Joint Hölder continuity of the solutions to a class of SPDEs arising from multi-dimensional superprocesses in the random environment

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Overview

1. Superprocesses in the random environment
   - Framework and one dimensional results
   - Formulation of the branching mechanism
   - Lebesgue density and SPDEs
2. Moment estimates of the one-particle motion
   - Non-degeneracy of the Malliavin matrix
   - Moment estimates
3. Hölder continuity of the density

Branching particle system

Particle motion:

Consider a branching particle system in the random environment. The motion of each particle is described by the SDE:

$$\xi^\alpha(t) = \xi^\alpha_r + B^\alpha(t) - B^\alpha(r) + \int_r^t \int_{\mathbb{R}^d} h(y - \xi^\alpha(s)) W(ds, dy).$$

- $\alpha = (\alpha_0, \ldots, \alpha_N) \in \mathbb{N} \times \{1, 2\}^N$: label of the N-th generation particles.
- $\xi^\alpha_r \in \mathbb{R}^d$: the birth position at time $r \geq 0$.
- B^α: independent d-dimensional Brownian motions.
- W: d-dimensional space-time white noise, independent of B^α.
- W is regarded as the random environment.

Branching events:

- For any $n \in \mathbb{N}$, the branching happens at fixed time $\frac{k}{n}$, $k = 1, 2, \dots$.
- At any branching time, each particle dies, and randomly generates at most 2 offspring.
- New particles continue the moving/branching mechanism.
- In the n-th approximation, denote by $\xi^{\alpha,n}$ the path of particle motion.
- Equip the system an empirical measure

$$X_t^n = \frac{1}{n} \sum_{\alpha \sim_n t} \delta_{\xi_t^{\alpha,n}},$$

where the sum is among all alive particles at time t.

Some notations:

- $\rho \in C_b^2(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$, given by

$$\rho(x) = \int_{\mathbb{R}^d} h(x-z)h^*(z)dz.$$

- $A : C_b^2(\mathbb{R}^d) \to C_b(\mathbb{R}^d)$ given by

$$A\phi(x) = \frac{1}{2}\sum_{i,j=1}^{d} \left(\rho^{ij}(0)\partial_{ij}\phi(x)\right) + \frac{1}{2}\Delta\phi(x),$$

is the generator of one particle motion.

One dimensional model (known results)

Under the Dawson-Watanabe branching mechanism:

Each particle independently splits into 2 or dies in probabilities $(\frac{1}{2}/\frac{1}{2})$.

Theorem (Wang 97, 98)

Suppose $X_0^n \Rightarrow X_0 \in M_F(\mathbb{R})$. Then $X_t^n \to X_t$, that uniquely solves the martingale problem (MP): for any $\phi \in C_b^2(\mathbb{R})$,

$$M_t(\phi) := X_t(\phi) - X_0(\phi) - \int_0^t X_s(A\phi)ds$$

is a continuous square integrable martingale with quadratic variation

$$\langle M(\phi)\rangle_t = \int_0^t \nabla\phi(x)\rho(x-y)\nabla\phi(y)X_s(dx)X_s(dy)ds + \int_0^t X_s(\phi^2)ds.$$

Theorem (Dawson et al. 00)

X_t has a density $u_t(x)$ almost surely, that solves the SPDE weakly:

$$u_t(x) = \mu(x) + \int_0^t A^* u_s(x) ds - \int_0^t \int_{\mathbb{R}} \nabla_x [h(y-x) u_s(x)] W(ds, dy) + \int_0^t \sqrt{u_s(x)} \frac{V(ds, dx)}{dx}.$$

where μ is the initial "density" that can be any L^1 function or distribution on $\mathbb{R}$, V is a space-time white noise independent of W.

Theorem (Li et al. 12, Hu et al. 14)

Suppose $\mu \in L^2(\mathbb{R})$ bounded, then u is almost surely jointly Hölder continuous with time exponent $\frac{1}{4}$, and spatial exponent $\frac{1}{2}$.

