Full cones swept out by minimal rational curves on irreducible Hermitian symmetric spaces as examples of varieties underlying geometric substructures

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§1. Introduction and motivation

Starting with Hwang-Mok [HM98], a program of study on uniruled projective manifolds from a differential-geometric perspective was launched based on the geometry of their varieties of minimal rational tangents (VMRTs). To explain the latter notion, fixing an ample line bundle L on a uniruled projective manifold X we introduced the notion of a minimal rational curve, by which we mean a *free* rational curve on X of minimal degree with respect to L among all free rational curves on X. Denoting by $\mathrm{Chow}(X)$ the Chow space of X and by $\mathrm{Chow}^\sharp(X)$ its normalization, a minimal rational component $\mathcal{K}$ on X is a nonsingular Zariski dense open subset of *some* irreducible component $\mathcal{Q}$ of $\mathrm{Chow}^\sharp(X)$ such that each member of $\mathcal{Q}$ is a rational 1-cycle and such that $\mathcal{K} \subset \mathcal{Q}$ is precisely the subset consisting of minimal rational curves. The VMRT of a uniruled projective manifold $(X, \mathcal{K})$ equipped with a minimal rational component, denoted by $\mathscr{C}_x(X) \subset \mathbb{P}T_x(X)$ at a general point $x \in X$, is the collection of projectivizations of vectors tangent to minimal rational curves passing through x. It is our perspective that, given $(X, \mathcal{K})$, the underlying VMRT structure $\pi : \mathscr{C}(X) \to X$ is a rich geometric object which encodes a lot of the information about X as a projective manifold. (For a reference to the early part of the theory we refer the reader to Hwang-Mok [HM99]). Assuming furthermore that X is of Picard number 1 and that the Gauss map of $\mathscr{C}_x(X) \subset \mathbb{P}T_x(X)$ is immersive at a general smooth point, we proved in Hwang-Mok [HM01] a general result called the Cartan-Fubini Extension Theorem ascertaining that X is uniquely determined as a projective manifold by $\pi|_U : \mathscr{C}(X)|_U \to U$ for any nonempty connected open subset U of X in the *complex* topology. In other words, given Fano manifolds $(X, \mathcal{K})$ and $(X', \mathcal{K}')$ of Picard number 1 equipped with minimal rational components, nonempty connected open subsets $U \subset X$ and $U' \subset X'$, and a biholomorphic map $f : U \xrightarrow{\cong} U'$ such that $[df](\mathscr{C}(X)|_U) = \mathscr{C}(X')|_{U'}$, we proved that f extends to a global biholomorphism $F : X \xrightarrow{\cong} X'$.

The study of $\pi : \mathscr{C}(X) \to X$, $\mathscr{C}(X) \subset \mathbb{P}T(X)$ as a fibered subspace of $\mathbb{P}T(X)$ leads to a rich geometric theory, especially when $\mathscr{C}_x(X) \subset \mathbb{P}T_x(X)$ are mutually projectively equivalent to each other as x varies over general points of X. In the latter case Hwang [Hw10] [Hw12] [Hw15] studied the Cartanian geometry of isotrivial families of VMRTs and obtained quite a number of rigidity results notably concerning the flatness of the VMRT structure, where one works primarily with the geometric structure defined on connected open subsets in the complex topology. On the other hand, when we fix a uniruled projective manifold $(X, \mathcal{K})$, we have the class of projective subvarieties of X uniruled by $\mathcal{K}$. Especially, restricting $\pi : \mathscr{C}(X) \to X$ to some appropriate connected open subset $W \subset X$ and considering a submanifold $S \subset W$ such that the intersection $\mathscr{C}_x(S) := \mathscr{C}_x(X) \cap \mathbb{P}T_x(S)$ for $x \in U$ defines a

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sub-VMRT structure (cf. Mok-Zhang [MZ17, §5]), the study of $S \subset W \subset X$ may be regarded as an analogue of the study of Riemannian submanifolds of a given Riemannian manifold. Here, a principal rigidity problem is the characterization of special uniruled projective subvarieties $Z \subset X$ among $S \subset W \subset X$ on which the sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ is in some precise sense modeled on $\big(\mathscr{C}_x(Z) \subset \mathscr{C}_x(X)\big)$. Classical examples of such a rigidity problem include the question of characterizing Grassmann submanifolds of Grassmann manifolds, in which $Z := G(r,s) \subset G(p,q) =: X$, where $r \le p, s \le q, \min(r,s) \ge 2$, and $G(r,s) \subset G(p,q)$ is the obvious inclusion map. In such examples for rational homogeneous spaces $X = G/P$ of Picard number 1, realizing X as a projective submanifold by means of the first canonical embedding, $Z \subset X$ is a special smooth linear section uniruled by projective lines, and $\mathscr{C}(Z) = \mathscr{C}(X) \cap \mathbb{P}T(Z)$ underlies the VMRT structure of Z equipped with the uniruling by projective lines of X lying on Z.

When $X = G/P$ is Hermitian symmetric, and $Z \subset X$ is a smooth Schubert cycle, the problem of *recognizing* $Z \subset X$ from the isomorphism class of its tangent spaces modulo G-action is a crucial step in determining whether $Z \subset X$ is Schur rigid in the sense that for any integer $r \ge 1$, any algebraic cycle homologous to rZ is necessarily a sum $\gamma_1 Z + \cdots + \gamma_r Z$, $\gamma_k \in G$ for $1 \le k \le r$, and the problem on Schur rigidity for smooth Schubert cycles was solved in special cases by Walters [Wa97] and Bryant [Br01] and in general by Hong [Ho07]. The same problem for singular Schubert cycles in case the ambient space is a Grassmannian was treated by Hong [Ho05] and the general case where X is an irreducible Hermitian symmetric space of the compact type is settled by Robles-The [RT12]. In the cited works crucial to the proofs are algebraic results ascertaining the vanishing of certain cohomology groups defined in terms of Lie algebras. From the perspective of the theory of VMRTs it is natural to consider the relevant recognition problem as a differential-geometric problem. In this vein Hong-Mok [Ho10] extended Cartan-Fubini Extension Theorem to the non-equidimensional situation for germs $f : (Z, z_0) \to (X; x_0)$ of VMRT-respecting maps under the hypotheses that a general point $[\alpha]$ of $[df](\mathscr{C}_z(Z)) \subset \mathscr{C}_{f(z)}(X)$ is a smooth point and that a certain non-degeneracy condition on the second fundamental form of $\mathscr{C}_x(X) \subset \mathbb{P}T_x(X)$ at $[\alpha]$ is satisfied. When the rational homogeneous space X of Picard number 1 is defined by a marked Dynkin diagram $(\mathscr{D}(\mathfrak{g}), \gamma)$ and $Z \subset X$ is defined by a marked Dynkin sub-diagram $(\mathscr{D}(\mathfrak{g}_0), \gamma_0)$ of $(\mathscr{D}(\mathfrak{g}), \gamma)$, applying non-equidimensional Cartan-Fubini extension Hong-Mok [HoM10] proved that f extends to a standard embedding in the case where Z is nonlinear and the marking is at a long simple root (cf. also Mok [Mo08a] for the case of the Grassmannian). The same problem for the maximal linear case and for the case of markings at a short simple root were settled by Hong-Park [HoP11]. Using the geometric theory of VMRTs, Hong-Mok [HoM13] settled the problem on homological rigidity (i.e., the special case of Schur rigidity with the restriction $r = 1$) for smooth Schubert cycles $Z \subset X$. In Mok-Zhang [MZ17] we defined admissible pairs (X_0, X) of rational homogeneous spaces of Picard number 1 and the notion of rigid pairs (X_0, X) among them, and introduced the general theory of sub-VMRT structures on uniruled projective manifolds. Especially, when $X_0 \subset X$ is nonlinear and (X_0, X) is of sub-diagram type we proved rigidity of the pair (X_0, X) in the sense that $X_0 \subset X$ can be recognized by the isomorphism class of tangent subspaces $T_x(X_0)$ modulo the action of $\mathrm{Aut}(X)$.

