omega_x \in \bigwedge^n T(G)_x^*$ ($T(G)_x$ the tangent space at x) such that if $X_1, ..., X_n$ are vector fields on G then the map

$$x \longmapsto \omega_x((X_1)_x, ..., (X_n)_x)$$

is of class C^∞. The condition $L_g^*\omega = \omega$ means that

$$\omega_{gx}((dL_g)_x(X_1)_x, ..., (dL_g)_x(X_n)_x) = \omega_x((X_1)_x, ..., (X_n)_x)$$

for all $x \in G$ (here d, as usual, stands for differential) The standard way of constructing such an ω is to choose a basis $X_1, ..., X_n$ looked upon as left invariant vector fields ($(dL_g)_x X_x = X_{gx}$) and choosing $\eta \in \bigwedge^n Lie(G)^*$ such that $\eta(X_1, ..., X_n) = 1$ then we identify $Lie(G)$ with $T(G)_1$. We set for $Y_1, ..., Y_n$ vector fields on G

$$\omega_x((Y_1)_x, ..., (Y_n)_x) = \eta((dL_x)^{-1}(Y_1)_x, ..., (dL_x)^{-1}(Y_n)_x).$$

We fix an orientation on G (we can choose the one corresponding to ω as above). Then if $f \in C_c(G)$ we can integrate f with respect to ω defining

$$\mu(f) = \int_G f\omega.$$

This defines a Haar measure on G. The point here is that the C^∞ structure has been taken into account.

Definition 91 *Let (π, V) be a representation of G on a locally convex topological space. Then a vector $v \in V$ is said to be a C^∞-vector if the map $g \longmapsto \pi(g)v$ is a C^∞ map of G into V.*

The following observation is due to Gårding.

Lemma 92 *Let (π, H) be a Hilbert representation of G. If $f \in C_c^\infty(G)(= C^\infty(G) \cap C_c(G))$ and if $v \in H$ then $\pi(f)v$ is a C^∞ vector.*

Proof. Let U be a relatively compact subset of G containing 1. Let $L^1(U)$ denote the subspace of all L^1-functions on G that are limits of elements of $C_c(G)$ with support in U. Let V be an open subset of U such that it is invariant under inverse and such that $VV \subset U$. Then we have

1. If $f \in C_c^\infty(V)$ then the map of V to $L^1(U)$ given by $x \longmapsto F(x) = L(x)f$ is of class C^∞. Indeed, if $X \in Lie(G)$ then we set $L(X)f(g) = \frac{d}{dt}_{|t=0} f(\exp(-tX)g)$ for $g \in G$. Taylor's theorem with remainder implies that there exists $\epsilon > 0$ and a function, E, of t, g for $|t| \leq \epsilon$ such that $|E(t,g)| \leq \phi(g)$ with $\phi \in C_c(G)$ for $|t| \leq \varepsilon$ and

$$f(\exp(-tX)g) = f(g) + tL(X)f(g) + t^2E(t,g)$$

for $|t| \leq \epsilon$ and $g \in V$. This implies that

$$\|L(x)L(X)f - (1/t)(L(x\exp(tX))f - L(x)f)\|_1 =$$

$$\|L(X)f - (1/t)(L(x\exp(tX))f - L(x)f)\|_1 \leq |t|C$$

for $|t| \leq \epsilon$ for $C > 0$ and appropriate constant. Hence the function F is of class C^1. This argument can be iterated to prove the result. We have seen (at least implicitly) that the map $f \longmapsto \pi(f)$ is continuous from $L^1(U)$ to H. Thus, since linear continuous maps are smooth, we see that if $v \in H$ then the map from V to H given by $x \longmapsto \pi(L(x)f)v$ is of class C^∞. Now $\pi(L(x)f)v = \pi(x)\pi(f)v$. The lemma now follows using a partition of unity argument. ■

We denote by V^∞ the space of all C^∞ vectors in V. Now arguing as in the proof of Lemma 73 we have

Theorem 93 *Let (π, H) be a Hilbert representation of G. Then the space of C^∞ vectors in H is dense.*

Proof. There exists a Delta sequence $\{u_j\}$ in G with each $u_j \in C_c^\infty(G)$. We have shown

$$\lim_{j\to\infty} \langle \pi(u_j)v, w\rangle = \langle v, w\rangle$$

for all $v, w \in H$. Now suppose that $w \in (H^\infty)^\perp$ then since $\pi(u_j)w \in H^\infty$ for all j we have

$$0 = \lim_{j\to\infty} \langle \pi(u_j)w, w\rangle = \langle w, w\rangle.$$

■

If (π, H) is a Hilbert representation of G and if $v \in H^\infty$ then we define for $X \in Lie(G)$

$$d\pi(X)v = \frac{d}{dt}\pi(\exp tX)v_{|t=0}.$$

We have

- $d\pi(X)H^\infty \subset H^\infty$ for all $X \in Lie(G)$,$\pi(g)H^\infty \subset H^\infty$ for $g \in G$.
- $d\pi(aX + bY) = ad\pi(X) + bd\pi(Y)$, $a, b \in \mathbb{R}$, $X, Y \in Lie(G)$.
- $d\pi([X, Y]) = d\pi(X)d\pi(Y) - d\pi(Y)d\pi(X)$, for all $X, Y \in Lie(G)$.
- If $g \in G$, $X \in Lie(G)$ then $\pi(g)\pi(X)v = \pi(Ad(g)X)\pi(g)v$.

We will simplify notation and write $\pi(X)$ for $d\pi(X)$. The first assertion is clear from the definition of C^∞. The second follows from

$$\exp(tX)\exp(tY) = \exp(t(X + Y) + O(t^2)).$$

The third follows from

$$\exp(tX)\exp(Y)\exp(-tX) = \exp(Y + t[X,Y] + O(t^2)).$$

The fourth follows from

$$\exp(tAd(g)X) = g(\exp tX)g^{-1}.$$

The three bullet items imply that (π, H^∞) defines a representation of $Lie(G)$. The fourth is a compatibility condition that will play a role later. We will also consider this to be a representation of the complexification of $Lie(G)$, that is $Lie(G)_\mathbb{C}$. Set $\mathfrak{g}_\mathbb{C} = Lie(G_\mathbb{C})$. Then (π, H^∞) extends to a representation of the universal enveloping algebra, $U(\mathfrak{g}_\mathbb{C})$. We define $Z_G(\mathfrak{g}_\mathbb{C})$ to be the subalgebra if $U(\mathfrak{g}_\mathbb{C})$ consisting of those z such that $Ad(g)z = z$ for all $g \in G$. We define an involution denoted * on $U(\mathfrak{g}_\mathbb{C})$ by the following rules

- $(z1)^* = \bar{z}1$.
- $X^* = -X$ for $X \in Lie(G)$.
- $(xy)^* = y^*x^*$ for $x, y \in U(\mathfrak{g}_\mathbb{C})$.

We note that the anti-homomorphism $x \longmapsto x^*$ exist by the universal problem solved by the universal enveloping algebra and also the naturality implies that if $g \in G$ then $(Ad(g)x)^* = Ad(g)(x^*)$.

Lemma 94 *If (π, H) is a unitary representation of G and if $v, w \in H^\infty$ and $x \in U(\mathfrak{g}_\mathbb{C})$ then*

$$\langle \pi(x)v, w \rangle = \langle v, \pi(x^*)w \rangle .$$

Proof. We note that if $X \in Lie(G)$ and $v, w \in H$ then

$$\langle \pi(\exp tX)v, w \rangle = \langle v, \pi(\exp(-tX))w \rangle$$

for all $t \in \mathbb{R}$. If $v, w \in H^\infty$ then both sides of the equation are differentiable in t. Taking the derivative at $t = 0$ yields

$$\langle \pi(X)v, w \rangle = \langle v, \pi(-X)w \rangle = \langle \pi(x)v, w \rangle = \langle v, \pi(X^*)w \rangle .$$

Now use the fact the $Lie(G)$ generates $U(\mathfrak{g}_\mathbb{C})$ over $\mathbb{C}$. ∎

We now come to the promised application of our generalized Schur's lemma.

