Cohomologies for a polarizble variation of Hodge structures over quasi-compact Kähler manifolds

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In this Talk I shall report my recent joint work with J. Jost and K. Zuo. Let Y be a compact Kähler manifold, S a normal crossing divisor in Y, denote $Y \setminus S$ by U; let $\mathbb{V}$ be a polarized variation of Hodge structure with weight m defined over $\mathbb{R}$ over U, with unipotent local monodromies around S. Let

$$E = \bigoplus_{p+q=m} E^{p,q},\ \theta = \sum \theta^{p,q}$$

denote the Higgs bundle corresponding to $\mathbb{V}$, here $\theta^{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_Y(\log S)$. Using the Pioncaré-like metric on U, the Hodge metric on $\mathbb{V}$, and the asymptotic behavior of θ near S, we can define an algebraic subcomplex by taking the sheaves of local sections satisfying certain algebraic condition:

$$E_{(2)} \xrightarrow{\theta} (E \otimes \Omega^1_Y(\log S))_{(2)} \xrightarrow{\theta} (E \otimes \Omega^2_Y(\log S))_{(2)} \cdots ,$$

which actually corresponds to the subcomplex of L_2-local sections in the complex of sheaves of holomorphic differential forms with the differential θ on U. On the other hand, using a construction of Deligne, one can also define a complex of fine sheaves as follows:

$$[Gr^*_F A^0(E)]_{(2)} \xrightarrow{D''} [Gr^*_F A^1(E)]_{(2)} \xrightarrow{D''} [Gr^*_F A^1(E)]_{(2)} \xrightarrow{D''} \cdots).$$

Here, $[Gr^*_F A^k(E)]_{(2)} = \oplus_{p\geq 0}(\oplus_{r+s=k}(A^{r,s} \otimes E^{p-r,m-p+r})_{(2)})$ where $(A^{r,s} \otimes E^{p-r,m-p+r})_{(2)}$ is the fine sheaves of local L^2 section of type (r,s) valued in $E^{p-r,m-p+r}$, $D'' = \overline{\partial} + \theta$.
Then we will prove the following

Main Theorem: *There exists a natural isomorphism*

$$H^*_{\mathrm{D}''}(\Gamma([Gr^*_F A^0(E)]_{(2)}) \xrightarrow{D''} \Gamma([Gr^*_F A^1(E)]_{(2)}) \cdots)$$
$$\simeq \mathbb{H}^*(E_{(2)} \xrightarrow{\theta} (E \otimes \Omega^1_Y(\log S))_{(2)} \cdots).$$

Here $\mathbb{H}^*(\cdots)$ *is the hypercohomology of the corresponding complex.*

Using the Kähler identity of the Laplacians for variation of Hodge structure, one can easily see that the above D''-cohomology is just the usual L^2-cohomology with coefficient in VHS, which, by a theorem of Cattani-Kaplan-Schmid (For this, see E. Cattani, A. Kaplan, and W. Schmid, L^2 *and intersection cohomologies for a polarizable variation of Hodge structure,* Inventiones Math., **87**, 1987, 217-252.), is isomorphic to the intersection cohomology $H^*_{\mathrm{int}}(Y, \mathbb{V})$. Thus we have

Corollary. *There exists a natural isomorphism*

$$H^*_{\mathrm{int}}(Y, \mathbb{V}) \simeq \mathbb{H}^*(E_{(2)} \xrightarrow{\theta} (E \otimes \Omega^1_Y(\log S))_{(2)} \cdots).$$

We will also give some applications of the above results in this Talk.

Complex multiplication, Griffiths-Yukawa couplings, and rigidity for families of hypersurfaces

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Report on joint work with K. Zuo

Let M_h denote a moduli scheme of canonically polarized manifolds of general type, or of minimal models of Kodaira dimension zero and of dimension $n-1$. Let $\varphi : S \to M_h$ be a morphism induced by a family $f : \mathcal{X} \to S$. The variation of polarized Hodge structures $R^{n+1} f_* \mathbb{Q}_{\mathcal{X}}$, and the corresponding Higgs bundle

$$\big(E = \bigoplus_{p+q=n-1} E^{p,q},\ \theta = \bigoplus_{p+q=n-1} \theta_{p,q}\big),$$

with $E^{p,q} = R^q f_* \Omega^p_{\mathcal{X}/S}$ and with Higgs field

$$\theta_{p,q} : E^{p,q} \to E^{p-1,q+1} \otimes \Omega^1_S,$$

define morphisms

$$\theta^i : E^{0,n-1} \longrightarrow E^{1,n-2} \otimes \Omega^1_S \longrightarrow E^{2,n-3} \otimes S^2(\Omega^1_S) \longrightarrow \cdots \longrightarrow E^{i,n-1-i} \otimes S^i(\Omega^1_S).$$

For $i = n-1$ one obtains the Griffiths-Yukawa coupling

$$\theta^{n-1} : E^{0,n-1} \longrightarrow E^{n-1,0} \otimes S^{n-1}(\Omega^1_S) = E^{0,n-1\vee} \otimes S^{n-1}(\Omega^1_S).$$

We define its length to be

$$\varsigma(f) = \mathrm{Min}\{i \geq 1;\ \theta^i = 0\} - 1.$$

If $\varsigma(f) = n-1$, then the family is known to be rigid. Little is known about the existence of families with $\varsigma(f) < n-1$, and the geometric implications of this condition are not well understood.

Kang and I tried to study the Griffiths-Yukawa coupling for families of hypersurfaces of degree d in $\mathbb{P}^n$. Although we are mainly interested in the case $d = n+1$, i.e. in the Yukawa coupling of Calabi-Yau hypersurfaces, it turns out to be necessary to study the case $d > n+1$, as well.

Let $\mathcal{M}_{d,n}$ be the moduli stack of hypersurfaces $X \subset \mathbb{P}^n$ of degree $d \geq n+1$, and let $\mathcal{M}^{(1)}_{d,n}$ be the sub-stack, parameterizing hypersurfaces obtained as a d fold cyclic covering of $\mathbb{P}^{n-1}$ ramified over a hypersurface of degree d. Iterating this construction, one obtains $\mathcal{M}^{(\nu)}_{d,n}$.

We show that $\mathcal{M}^{(1)}_{d,n}$ is rigid in $\mathcal{M}_{d,n}$, although for $d < 2n$ the Griffiths-Yukawa coupling degenerates. However, for all $d \geq n+1$ the sub-stack $\mathcal{M}^{(2)}_{d,n}$ deforms.

We calculate the exact length of the Griffiths-Yukawa coupling over $\mathcal{M}^{(\nu)}_{d,n}$. As it will turn out, all values between 1 and $n-1$ really occur.

As a byproduct of the calculation of the variation of Hodge structures, for those families, we construct a 4-dimensional family of quintic hypersurfaces $g : \mathcal{Z} \to T$ in $\mathbb{P}^4$, and a dense set of points t in T, such that $g^{-1}(t)$ has complex multiplication, i.e. such that the special Mumford Tate group is abelian.