THE UNIVERSITY OF HONG KONG

Institute of Mathematical Research
Department of Mathematics

LECTURE SERIES

on

Some Aspects of the Conjecture of André-Oort

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Abstract

In this series of lectures we shall give the necessary background for studying and understanding the André-Oort conjecture. The conjecture can be stated for arbitrary Shimura varieties. We restrict ourselves however to simple cases which need only some classical mathematical background. We begin with studying the situation when the underlying Shimura variety is of the simplest possible form, namely a product $\mathbb{P}^1 \times \mathbb{P}^1$ of two projective lines. However here we meet already the prototype of the problem. This space $\mathbb{P}^1 \times \mathbb{P}^1$ can be seen as the space classifying those abelian surfaces which are products of two elliptic curves. To understand this moduli space, as it is called, we shall briefly introduce the concept of elliptic curves together with the concept of complex multiplication.

The next step in the complexity of the problem will be to replace the product of two elliptic curves by arbitrary abelian surfaces and study their moduli space $\mathcal{M}_2$ which is the Siegel upper half plane $\mathbb{H}_2$ consisting of complex 2 × 2-matrices subject to some extra condition and taken modulo the symplectic group Sp(2, $\mathbb{Z}$). It has dimension $\frac{n^2+n}{2}$ for general n which becomes 3 in the case of $\mathcal{M}_2$. Again we shall briefly introduce this space and this gives our second example. Its points classify isomorphism classes of abelian varieties of dimension 2, so-called abelian surfaces which we shall also introduce in an elementary way.

The space $\mathbb{P}^1 \times \mathbb{P}^1$ mentioned above is a very particular case for such a moduli space and sits as an algebraic surface in $\mathcal{M}_2$. Another example to which we shall pay some attention are the famous Hilbert modular surface which were introduced and first studied by Hilbert. They classify abelian surfaces with a special type of endomorphism algebra which appears as a particular case in the classification theory of the endomorphism algebra of an abelian variety. There are four possible types which were found by Albert when establishing the classification. Going through the classification one sees that each class ϕ of endomorphism algebra gives rise to a subvariety $\mathcal{S}_\Phi$ of $\mathcal{M}_2$. There are infinitely many such subspaces which are called Shimura (sub-)varieties. We shall carefully and in an elementary way introduce these objects and this then lays the ground for looking into the original conjecture of André-Oort. The conjecture gives some very interesting statement about the geometric nature of the Zariski closure of a set, finite or infinite, of Shimura subvarieties of $\mathcal{M}_2$. When the Shimura subvarieties are all of dimension zero and then are just points in $\mathcal{M}_2$ we are in the situation of the original conjecture. It will turn out that they will be again a finite collection of Shimura subvarieties. This is in our situation the upshot of the conjecture of André-Oort.

In these lectures we shall explain the conjecture but not go into any proof of it. There are several reasons. The conjecture is only proved in the case of the Shimura variety $\mathbb{P}^1 \times \mathbb{P}^1$ and there is so far no proof in the general situation. In the $\mathbb{P}^1 \times \mathbb{P}^1$-case there are three very different approaches which are all too involved as one can go into them in short time. One is by André, one by Pila and one by Kühne. The first two proofs are not effective whereas Kühne's proof relies on the Baker Theory and is fully effective. We leave it then to Kühne to give some account of his proof. In the case of arbitrary dimension one only knows that the Generalized Riemann Hypothesis implies the André -Oort conjecture. This has been shown in a paper by Yafaev and Klingler which however is since years in the refereeing process. But it is unknown whether the hypothesis is true or not.

Date:	**May 2, 9, 16, 23, 30; June 6; July 4, 11, 2012 (Wednesdays)**
Time:	**3:00 - 5:00pm**
Place:	**Room 210, Run Run Shaw Bldg., HKU**

All are welcome (especially graduate students)

The lectures assume only basic classical mathematical background and will be accessible to graduate students in any area of pure mathematics.
The speaker will proceed in a relaxed manner and questions and discussions are welcome throughout the lectures.