In this article, by way of the examination of a special class of singular Schubert cycles on irreducible Hermitian symmetric spaces of the compact type, we explore the application of the general theory of sub-VMRT structures of Mok-Zhang [MZ17] to rigidity problems. Specifically we consider germs of complex submanifolds modeled on full cones of minimal rational curves irreducible Hermitian symmetric spaces X of rank ≥ 2 other than Lagrangian Grassmannians. Here by a full cone of minimal rational curves we mean the union $\mathcal{V} = \mathcal{V}(x)$ of minimal rational curves emanating from a point $x \in X$, noting that the associated sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$ have singular and possibly reducible fibers. For these Schubert cycles $\mathcal{V} \subset X$ we apply the methods and results of [MZ17] to study sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$ on complex submanifolds $S \subset W$ of connected open subsets $W \subset X$ in the complex topology. Since the problem of Schur rigidity on possibly singular Schubert cycles $Z \subset X$ has been completely settled in Robles-The [RT12] in the case where X is an irreducible Hermitian symmetric space of the compact type, our focus is rather on the methodology, proving results by means of checking nondegeneracy conditions arising from the theory of sub-VMRT structures, a method that is potentially applicable to rational homogeneous spaces and horospherical varieties of Picard number 1. It can be checked from [RT12] that with the exception of the case where X is the hyperquadric Q^n, $n \geq 3$, or the rank-2 Grassmann manifold $G(2, q)$, $q \geq 2$, Schur rigidity holds for any full cone $\mathcal{V} = \mathcal{V}(x) \subset X$ of minimal rational curves (including the Lagrangian Grassmannian). We are interested in general in the Recognition Problem for Schubert cycles on rational homogeneous spaces of Picard number 1 and, by way of illustration, we will show that in the special cases considered in the current article, methods of [MZ17] apply to prove results of linear saturation and algebraicity of $S \subset W$ modeled on $\mathcal{V} \subset X$, and that, under the additional assumption that the sub-VMRT structure on $S \subset W$ is *intrinsically flat*, it remains the case that S is linearly saturated and algebraic (as a germ) for the cases of the hyperquadric and rank-2 Grassmannians.

On top of providing examples for illustration, the set of full cones of minimal rational curves $\mathcal{V}$ is also important for the study of holomorphic isometries of complex unit balls into irreducible bounded symmetric domains. In fact, taking $\Omega \subset X$ to be the Hermitian symmetric space of the noncompact type dual to X and embedded in X by means of the Borel embedding, and taking $q \in \partial\Omega$ to be a regular boundary point, the author has proven in [Mo16a] that $V(q) := \mathcal{V}(q) \cap \Omega$ is the image of a holomorphic isometric embedding of the complex hyperbolic space form into Ω equipped with a canonical Kähler-Einstein metric, and Recognition Problem for $\mathcal{V} \subset X$ enters into the picture in the uniqueness question for holomorphic isometric embedding of the complex unit ball of maximal admissible dimension. In conjunction with results of the current article, the Recognition Problem for $\mathcal{V} \subset X$ has been settled by differential-geometric means by Mok-Yang [MY17] by way of the Thickening Lemma of Mok-Zhang [MZ17, Proposition 6.1] and a process of reconstruction analogous to that in Mok [Mo08a] and Hong-Mok [HoM10] [HoM13]. We believe that the theory of sub-VMRT structures on uniruled projective manifolds, beyond its applicability to the study of uniruled projective subvarieties, also provides a useful link for the study of transcendental problems such as those on bounded symmetric domains. For the explanation of this perspective we refer the reader to Mok [Mo16b] on the theory of geometric structures and sub-structures.

§2. Background materials and results

We provide here a number of basic definitions and results taken from Mok-Zhang [MZ17] necessary for the current article. Let $(X, \mathcal{K})$ be a uniruled projective manifold equipped with a minimal rational component, $\mathcal{K} \subset \mathcal{Q}$ be the compactification of $\mathcal{K}$ by the normalization of some Chow component of X, $B \subset X$ be the bad locus of $(X, \mathcal{K})$, i.e., the minimal subset of X outside of which every member of $\mathcal{Q}$ passing through x must necessarily belong to $\mathcal{K}$, and $\pi : \mathscr{C}(X) \to X$, $\mathscr{C}(X) \subset \mathbb{P}T(X - B)$ be the VMRT structure of $(X, \mathcal{K})$. We denote by $B' \supset B$ the minimal subset outside of which the tangent map is a birational finite morphism (cf. Mok-Zhang [MZ17,§5]) and call B the enhanced bad locus of $(X, \mathcal{K})$. Let $W \subset X - B'$ be a connected open subset in the complex topology, $S \subset W$ be a complex submanifold, and define $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$. We have the following definition of sub-VMRT structures given in Mok-Zhang [MZ17, Definition 5.1].

Definition 2.1 *We say that $\varpi := \pi|_{\mathscr{C}(S)} : \mathscr{C}(S) \to S$ is a sub-VMRT structure on $(X, \mathcal{K})$ if and only if* (a) *the restriction of ϖ to each irreducible component of $\mathscr{C}(S)$ is surjective, and* (b) *at a general point $x \in S$ and for any irreducible component Γ_x of $\mathscr{C}_x(S)$, we have $\Gamma_x \not\subset \mathrm{Sing}(\mathscr{C}_x(X))$.*

Next we will need to consider pairs consisting of VMRTs and their linear sections. We introduce the notion of proper pairs of projective subvarieties, as follows (cf. Mok-Zhang [MZ17, Definition 5.2]).

Definition 2.2. *Let V be a Euclidean space and $\mathcal{A} \subset \mathbb{P}(V)$ be an irreducible subvariety. We say that $(\mathcal{B}, \mathcal{A})$ is a proper pair if and only if $\mathcal{B}$ is a linear section of $\mathcal{A}$, and for each irreducible component Γ of $\mathcal{B}$, $\Gamma \not\subset \mathrm{Sing}(\mathcal{A})$.*

Note that for a uniruled projective manifold X and a complex submanifold $S \subset W \subset X - B'$ inheriting a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ as in Definition 2.1, at a general point $x \in S$, $(\mathscr{C}_x(S), \mathscr{C}_x(X))$ is a proper pair of subvarieties. We introduce next two nondegeneracy conditions in terms of second fundamental forms of VMRTs. They concern nondegeneracy for mappings and nondegeneracy for substructures. For the formulation recall that for a finite-dimensional vector space V and for a subset $Z \subset \mathbb{P}(V)$, denoting by $\lambda : V - \{0\} \to \mathbb{P}(V)$ the canonical projection we write $\widetilde{Z} := \lambda^{-1}(Z) \subset V - \{0\}$ for the affinization of Z. The following two definitions are adaptations of Mok-Zhang [MZ17, Definition 5.3].

Definition 2.3. *Let V be a finite-dimensional vector space, $E \subsetneq V$ be a vector subspace and $(\mathcal{B}, \mathcal{A})$ be a proper pair of projective subvarieties in $\mathbb{P}(V)$, $\mathcal{B} := \mathcal{A} \cap \mathbb{P}(E) \subset \mathcal{A} \subset \mathbb{P}(V)$. Assume that $\mathcal{A}$ is irreducible. Let $\xi \in \widetilde{\mathcal{B}}$ be a smooth point of both $\widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{B}}$, and let $\sigma : S^2 T_\xi(\widetilde{\mathcal{A}}) \to V/T_\xi(\widetilde{\mathcal{A}})$ be the second fundamental form of $\widetilde{\mathcal{A}}$ in V with respect to the Euclidean flat connection on V. We say that $(\mathcal{B}, \mathcal{A})$ is nondegenerate for mappings if and only if for each irreducible component Γ of $\mathcal{B}$ and for a general point $\chi \in \Gamma$, we have*

$$\left\{\eta \in T_\chi(\widetilde{\mathcal{A}}) : \sigma(\eta, \xi) = 0 \text{ for any } \xi \in T_\chi(\widetilde{\mathcal{B}})\right\} = \mathbb{C}\chi\,.$$

Definition 2.4. *In the notation of Definition 2.3 write furthermore $V' \subset V$ for the linear span of $\widetilde{\mathcal{A}}$ and define $E' := E \cap V'$. Let $\nu : V/T_\xi(\widetilde{\mathcal{A}}) \to V/(T_\xi(\widetilde{\mathcal{A}}) + E')$ be* *the canonical projection and define* $\tau : S^2T_\xi(\widetilde{\mathcal{A}}) \to V/(T_\xi(\widetilde{\mathcal{A}}) + E')$ *by* $\tau := \nu \circ \sigma$. *We say that* $(\mathcal{B}, \mathcal{A}; E)$ *is nondegenerate for substructures if and only if for each irreducible component* Γ *of* $\mathcal{B}$ *and for a general point* $\chi \in \Gamma$, *we have*

$$\left\{\eta \in T_\chi(\widetilde{\mathcal{A}}) : \tau(\eta, \xi) = 0 \text{ for any } \xi \in T_\chi(\widetilde{\mathcal{B}})\right\} = T_\chi(\widetilde{\mathcal{B}}).$$

When $E' = E \cap V'$ is the same as the linear span of $\widetilde{\mathcal{B}}$ we drop the reference to E, with the understanding that the projection map ν is defined by using the linear span of $\widetilde{\mathcal{B}}$ as E'.