Remark:

- The density SPDE doesn't have a mild representation. The mild representation is formally written as

$$u_t(x) = \int_{\mathbb{R}} p(t, x-y)\mu(y)dy + \int_0^t \int_{\mathbb{R}} \sqrt{u_s(y)}p(t-s, x-y)V(dsdy)$$
$$+ \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} h(y-z)u_s(z)\partial_z p(t-s, x-z)dzW(dsdy),$$

  then the last term is not integrable at t.

- When $d \geq 2$, it is proved (Dawson and Hochberg 79) that the super Brownian motion ($h \equiv 0$) involves a singular measure.

New branching mechanism (Mytnik 96, Sturm 03)

Let $\{\eta(x) \in \mathbb{R} : x \in \mathbb{R}^d\}$ be the random field on $\mathbb{R}^d$:

- η is symmetric: for any x, the distribution of $\eta(x)$ is symmetric.
- $\exists\ p > 2$, s.t. $\sup_x \mathbb{E}|\eta(x)|^p < \infty$.
- The correlation in different points:

$$\mathbb{E}(\eta(x)\eta(y)) = \kappa(x, y) \in C_b^2(\mathbb{R}^{2d}).$$

For any $n \in \mathbb{N}$, let $\eta_n = (\sqrt{n} \wedge \eta) \vee -\sqrt{n}$. The offspring distribution is described by ξ_n:

In the n-th approximation:

- Let $\{\eta_n^i\}_{i\in\mathbb{Z}_+}$ be i.i.d. copies of η_n.
- At the branching time $\frac{i+1}{n}$, the offspring distribution of the i-th generation particle, conditioned on η_n^i and its position x, is given by

$$P\left(N^{\alpha,n}=2\middle|\,\eta_n,x\right)=\frac{1}{\sqrt{n}}\eta_n^+(x),$$

$$P\left(N^{\alpha,n}=0\middle|\,\eta_n,x\right)=\frac{1}{\sqrt{n}}\eta_n^-(x),$$

$$P\left(N^{\alpha,n}=1\middle|\,\eta_n,x\right)=1-\frac{1}{\sqrt{n}}|\eta_n|(x).$$

Lebesgue density and SPDEs

Theorem (Hu, nualart, X.)

Suppose $X_0^n \to X_0 \in M_F(R^d)$ *that has a bounded Lebesgue density* μ. *Then,*

- X^n *converges weakly to* X *in* $D([0,T], M_F(\mathbb{R}^d))$.
- X_t *has a Lebesgue density* u_t *for all* $t \in [0,T]$.
- u_t *is the unique weak solution to the following SPDE:*

$$u_t = \mu + \int_0^t A^* u_s ds + \int_0^t \int_{\mathbb{R}^d} \nabla^* \cdot [u_s(y)h(y-x)] W(ds,dy) + \int_0^t u_s(x) \frac{V(ds,dx)}{dx}. \quad (1)$$

Here A^* *is the adjoint of* A, V *is an independent noise, that is white in time, and colored in space with correlation* κ.

Martingale Problems

- By the typical tightness argument, we show that $\{X^n\}_{n\in\mathbb{N}}$ is tight in $D([0,T];M_F(\mathbb{R}^d))$:
  - By Itô's formula, $\phi \in C_b^2(\mathbb{R}^d)$, one can decompose

    $$X_t^n(\phi) := \int_{\mathbb{R}^d} \phi(x) X_t^n(dx) = X_0^n(\phi) + Z_t^n(\phi) + B_t^n(\phi) + U^n(\phi) + M^{r,n}(\phi),$$

    where $Z_t^n(\phi)$ denotes the drift term, $B_t^n(\phi)$ denotes the martingale from B^α, $U^n(\phi)$ denotes the random environment martingale, and $M^{r,n}(\phi)$ denotes the branching martingale.
  - $X^n(\phi)$, $Z^n(\phi)$, $M^{b,n}(\phi)$, $U^n(\phi)$ are C-Tight in $D([0,T],\mathbb{R})$.

    $B_t^n(\phi) \to 0$ in $L^2(\Omega)$ for all $t \in [0,T]$.
  - (Dawson) X^n is C-tight in $D([0,T],M_F(\mathbb{R}^d))$.

