Extremal bounded holomorpohic functions and an embedding theorem for arithmetic varieties of rank ≥ 2

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Abstract

Let Ω be a bounded symmetric domain of rank ≥ 2, and $\Gamma \subset Aut(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. On the quasi-projective manifold X there is canonical Kähler-Einstein metric, which is of nonpositive holomorphic bisectional curvature. In the case of compact X we established a Hermitian metric rigidity theorem which in the locally irreducible case says that any Hermitian metric of nonpositive curvature in the sense of Griffiths is necessarily a constant multiple of the Kähler-Einstein metric. This was generalized by W.-K. To to include the case where X is noncompact (but of finite Kähler-Einstein volume). This phenomenon is called Hermitian metric rigidity, which implies a rigidity theorem for holomorphic mappings from X to Hermitian manifolds (N, h) of nonpositive curvature in the sense of Griffiths. In case (N, h) is Kähler and X is locally irreducible it says that any holomorphic mapping $f : X \to N$ is a totally-geometric isometric immersion up to scaling. We will call this the "rigidity phenomenon for holomorphic mappings". This was the state of affairs in 1989.

From the function-theoretic perspective the result is less than satisfactory. For instance, one should be able to treat target manifolds which are covered by bounded domains, where one expects to have new input to the problem from the study of extremal bounded holomorphic functions and ergodicity phenomena. First of all, we have obtained generalizations of Hermitian metric rigidity to the situation where N admits a continuous complex Finsler metric of nonpositive curvature, a phenomenon which we call Finsler metric rigidity. While this has some bearings on the analogous situation for the rigidity phenomenon on holomorphic mappings, one does not obtain a complete "Finsler" generalization of the rigidity phenomenon for holomorphic mappings. However, when the complex Finsler (pseudo)metric is the Carathéodory (pseudo)metric, we have established a strong rigidity phenomenon for holomorphic mappings, to the effect that in case X is locally irreducible we can show that any nonconstant holomorphic mapping $f : X \to N$ into a complex manifold N necessarily lifts to a totally-geodesic holomorphic *embedding* $F : \Omega \to \tilde{N}$ to universal covering spaces, provided that there exists at least one *bounded* holomorphic function h on $\tilde{N}$ such that F^*h is nonconstant. We will also examine the special case where f induces an isomorphism on fundamental groups.