Theorem 95 *Let (π, H) be an irreducible unitary representation of G then there exists an algebra homomorphism $\eta_\pi : Z_G(\mathfrak{g}_\mathbb{C}) \to \mathbb{C}$ such that $\pi(z)v = \eta_\pi(z)v$ for all $v \in H^\infty$.*

Proof. In Proposition 88 take $D = D'$ to be H^∞. If $z \in Z_G(\mathfrak{g}_{\mathbb{C}})$ then take $T = \pi(z)$. We note that if $v \in H^\infty$ then

$$\pi(g)T\pi(g)^{-1}v = \pi(g)\pi(z)\pi(g)^{-1}v$$

$$= \pi(Ad(g)z)v = \pi(z)v = Tv.$$

Also take $S = \pi(z^*)$. Then the previous lemma implies that the hypotheses of Proposition 88 are satisfied. Thus $\pi(z)$ acts as a scalar on H^∞. Denote this scalar by $\eta_\pi(z)$. ■

Definition 96 *The homomorphism η_π is called the* infinitesimal character *of (π, H).*

If (π, H) is a Hilbert representation of G and if $x \in U(\mathfrak{g}_{\mathbb{C}})$ then we denote by p_x the semi-norm on H^∞defined by $p_x(v) = \|\pi(x)v\|$. We give H^∞ the corresponding locally convex topology. Notice that if $\{x_i\}$ is a basis of $U(\mathfrak{g}_{\mathbb{C}})$ then the semi-norms $\{p_{x_i}\}$ suffice to define the topology. The following result uses basic calculus in its proof. We will just refer to *Real Reductive Groups I*, Academic Press, 1988, Lemma 1.6.2 since the result will not be used in a serious way here.

Lemma 97 *The space H^∞ is a Fréchet space with respect to the locally convex topology given above. Furthermore, (π, H^∞), is a smooth Fréchet representation (i.e. if $v \in H^\infty$ then the map $g \longmapsto \pi(g)v$ defines a C^∞ map from G to H^∞.*

1.2.2 Square integrable representations.

Throughout this section we will assume that G is unimodular.

Definition 98 *Let G be a locally compact, separable topological group. Then an irreducible unitary representation, (π, H), of G is said to be square integrable if there exists a non-zero $v \in H$ such that the matrix coefficient $c_{v,v}$ is square integrable (recall $c_{v,w}(g) = \langle \pi(g)v, w\rangle$).*

We will now spend a substantial part of this section giving an important family of examples of square integrable representations. We will be using some results from later lectures. Let $D = \{z \in \mathbb{C} | |z| < 1\}$. We use ordinary Lebesgue measure on D thinking of $z = x + iy$ as $(x, y) \in \mathbb{R}^2$. We write $dz = dx + idy$ and $d\bar{z} = dx - idy$ then $dx \bigwedge dy = \frac{1}{2i} d\bar{z} \bigwedge dz$. Let G be the group $SU(1,1)$. That is if $g \in G$ then

$$g = \begin{bmatrix} a & b \\ \bar{b} & \bar{a} \end{bmatrix}, |a|^2 - |b|^2 = 1. \tag{1}$$

Here the entries $a, b \in \mathbb{C}$ but the group is a real Lie group. We define an action of G on D by

$$g \cdot z = \frac{az + b}{\bar{b}z + \bar{a}}$$

where g is given as in (1). We will use the following formulas.

$$1-\left|\frac{az+b}{\bar{b}z+\bar{a}}\right|^2=\frac{1-|z|^2}{|\bar{b}z+\bar{a}|^2}. \tag{2}$$

$$d(g\cdot z)=\frac{dz}{(\bar{b}z+\bar{a})^2}. \tag{3}$$

Using (2),(3) we have

$$\int_D \phi(g\cdot z)\frac{d\bar{z}dz}{(1-|z|^2)^2}=\int_D \phi(g\cdot z)\frac{d(\overline{g\cdot z})d(g\cdot z)}{(1-|g\cdot z|^2)^2}=\int_D \phi(z)\frac{d\bar{z}dz}{(1-|z|^2)^2}.$$

Thus

$$\mu(\phi)=\frac{1}{2i}\int_D \phi(z)\frac{d\bar{z}dz}{(1-|z|^2)^2} \tag{4}$$

defines a G-invariant measure on D.

Let H^k be the space of all holomorphic functions $f:D\to\mathbb{C}$ such that

$$\frac{1}{2i}\int_D |f(z)|^2(1-|z|^2)^k\frac{d\bar{z}dz}{(1-|z|^2)^2}<\infty.$$

If $f_1,f_2\in H^k$ then we set

$$\langle f_1,f_2\rangle_k=\frac{1}{2i}\int_D f_1(z)\overline{f_2(z)}(1-|z|^2)^k\frac{d\bar{z}dz}{(1-|z|^2)^2}. \tag{5}$$

We define for $f\in H^k$, and g as in (1)

$$\pi_k(g)f(z)=(-\bar{b}z+a)^{-k}f(g^{-1}\cdot z). \tag{6}$$

Then using formulas (2),(3) as we did in the proof of the invariance of μ. We find that

$$\langle \pi_k(g)f_1,\pi_k(g)f_2\rangle=\langle f_1,f_2\rangle\,, f_1,f_2\in H^k, g\in G.$$

Proposition 99 *If $k\geq 2$ then H^k is a Hilbert space and if $k\in\mathbb{Z}$, $k\geq 2,(\pi_k,H^k)$ is a square integrable representation of G.*

Proof. We first show that the space H^k is complete. For this we observe that if $z_o\in D$ and if $r=\frac{1-|z_o|}{2}$ then the set $\overline{D}_r=\{z\in\mathbb{C}||z-z_o|\leq r\}\subset D$. Then if $k\geq 2$ we have

$$(1-|z|^2)^{k-2}\geq(1-\frac{1}{4}(1+|z_o|)^2)^{k-2}\text{for all } z\in\overline{D}_r.$$

Thus we see that

$$\langle f, f\rangle_k \geq \frac{(1-\frac{1}{4}(1+|z_o|)^2)^{k-2}}{2i}\int_{\overline{D}_r}|f(z)|^2 d\bar{z}dz.$$

On $\overline{D}_r$ the holomorphic function f is given as a series

$$f(z)=\sum_{n=0}^{\infty}a_n(z-z_o)^n.$$

Then

$$\frac{1}{2i}\int_{\overline{D}_r}|f(z)|^2 d\bar{z}dz=\int_0^r\int_0^{2\pi}\sum_{n,m\geq 0}a_n\overline{a_m}s^n s^m e^{i(n-m)\theta}d\theta s ds=$$

$$2\pi\sum_{n\geq 0}|a_n|^2\int_0^r s^{2n+1}dr=2\pi\sum_{n\geq 0}|a_n|^2\frac{r^{2n+2}}{2n+2}\geq 2\pi|a_0|^2r^2.$$

We therefore see that

$$\langle f, f\rangle_k \geq (1-\frac{1}{4}(1+|z_o|)^2)^{k-2}2\pi|f(z_o)|^2.$$

This implies the completeness, since if $\{f_j\}$ is Cauchy in H_k then it is Cauchy relative to the topology of uniform convergence on compacta. This implies that there is a continuous function on D, f, such that $\lim_{j\to\infty}f_j(z)=f(z)$ uniformly on compacta of D. But then f is holomorphic on D and it is easy to check that it is in H^k.

