Two Vanishing Theorems for the Cohomology of Arithmetic Manifolds

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Abstract

Consider a locally symmetric manifold $\Gamma\backslash X = \Gamma\backslash G/K$, and a local system E arising from a finite dimensional algebraic representation of G denoted by the same letter. We ask the simple question of whether the j-th cohomology $H^j(\Gamma\backslash X, E)$ vanishes for a given degree j. The well known formula of Matsushima together with the explicit description of unitary representations with non-zero cohomology, as given by Vogan and Zuckerman, provides *a priori* vanishing results independent of the choice of Γ or E. There is now substantial evidence that these results cannot be improved for arbitrary Γ.

In this talk we will describe two vanishing results beyond what is implied by the combination of Matsushima and Vogan-Zuckerman. The first, joint with J. Schwermer, asserts that if E has regular highest weight then $H^j(\Gamma\backslash X, E) = 0$ for any $j < q_0(G)$, where

$$q_0(G) = \frac{1}{2}(\dim X - rk(G) + rk(K))$$

is roughly the middle dimension of X. We will formulate a precise conjecture which implies that the above is best possible.

Let Γ be a congruence subgroup arising from a quasi-split classical group defined over some number field. Our second result states that the *cuspidal cohomology* of Γ with an arbitrary coefficient system E vanishes in degrees below $q_0(G)/2$. This should be viewed as the analogue of a well known vanishing result for cuspidal cohomologies of the general linear groups. The proof depends on a fairly old result asserting the non-existence of singular cusp forms.