We will use interchangeably the second fundamental form of $\widetilde{\mathcal{A}}$ in V, and the projective second fundamental form of $\mathcal{A}$ in $\mathbb{P}V$, denoting both by σ. More precisely, for a smooth point χ of $\mathcal{A}$, we have $T_{[\chi]}(V) \cong V/\mathbb{C}\chi$, and, for $\xi, \eta \in T_\chi(\widetilde{\mathcal{A}})$, $\xi' := \xi + \mathbb{C}\chi \in V/\mathbb{C}\chi, \eta' := \eta + \mathbb{C}\chi \in V/\mathbb{C}\chi$, we have $\sigma_\chi : S^2T_\chi(\widetilde{\mathcal{A}}) \to V/T_\chi(\widetilde{\mathcal{A}})$, $\sigma_{[\chi]} : S^2T_{[\chi]}(\mathcal{A}) \to T_{[\chi]}(\mathbb{P}V)/T_{[\chi]}(\mathcal{A}) = (V/\mathbb{C}\chi)/(T_\chi(\widetilde{\mathcal{A}})/\mathbb{C}\chi) \cong V/T_\chi(\widetilde{\mathcal{A}})$, and, identifying the two normal spaces by means of the latter isomorphism, we have $\sigma_{[\chi]}(\xi', \eta') = \sigma_\chi(\xi, \eta)$. The same consideration applies analogously to the vector-valued quadratic form $\tau_\chi : S^2T_\chi(\widetilde{\mathcal{A}}) \to V/(T_\chi(\widetilde{\mathcal{A}}) + E')$ and its projectivized form $\tau_{[\chi]} : S^2T_\chi(\mathcal{A}) \to T_{[\chi]}(\mathbb{P}V)/(T_{[\chi]}(\mathcal{A}) + T_{[\chi]}(\mathbb{P}(E'))) = (V/\mathbb{C}\chi)/((T_\chi(\widetilde{\mathcal{A}}) + E')/\mathbb{C}\chi) \cong V/(T_\chi(\widetilde{\mathcal{A}}) + E')$, and we have $\tau_{[\chi]}(\xi', \eta') = \tau_\chi(\xi, \eta)$.

For the study of rigidity properties of sub-VMRT structures, on top of non-degeneracy conditions formulated in terms of second fundamental forms, we also need a condition regarding the intersection $\mathscr{C}_x(S) := \mathscr{C}_x(X) \cap \mathbb{P}T_x(S)$, to be called Condition (T), as follows (cf. [MZ17, Definition 5.4]).

Definition 2.5. *Let* $\varpi : \mathscr{C}(S) \to S$, $\mathscr{C}(S) := \mathscr{C}(X) \cap \mathbb{P}T(S)$, *be a sub-VMRT structure on* $S \subset W \subset X - B'$ *as in Definition 2.1. For a point* $x \in S$, *and* $[\alpha] \in \mathrm{Reg}(\mathscr{C}_x(S)) \cap \mathrm{Reg}(\mathscr{C}_x(X))$, *we say that* $(\mathscr{C}_x(S), [\alpha])$, *or equivalently* $(\widetilde{\mathscr{C}_x}(S), \alpha)$, *satisfies* Condition (T) *if and only if* $T_\alpha(\widetilde{\mathscr{C}_x}(S)) = T_\alpha(\widetilde{\mathscr{C}_x}(X)) \cap T_x(S)$. *We say that* $\varpi : \mathscr{C}(S) \to S$ *satisfies* Condition (T) *at* x *if and only if* $(\widetilde{\mathscr{C}_x}(S), [\alpha])$ *satisfies* Condition (T) *for a general point* $[\alpha]$ *of each irreducible component of* $\mathrm{Reg}(\mathscr{C}_x(S)) \cap \mathrm{Reg}(\mathscr{C}_x(X))$. *We say that* $\varpi : \mathscr{C}(S) \to S$ *satisfies* Condition (T) *if and only if it satisfies the condition at a general point* $x \in S$.

The following two results are the principal results of Mok-Zhang [MZ17] relevant to the current article which are adaptations of [MZ17, Theorem 1.4] and [MZ17, Main Theorem 2]. In the notation of the preceding paragraphs recall that $\varpi := \pi\big|_{\mathscr{C}(S)}$, and $\varpi : \mathscr{C}(S) \to S$ is a sub-VMRT structure.

Theorem 2.1. *Suppose the VMRT structure* $\varpi : \mathscr{C}(S) \to S$ *on* $S \subset W \subset X - B'$ *satisfies* Condition (T). *Assume furthermore that for a general point* x *on* S *and for each of the irreducible components* $\Gamma_{k,x}$ *of* $\mathscr{C}_x(S)$, $1 \le k \le m$, *the pair* $(\Gamma_{k,x}, \mathscr{C}_x(X))$ *is nondegenerate for substructures. Then,* S *is rationally saturated with respect to* $(X, \mathcal{K})$.

By the concluding sentence of Theorem 2.1 we mean that for *any* minimal rational curve ℓ belonging to $\mathcal{K}$ such that ℓ is tangent to S at some point $x \in \ell \cap S$, the germ $(\ell; x)$ of holomorphic curve must lie on $(S; x)$. When the ambient uniruled projective X is of Picard number 1 (and hence Fano) and uniruled by lines, i.e., by minimal rational curves whose homology classes are generators of $H_2(X, \mathbb{Z}) \cong \mathbb{Z}$, we have in [MZ17, Main Theorem 2] the following algebraicity result for germs of sub-VMRT structures on X.

Theorem 2.2. *In Theorem 2.1 suppose furthermore that* $(X, \mathcal{K})$ *is a projective manifold of Picard number 1 uniruled by lines and that the distribution* $\mathscr{D}$ *on* S *defined by* $\mathscr{D}_x := \mathrm{Span}(\widetilde{\mathscr{C}_x(S)})$ *is bracket generating. Then, there exists an irreducible subvariety* $Z \subset X$ *such that* $S \subset Z$ *and such that* $\dim(Z) = \dim(S)$.

Note that the bracket-generating condition on $\mathscr{D}$ is trivially satisfied whenever $\mathscr{C}_x(S) \subset \mathbb{P}T_x(S)$ is linearly nondegenerate at a general point $x \in S$. When $(X, \mathcal{K})$ is a Fano manifold equipped with a uniruling by lines, a rationally saturated sub-VMRT structure S on X is said to be linearly saturated.

§3. Sub-VMRTs structures modeled on full cones of minimal rational curves in the Hermitian symmetric case

We discuss here some examples on which Main Theorem 2 and its proof apply to show that sub-VMRT structures modeled on them are algebraic, possibly under additional assumptions. These are subvarieties with isolated singularities of irreducible Hermitian symmetric spaces of the compact type. They are examples of singular Schubert cycles for which methods of VMRT geometry especially sub-VMRT structures apply to study rigidity problems in the spirit of Mok [Mo08a], Hong-Mok [HoM10] [HoM13] and Mok-Zhang [MZ17], and they are also particularly interesting in view of their relation to holomorphic isometries in Kähler geometry as in Mok [Mo16a]. We will see from these examples that in case nondegeneracy of substructures fails, it may still happen that underlying complex submanifolds $S \subset W$ of sub-VMRT structures arising from *VMRT-respecting maps* remain always linearly saturated and algebraic as germs of manifolds.