Notice that the function $f(z)\equiv 1$ is in H^k if $k\geq 2$. We calculate the matrix coefficient (g as in (1))

$$\langle\pi_k(g)1,1\rangle=\frac{1}{2i}\int_D(-\bar{b}z+a)^{-k}(1-|z|^2)^{k-2}d\bar{z}dz=$$

$$\int_0^1\int_0^{2\pi}(-\bar{b}re^{i\theta}+a)^{-k}(1-r^2)^{k-2}d\theta r dr=a^{-k}\int_0^1 r(1-r^2)^{k-2}\int_0^{2\pi}(-\frac{\bar{b}}{a}re^{i\theta}+1)^{-k}d\theta dr.$$

We observe that since $|a|^2-|b|^2=1$, $|\frac{\bar{b}}{a}|\leq 1$. Thus if $0\leq r<1$ then the function

$$\phi(z)=(-\frac{\bar{b}}{a}rz+1)^{-k}$$

is holomorphic in z for $|z|<\frac{1}{r}$. This implies that

$$\int_0^{2\pi}\phi(e^{i\theta})d\theta=2\pi\phi(1)=2\pi.$$

We therefore see that

$$\langle\pi_k(g)1,1\rangle_k=a^{-k}2\pi\int_0^1 r(1-r^2)^{k-2}dr=c_k a^{-k}.$$

Notice that this is a continuous function of g. Let $f(g) = a^{-k}$ we will show that

$$\int_G |f(g)|^2 dg < \infty.$$

For this we need a formula for the Haar integral analogous to the formula for polar coordinates. Set

$$K = \{k(\theta)|k(\theta) = \begin{bmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{bmatrix}, \theta \in \mathbb{R}\}$$

and

$$A^+ = \{a_t|a_t = \begin{bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{bmatrix}, t \in \mathbb{R}, t \geq 0\}.$$

Then $G = KA^+K$ (exercise we will see it in general later). Furthermore, if ϕ is summable on G then up to constants of normalization

$$\int_G \phi(g)dg = \int_0^{2\pi} \int_0^{\infty} \int_0^{2\pi} \phi(k(\theta_1)a_t k(\theta_2)) \sinh(2t) d\theta_1 dt d\theta_2.$$

This can be proved by observing that $K = \{g \in G|g \cdot 0 = 0\}$. Thus if $f \in C_c(G)$ then

$$\bar{f}(gK) = \int_0^{2\pi} f(gk(\theta))d\theta$$

Defines a function on $D = G \cdot 0$. If we write out the invariant measure given in formula (4) above in polar coordinates and consider the change of variables $r \longmapsto \tanh t$,$t > 0$ the formula follows. Now

$$f(k(\theta_1)a_t k(\theta_2)) = (e^{i\theta_1} \cosh t e^{i\theta_2})^{-k}.$$

Thus

$$\int_G |f(g)|^2 dg = (2\pi)^2 \int_0^{\infty} (\cosh t)^{-2k} \sinh(2t) dt =$$

$$4\pi^2 \int_0^{\infty} (\cosh t)^{-2k} \cosh(t) \sinh(t) dt = \frac{2\pi^2}{k-1}.$$

This shows that $c_{1,1}$ is square integrable. The exercise below proves that π_k is a representation. We will prove the irreducibility later. ∎

For later reference we note that $c_k = \langle 1, 1 \rangle_k$ thus we have (up to normalization of measures)

$$\int_G |\langle \pi(g)1, 1 \rangle_k|^2 dg = \frac{2\pi^2}{k-1} \langle 1, 1 \rangle_k^2 . \tag{7}$$

Exercise 100 *Calculate* $\langle \pi_k(g)z^l, z^m \rangle$ *for* $l, m = 0, 1, 2, ...$ *as above and show that it is a continuous function of* g. *Show that the span of the functions* $1, z, z^2, ...$ *is dense in* H^k *for* $k \geq 2$. *Now use an appropriate extension of Lemma 73 (allowing the element* w *in* 2. *to be taken from a dense subspace) to show that* (π_k, H^k) *is a representation for* $k \geq 2$.

Exercise 101 *Give the details of the proof of the integration formula for Haar measure on* G *as sketched in the above proof.*

We will now prove two basic results about square integrable representations. We will consider $L^2(G)$ as a unitary representation under the *right* regular action.

Proposition 102 *Let* (π, H) *be a square integrable representation of* G. *Then every matrix entry* $(c_{v,w},\ v, w \in H)$ *is square integrable. Furthermore, there exists an element* $T \in L_G(H, L^2(G))$ *with closed range consisting of continuous functions that is a unitary bijection onto its range. The map* T *can be implemented as follows: fix* v_o *in* H *a unit vector then* $T(w) = c_{w,v_o}$.

Proof. Fix v_o a unit vector in H such that c_{v_o,v_o} is in $L^2(G)$. Let D denote the space of all $v \in H$ such that $c_{v,v_o} \in L^2(G)$. We note that

$$c_{\pi(g)v,w} = R_g c_{v,w}.$$

Thus D is an invariant non-zero subspace. Since $v_o \in D$ the irreducibility implies that D is a dense subspace. On D we put the pre-Hilbert space structure

$$(v, w) = \langle v, w \rangle + \langle c_{v,v_o}, c_{w,v_o} \rangle .$$

The last inner product is the L^2-inner product the first one on the right hand side is the inner product on H.

We now come to the key point.

(*) D is complete with respect to $(..., ...)$.

Indeed, if $\{v_j\}$ is a Cauchy sequence in D then it is Cauchy in H and $\{c_{v_j,v_o}\}$ is Cauchy in $L^2(G)$. Since H is complete there exists $v \in H$ such that $\lim_{j\to\infty} v_j = v$. Since $L^2(G)$ is complete by definition there exists $f \in L^2(G)$ such that $\lim_{j\to\infty} c_{v_j,v_o} = f$ in L^2. We note that

$$|c_{v_j,v_o}(g) - c_{v,v_o}(g)| \leq \|v_j - v\|, g \in G.$$

Let U be an open subset of G such that $\bar{U}$ is compact and let $\phi \in C_c(G)$ be such that $\phi(x) \geq 0$ for all $x \in G$ and $\phi(x) = 1$ if $x \in \bar{U}$. We note that the operator of multiplication by ϕ on $C_c(G)$ extends to a bounded operator $T_\phi : L^2(G) \to L^2(G)$. Now we have $\lim_{j\to\infty} \phi c_{v_j,v_o} = \phi c_{v,v_o}$ in $L^2(G)$ by the above uniform convergence. We also have

$$\lim_{j\to\infty} \phi c_{v_j,v_o} = T_\phi f$$

in $L^2(G)$. Hence we have $\phi c_{v,v_o} = T_\phi f$. This implies that f is represented by the continuous function c_{v,v_o}. But then $v \in D$.

We note that if $g \in G$ and $v, w \in D$ then

$$(\pi(g)v, \pi(g)w) = (v, w).$$

Thus the operators $\pi(g)_{|D}$ define unitary operators $\rho(g)$ on D with respect to $(...,...)$. Let $S(v) = v$ for $v \in D$ but looked upon as a map of the Hilbert space D into H. Then

$$\langle S(v), S(v) \rangle \leq (v, v)$$

for all $v \in D$. This implies that S extends to a bounded operator from the Hilbert space completion, D, into H. Furthermore, $S \circ \rho(g) = \pi(g) \circ S$. Let $S^* : H \to D$ denote the adjoint operator. Then $S^* \circ \pi(g) = \rho(g) \circ S^*$ for all $g \in G$. We therefore see that $SS^* \in L_G(H, H)$. Schur's lemma implies that $SS^* = \lambda I$ and it is clear that λ is real and $\lambda > 0$. Now if $v \in H$ then $S^*(v) \in D$ so $\lambda v = S(S^* v) = S^* v$. But then $v \in D$. Hence $D = H$. We also note that this implies that

$$(v, v) \leq \frac{1}{\lambda^2} \langle v, v \rangle$$

for all $v \in H$. Thus $\|c_{v,v_o}\|_2^2 \leq \frac{1-\lambda^2}{\lambda^2} \|v\|^2$. Define $T(v) = c_{v,v_o}$. To complete the proof we note that all we used about v_o in the proof above was that the set $\{w \in H | c_{v,v_o} \in L^2(G)\}$ is non-zero. By the above this is true for every $v \in H$ since v_o is in the corresponding set. ■

The next theorem is a general form of the Schur orthogonality relations.

