THE UNIVERSITY OF HONG KONG

Institute of Mathematical Research

Department of Mathematics

CONFERENCE ON NUMBER THEORY

November 4, 2011
Room 210, Run Run Shaw Building, HKU

9:55 – 10:00	Welcoming speech by Prof. Ngaiming Mok (Director of IMR, HKU)
10:00 – 10:55	Irina Rezvyakov (Steklov Mathematical Institute, Moscow) *An additive problem with Fourier coefficients of automorphic cusp forms and its application to the zeros on the critical line of associated L-functions*
11:00 – 11:55	Maosheng Xiong (Hong Kong University of Science and Technology) *On the distribution of Selmer groups for a family of elliptic curves*
Lunch Break	
14:30 – 15:25	Ilya Shkredov (Steklov Mathematical Institute, Moscow) *Roth's theorem in many variables*
15:30 – 16:25	Kai-Man Tsang (University of Hong Kong, Hong Kong) *On a mean value theorem for the Riemann zeta-function*
16:45 – 17:40	Yuk-Kam Lau (University of Hong Kong, Hong Kong) *On a variance of Hecke eigenvalues in arithmetic progressions*
18:30	*Dinner*

Organizers: Y.K. Lau, K.M. Tsang

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Abstract

Yuk-Kam Lau
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On a variance of Hecke eigenvalues in arithmetic progressions

Let f be a normalized Hecke eigenform of even weight k for the full modular group and $T_n f = \lambda_f(n) f$ for each Hecke operator T_n. We shall discuss a variance associated with $\lambda_f(n)$ in an arithmetic progression, defined as

$$\mathcal{A}(X, q, \lambda_f) := \sum_{b=1}^{q} \left| \sum_{\substack{n \le X \\ n \equiv b \,(\mathrm{mod}\, q)}} \lambda_f(n) \right|^2 .$$

Irina Rezvyakova
Steklov Mathematical Institute,
Moscow

An additive problem with Fourier coefficients of automorphic cusp forms and its application to the zeros on the critical line of associated L-functions.

Let $f(z)$ be an automorphic cusp form of integral weight $k \ge 1$ for the group $\Gamma_0(D)$ with a character χ modulo D, which is an eigenfunction of all the Hecke operators T_n for $n = 1, 2, \ldots$, where

$$T_n f(z) = \frac{1}{n} \sum_{ad=n} \chi(a) a^k \sum_{0 \le b < d} f\left(\frac{az+b}{d}\right).$$

This entails, that $f(z)$ has an expansion

$$f(z) = \sum_{n=1}^{+\infty} a(n) e^{2\pi i n z} \quad \text{for} \quad \operatorname{Re} z > 0$$

and that for $\operatorname{Re} s > 1$, the Dirichlet series

$$L(s) = L_f(s) = \sum_{n=1}^{+\infty} \frac{r(n)}{n^s}$$

with

$$r(n) = a(n) n^{\frac{1-k}{2}},$$

satisfies the identity

$$L(s) = \prod_p \left(1 - \frac{r(p)}{p^s} + \frac{\chi(p)}{p^{2s}}\right)^{-1}$$

(here the product is carried over all consecutive prime numbers).

We know the following estimate:

$$|r(n)| \leq \tau(n),$$

where $\tau(n)$ is the number of divisors of n. This inequality was previously known as the Ramanujan - Petersson conjecture until its truth was proved by P. Deligne in 1974 for $k \geq 2$, and by P. Deligne and J. -P. Serre for $k = 1$.

In our talk we will consider and give a proof of the following additive type problem.

Theorem 1 *Let N, m_1, m_2, l be integers satisfying the conditions $N \gg 1$, $(m_1, m_2) = 1$, $m_1^8 m_2^9 \leq N, \quad l \leq N^{10/11}$,*

$$S = \sum_{n=1}^{N-1} r(n) \overline{r\left(\frac{m_1 n + l}{m_2}\right)}$$

(where we assume that the function $r(\cdot)$ vanishes at non-integer argument). Then for arbitrary $\varepsilon > 0$ the following estimate holds:

$$S \ll_\varepsilon N^{10/11+\varepsilon} m_1^{8/11} m_2^{-2/11}.$$

Our proof of this estimate relies on M. Jutila's circle method with overlapping intervals.

We also plan to introduce our result on zeros on the intervals of the critical line of L-functions associated with automorphic cusp forms and discuss a reduction of its proof to a non-trivial estimate of the sum S from the Theorem.

Ilya Shkredov
Steklov Mathematical Institute,
Moscow

Roth's theorem in many variables

In 1953 using the Hardy–Littlewood method K.F. Roth proved his celebrated theorem on sets having no arithmetic progressions of length three. If $A \subseteq \{1, \ldots, N\}$ is such a set, that is a set without solutions of the equation $x_1 + x_2 = 2y$, $x_1, x_2, y \in A$ are distinct then $|A| = O(N/\log\log N)$. The result was improved several times by E. Szemerédi, D.R. Heath–Brown, J. Bourgain and T. Sanders. At the moment the best result is due to Sanders who obtain $|A| = O(N/(\log N)^{1-\epsilon})$. On the other hand the best lower bound was proved in 1947 by F.A. Behrend. He constructed a set A having no arithmetic progressions of length three such that $|A| \gg Ne^{-C(\log N)^{1/2}}$.

We prove, in particular, that if $A \subseteq \{1, \ldots, N\}$ has no nontrivial solution to the equation $x_1 + x_2 + x_3 + x_4 + x_5 = 5y$ then $|A| \ll Ne^{-c(\log N)^{1/6-\varepsilon}}$, $c > 0$. In view of Behrend's construction this estimate is close to best possible.

Kai-Man Tsang
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On a mean value theorem for the Riemann zeta-function

The error term $E(T)$ in the mean square formula

$$\int_0^T \left|\zeta\Big(\frac{1}{2} + it\Big)\right|^2 dt = T\log\frac{T}{2\pi} + (2\gamma - 1)T + E(T)$$

for the Riemann zeta-function on the critical line has been studied extensively alongside other famous error terms in analytic number theory, such as the error term in the Dirichlet divisor problem. The behaviour of $E(T)$ is very interesting and intriguing, and it has attracted the attention of many researchers. Asymptotics for the mean square of $E(T)$ had been obtained long time ago, with successive better bounds for the error term. In this talk, I shall discuss a new mean value theorem for $E(T)$ and thereby probes into the finer shape of the error term for the mean square of $E(T)$. In particular, we find that the error term in the mean square of $E(T)$ is $\Omega_-(x\log^2 x \log\log x)$.

Maosheng Xiong
Department of Mathematics
Hong Kong University of Science and Technology
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On the distribution of Selmer groups for a family of elliptic curves

In the arithmetic of elliptic curves, to find the rank is of fundamental importance. Ranks can be bounded by Selmer groups, which are easier to handle and also contain information on the mysterious Tate-Shafarevich groups.

The purpose of this talk is to survey various results related with the title of this talk. We will focus on the recent work of the author and his collaborators on the distribution of Selmer groups arising from a 2-isogeny for quadratic twists of an elliptic curve which possesses a non-trivial 2-torsion point over the rationals. It turns out that whether or not the elliptic curve has full 2-torsions over the rationals influences the distribution. Combining with other people's results on Selmer groups, this implies in many families that the corresponding Tate-Shafarevich group is quite large in average.