Let $(X, \mathcal{K})$ be an irreducible Hermitian symmetric space of the compact type of rank ≥ 2, equipped with the uniruling by projective lines. Write $X = G/P$, where G is the identity component of the group $\mathrm{Aut}(X)$ of biholomorphic automorphisms and $P \subset G$ is a parabolic subgroup. Let $x \in X$ and denote by $\mathcal{V}(x)$ the union of minimal rational curves passing through x. Under the natural action of G, $\mathcal{V}(x)$ is fixed by $\gamma \in G$ if and only if $\gamma \in P$, from which it follows that $\mathcal{V}(x)$ is a Schubert cycle. At a point $y \in \mathcal{V}(x)$ distinct from x let ℓ be the projective line joining x and y, $T_y(\ell) := \mathbb{C}\alpha$ and consider $\mathscr{C}_y(\mathcal{V}(x)) := \mathbb{P}T_y(\mathcal{V}(x)) \cap \mathscr{C}_y(X)$. We may assume that x and y lie on a Harish-Chandra coordinate chart so that T_y and T_x are identified by parallel transport with respect to the Euclidean flat connection. Hence, also $\mathscr{C}_y(X)$ and $\mathscr{C}_x(X)$ are identified with each other. Then, $\xi \in \mathscr{C}_y(\mathcal{V}(x))$ if and only if there exists a projective line ℓ' on $\mathcal{V}(x)$ passing through y such that $T_y(\ell') = \mathbb{C}\xi$. For $[\xi] \neq [\alpha_y]$ this occurs if and only if there exists a projective plane Π containing y such that $T_y(\Pi) = \mathbb{C}\alpha_y + \mathbb{C}\xi$. We have $T_y(\mathcal{V}(x)) = P_{\alpha_y}$ since Harish-Chandra coordinates are privileged coordinates (cf. Mok-Zhang [MZ17, Definition 2.1]). Thus, $\mathscr{C}_y(\mathcal{V}(x)) := \mathbb{P}(P_{\alpha_y}) \cap \mathscr{C}_y(X)$. The pairs $(\mathscr{C}_y(\mathcal{V}(x), \mathscr{C}_y(X))$ are thus constant along $\ell - \{x\}$ in the Harish-Chandra coordinate chart, hence the pairs $\big(\mathscr{C}_y(\mathcal{V}(x)) \subset \mathscr{C}_y(X)\big)$ are projectively equivalent to each other for $y \in \mathcal{V}(x) - \{x\}$. When $X = G^{III}(n, n)$ are Lagrangian Grassmannians of rank $n \geq 2$, $\mathscr{C}_y(\mathcal{V}(x)) = \{[\alpha_y]\}$ is a single point. At $0 \in X$, let $[\alpha] \in \mathscr{C}_0(X)$ and define $\mathscr{S}_{[\alpha]} := \mathbb{P}(P_\alpha) \cap \mathscr{C}_0(X)$. For $X \not\cong G^{III}(n, n)$ we consider now sub-VMRT structures modeled on $(\mathcal{V}(x), X)$, i.e., on the pair $(\mathscr{S}_{[\alpha]}, \mathscr{C}_0(X))$ for any $[\alpha] \in \mathscr{C}_0(X)$, and examine the question whether a sub-

VMRT structure $\varpi : \mathscr{C}(S) \to S$ modeled on $(\mathscr{S}_{[\alpha]}, \mathscr{C}_0(X))$ is necessarily linearly saturated. As a preparation we prove

Lemma 3.1. *Let X be an irreducible Hermitian symmetric space of rank ≥ 2 not biholomorphic to a Lagrangian Grassmannian, $0 \in X$, and $[\alpha] \in \mathscr{C}_0(X)$. Then, $\mathscr{S}_{[\alpha]} \subset \mathbb{P}(P_\alpha)$ is the cone with vertex $[\alpha]$ over a projective variety $\mathcal{J} \subset \mathbb{P}T_{[\alpha]}(\mathbb{P}(P_\alpha)) = \mathbb{P}(P_\alpha/\mathbb{C}\alpha) = \mathbb{P}T_{[\alpha]}(\mathscr{C}_0(X))$ which is the VMRT of $\mathscr{C}_0(X)$, i.e., $\mathcal{J} = \mathscr{C}_{[\alpha]}(\mathscr{C}_0(X))$.*

Proof Note that also the 3-dimensional hyperquadric Q^3 is excluded as it is biholomorphic to the Lagrangian Grassmannian of rank 2. To prove the lemma observe that $\mathscr{C}_0(X)$ admits at $[\alpha]$ a quadratic expansion (cf. Hwang-Mok [HM99, (4.2)]). More precisely, writing $T_0(X) = \mathbb{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$, where $\mathbb{C}\alpha \oplus \mathcal{H}_\alpha = P_\alpha$, there exists a quadratic vector-valued symmetric bilinear form τ on $T_{[\alpha]}(\mathscr{C}_0(X)) = P_\alpha/\mathbb{C}\alpha \cong \mathcal{H}_\alpha$, $\tau : S^2\mathcal{H}_\alpha \to \mathcal{N}_\alpha$, such that, identifying $\mathbb{P}T_0(X)$ as the Zariski closure of $\mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$, $\mathscr{C}_0(X)$ is the Zariski closure of the graph of $\varphi : \mathcal{H}_\alpha \to \mathcal{N}_\alpha$ given by $\varphi(\xi) = \tau(\xi, \xi)$. Thus, $\mathcal{H}_\alpha \cap \mathscr{C}_{[\alpha]}(X)$ is the union of complex lines $\mathbb{C}\xi$ satisfying $\tau(\xi, \xi) = 0$, and its Zariski closure $\mathscr{S}_{[\alpha]}$ is the union of projective lines Λ on $\mathbb{P}(P_\alpha/\mathbb{C}\alpha)$ passing through $[\alpha]$. Hence, $\mathscr{S}_{[\alpha]}$ is the cone with vertex $[\alpha]$ over $\mathcal{J} = \mathscr{C}_{[\alpha]}(\mathscr{C}_0(X)) \subset \mathbb{P}T_{[\alpha]}(\mathscr{C}_0(X)) = \mathbb{P}(P_\alpha/\mathbb{C}\alpha)$, as desired. □

Since $\mathscr{C}_{[\alpha]}(\mathscr{C}_0(X)) \subset \mathbb{P}T_{[\alpha]}(\mathscr{C}_0(X))$ is nonlinear and homogeneous, the second fundamental form on $\mathscr{C}_{[\alpha]}(\mathscr{C}_0(X))$ has trivial kernels. From the description of $\mathscr{S}_{[\alpha]}$ as a cone of projective lines over $\mathscr{C}_{[\alpha]}(\mathscr{C}_0(X))$ we have readily

Corollary 3.1. *Denoting by ζ the second fundamental form of $\mathscr{S}_{[\alpha]} \subset \mathbb{P}(P_\alpha)$ at a smooth point $[\beta] \in \mathscr{S}_{[\alpha]}$, we have $\mathrm{Ker}(\zeta_{[\beta]}(\,\cdot\,, T_{[\beta]}(\mathscr{S}_{[\alpha]}))) = T_{[\beta]}(\Lambda)$, where Λ is the projective line on $\mathscr{S}_{[\alpha]}$ containing $[\alpha]$ and $[\beta]$.*

For the parabolic subgroup P at $0 \in X$, let $Q = Q(\alpha) \subset P$ be the subgroup which fixes $[\alpha] \in \mathbb{P}T_0(X)$. Let $J \subset Q$ be a Levi factor, $J \subset K^{\mathbb{C}}$. Since J fixes $[\alpha]$ it acts on $V := T_{[\alpha]}(\mathbb{P}(P_\alpha))$. By examining the VMRTs, which are irreducible Hermitian symmetric spaces of the compact type except in the case of the Grassmannian, it follows that the action of J on V is irreducible excepting the Grassmannians $G(p,q)$ with $p, q \geq 2$, where $\mathscr{C}_{[\alpha]}(G(p,q)) = \varsigma(\mathbb{P}^{p-1} \times \mathbb{P}^{q-1})$ for the Segre embedding ς, in which case V splits into the direct sum of two irreducible components. For the latter cases writing $T_0(G(p,q)) = U_0 \otimes V_0$, $\alpha = u \otimes v$, $\mathscr{C}_{[\alpha]}(\mathscr{C}_0(G(p,q)))$ is the disjoint union of a copy of $\mathbb{P}^{p-2}$ and a copy of $\mathbb{P}^{q-2}$, $\mathscr{S}_{[\alpha]} = \mathbb{P}(U_0 \otimes \mathbb{C}v) \cup \mathbb{P}(\mathbb{C}u \otimes V_0)$, the two irreducible components intersecting at one point $[\alpha] = [u \otimes v]$, and obviously $\mathscr{S}_{[\alpha]}$ is linearly nondegenerate in $\mathbb{P}(P_\alpha)$, $P_\alpha = U_0 \otimes \mathbb{C}v \oplus \mathbb{C}u \otimes V_0$. For X being considered other than a Grassmannian of rank ≥ 2, $\mathscr{S}_{[\alpha]} \subset \mathbb{P}(P_\alpha)$ is necessarily linearly nondegenerate by the irreducibility of V under the action of J.