Theorem 103 *Let (π, H) and (ρ, V) be square integrable representations of G. If π and ρ are inequivalent then their matrix coefficients are orthogonal. There exists a positive real number $d(\pi)$ (which depends only on π and the normalization of Haar measure) such that if $v_1, v_2, w_1, w_2 \in H$ then*

$$\int_G \langle \pi(g)v_1, w_1 \rangle \overline{\langle \pi(g)v_2, w_2 \rangle} dg = \frac{1}{d(\pi)} \langle v_1, v_2 \rangle \langle w_2, w_1 \rangle .$$

Proof. Assume that $h_o \in H$ and $v_o \in V$ are unit vectors and that there exists $h \in H$, $v \in V$ such that

$$\int_G \langle \pi(g)h, h_o \rangle \overline{\langle \rho(g)v, v_o \rangle} dg \neq 0.$$

Let $T : H \to L^2(G)$ and $S : V \to L^2(G)$ be as in the proof of the preceding proposition. That is $T(x) = c_{x,h_o}$ and $S(y) = c_{y,v_o}$. Then we showed that T and S respectively define injective intertwining operators from H and V into $L^2(H)$ with closed range. Consider

$$(x, y) = \langle T(x), S(y) \rangle .$$

Then $(h, v) \neq 0$ and $(\pi(g)x, \rho(g)y) = (x, y)$. Finally the pairing is continuous in x, y. Thus the Riesz representation theorem, for H, implies that $(x, y) = \langle x, A(y) \rangle$ with $A : V \to H$ a bounded operator. It is easy to see that $A \in L_G(V, H)$. Since $A \neq 0$. We see that $\ker A = 0$. We also see that $\operatorname{Im} A$ is dense in H. We also observe that $A^*A \in L_G(V, V)$ and $AA^* \in L_G(H, H)$. Thus each is a scalar by Schur's lemma. We conclude that there is a scalar $s > 0$ such that sA is a unitary bijection, We therefore conclude that π and ρ are unitarily equivalent.

To prove the last part we see that

$$\int_G \langle \pi(g)v_1, w_1 \rangle \overline{\langle \pi(g)v_2, w_2 \rangle} dg = a(w_2, w_1) \langle v_1, v_2 \rangle$$

and

$$\int_G \langle \pi(g)v_1, w_1 \rangle \overline{\langle \pi(g)v_2, w_2 \rangle} dg = b(v_1, v_2) \langle w_2, w_1 \rangle$$

for v_1, v_2, w_1, w_2. This implies that $a(w_2, w_1)$ is a positive multiple of $\langle w_2, w_1 \rangle$. We call the multiple $\frac{1}{d(\pi)}$. ■

Definition 104 *We call the number $d(\pi)$ the* formal degree *of π.*

Example 105 *If (π_k, H^k) is as above for $SU(1,1)$ then $d(\pi_k) = \frac{k-1}{2\pi^2}$.*

1.2.3 Representations of compact groups

Unless otherwise specified, throughout this section G will denote a compact, separable topological group. The following result is Weyl's "unitarian trick".

Lemma 106 *Let (π, H) be a Hilbert representation of G. Then there exists an inner product, $(..., ...)$ on H that induces the same topology on H and such that relative to that inner product the representation is unitary.*

Proof. We note that since G is compact there exists $C < \infty$ such that

$$\|\pi(g)\| \leq C$$

for all $g \in G$ (see Lemma 73). We set for $v, w \in H$

$$(v, w) = \int_G \langle \pi(g)v, \pi(g)w \rangle \, dg.$$

Then the fact that G is unimodular implies that

$$(\pi(g)v, \pi(g)w) = (v, w), g \in G, v, w \in H.$$

We also note that $(v, v) > 0$ if $v \neq 0$. Since $\pi(g)\pi(g^{-1}) = I$ we have

$$1 \leq \|\pi(g)\| \left\|\pi(g)^{-1}\right\| \leq C \|\pi(g)\|.$$

Hence

$$C^{-1} \leq \|\pi(g)\| \leq C.$$

This implies that

$$C^{-1} \langle v, v \rangle \leq (v, v) \leq C \langle v, v \rangle .$$

■

Clearly, an irreducible unitary representation of G is square integrable. We have

Theorem 107 *Let (π, H) be an irreducible Hilbert representation of G. Then $\dim H < \infty$. If (π, H) is unitary and we normalize the Haar measure, μ, on G such that $\mu(1) = 1$ then $d(\pi) = \dim H$ (recall $d(\pi)$ is the formal degree).*

Proof. We may assume that (π, H) is unitary. Then it is square integrable. There is therefore an injective intertwining operator $T : H \to L^2(G)$ with closed image contained in $C(G)$. We look upon $C(G)$ as a Banach space under the sup-norm, $p_G(f) = \max_{x \in G} |f(x)|$. Let V denote the closure of $T(H)$ in $C(G)$. Then if we normalize the Haar measure as in the statement of the theorem it is clear that

$$\|f\|_2 \leq p_G(f).$$

Thus the map $f \longmapsto f$ of $T(H)$ to $T(H)$ extends to a continuous linear map of V to $T(H)$ (since $T(H)$ is closed in $L^2(G)$). The closed graph theorem (Yoshida, p. 79, Theorem 1) implies that this map is continuous. Hence there exists $C < \infty$ such that if $f \in T(H)$ then

$$p_G(f) \leq C \|f\|_2 .$$

We will show that this implies that $T(H)$, hence H, is finite dimensional.

Let $f_1, ..., f_d$ be orthonormal in $T(H)$ then if $\lambda_i \in \mathbb{C}$ we have

$$|\sum_i \lambda_i f_i(x)| \leq p_G(\sum_i \lambda_i f_i) \leq C \left\| \sum_i \lambda_i f_i \right\|_2 = C \left(\sum_i |\lambda_i|^2 \right)^{\frac{1}{2}} .$$

We apply this with $\lambda_i = \overline{f_i(x)}$. We conclude that

$$\sum_i |f_i(x)|^2 \leq C \left(\sum_i |f_i(x)|^2 \right)^{\frac{1}{2}} .$$

Hence

$$\sum_i |f_i(x)|^2 \leq C^2.$$

Integrating both sides of the equation over G yields $d \leq C^2$.

We now calculate the formal degree. Let $v_1, ..., v_n$ be an orthonormal basis of H then the matrix $\left[c_{v_i,v_j}(g)\right]$ is unitary for all $g \in G$. Hence

$$\sum_{i,j} |c_{v_i,v_j}(g)|^2 = n$$

for all $g \in G$. If we integrate both sides of this equation and take into account the Schur orthogonality relations we have

$$\frac{1}{d(\pi)} n^2 = n.$$

■

If $\{H_n\}_{1 \leq n < N}$ with $N \leq \infty$ is a sequence of Hilbert spaces then we write $\widehat{\bigoplus}_{n<N} H_n$ for the space of all sequences $\{x_n\}_{n<N}$ such that $\sum_n \|x_n\|^2 < \infty$ we define

$$\langle \{x_n\}, \{y_n\} \rangle = \sum \langle x_n, y_n \rangle .$$

This endows $\widehat{\bigoplus}_{n<N} H_n$ with a Hilbert space structure. This construction defines the Hilbert space direct sum. Notice that it is a completion of the algebraic direct sum.

Definition 108 *Let B a locally compact topological group and for each $n, 1 \leq n < N$ we have a unitary representation (π_n, H_n) of B then the unitary direct sum of these representations is the representation $(\widehat{\bigoplus}_{n<N} \pi_n, \widehat{\bigoplus}_{n<N} H_n)$ of B with*

$$\left(\widehat{\bigoplus}_{n<N} \pi_n(g) \{x_n\} \right) = \{\pi_n(g) x_n\}.$$

Definition 109 *Let B be a locally compact, separable topological group then a unitary representation (π, H) of B is said to be of* class CC *if $\pi(f)$ is completely continuous for all $f \in C_c(B)$. We say that B is a* CCR group *if every irreducible unitary representation of B is of class CC.*

One of Harish-Chandra's basic theorems is that all real reductive groups are CCR groups.