    (Mitoma) Z^n, M^n, U^n, and B^n are C-tight in $D([0,T],\mathcal{S}'(\mathbb{R}^d))$.

- Any limit X of a convergent subsequence X^{n_k} is a solution to the MP: for any $\phi \in C_b^2(\mathbb{R}^d)$,

$$M_t(\phi) = X_t(\phi) - X_0(\phi) - \int_0^t X_s(A\phi)ds \tag{2}$$

is a continuous square integrable martingale with quadratic variation

$$\begin{aligned}\langle M(\phi)\rangle_t =& \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \nabla\phi(x)^* \rho(x-y)\nabla\phi(y) X_s(dx)X_s(dy)ds \\ &+ \int_0^t \int_{\mathbb{R}^d \times \mathbb{R}^d} \kappa(x,y)\phi(x)\phi(y)X_s(dx)X_s(dy)ds. \end{aligned} \tag{3}$$

Absolute continuity w.r.t. the Lebesgue measure

Lemma

Let X_t be any solution to the MP (2) - (3). $X_0 \in M_F(\mathbb{R}^d)$ has a bounded Lebesgue density μ. Then X_t also has Lebesgue density a.s.

Sketch of the proof: We prove the theorem by showing

$$\sup_{h\in(0,1)} \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\left|X_t(p_h^x)\right|^2 dxdt < \infty,$$

$$\lim_{h_1,h_2\downarrow 0} \int_0^T \int_{\mathbb{R}^d} \mathbb{E}\left|X_t(p_{h_1}^x) - X_t(p_{h_2}^x)\right|^2 dxdt = 0,$$

where p_h^x is the heat kernel:

$$p_h^x(\cdot) = (4\pi h)^{-\frac{d}{2}} \exp\Big(-\frac{|x-\cdot|^2}{4h}\Big).$$

- $\{X_t(p_h^x)\}_{h>0}$ is Cauchy in $L^2(\Omega\times[0,T]\times\mathbb{R}^d)$.
- The limit $u_t(x)$ is the Lebesgue density of X_t a.s.

By using the moment duality, one can obtain the moment formula:

Lemma (Moment formula)

For any $f \in C_b^2(\mathbb{R}^{nd})$,

$$\mathbb{E}X_t^{\otimes n}(f) = X_0^{\otimes n}(v(t,\cdot)),$$

where v is the solution to the PDE

$$\partial_t v(t,x) - \Big[A^{(n)} v(t,x) + \frac{1}{2} \sum_{\substack{1 \le i,j \le n \\ i \neq j}} \kappa(x_i, x_j) v(t,x)\Big] = 0, \tag{4}$$

with $v(0,x) = f(x)$, $A^{(n)}$ is the generator of the n-particle motion:

$$A^{(n)} f = \frac{1}{2}\Delta f + \frac{1}{2} \sum_{i_1,i_2=1}^{n} \sum_{j_1,j_2=1}^{d} \rho^{j_1 j_2}(x_{i_1} - x_{i_2}) \frac{\partial^2 f}{\partial x_{i_1}^{j_1} \partial x_{i_2}^{j_2}}(x_1, \dots, x_n).$$

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Density Equation and Conditional Mild Formulation

- Any solution to the MP has a Lebesgue density.
- Every limit process of the particle approximation is a solution to the MP.

Thus every limit process has a density. It solves the SPDE (1) weakly.

- **Uniqueness:** Let $d_t(x) = u_t^1(x) - u_t^2(x)$, then d is a solution to the SPDE with initial condition $\mu \equiv 0$, and thus a solution to the MP. By the moment duality,

  $$\mathbb{E}\langle d_t, \phi \rangle = \mathbb{E}\langle \mu, \phi_t \rangle = 0,$$

  for all $\phi \in C_b^2(\mathbb{R}^d)$, which means $d \equiv 0$, a.s..

The uniqueness of the solution to the SPDE implies the convergence of empirical measures.

