Action of surface groups on affine spaces

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Abstract

Let Γ be the fundamental group of a compact surface. Let λ_q be the irreducible q-dimensional representation of $SL(2,\mathbb{R})$ in $SL(q,\mathbb{R})$. We shall say a representation ρ of Γ in $SL(q,\mathbb{R})$ is *Fuchsian* (or q-*Fuchsian*) if $\rho = \lambda_q \circ \iota$, where ι is a discrete faithful representation of Γ in $SL(2,\mathbb{R})$. We shall also say by extension the image of ρ is *Fuchsian*, and that an affine action of a surface group is *Fuchsian*, if its linear part is Fuchsian.

Our main result is the following theorem

Theorem 0.1 *A finite dimensional affine Fuchsian action of the fundamental group of a compact surface is not proper.*

In even dimensions, this is an easy remark. For dimension $4p+1$, this theorem follows from the use of *Margulis invariant*, also due to Margulis. These invariant and lemma were introduced in the work of Margulis, and later generalized with his coauthors H. Abels and G. Soifer and also by T. Drumm. Therefore, our proof shall concentrate on dimensions $4p+3$.

This case bears special features: one should notice that G. Margulis has exhibited proper affine actions of free group (with two generators) on $\mathbb{R}^3$, constructions later explained by T. Drumm and by V. Charette and W. Goldman. Therefore, surface groups behave differently than free groups in these dimensions.

When $\dim(E) = 3$, our result is a theorem of G. Mess, for which G. Margulis and W. Goldman have obtained a different proof using Margulis invariant and Teichmüller theory. Our proof is based on similar ideas, but uses instead of Teichmüller theory a result on Anosov flows and a holomorphic interpretation of Margulis invariant, hence generalizing to higher dimensions.