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On isometric embeddings of the Poincaré disk

Let $f : (D, 0) \to (D', 0)$ be a germ of holomorphic isometry up to a normalizing constant between two bounded symmetric domains equipped with the Bergman metric. We pose the question of characterizing such maps and of finding conditions which force such maps to be totally geodesic. The special case of the problem where D is the unit disk, D' is a polydisk, and f satisfies some supplementary conditions, was studied by Clozel-Ullmo in connection to an arithmetic problem. There first of all they proved that f extends algebraically by making use of real algebraic functional identities arising from Kähler potentials.

By the standard procedure of polarizing real-analytic identities one obtains an infinite number of holomorphic identities. They define subvarieties of $D \times D'$ reminiscent of Segre varieties. We show that, in the event that there are nontrivial deformations of solutions of the holomorphic identities, the germ of holomorphic isometry must take values on intersections of hyperplane sections of the embedding of the domain into the infinite-dimensional projective space $\mathbb{P}^\infty$ defined by an orthonormal basis of $H^2(D')$. Such hyperplane sections correspond to zero sets of extremal holomorphic functions, which can be read off from the Bergman kernel $K(z, \overline{w})$. As a consequece, we show that $f : (D, 0) \to (D', 0)$ extends algebraically.

The case of isometric embeddings of the Poincaré disk Δ is of special interest, since there is an ample supply of geodesic disks on a bounded symmetric domain. We show that any $f : (\Delta, 0) \to (D', 0)$, which extends algebraically, must be asymptotically totally geodesic at a good boundary point. However, there are examples of holomorphic isometries of the Poincaré disk into the polydisk where singularities develop in the algebraic extension. In the case of holomorphic isometries into the polydisk, it can be shown that f is totally geodesic if and only if there are no singularities of the algebraic extension on the boundary circle.