Conditional mild representation: (Li et al. 12)
The SPDE (1) has a unique weak solution that is the unique strong solution to the equation:

$$u_t(x) = \int_{\mathbb{R}^d} \mu(z) p^W(r, z; t, x) dz + \int_0^t \int_{\mathbb{R}^d} p^W(r, z; t, x) u_s(z) V(ds, dz),$$

where p^W is the transition density density of one-particle motion conditional on W.

The well-posedness of the equation can be proved by using the moment estimates of p^W below.

Moment estimate of one-particle motion

Recall the one-particle motion $\xi_t = \xi_t^{r,x}$, described by the SDE:

$$\xi_t = x + B_t - B_r + \int_0^t \int_{\mathbb{R}^d} h(y - \xi_u) W(du, dy),$$

where B is a Brownian motion, W is a space-time white noise, independent of B, and $h \in H_2^3(\mathbb{R}^d; \mathbb{R}^d \otimes \mathbb{R}^d)$.

Lemma

Let $p^W(r, x; t, y)$ be the transition density of ξ_t conditioned on W. Then

$$\|p^W(r, x; t, y)\|_{2p} \leq C \exp\Big(-\frac{|x-y|^2}{t-r}\Big)(t-r)^{-\frac{d}{2}},$$

$$\int_{\mathbb{R}^d} \Big\|p^W(r, z; t, y_2) - p^W(r, z; t, y_1)\Big\|_{2p} dz \leq C(t-r)^{-\frac{1}{2}\beta} |y_2 - y_1|^\beta,$$

$$\int_{\mathbb{R}^d} \Big\|p^W(r, z; t, y) - p^W(r, z; s, y)\Big\|_{2p} dz \leq C(s-r)^{-\frac{1}{2}\beta}(t-s)^{\frac{1}{2}\beta}.$$

Preliminaries and notations

- Let $B = \{B_t : 0 \leq t \leq T\}$ be the standard d-dimensional Brownian motion.

  $H = L^2([0, T]; \mathbb{R}^d)$ is then the basic Hilbert space with B.

- Denote by D the Malliavin derivative operator and δ the divergence operator.

  For any n-dimensional random variable F, denoted by γ_F, σ_F the $n \times n$ Malliavin matrix and its inverse respectively, if exists.

- The integration by parts formula:

$$\mathbb{E}(\partial_i \phi(F) G) = \sum_{j=1}^{n} \mathbb{E}\left[\phi(F)\delta(G\sigma_F^{ij} DF_j)\right],$$

for all $F \in \mathbb{D}^{1,2}(\mathbb{R}^n)$, and $G \in \mathbb{D}^{1,2}(\mathbb{R})$.

Lemma (Malliavin and Thalmaier 05, Bally and Caramellino 11)

Let $F \in \mathbb{D}^{2,p}(\mathbb{R}^n)$, and $\sigma_F^{ij} \in L^p(\Omega)$ for all $p \geq 1$. Then F has the Lebesgue density:

$$p(x) = -\mathbb{E} \sum_{i,j=1}^{n} \mathbf{1}_{B(x,\rho)} \partial_i Q_n(F - x) \delta\left[\phi_\rho^x(F) \sigma_F^{ij} DF_j\right],$$

where $\partial_i Q_n(x) = C_n \frac{x_i}{|x|^{n-1}}$, and $\phi_\rho^x \in C^1(\mathbb{R}^d)$: $1_{B(x,\rho)} \leq \phi_\rho^x \leq 1_{B(x,2\rho)}$ and $|\nabla \phi_\rho^x| \leq 1/\rho$.

For simplification, we denote by

$$H_{(i)}(F, G) := \sum_{j=1}^{n} \delta(G \sigma_F^{ij} DF_j)$$

Non-degeneracy of the Malliavin matrix

- By differentiating the equation of particle motion:

$$D_\theta^{(k)}\xi_t = I - \int_\theta^t \int_{\mathbb{R}^d} D_\theta^{(k)}\xi_s^* dM_s,$$

  where M is a matrix-valued martingale, with entries

$$M_t^{ij} := \sum_{k=1}^d \int_r^t \int_{\mathbb{R}^d} \partial_i h^{jk}(y-\xi_t) W^k(ds,dy).$$