§4 Algebraicity of germs of sub-VMRT structures modeled on certain full cones of minimal rational curves

We consider now sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$ of the VMRT structure $\pi : \mathscr{C}(X) \to X$ on X where $\big(\mathscr{C}_x(S) \subset \mathscr{C}_x(X)\big)$ is projectively equivalent to $\big(\mathscr{S}_{[\alpha]} \subset \mathscr{C}_0(X)\big)$. Recall that X is irreducible and of rank ≥ 2, and $X \not\cong G^{III}(n,n), n \geq 2$. We have

Theorem 4.1. *The proper pair of projective subvarieties $(\mathscr{S}_{[\alpha]}, \mathscr{C}_0(X))$ of $\mathbb{P}T_0(X)$ is nondegenerate for substructures, excepting in the cases of hyperquadrics Q^n,*

$n \geq 3$, and of Grassmannians $G(2,q)$, $q \geq 2$, where nondegeneracy for substructures fails. Excluding those cases, any locally closed complex submanifold $S \subset X$ inheriting a sub-VMRT structure $\varpi : \mathscr{C}(S) \to S$ with fibers $\big(\mathscr{C}_x(S) \subset \mathscr{C}_x(X)\big)$ projectively equivalent to $\big(\mathscr{S}_{[\alpha]} \subset \mathscr{C}_0(X)\big)$ is linearly saturated, i.e., it is uniruled by open subsets of projective lines. Moreover, there exists a subvariety $Z \subset X$ such that $S \subset Z$ and $\dim(Z) = \dim(S)$.

In the sequel for brevity we will say that S is algebraic as a germ of submanifold (at any point $x \in S$) whenever there exists an irreducible subvariety $Z \subset X$ such that $S \subset Z$ and $\dim(Z) = \dim(S)$. For the proof of Theorem 4.1 we will study the symmetric bilinear form $\tau_{[\beta]} : S^2T_{[\beta]}(\mathscr{C}_0(X)) \to T_{[\beta]}(\mathbb{P}T_0(X))\big/(T_{[\beta]}(\mathscr{C}_0(X)) + T_{[\beta]}\mathbb{P}(P_\alpha))$ for the pair $\big(\mathscr{S}_{[\alpha]} \subset \mathscr{C}_0(X)\big)$ at a smooth point $[\beta] \in \mathscr{S}_{[\alpha]}$, $\tau_{[\beta]} := \nu_{[\beta]} \circ \sigma_{[\beta]}$, following Definition 2.4. We prove first of all the following result related to Corollary 3.1.

Proposition 4.1. *Denoting by $\sigma_{[\beta]} : S^2T_{[\beta]}(\mathscr{C}_0(X)) \to T_{[\beta]}(\mathbb{P}T_0(X))\big/(T_{[\beta]}(\mathscr{C}_0(X))$ the second fundamental form at $[\beta] \in \mathrm{Reg}(\mathscr{S}_{[\alpha]})$, and by $\Lambda \subset \mathscr{S}_{[\alpha]}$ the projective line containing $[\alpha]$ and $[\beta]$, we have* $\mathrm{Ker}(\sigma_{[\beta]}(\,\cdot\,, T_{[\beta]}(\mathscr{S}_{[\alpha]}))) = T_{[\beta]}(\Lambda)$.

We will give a differential-geometric proof of Proposition 4.1 basing on a characterization of projective submanifolds with parallel second fundamental form. On $\mathbb{P}^N$ denote by θ the Fubini-Study metric. We have

Theorem 4.2 (Nakagawa-Takagi [NT76]). *A linearly nondegenerate Kähler projective submanifold $(M, \theta|_M) \hookrightarrow (\mathbb{P}^N, \theta)$ has parallel second fundamental form $\sigma : S^2T_M \to T_{\mathbb{P}^N}|_M/T_M$ if and only if M is biholomorphic to a Hermitian symmetric space (S, g) of the compact type of rank ≤ 2 and $(M, \theta|_M)$ is either the image of S under a holomorphic isometric minimal embedding, or S is a projective space and M is its image under the Veronese embedding.*

Note here that there is the projective second fundamental form σ, which is a holomorphic bundle homomorphism, and there is also the second fundamental form σ' with respect to the Riemannian connection of the Kähler manifold $(\mathbb{P}T_0(X), \theta)$. If we regard σ' as taking values in the *holomorphic* normal bundle $T_{\mathbb{P}^N}|_M/T_M$ (in place of the orthogonal complement of T_M in $T_{\mathbb{P}^N}|_M$, then σ' agrees with σ. Here and in the sequel we use the same symbol σ for the two second fundamental forms, noting that parallelism is always defined in terms of the Riemannian connection. The relevance of Theorem 4.2 to VMRT geometry lies in the fact that the set of linearly nondegenerate projective manifolds with parallel second fundamental form is in one-to-one correspondence with the set of projective submanifolds given by VMRTs of irreducible Hermitian symmetric spaces X of the compact type, as given in Mok [Mo89, Appendix III.2].

Proof of Proposition 4.1 $\mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$ is the VMRT of an irreducible Hermitian symmetric space of the compact type. $\mathscr{C}_0(X)$ is itself a Hermitian symmetric space of rank 2 and $\mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$ is the minimal embedding, which is a holomorphic isometric embedding into $(\mathbb{P}T_0(X), \theta)$ for some choice of Fubini-Study metric θ. $(\mathscr{C}_0(X), \theta|_{\mathscr{C}_0(X)})$ is of nonnegative holomorphic bisectional curvature.

$\mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$ is a homogeneous projective submanifold uniruled by projective lines. For $[\beta] \in \mathrm{Reg}(\mathscr{S}_{[\alpha]})$, $\Lambda := \Lambda(\alpha, \beta) := \mathbb{P}(\mathbb{C}\alpha + \mathbb{C}\beta)$ is a minimal rational curve on $\mathscr{C}_0(X)$, hence a standard rational curve. Let $T(\mathscr{C}_0(X))|_\Lambda \cong$

$\mathcal{O}(2) \oplus (\mathcal{O}(1))^a \oplus \mathcal{O}^b$ be the Grothendieck splitting over Λ, $\dim(\mathscr{S}_{[\alpha]}) = 1 + a$. Denoting by $Q(\Lambda) = \mathcal{O}(2) \oplus (\mathcal{O}(1))^a \subset T(\mathscr{C}_0(X))|_\Lambda$ the positive part, we have $T_{[\beta]}(\mathscr{S}_{[\alpha]}) = Q_{[\beta]}(\Lambda)$, the fiber of $Q(\Lambda)$ at $[\beta]$. Moreover, we have

Lemma 4.1. *Equipping $\mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$ with $\theta|_{\mathscr{C}_0(X)}$, for the projective line $\Lambda \subset \mathscr{C}_0(X)$, $Q(\Lambda) \subset T(\mathscr{C}_0(X))|_\Lambda$ is a parallel vector subbundle.*

Proof $T^*(\mathscr{C}_0(X))|_\Lambda \cong \mathcal{O}(-2) \oplus (\mathcal{O}(-1))^a \oplus \mathcal{O}^b$, and $\mathcal{O}^b \subset T^*(\mathscr{C}_0(X))|_\Lambda$ is a trivial bundle equipped by restriction with a Hermitian metric of nonpositive curvature in the sense of Griffiths, hence must be parallel due to monotonicity of curvatures (cf. Mok [Mo89, (3.2)]). Thus, $Q(\Lambda) \subset T(\mathscr{C}_0(X))|_\Lambda$, being the annihilator of $\mathcal{O}^b \subset T^*(\mathscr{C}_0(X))|_\Lambda$ must also be parallel, as desired. □