The next result is a generalization of the Peter-Weyl theorem and is basic to the theory of automorphic forms (it applies to the so-called cuspidal spectrum). In the course of the proof of the result we will be using the fact that if B is unimodular than

$$\int_B f(b) db = \int_B f(b^{-1}) db.$$

Theorem 110 *Let B be a locally compact, separable topological group and let (π, H) be a unitary representation of B of class CC. Then (π, H) is unitarily equivalent with a unitary direct sum of irreducible representations of B.*

Before we prove the theorem we will describe the general form of the main application. Assume that B is unimodular. Let $X = B/C$ with C a closed unimodular subgroup of B and assume that X is compact and that there exists a B-invariant measure on X, λ (one can show that this is not really an assumption under our hypotheses on B and C). We will write the measure as

$$\lambda(f) = \int_X f(x)dx$$

as usual. Let $\phi \in C_c(B)$ then we can choose Haar measure on B and C such that if we set $\bar{\phi}(bC) = \int_C \phi(bc)dc$ (the integration with respect to Haar measure on C) then

$$\int_B \phi(b)db = \int_X \bar{\phi}(x)dx.$$

Let $H = L^2(X)$ and $\pi(b) = L_b$. We calculate

$$\begin{aligned}
\pi(\phi)f(x) &= \int_B \phi(b)f(b^{-1}x)db = \int_B \phi(b)f(b^{-1}gC)db \\
&= \int_B \phi(b)f((g^{-1}b)^{-1}C)db = \int_B \phi(gb)f(b^{-1}C)db \\
&= \int_B \phi(gb^{-1})f(bC)db = \int_X \int_C \phi(gcb^{-1})f(bC)dcd(bC).
\end{aligned}$$

Let

$$k_\phi(gC, bC) = \int_C \phi(gcb^{-1})dc.$$

The function $k_\phi \in C(X \times X)$ and

$$\pi(\phi)f(x) = \int_X k_\phi(x,y)f(y)dy.$$

on $L^2(X)$. The lemma below implies that $(\pi, L^2(X))$ is of class CC.

Lemma 111 *Let Y be a locally compact, separable, topological space and let λ be a regular measure on Y. Let $k \in L^2(Y \times Y)$ (with respect to the product measure). If we define $T : L^2(Y) \to L^2(Y)$ by $T(f)(x) = \lambda(k(x,\cdot)f)$.then T defines a compact operator.*

Proof. Let $\{\phi_n\}$ be an orthonormal basis of $L^2(Y)$ consisting of continuous functions ($L^2(Y)$ is separable since Y is separable). Define $u_{n,m}(x,y) = \phi_n(x)\overline{\phi_m(y)}$. Then $\{u_{n,m}\}$ is an orthonormal basis of $L^2(Y \times Y)$. Now

$$k = \sum_{n,m} a_{n.m}u_{n,m}$$

in $L^2(Y \times Y)$. Set $k_N = \sum_{n,m \leq N} a_{n.m} u_{n,m}$. Then the operator

$$T_N(f)(x) = \int_Y k_N(x,y) f(y) dy$$

is of finite rank hence compact. Also

$$(T - T_N)f = \sum_{m,n>N} a_{n,m} \phi_n \langle f, \phi_m \rangle .$$

An application of the Schwarz inequality yields

$$\|T - T_N\|^2 \leq \sum_{m,n>N} |a_{n,m}|^2.$$

Thus T is in the norm closure of the finite rank operators. Hence it is compact. ■

We will now prove the theorem. Let $\mathcal{S}$ denote the set of all closed invariant subspaces, V, of H such that V is a Hilbert space direct sum of irreducible subrepresentations ordered by inclusion. If $\{V_\alpha\}$ is a linearly ordered subset of $\mathcal{S}$ then the closure of $\bigcup_\alpha V_\alpha$ is in $\mathcal{S}$ (exercise). Hence Zorn's lemma implies that there is a maximal element V in $\mathcal{S}$. We will now prove that $V = H$ and thereby prove the Theorem. Let $W = V^\perp$. If $u \in C_c(B)$ then $\pi(u)W = W$. Let $w \in W$ be a unit vector. Let $\{u_n\}$ be a delta sequence such that $u_n(x^{-1}) = u_n(x)$ for all $x \in B$. Then $\pi(u_n)^* = \pi(u_n)$ (exercise)for all n. Now $\lim_{n\to\infty} \pi(u_n)w = w$. Hence there exists n such that $\pi(u_n)w \neq 0$. Fix $T = \pi(u_n)_{|W}$. Then T is a compact, non-zero self-adjoint operator on W. Lemma 64 implies that T has a nonzero eigenvalue on W. Let Z denote the corresponding eigenspace. Then Lemma 64 also implies that $\dim Z < \infty$. Let $m > 0$ denote the positive minimal dimension of an intersection of a closed B-invariant subspace with Z. Fix M an intersection of this type with $\dim M = m$. Let U denote the intersection of all closed invariant spaces Y such than $Y \cap Z = M$. Then U is closed and invariant. If N is a closed invariant subspace of U then both N and $N^\perp$ are T invariant. Thus $M = M \cap N \bigoplus M \cap N^\perp$. But then $M \cap N = M$ or $M \cap N^\perp = M$. If $M \cap N = N$ (resp.$M \cap N^\perp = M$) then $N = U$ (resp. $N^\perp = U$) by definition of U. Thus U is a closed, invariant, irreducible subspace of W and thus $V \bigoplus U$ is in $\mathcal{S}$. This contradicts the definition of V. Hence $W = 0$.

Let $\widehat{G}$ denote the set of equivalence classes of irreducible finite dimensional representations of G. For each $\gamma \in \widehat{G}$ we fix $(\tau_\gamma, V_\gamma) \in \gamma$ which we assume is unitary. If (π, V) is a representation of G then we set $V(\gamma)$ equal to the sum of the closed, G-invariant, irreducible subspaces in the class of γ.

Definition 112 *The space $V(\gamma)$ is called the* γ-isotypic component *of V.*

We will now concentrate on $L^2(G)$ we first note that since G is compact the discussion after the statement of Theorem 110 implies that the right (or the left) regular representation is of class CC. If $\gamma \in \widehat{G}$ then we define a map $A_\gamma : V_\gamma^* \bigotimes V_\gamma \to L^2(G)$ by

$$A_\gamma(\lambda \bigotimes v)(g) = \lambda(\pi(g)v).$$

Set $d(\gamma) = \dim V_\gamma$. If $\lambda \in V_\gamma^*$ then there exists $v_\lambda \in V_\gamma$ such that $\lambda(v) = \langle v, v_\lambda \rangle$ for all v. We define $\langle \lambda_1, \lambda_2 \rangle = \langle v_{\lambda_2}, v_{\lambda_1} \rangle$. Then the Schur orthogonality relations imply that $\sqrt{d(\gamma)}A_\gamma$ is a unitary operator from $V_\gamma^* \bigotimes V_\gamma$ onto its image. We also observe that $A_\gamma(\lambda \circ \tau_\gamma(g)^{-1} \bigotimes v) = L_g A_\gamma(\lambda \bigotimes v)$.

The next result is the Peter-Weyl theorem.

