Abstract

The seminorm form of Morrey’s inequality is summarized as follows: Let \( u \in L^1_{\text{loc}}(\mathbb{R}^n) \) be such that \( Du \in L^p(\mathbb{R}^n) \) and \( p > n \). Then there is some \( C > 0 \) depending only on \( n \) and \( p \) such that

\[
C\|Du\|_p \geq [u]_{C^{0,1-n/p}}
\]

(0.1)

where \([u]_{C^{0,1-n/p}}\) is the \( C^{0,1-n/p} \)-Hölder seminorm given by \([u]_{C^{0,1-n/p}} := \sup_{x \neq y} \left\{ \frac{|u(x) - u(y)|}{|x - y|^{1-n/p}} \right\}\). This inequality was (essentially) proven 80 years ago by C. B. Morrey Jr. However, until recently, nothing was known about extremals or the sharp constant of Morrey’s inequality. In a recent project, R. Hynd and I proved the existence of extremals and some of their qualitative characteristics. The key to our results is to show that a function, \( v \), is an extremal of Morrey’s Inequality if and only if it satisfies a PDE:

\[
-\Delta_p v = c(\delta_x - \delta_y)
\]

(0.2)

where \( \delta_x \) and \( \delta_y \) are dirac masses at some \( x, y \in \mathbb{R}^n \) and \( c \) is any nonzero constant. The points \( x \) and \( y \) in (0.2) are essential in the structure of \( v \). In a recent project, we show that \( x \) and \( y \) are the unique pair of points where \( v \) achieves its \( C^{0,1-n/p} \)-Hölder seminorm, they are the points where \( v \) achieves its absolute maximum and minimum, and \( v \) is analytic except at \( x \) and \( y \). Moreover, using the PDE, (0.2), we are able to show that extremals of Morrey’s inequality are cylindrically symmetric (if \( n \geq 3 \)) or evenly symmetric (if \( n = 2 \)) about the line containing \( x \) and \( y \), reflectionally antisymmetric (up to addition by a constant), and unique up to operations that are invariant on the ratio of the seminorms in (0.1). We also give explicit solutions for extremals when \( n = 1 \) and some numerical approximations of extremals for \( n = 2 \) and \( p = 4 \). This work is a collaboration with R. Hynd.