Proof of Proposition 4.1 cont. Recalling that $\mathscr{S}_{[\alpha]} = \mathscr{C}_0(X) \cap \mathbb{P}(P_\alpha) \subset \mathscr{C}_0(X)$ is a linear section smooth at $[\beta]$, we have $\mathrm{Ker}(\sigma_{[\beta]}(\,\cdot\,, T_{[\beta]}(\mathscr{S}_{[\alpha]}))) \cap T_{[\beta]}(\mathscr{S}_{[\alpha]}) = \mathrm{Ker}(\zeta_{[\beta]}(\cdot, T_{[\beta]}(\mathscr{S}_{[\alpha]}))) = T_{[\beta]}(\Lambda)$, by Corollary 3.1. Suppose $\mathrm{Ker}(\sigma_{[\beta]}(\cdot, T_{[\beta]}(\mathscr{S}_{[\alpha]}))) \neq T_{[\beta]}(\Lambda)$. Since $T_{[\beta]}(\mathscr{S}_{[\alpha]}) = Q_{[\beta]}(\Lambda)$, there exists $\eta \in T_{[\beta]}(\mathscr{C}_0(X)) - Q_{[\beta]}(\Lambda)$ such that $\sigma_{[\beta]}(\eta, Q_{[\beta]}(\Lambda)) = 0$. For $[\gamma] \in \Lambda$ define $U_{[\gamma]} := \mathrm{Ker}(\sigma_{[\gamma]}(\,\cdot\,, Q_{[\gamma]}(\Lambda))) \subset T_{[\gamma]}(\mathscr{C}_0(X))$. Since $Q(\Lambda) \subset T(\mathscr{C}_0(X))|_\Lambda$ is a parallel subbundle by Lemma 3.2, and σ is parallel with respect to $(\mathbb{P}T_0(X), \theta)$ by Nakagawa-Takagi [NT76] (Theorem 4.2 here), it follows that $U \subset T(\mathscr{C}_0(X))|_\Lambda$ is a parallel subbundle. By Corollary 3.1 we have $U \cap Q(\Lambda) = T(\Lambda)$, hence $U = T(\Lambda) \oplus V$ for some parallel subbundle $V \subset T(\mathscr{C}_0(X))|_\Lambda$ transversal to $Q(\Lambda)$, so that $V \cong \mathcal{O}^c$ for some integer c, $1 \leq c \leq b$. Hence $\sigma_{[\beta]}$ induces a *parallel* bundle homomorphism $\varphi : V \otimes (T(\mathscr{C}_0(X))|_\Lambda / Q(\Lambda)) \to (T(\mathbb{P}T_0(X))/T(\mathscr{C}_0(X)))|_\Lambda := N$. Now from Grothendieck splitting $T(\mathscr{C}_0(X))|_\Lambda / Q(\Lambda) \cong \mathcal{O}^b$, while the normal bundle N, being a quotient bundle of $T(\mathbb{P}T_0(X))|_\Lambda \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^{n-2}$, is necessarily a direct sum of positive line bundles over Λ. By comparing degrees the parallel bundle homomorphism $\varphi : \mathcal{O}^c \otimes \mathcal{O}^b \to N$ must necessarily be 0. In particular, $\sigma_{[\beta]}(\eta, \xi) = 0$ for any $\xi \in T_{[\beta]}(\mathscr{C}_0(X))$, contradicting the standard fact that $\mathrm{Ker}(\,\sigma_{[\beta]}(\,\cdot\,, T_{[\beta]}(\mathscr{C}_0(X)))) = 0$ on the homogeneous nonlinear submanifold $\mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$, proving Proposition 4.1. □

Remark By the process of reconstructing S by adjunction of minimal rational curves following Mok [Mo08a], and Hong-Mok [HoM10] [HoM13]) it can be established that $Z = \gamma(\mathcal{V})$ for some $\gamma \in \mathrm{Aut}(X)$. The arguments will be given in Mok-Yang [MY17] in the proof of uniqueness results of holomorphic isometric embeddings of the complex unit ball of maximal admissible dimension into irreducible bounded symmetric domains of rank ≥ 2..

To apply Theorem 2.1 and Theorem 2.2 we need to check Condition (T) as defined in Mok-Zhang [MZ17, Definition 5.4] and recalled here in Definition 2.5 for the pair $(\mathscr{S}_{[\alpha]}, \mathscr{C}_0(X))$. We have

Proposition 4.2 *Let X be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2 not biholomorphic to a Lagrangian Grassmannian. Then $(\mathscr{S}_{[\alpha]}, \mathscr{C}_0(X))$ satisfies* Condition (T). *More precisely, for any nonzero vector $\beta \in \widetilde{\mathscr{S}_{[\alpha]}}$, $[\beta] \neq [\alpha]$, $T_\beta(\widetilde{\mathscr{S}_{[\alpha]}}) = T_\beta(\widetilde{\mathscr{C}_0}(X)) \cap T_0(\mathcal{V}) = P_\beta \cap P_\alpha$.*

Proof Equivalently, under the hypothesis of the proposition we are going to show that $T_{[\beta]}(\mathscr{S}_{[\alpha]}) = T_{[\beta]}(\mathscr{C}_0(X)) \cap T_{[\beta]}(\mathbb{P}T_0(\mathcal{V}))$, which is the same as $(P_\beta \cap P_\alpha)/\mathbb{C}\beta$. Clearly, $[\beta]$ is a smooth point of $\mathscr{S}_{[\alpha]}$ and $T_{[\beta]}(\mathscr{S}_{[\alpha]}) = T_{[\beta]}(\mathscr{C}_0(X) \cap \mathbb{P}T_0(\mathcal{V})) \subset$

$T_{[\beta]}(\mathscr{C}_0(X)) \cap T_{[\beta]}\mathbb{P}T_0(\mathcal{V}) = (P_\beta \cap P_\alpha)/\mathbb{C}\beta$. Recall that $\Lambda \subset \mathbb{P}T_0(X)$ denotes the projective line containing both $[\alpha]$ and $[\beta]$. Let $\gamma \in T_0(X)$ be a nonzero vector such that $(\gamma + \mathbb{C}\beta)/\mathbb{C}\beta \in (P_\beta \cap P_\alpha)/\mathbb{C}\beta$. We have to prove that $(\gamma + \mathbb{C}\beta)/\mathbb{C}\beta$ is tangent to $\mathscr{S}_{[\alpha]}$. When $\gamma \in \mathbb{C}\alpha + \mathbb{C}\beta$, $(\gamma + \mathbb{C}\beta)/\mathbb{C}\beta$ is tangent to Λ at $[\beta]$ and *a fortiori* tangent to $\mathscr{S}_{[\alpha]}$, and it remains to consider the case where α, β and γ are linearly independent. Let Π be the projective 2-plane on $\mathbb{P}T_0(X)$ spanned by $[\alpha], [\beta]$ and $[\gamma]$. We have $\Lambda \subset \Pi$ and Π is tangent to $\mathscr{C}_0(X)$ both at $[\alpha]$ and at $[\beta]$, and our task is to prove that Π is tangent to $\mathscr{S}_{[\alpha]}$ at $[\beta]$. For this purpose it is sufficient to prove that Π is tangent to $\mathscr{S}_{[\alpha]}$ along $\Lambda - \{[\alpha]\} \subset \mathrm{Reg}(\mathscr{S}_{[\alpha]})$.

Recall from Theorem 4.2 (by Nakagawa-Takagi [NT76]) that the second fundamental form $\sigma := \sigma_{\mathscr{C}_0(X)|\mathbb{P}T_0(X)}$ with respect to the Kähler manifold $(\mathbb{P}T_0(X), \theta)$ is holomorphic and parallel. In particular, $\sigma|_\Lambda : S^2T(\mathscr{C}_0(X))|_\Lambda \to N_{\mathscr{C}_0(X)|\mathbb{P}T_0(X)}|_\Lambda = N$ is holomorphic and parallel. Moreover, σ is surjective since $\mathscr{C}_0(X) \subset \mathbb{P}T_0(X)$ is linearly nondegenerate and it is the closure of the graph of a vector-valued holomorphic quadratic form in terms of Harish-Chandra coordinates. Write $E := \mathrm{Ker}(\sigma|_\Lambda) \subset S^2T(\mathscr{C}_0(X))|_\Lambda$. By the parallelism of σ, $E \subset S^2T(\mathscr{C}_0(X))|_\Lambda$ is a parallel subbundle, hence there is a holomorphic direct sum decomposition $S^2T(\mathscr{C}_0(X))|_\Lambda = E \oplus F$, $F = E^\perp$, and $N \cong F$. From the Grothendieck decomposition we have $T(\mathscr{C}_0(X))|_\Lambda \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^a \oplus \mathcal{O}^b$, and it follows that N is a direct sum of non-negative holomorphic line bundles of degree ≤ 2. Observe that for any $[\delta] \in \Lambda$ distinct from $[\alpha]$, and for any $\xi \in T_{[\delta]}(\Lambda)$ and $\eta \in T_{[\delta]}(\mathscr{S}_{[\alpha]})$, we have $\sigma_{[\delta]}(\xi, \eta) = 0$ (which is the easier part of Proposition 4.1). Note that $N_{\Lambda|\Pi} \cong \mathcal{O}(1)$. Let now $\tau \in \Gamma(\Lambda, N_{\Lambda|\Pi})$ be a nonzero section vanishing at some point $[\delta] \in \Lambda$ other than $[\alpha]$ and $[\beta]$. Then τ induces a holomorphic section $\tau^\flat \in \Gamma(\Lambda, N)$ which vanishes at the three distinct points $[\alpha], [\beta]$ and $[\delta]$. Hence, $\tau^\flat \equiv 0$ since N is a direct sum of holomorphic line bundles of degree ≤ 2. The proof of Proposition 4.2 is complete. □

Remark In the proof actually N is a quotient bundle of $N_{\Lambda|\mathbb{P}T_0(X)}$, which is a direct sum of $\mathcal{O}(1)$, hence ample, and it follows that N is a direct sum of positive holomorphic line bundles of degrees 1 or 2.