Theorem 113 *The γ-isotypic component of $L^2(G)$ is the image of A_γ. Furthermore, $L^2(G)$ is the Hilbert space direct sum of the spaces $L^2(G)(\gamma)$.*

Proof. Let V be a closed, invariant, irreducible subspace of $L^2(G)$. Then in particular it is an irreducible unitary representation hence Theorem 107 implies that $\dim V < \infty$. If $u \in C(G)$ then $\pi(u)V \subset V$. We have seen that the span of the elements $\pi(u)v$ with $u \in C(G)$, $v \in V$ is dense in V. Hence it is equal to V. We leave it to the reader to check that this implies that $V \subset C(G)$. Define $\lambda(f) = f(1)$ for $f \in V$. Then $\lambda \in V^*$ and $\lambda(\pi(g)f) = f(g)$. Let $T : V_\gamma \to V$ be a bijective intertwining operator. Let $\xi = \lambda \circ T^{-1}$. Then if $T(v) = f$, $f = A_\gamma(\xi \bigotimes v)$. The last assertion now follows from Theorem 110. ■

Definition 114 *If (τ, V) is a finite dimensional representation of G then its* character *is defined to be the function $\chi_V(g) = tr(\tau(g))$.*

We note that $\chi_V \in C(G)$ and that $\chi_V(xgx^{-1}) = \chi(g)$ for all $x, g \in G$. We also observe that if (τ_1, V_1) and (τ_2, V_2) are equivalent then $\chi_{V_1} = \chi_{V_2}$. We will now show that the converse is also true. We first observe that this implies that if $V_1, V_2 \in \gamma \in \widehat{G}$ then $\chi_{V_1} = \chi_{V_2}$. This common value will be denoted χ_γ. We also set $\alpha_\gamma = d(\gamma)\overline{\chi}_\gamma$(complex conjugate). We note that the Schur orthogonality relations imply that

$$\alpha_\gamma * \alpha_\tau = \delta_{\gamma,\tau}\alpha_\gamma$$

for $\gamma, \tau \in \widehat{G}$. Also, since $\alpha_\gamma(xgx^{-1}) = \alpha_\gamma(g)$ for $x, g \in G$ we have

$$\pi(\alpha_\gamma)\pi(g) = \pi(g)\pi(\alpha_\gamma)$$

for all $\gamma \in \widehat{G}$.

Lemma 115 *The orthogonal projection of $L^2(G)$ onto $L^2(G)(\gamma)$ is the operator $P_\gamma = \pi(\alpha_\gamma)$.*

Proof. Let $v_1, ..., v_d$ be an orthonormal basis of V_γ. Then $\alpha_\gamma = d(\gamma) \sum \overline{c_{v_i,v_i}}$. We thus have

$$\pi(\alpha_\gamma)f(x) = d(\gamma) \sum \int_G \overline{c_{v_i,v_i}}(g)f(xg)dg = d(\gamma) \int_G \overline{c_{v_i,v_i}}(g)L_{x^{-1}}f(g)dg.$$

If $\mu \in \widehat{G}$, $\mu \neq \gamma$ and f is in the image of A_μ then the above integral is 0 by the Schur orthogonality relations and the observation preceding Theorem 113. If $\mu = \gamma$ then assuming that $f(g) = \langle \tau_\gamma(g)v, w\rangle$ for some v, w in V_γ we have $f(xg) = \left\langle \tau_\gamma(g)v, \tau_\gamma(x)^{-1}w\right\rangle$. The Schur orthogonality relations yield

$$\begin{aligned} d(\gamma)\int_G \overline{c_{v_i,v_i}}(g)L_{x^{-1}}f(g)dg &= d(\gamma)\sum\int_G \overline{\langle \tau_\gamma(g)v_i, v_i\rangle}\left\langle \tau_\gamma(g)v, \tau_\gamma(x)^{-1}w\right\rangle dg = \\ \sum \langle v, v_i\rangle \left\langle v_i, \tau_\gamma(x)^{-1}w\right\rangle &= \left\langle v, \tau_\gamma(x)^{-1}w\right\rangle = \langle \tau_\gamma(x)v, w\rangle = f(x). \end{aligned}$$

■

Let (π, H) be a Hilbert representation of G. By Lemma 106 we may assume that the representation is unitary. If $\gamma \in \widehat{G}$ then we set $E_\gamma = \pi(\alpha_\gamma)$. Then if $v, w \in H$ we have

$$\langle E_\gamma \pi(g)v, w\rangle = (P_\gamma c_{v,w})(g).$$

Thus if $E_\gamma v = 0$ for all $\gamma \in \widehat{G}$ then $c_{v,w} = 0$ for all $v \in H$. Hence $v = 0$. If $v \in H(\gamma)$ then $c_{v,w} \in L^2(G)(\gamma)$. Thus we see that E_γ is the orthogonal projection of H onto $H(\gamma)$.

We conclude

Proposition 116 *Let (π, H) be a Hilbert representation of G then the algebraic sum of the spaces $H(\gamma)$, $\gamma \in \widehat{G}$ is dense in H. Furthermore, if (π, H) is unitary then H is the Hilbert space direct sum of the spaces $H(\gamma), \gamma \in \widehat{G}$.*

1.2.4 The definition of a $(\mathfrak{g}, K)$ module.

In this section G will denote a Lie group with a finite number of connected components. Let K be a compact subgroup of G. Set $\mathfrak{g} = Lie(G)$ and $\mathfrak{k} = Lie(K)$. The most important special case is when K is a maximal compact subgroup of G.

Definition 117 *A $(\mathfrak{g}, K)$ module is a vector space, V, over $\mathbb{C}$ that is a module for the Lie algebra $\mathfrak{g}$ and a module for K (as an abstract group) such that*

1. $k \cdot X \cdot v = (Ad(k)X) \cdot k \cdot v$ for $k \in K$, $X \in \mathfrak{g}$, $v \in V$.

2. If $v \in V$ then $W_v = span_{\mathbb{C}}\{k\cdot v | k \in K\}$ is a finite dimensional vector space such that the map $k \to k \cdot w$, is C^∞ as a map from K to W_v for all $w \in W_v$.

3. If $Y \in \mathfrak{k}$ and $v \in V$ then $\frac{d}{dt}_{|t=0} \exp(tY) \cdot v = Yv$ (here the differentiation is as a map into W_v.

Our main class of example of $(\mathfrak{g}, K)$ modules are given as follows. Let (π, H) be a Hilbert representation of G. Let H^∞ be the space of C^∞ vectors of H. We have seen that this space is dense in H. Then the material after Theorem 93 implies that condition 1. is satisfied. As is condition 3. (but as a map into H). We set $H^\infty_{(K)}$ equal to the space of all $v \in H^\infty$ that satisfy 2. Then if $v \in H^\infty_{(K)}$ it satisfies 3. The only condition missing is that $\mathfrak{g}$ still acts.

Lemma 118 *If $X \in \mathfrak{g}$ then $XH^{\infty}_{(K)} \subset H^{\infty}_{(K)}$.*

Proof. Let $v \in H^{\infty}_{(K)}$ then we have a map of $\mathfrak{g} \bigotimes W_v \to H^{\infty}$ given by $X \bigotimes w \longmapsto Xw$. The compatibility condition 1. implies that the image of this map is a K-invariant finite dimensional space. It clearly contains Xv. Thus $Xv \in H^{\infty}_{(K)}$. ■

Definition 119 *The $(\mathfrak{g}, K)$ module $H^{\infty}_{(K)}$ is called the underlying $(\mathfrak{g}, K)$-module of (π, H).*

The $(\mathfrak{g}, K)$-modules form a full subcategory $\mathcal{C}(\mathfrak{g}, K)$ of the category of $\mathfrak{g}$ and K modules. That is $\mathrm{Hom}_{(\mathfrak{g},K)}(V, W) = \mathrm{Hom}_{\mathfrak{g}}(V, W) \cap \mathrm{Hom}_K(V, W)$. We say that a $(\mathfrak{g}, K)$-module V is irreducible if the only $\mathfrak{g}$ and K invariant subspaces are V and 0.

If V is a $(\mathfrak{g}, K)$-module and if $\gamma \in \widehat{K}$ then we set $V(\gamma)$ equal to the span of all v such that the representation $W_v \in \gamma$.

