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Hypersurfaces

Abstract

Affine maximal hypersurfaces are extremals of the interior variation of the affinely invariant volume. The corresponding Euler-Lagrange equation is a fourth order PDE. Originally, these hypersurfaces are called "affine minimal hypersurfaces". Calabi calculated the second variation and proposed to call them "affine maximal".

1. Calabi Conjecture

For affine maximal surfaces, there are different versions of so called affine Bernstein conjectures, stated by Calabi and Chern. The conjectures differ in the assumptions on the completeness of the affine maximal surface considered. While Chern assumed that the surface is a convex graph over R^2, which means that the surface is Euclidean complete, Calabi assumed that the surface is complete with respect to the Blaschke metric. Both conjectures were generalized to higher dimensions. Trudinger and Wang X. J present a proof of Chern's conjecture for 2-dimension. Recently we gave an affirmative answer to Calabi's conjecture. Precisely, we prove the following theorem:

Theorem 1 Let $x : M \to A^3$ be a locally strongly convex affine maximal surface. If M is complete with respect to the Balschke metric, then M must be an elliptic paraboloid.

2. Affine Maximal Graph

Let $x : M \to A^{n+1}$ be the graph of some strictly convex function $x_{n+1} = f(x_1, \cdots, x_n)$ defined in a convex domain $\Omega \subset A^n$. Following E. Calabi and A.V. Pogorelov, we consider the Riemannian metric $G^\#$ on M, defined by

$$G^\# = \sum f_{ij} dx_i dx_j,$$

where $f_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$. This is a very natural metric for a convex graph. We want to raise the following conjecture:

Theorem 2. *Let $x_{n+1} = f(x_1, \cdots, x_n)$ be a strictly convex function defined in a convex domain $\Omega \subset A^n$. If $M = \{(x_1, \cdots, x_n, f(x_1, \cdots, x_n)) | (x_1, \cdots, x_n) \in \omega\}$ is an affine maximal hypersurface, and if M is complete with respect to the metric $G^\#$, then, in the case $n = 2$ or $n = 3, M$ must be an elliptic paraboloid.*