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1 In the notation of Definition 1.4, for a smooth point $[\beta]$ on $\mathscr{S}_{[\alpha]}$, we have $\tau_{[\beta]} = \nu_{[\beta]} \circ \sigma_{[\beta]}$, where $\nu_{[\beta]} : T_0(\mathbb{P}T_0(X))/T_{[\beta]}(\mathscr{C}_0(X)) \to T_0(\mathbb{P}T_0(X))/(T_{[\beta]}(\mathscr{C}_0(X)) + T_{[\beta]}(\mathscr{S}_{[\alpha]}))$ is the canonical projection, $T_{[\beta]}(\mathbb{P}(P_\alpha)) \cong P_\alpha/\mathbb{C}\beta$. It remains to check whether $\mathrm{Ker}(\tau_{[\beta]}(\,\cdot\,, T_{[\beta]}(\mathbb{P}(P_\alpha))) \subset T_{[\beta]}(\mathscr{S}_{[\alpha]})$ holds. Using notation as in §3 and noting that in the Hermitian symmetric case we have $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$, for a root $\rho \in \Phi^+$ we define $\Psi_\rho := \{\lambda \in \Phi^+ : \rho - \lambda \in \Phi\}$. We may take $\alpha = E_\mu$, where $\mu \in \Phi^+$ is the highest root. Recall that $P \subset G$ is the parabolic subgroup at $0 \in X$ so that $X = G/P$, and $Q(\alpha) \subset P$ is the subgroup which fixes $\mathbb{C}\alpha$. Since X is not biholomorphic to a Lagrangian Grassmannian, there exists $\nu \in \Psi_\mu$ such that the projective line $\Lambda := \mathbb{P}(\mathbb{C}E_\mu + \mathbb{C}E_\nu) \subset \mathbb{P}T_0(X)$ lies on $\mathscr{C}_0(X)$, which is the case if and only if $\nu \in \Psi_\mu$ is a long root. If X is not biholomorphic to a Grassmannian, then the space of lines $\Lambda \subset \mathscr{C}_0(X)$ passing through $[\alpha]$ is given by $\mathscr{C}_{[\alpha]}(\mathscr{C}_0(X))$, which is itself an irreducible Hermitian symmetric space of the compact type on which $Q(\alpha)$ acts transitively. We may therefore take $\beta = E_\nu$, so that $\mu, \nu \in \Phi^+$ are long roots, $\nu \in \Psi_\mu$, $\mu \in \Psi_\nu$. In the case where X is the

Grassmannian $G(p,q)$, $p,q \geq 2$, taking $\alpha = E_\mu$, $\mathscr{S}_{[\alpha]}$ is the union of two projective subspaces of dimension $p-1$ resp. $q-1$ intersecting at $[\alpha]$, and we have to consider $[\beta] \neq [\alpha]$ belonging to either of them. In both cases, modulo the action of $Q(\alpha)$ clearly we may take $\beta = E_\nu$ for some root $\nu \in \Phi^+$.

We take now $\alpha = E_\mu$ and $\beta = E_\nu$, $[\beta] \in \mathscr{S}_{[\alpha]}$ being a smooth point. We have $T_{[\beta]}(\mathscr{C}_0(X)) = P_\beta/\mathbb{C}\beta$. Identifying $P_\beta/\mathbb{C}\beta$ as the orthogonal complement of β in P_β, $T_{[\beta]}(\mathscr{C}_0(X)) = \text{Span}\{E_\rho : \rho \in \Psi_\nu\}$, while $T_{[\beta]}(\mathscr{S}_{[\alpha]}) = (P_\alpha \cap P_\beta)/\mathbb{C}\beta = \text{Span}\{E_\rho : \rho \in (\Psi_\nu \cap \Psi_\mu) \cup \{\mu\}\}$. As in the proof of Mok-Zhang [MZ17, Lemma 3.4], to show that $(\mathscr{C}_{[\beta]}(\mathscr{S}_{[\alpha]}), \mathscr{C}_{[\beta]}(\mathscr{C}_0(X)))$ is nondegenerate for substructures, it suffices to check that for any root vector E_π, $\pi \in \Psi_\nu - ((\Psi_\nu \cap \Psi_\mu) \cup \{\mu\}) = \Psi_\nu - (\Psi_\mu \cup \{\mu\})$, there must exist $\omega \in (\Psi_\nu \cap \Psi_\mu) \cup \{\mu\}$ such that $\tau_{[\beta]}(E_\pi, E_\omega) \neq 0$. By Proposition 4.1, there exists $\omega \in (\Psi_\nu \cap \Psi_\mu) \cup \{\mu\}$ such that $\sigma_{[\beta]}(E_\pi, E_\omega) \neq 0$, i.e., $\lambda := \pi + \omega - \nu \in \Phi^+$. We have $\tau_{[\beta]}(E_\pi, E_\omega) \neq 0$ if and only if $E_\lambda \mod \mathbb{C}\beta \notin T_{[\beta]}(\mathbb{P}(P_\alpha)) \cong P_\alpha/\mathbb{C}\beta$, i.e., $\lambda \notin \Psi_\mu \cup \{\mu\}$. Now $\lambda(H_\mu) = \pi(H_\mu) + \omega(H_\mu) - \nu(H_\mu)$. From Grothendieck splitting over minimal rational curves ℓ on X we see that $\mu(H_\mu) = 2$, $\rho(H_\mu) = 1$ for $\rho \in \Psi_\mu$ and $\rho(H_\mu) = 0$ for $\rho \notin \Psi_\mu \cup \{\mu\}$. (Thus, $\pi(H_\mu) = 0$.) We know that $\omega \in (\Psi_\nu \cap \Psi_\mu) \cup \{\mu\}$. In case $\omega = \mu$ we have $\lambda(H_\mu) = 1$ so that $\tau_{[\beta]}(E_\pi, E_\omega) = 0$. On the other hand, when $\omega, \nu \in \Psi_\mu$ we have $\lambda(H_\mu) = 0$, in which case $\lambda \notin \Psi_\mu \cup \{\mu\}$, so that $\tau_{[\beta]}(E_\pi, E_\omega) \neq 0$. We conclude therefore that $E_\pi \notin \text{Ker}(\tau_{[\beta]}(\,\cdot\,, T_{[\beta]}(\mathscr{S}_{[\alpha]})))$ whenever $\dim(\sigma_{[\beta]}(E_\pi, T_{[\beta]}(\mathscr{S}_{[\alpha]}))) \geq 2$.

Denote by $s(X)$ the minimum of $\dim(\sigma_{[\beta]}(E_\pi, T_{[\beta]}(\mathscr{C}_0(X))))$ as π ranges over Ψ_ν, which is independent of the choice of $[\beta] \in \text{Reg}(\mathscr{S}_{[\alpha]})$ by homogeneity. Note that $\sigma_{[\beta]}(E_{\pi_1}, E_{\pi_2}) = 0$ whenever $\pi_1, \pi_2 \in \Psi_\nu - (\Psi_\mu \cup \{\mu\})$ as $(\pi_1 + \pi_2 - \nu)(H_\mu) = -1 < 0$, so that $\sigma_{[\beta]}(E_\pi, T_{[\beta]}(\mathscr{S}_{[\alpha]})) = \sigma_{[\beta]}(E_\pi, T_{[\beta]}(\mathscr{C}_0(X)) = s(X)$. Writing $X(E_6)$ for the 16-dimensional Hermitian symmetric space of the compact type of type E_6, and $X(E_7)$ for the 27-dimensional one of type E_7, a straightforward checking gives $s(G(p,q)) = \min(p-1, q-1)$ $(p,q \geq 2)$, $s(G^{II}(n,n)) = n-3$ $(n \geq 4$, noting that $G^{II}(4,4) \cong Q^6)$, $s(Q^n) = 1$ $(n \geq 3)$, $s(X(E_6)) = 3$, $s(X(E_7)) = 5$, showing that $\tau_{[\beta]}(E_\pi, T_{[\beta]}(\mathscr{S}_{[\alpha]})) \neq 0$ for any $\pi \in \Psi_\nu$ with the exception of $X = Q^n$, $n \geq 3$, and $X = G(2,q)$, $q \geq 2$. Thus, excluding the latter cases $(\mathscr{S}_{[\alpha]}, \mathscr{C}_0(X))$ is nondegenerate for substructures.