Definition 120 *A $(\mathfrak{g}, K)$-module V is said to be admissible if $\dim V(\gamma) < \infty$ for all $\gamma \in \widehat{K}$.*

The following result is the easy direction in a necessary and sufficient condition.

Theorem 121 *Let (π, H) be a Hilbert representation of G such that the underlying $(\mathfrak{g}, K)$-module is admissible and irreducible. Then (π, H) is irreducible.*

Proof. Let $V = (H^{\infty})_{(K)}$. Suppose that V is reducible. Let W be a closed invariant subspace of H. Then $\pi(\alpha_{\gamma})W \subset W$ for all $\gamma \in \widehat{K}$. Since H^{∞} is dense in H and $\pi(\alpha_{\gamma})H^{\infty} \subset H^{\infty}$ this implies that $H^{\infty} \cap H(\gamma) = H(\gamma)$ since π is admissible. Now if $W(\gamma) = H(\gamma)$ for all $\gamma \in \widehat{K}$ then $W = H$. Also as a subrepresentation of H we have $W^{\infty} = W \cap H^{\infty}$. This implies that $(W^{\infty})_{(K)} \subset (H^{\infty})_{(K)}$. If the two spaces are equal then the above considerations imply that $(W^{\infty})_{(K)} = (H^{\infty})_{(K)}$. Assume $W \neq H$. Since $(W^{\infty})_{(K)} \subset (H^{\infty})_{(K)}$ is a $\mathfrak{g}$ and a K-invariant subspace and $(H^{\infty})_{(K)}$ is an irreducible $(\mathfrak{g}, K)$-module this implies that $(W^{\infty})_{(K)} = (0)$. Hence $W(\gamma) = 0$ for all $\gamma \in \widehat{K}$. But than $W = 0$. Hence π is irreducible as asserted, ■

This result allows us to finish the discussion of the holomorphic discrete series of $SU(1,1)$. Here we take K and much of our notation as in 1.2.2. Since the map $T \to K$, $e^{i\theta} \longmapsto k(\theta)$ defines an isomorphism of $T = \{z \in \mathbb{C} | |z| = 1\}$ with K. One sees easily that if $\eta_n(k(\theta)) = e^{in\theta}$ then $\widehat{K} = \{\eta_n | n \in \mathbb{Z}\}$. From the definition of (π_k, H^k) we have $\pi_k(k)z^l = \eta_{-k-2l}(k)z^l$. It is easily seen that if $j \neq \{-k-2l | l \in \mathbb{Z}, l \geq 0\}$ then $H^k(\eta_j) = 0$ and that $H^k(\eta_{-k-2l}) = \mathbb{C}z^l$ otherwise. Thus $V = (H^k)^{\infty}_{(K)}$ is just the space of all polynomials in one complex variable. We will now prove the irreducibility. Set

$$h = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, u = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}.$$

Then both are elements of $Lie(SU(1,1))$. Set $Z^+ = \frac{h-iu}{2}$ and $Z^- = \frac{h+iu}{2}$ then

$$Z^+ z^l = -l z^{l-1}, Z^- z^l = (k+l) z^{l+1}, l \geq 0.$$

Now suppose that W is an invariant non-zero subspace of V. Then $W(\gamma) \neq 0$ for some $\gamma \in \widehat{K}$. This implies that $z^l \in W$ for some $l \geq 0$. Now $(Z^+)^l z^l = l!1$. This $1 \in W$. But $(Z^-)^m 1 = k\,(k+1) \cdots (k+m-1) z^m$. Hence $z^m \in W$ for all $m \geq 0$ so $W = V$.

1.2.5 A class of induced representations.

Let G be a locally compact, separable, unimodular, topological group and let K be a compact subgroup and P a closed subgroup such that $PK = G$. For example, G is a real reductive group, K is a maximal compact subgroup of G and P is a standard parabolic subgroup.

We note that we have shown that we may choose Haar measure on G such that if $f \in C_c(G)$ then

$$\int_G f(g)dg = \int_{P \times K} f(pk) dp dk$$

where the integration in P is relative to left invariant measure and that in K is relative to normalized invariant measure (Proposition 48).

The following is also standard

Lemma 122 *If $f \in C(K \cap P \backslash K) = C(P \backslash G)$ then there exists $\varphi \in C_c(G)$ such that*

$$f(k) = \int_P \varphi(pk) dp.$$

Proof. Consider the map $p, k \longmapsto pk$ of $P \times K$ onto G. Then if $pk = p_1 k_1$ with $p, p_1 \in P$ and $k, k_1 \in K$ then $(p_1)^{-1} p = k_1 k^{-1} = m \in K \cap P$. Hence $p_1 = pm^{-1}$ and $k_1 = mk$. This implies that if we consider the left action $m(p,k) = (pm^{-1}, mk)$, $m \in K \cap P$ then G is homeomorphic with $K \cap P \backslash (P \times K)$. Let $\phi \in C_c(P/K \cap P)$ and define $\varphi(p,k) = \phi(p) f(k)$. Then $\varphi(pm.m^{-1}k) = \varphi(p,k)$ for all $p \in P$, $k \in K$, $m \in P \cap K$. Assume that

$$\int_P \phi(p) dp = 1.$$

Then the formula in the statement is satisfied. ■

The following integration formula is the key to parabolic induction for real (and p-adic) reductive groups. First we need some notation. If $u \in C(P)$ and $u(pm) = u(p)$ for $p \in P$ and $m \in P \cap K$ then we extend u to G by $u(pk) = u(p)$, for $p \in P$ and $k \in K$. If $h \in C(K)$ and $h(mk) = h(k)$ for $m \in K \cap P$ and $k \in K$ then we extend h to G by setting $h(pk) = h(k)$, $p \in P, k \in K$. Let δ be the modular function of P. Then since $\delta_{|K \cap P} = 1$ we may extend it to G as above.

Lemma 123 *Let* $f \in C(K \cap P \backslash K)$ *then*

$$\int_K f(k)dk = \int_K f(kg)\delta(kg)dk$$

for all $g \in G$.

Proof. Let φ be as in the previous lemma. Then

$$\int_K f(k)dk = \int_G \varphi(x)dx = \int_G \varphi(xg)dg.$$

If $x \in G$ then we write $x = p(x)k(x)$ for some choice of $p(x)$ and $k(x)$ the ambiguity of the choice will be irrelevant in the rest of the argument. We continue

$$\begin{aligned}\int_G \varphi(xg)dx &= \int_{P\times K} \varphi(pkg)dpdk = \int_{P\times K} \varphi(pp(kg)k(kg))dpdk \\ &= \int_{P\times K} \delta(p(kg))\varphi(pk(kg))dpdk = \int_K \delta(p(kg))f(k(kg))dk.\end{aligned}$$

Since $\delta(p(kg)) = \delta(kg)$ and $f(k(kg)) = f(kg)$ the lemma follows. ■

We can now define the class of induced representations that are important to the harmonic analysis of reductive groups. Let (σ, H_σ) be a Hilbert representation of P. Lemma 106 implies that we may assume that it is unitary when restricted to $K \cap P$. Let H_o^σ denote the space of all continuous functions

$$f : G \to H_\sigma$$

such that $f(pg) = \delta(p)^{\frac{1}{2}}\sigma(p)f(g)$ for $p \in P$ and $g \in G$. We note that if $f \in H_o^\sigma$ and $f_{|K} = 0$ then $f = 0$. We endow H_o^σ with a pre-Hilbert space structure by taking

$$\langle f_1, f_2\rangle = \int_K \langle f_1(k), f_2(k)\rangle\, dk$$

for $f_1, f_2 \in H_o^\sigma$ (here the inner product inside the integral is that of H_σ). Let H^σ denote the Hilbert space completion of H_o^σ. If $g \in G$ then we define the operator $\pi_\sigma(g)$ on H_o^σ by $\pi_\sigma(g)f(x) = f(xg)$.