- Denote by

$$g_k^{ij}(t,y) = \partial_i h^{jk}(y-\xi_t).$$

- Let $\lambda_\theta^{ij}(t) = \sum_{k=1}^d D_\theta^{(k)}\xi_t^i D_\theta^{(k)}\xi_t^j$, then

$$\lambda_\theta(t) = I - \int_\theta^t \lambda_\theta(s) dM_s - \int_\theta^t dM_s^* \cdot \lambda_\theta(s)$$
$$+ \sum_{k=1}^d \int_\theta^t \int_{\mathbb{R}^d} g_k^*(s,y)\lambda_\theta(s) g_k(s,y) dyds.$$

- It turns out that fix $\theta \in [r, t)$, λ is invertible, with $\lambda_\theta(t)^{-1} = \beta_\theta(t)$, satisfies the SDE:

$$\begin{aligned}\beta_\theta(t) =& I + \int_\theta^t \beta_\theta(s) dM_s^* + \int_\theta^t dM_s \cdot \beta_\theta(s) \\ &+ \sum_{k=1}^d \int_\theta^t \int_{\mathbb{R}^n} \left[g_k(s,y)^2 \beta_\theta(s) + g_k(s,y)\beta_\theta(s) g_k^*(s,y) \right. \\ &\left. + \beta_\theta(s) g_k^*(s,y)^2 \right] dyds.\end{aligned}$$

- By Jensen's inequality (Stroock 83.), almost surely we have

$$\|\sigma_t\|_{H.S.} = \left\| \left(\int_r^t \lambda_\theta(t) d\theta \right)^{-1} \right\|_{H.S.} \le \frac{1}{(t-r)^2} \left\| \int_r^t \lambda_\theta(t)^{-1} d\theta \right\|_{H.S.}$$

Sketch of the proof: Let $\rho = \sqrt{t-r}$. By the density lemma:

$$p^W(r,x;t,y) = -\sum_{i=1}^{d} \mathbb{E}^W \left(\mathbf{1}_{B_{(y,2\rho)}}(\xi_t)\partial_i Q_d(\xi_t - y) H_{(i)}(\xi_t, \phi_\rho^y(\xi_t)) \right).$$

- ξ_t is a Gaussian process:

$$\mathbb{P}(\xi_t \in B(y,2\rho)) \le C \exp(-\frac{k|x-y|^2}{t-r}).$$

- (Nualart and Nualart) By integration by parts formula:

$$\sup_{y\in\mathbb{R}^d} \|\partial_i Q_d(\xi_t - y)\|_p \le C \max_{1\le i\le d} \|H_{(i)}(\xi_t;1)\|_{\frac{p}{p-1}}^{d-1}.$$

- For $H_{(i)}(\xi_t, \phi_\rho^y(\xi_t))$, we have the identity by integration by parts formula:

$$H_{(i)}(\xi_t, \phi(\xi_t)) = \phi_\rho^y(\xi_t) H_{(i)}(\xi_t, 1) + \partial_i \phi_\rho^y(\xi_t).$$

- By Meyer's inequality and the estimates of Sobolev norms of ξ_t, and σ_t:

$$\|H_{(i)}(\xi_t, 1)\|_p \le C(t-r)^{-\frac{1}{2}}.$$

The moment estimation of $p^W(r, x; t, y)$ follows from Hölder's inequality.

Final result

Based on the conditional mild representation and moment estimates, we easily show:

Theorem (Hu, Nualart, X.)

For any $\beta_1, \beta_2 \in (0,1)$, $p > 1$, $0 < s < t \le T$, there exists $C = C_{p,T,\beta_1,\beta_2}$, s.t.

$$\|u(t,x) - u(s,y)\|_{2p} \le Cs^{-\frac{1}{2}} \left(|x-y|^{\beta_1} + (t-s)^{\frac{1}{2}\beta_2} \right).$$

Hence by Kolmogorov's continuity criteria, $u(t,x)$ is almost surely jointly Hölder continuous, with exponent $\beta_1 \in (0,1)$ in space and $\beta_2 \in (0, \frac{1}{2})$ in time.

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Thanks!