Lemma 124 *If* $g \in G$ *then* $\pi(g)$ *extends to a bounded operator on* H^σ. *Furthermore,* (π_σ, H^σ) *defines a Hilbert representation of* G *which is unitary if* (σ, H_σ) *is unitary.*

Proof. As above, we will write $x = p(x)k(x)$. If Ω is a compact subset of G then since the ambiguity is in $K \cap P$ we see that $p(\Omega) \subset \Omega'$ a compact subset of P. Thus there exists a constant $C_\Omega < \infty$ such that $\|\sigma(p(x))\| \le C_\Omega$ for all $x \in \Omega$. If $f \in H_o^\sigma$ then

$$\|\pi_\sigma(g)f\|^2 = \int_K \|f(kg)\|^2\, dk = \int_K \delta(p(kg))\, \|\sigma(p(kg))f(k(kg))\|^2\, dk. \tag{8}$$

Now if $g \in \Omega$ then this last expression is less than or equal to

$$C_\Omega^2 \int_K \delta(p(kg)) \|f(k(kg))\|^2 \, dk = C_\Omega^2 \|f\|^2$$

in light of the previous lemma. Thus $\|\pi_\sigma(g)\| \leq C_\Omega$. We note that (8) above combined with the integral formula above implies that $\pi_\sigma(g)$, for $g \in G$, is unitary if σ is unitary. We leave it to the reader to check that the matrix coefficients $g \longmapsto \langle \pi_\sigma(g)u, v\rangle$ are continuous for $u, v \in H_o^\sigma$. Thus Lemma 73 (properly extended) implies that π_σ defines a representation of G. ■

The representation (π_σ, H^σ) is usually denoted $Ind_P^G(\sigma)$ or $Ind_P^G(H_\sigma)$ and called an *induced* representation. We will now study the K-isotypic components of these representations under the hypothesis that the restriction of σ to $K \cap P$ has finite multiplicities that is if $\mu \in \widehat{K \cap P}$ then $\dim H_\sigma(\mu) < \infty$. We note that this restriction of $\pi_\sigma$to K is the K representation $Ind_{K\cap P}^K(\sigma_{|K\cap P})$. We set $M = K \cap P$ and $\tau = \sigma_{|K\cap P}$. We denote the induced representation to K by (ρ_τ, V^τ).

Lemma 125 *The representation (ρ, V^τ) is the unitary direct sum of the representations $Ind_M^K(H_\sigma(\mu))$ for $\mu \in \widehat{M}$.*

Proof. Set V_o^τ equal to the subspace of V^τ consisting of continuous elements. Let P_μ be the orthogonal projection of H_σ onto $H_\sigma(\mu)$. If $f \in V_o^\tau$ then set $f^\mu(k) = P_\mu f(k)$. Then $f^\mu \in V_o^\tau$ and $f = \sum f^\mu$ in V^τ. Clearly, $f^\mu \in Ind_M^K(H_\sigma(\mu))$. The lemma follows. ■

The next result is a generalization of the classical Frobenius reciprocity.

Proposition 126 *Let (ν, W) be a finite dimensional representation of K. Then the set of all $\mu \in \widehat{M}$ such that*

$$L_K(W, Ind_M^K(H_\sigma(\mu))) \neq 0$$

is finite. Furthermore $L_K(W, H^\sigma) = \sum_\mu L_K(W, Ind_M^K(H_\sigma(\mu)))$ (a finite sum by the above). Finally, there is a natural isomorphism of $L_K(W, Ind_M^K(H_\sigma(\mu)))$ with $L_M(W, H_\sigma(\mu))$.

Proof. Let $T \in L_K(W, Ind_M^K(H_\sigma(\mu)))$ and let $\gamma \in \widehat{K}$ be such that $W(\gamma) \neq 0$. Then $T(W(\gamma)) \subset Ind_M^K(H_\sigma(\mu))(\gamma) = \pi_\sigma(\alpha_\gamma) \, Ind_M^K(H_\sigma(\mu))$. This space us contained in H_o^σ. Since $\dim W < \infty$ this implies that $T(W)$ is contained in the continuous elements of $Ind_M^K(H_\sigma(\mu)) = W_{\sigma,\mu}$. Thus if $w \in W$ we can define $T(w)(1)$. We use the notation $\widehat{T}(w)$ for $T(w)(1)$. Then since $T(\nu(k)w)(1) = T(w)(k)$ for $k \in K$. The map $T \longmapsto \widehat{T}$ is injective. We also note that $\widehat{T} \in L_M(W, H_\sigma(\mu))$. Since $\dim W < \infty$ this implies the first assertion of the proposition. Suppose that $S \in L_M(W, H_\sigma(\mu))$ then define $\widetilde{S}(w)(k) = S(\nu(k)w)$. Then $\widetilde{S} \in L_K(W, Ind_M^K(H_\sigma(\mu)))$. It is clear that $\widetilde{\widehat{T}} = T$ and $\widehat{\widetilde{S}} = S$. ■

By analogy with the definition of the last section we will say that a representation (π, H) of G is K-*admissible* for K if $\dim H(\gamma) < \infty$ for all $\gamma \in \widehat{K}$. This is that same as saying that whenever (ν, W) is a finite dimensional representation of K, $\dim L_K(W,H) < \infty$.

The above results have the following implication

Corollary 127 *If (σ, H_σ) is $(K \cap P)$-admissible then $Ind_P^G(\sigma)$ is K-admissible.*

We look at an example. Let $G = SL(2,\mathbb{R})$. Let $K = SO(2)$ and let P denote the subgroup of upper triangular representations of G. Then $K \cap P = \{\pm I\}$. Set

$$A = \left\{ a_t = \begin{bmatrix} e^t & 0 \\ 0 & e^{-t} \end{bmatrix} | t \in \mathbb{R} \right\}$$

and

$$N = \left\{ n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} | x \in \mathbb{R} \right\}.$$

Then $P = K \cap P \cdot A \cdot N$ and every element can be written uniquely as $\pm a_t n_x$. Let σ denote one of the two characters of $K \cap P$. Let $\nu \in \mathbb{C}$ and set $\sigma_\nu(m a_t n_x) = e^{\nu t}\sigma(m)$ for $m \in K \cap P$. The series of representations $Ind_P^G(\sigma_\nu)$ is called the *principal series* of G. Notice that if ν is purely imaginary this representation is unitary. It turns out that for ν purely imaginary then the only representation of the form $Ind_P^G(\sigma_\nu)$ that is reducible is the one for σ non-trivial and $\nu = 0$. These representations combined with the holomorphic discrete series ($SL(2,\mathbb{R})$ is isomorphic with $SU(1,1)$) and the anti-holomorphic discrete series (take the other choice of $\sqrt{-1}$) give all of the irreducible unitary representations of G except for the complimentary series (which correspond to a new Hilbert space structure on $Ind_P^G(\sigma_\nu)$ for ν real and $-1 < \nu < 1$) and the trivial one dimensional representation.

Let G be a real reductive group, K a maximal compact subgroup of G and let P be a standard parabolic subgroup of G. Then $P = MAN$ the standard Langlands decomposition. Let $\mathfrak{n} = Lie(N)$ then $\delta(man) = \det(Ad(a)_{|\mathfrak{n}})$ $(= |\det(Ad(man)_{|Lie(P)}|$. Here we are using the fact that for a Lie group, B, the modular function is $|\det(Ad(b))|$.)

If (σ, H_σ) is a unitary, $K \cap P = K \cap M$-admissible representation of M and if $\nu \in Lie(A)^*_{\mathbb{C}}$ then we define $\sigma_\nu(m(\exp H)n) = e^{\nu(H)}\sigma(m)$. If ν is purely imaginary then the corresponding unitary representation is called a *principal series representation*. You will see that irreducible unitary representations of real reductive groups are admissible with respect to maximal compact subgroups and that the representations $Ind_P^G(\sigma_\nu)$ are generically irreducible if σ is irreducible. The underlying $(Lie(G), K)$-module is denoted $I_{P,\sigma,\nu}$. These modules are basic ingredients in the general theory...