When $X = Q^n, n \geq 3$, we have $\nu_{[\beta]} \equiv 0$ since $P_\beta + T_{[\beta]}(\mathscr{S}_{[\alpha]}) = T_{[\beta]}(\mathbb{P}T_0(X))$. When X is $G(2,q)$, $q \geq 2$, where $s(G(2,q)) = 1$, $\sigma_{[\beta]}(E_\pi, E_\mu) \neq 0$ also indeed occurs in the preceding arguments, so that $\text{Ker}(\tau_{[\beta]}(\,\cdot\,, T_{[\beta]}(\mathscr{S}_{[\alpha]}))) \supsetneq T_{[\beta]}(\mathscr{S}_{[\alpha]})$. In both cases $(\mathscr{S}_{[\alpha]}, \mathscr{C}_0(X))$ fails to be nondegenerate for substructures. □

Remarks

(a) By computing holomorphic bisectional curvatures on $(\mathscr{C}_0(X), \theta|_{\mathscr{C}_0(X)})$, conceptually $s(X)$ is the minimum of the number of flat direct summands in the Grothendieck splitting of $T(\mathscr{C}_0(X))$ over a projective line $\Lambda \subset \mathscr{C}_0(X)$.

(b) For the proof of Theorem 4.1 instead of the parallelism of σ one can also use the combinatorial argument as above. We gave the proof exploiting splitting types as the latter is more geometric and of independent interest.

(c) Robles-The [RT12] and Robles [Ro13] have completely determined the set of Schubert cycles on irreducible Hermitian symmetric spaces of the compact type which are Schur rigid. In [RT12] the authors used cohomological methods due to

Kostant [Ko63]. In the cases where the desired vanishing of cohomological groups fails, it was established in [Ro13] that the underlying Schubert cycle is flexible. More precisely, it was established in [Ro13, Theorem 4.1] that there are irreducible (projective algebraic) integral varieties of the associated Schubert system which are not translates of the given Schubert cycle. An integral variety of the associated Schubert system is precisely a subvariety where the isomorphism classes of tangent spaces at smooth points are equivalent to those at smooth points of the corresponding Schubert cycle under the action of automorphisms of the ambient Hermitian symmetric space. In particular, [Ro13] applies to the cases $\mathcal{V} = \mathcal{V}_{\rm exc}$ of the full cones of minimal rational tangents of hyperquadrics and rank-2 Grassmannians, which are the *exceptional* cases excluded in the statement of Theorem 4.1, to show that there exist projective algebraic subvarieties of X which are integral subvarieties S of the Schubert differential systems associated to the Schubert cycle $\mathcal{V}_{\rm exc}$, which implies that, writing $\mathscr{C}_x(S) := \mathscr{C}_x(X) \cap \mathbb{P}T_x(S)$, $\big(\mathscr{C}_x(S) \subset \mathscr{C}_x(X)\big)$ is projectively equivalent to $\big(\mathscr{S}_{[\alpha]} \subset \mathscr{C}_0(X)\big)$ at a general smooth point $x \in S$. It is however not clear how the construction there leads to *transcendental* sub-VMRT structures modeled on to $\big(\mathscr{S}_{[\alpha]} \subset \mathscr{C}_0(X)\big)$ (equivalently modeled on $\big(\mathcal{V} \subset X\big)$). In an indirect way, by the method of reconstruction of $\mathcal{V}$ by parallel transport along minimal rational curves (Mok-Yang [MY17]), linear saturation of the smooth locus $\mathrm{Reg}(S)$ of S would imply that S is $\gamma(\mathcal{V})$ for some $\gamma \in \mathrm{Aut}(X)$, hence the relevant examples of [Ro13] for $\mathcal{V} = \mathcal{V}_{\rm exc}$ must fail to be linearly saturated.

In §6 we will construct examples of *transcendental* integral varieties of the Schubert differential system associated to $\mathcal{V}_{\rm exc}$ in the cases where the ambient space X is the hyperquadric Q^n of dimension $n \geq 4$.

§5 Intrinsically flat sub-VMRT structures modeled on certain full cones of minimal rational curves

In the statement of Theorem 4.1, $(\mathscr{S}_\alpha, \mathscr{C}_0(X))$ fails to be nondegenerate for substructures in the case where X is either a hyperquadric $Q^n, n \geq 3$ or a rank-2 Grassmannian $G(2,q)$, $q \geq 2$. We will show that for certain sub-VMRT structures $\varpi : \mathscr{C}(S) \to S$ modeled on $\big(\mathscr{S}_\alpha \subset \mathscr{C}_0(X)\big)$, S remains linearly saturated and algebraic as a germ of submanifold at any point $x \in S$. Let $(Z, \mathcal{H})$ and $(X, \mathcal{K})$ be uniruled projective manifolds equipped with minimal rational components. Denote by $\mathscr{C}(Z) \subset \mathbb{P}T(Z)$ resp. $\mathscr{C}(X) \subset \mathbb{P}T(X)$ the VMRT structures of $(Z, \mathcal{H})$ resp. $(X, \mathcal{K})$. Let $A \subset Z$ resp. $B \subset X$ be the bad locus of $(Z, \mathcal{H})$ resp. $(X, \mathcal{K})$, $W \subset X - B$ be an open subset, $S \subset W$ be a complex submanifold such that $S = f(U)$ for some holomorphic embedding $f : U \to X - B$ from a connected open subset $U \subset Z - A$ which respects VMRTs at a general point of U. Defining $\mathscr{C}(S) = \mathscr{C}(X) \cap \mathbb{P}T(S)$, assume that the canonical projection $\varpi : \mathscr{C}(S) \to S$ is a sub-VMRT structure in the sense of Definition 2.1. By the proofs of Hong-Mok [HoM10, Proposition 2.1], Hong-Park [HoP11, Proposition 2.3] and Mok-Zhang [MZ17, Proposition 5.2] here, it remains the case that S is linearly saturated with respect to $(X, \mathcal{K})$ under the weaker assumption (†) $\mathrm{Ker}(\sigma_{[\chi]}(\,\cdot\,, T_{[\chi]}(\mathscr{C}(S)) \subset T_{[\chi]}(\mathscr{C}(S)))$ at a general point $[\chi]$ of any irreducible component of $\mathscr{C}_x(S)$ for a general point $x \in S$, where the second fundamental form $\sigma_{[\chi]}$ is used in place of $\tau_{[\chi]} = \nu_{[\chi]} \circ \sigma_{[\chi]}$. Following [HoP11] and relating to the notion of substructures in this article we will say that $\varpi : \mathscr{C}(S) \to S$ is *weakly nondegenerate for substructures* whenever (†) holds.

Note that the condition (†) depends only on S and $(Z, \mathcal{H})$ disappears from the definition of weak nondegeneracy for substructures. The assumption that S arises from a map allows one to make use of Hessians $\nabla^2 f$ in place of $\tau = \nu \circ \sigma$, i.e., without taking quotients modulo $T(S)$, and this explains why a weaker nondegeneracy condition is sufficient for proving linear saturation. In place of a uniruled projective manifold $(X, \mathcal{K})$ the notions and proofs go through even in the case when Z has singularities, provided that we consider a 'minimal rational component' $\mathcal{H}$ on Z such that a general member of $\mathcal{H}$ is a standard minimal rational curve on Z lying on $\mathrm{Reg}(Z)$. This is the case when $(X, \mathcal{K})$ stands for an irreducible Hermitian symmetric space of rank ≥ 2 other than a Lagrangian Grassmannian, $Z = \mathcal{V}$ stands for a full cone of minimal rational curves on X, and $\mathcal{H}$ resp. $\mathcal{K}$ stands for the space of projective lines lying on $\mathcal{V}$ resp X. We will refer to $\pi_Z : \mathscr{C}(Z) \to Z$ thus obtained as a generalized VMRT-structure. Such a structure is said to be *flat* whenever there exist local holomorphic coordinates with respect to which the generalized VMRTs $\mathscr{C}(Z)$ form a constant family. If $\varpi : \mathscr{C}(S) \to S$ arises as the image of a generalized VMRT structure under a VMRT-respecting map, we call it an *intrinsically flat* sub-VMRT structure. We have

Lemma 5.1. *For the cone $\mathcal{V} := \mathcal{V}(x) \subset X$ of minimal rational curves at some $x \in X$, the generalized VMRT-structure $\pi_{\mathcal{V}} : \mathscr{C}(\mathcal{V}) \to \mathcal{V}$ is intrinsically flat.*

Proof Let $\mathbb{C}^n \subset X$ be a Harish-Chandra coordinate chart. Let $\alpha \in \widetilde{\mathscr{C}_0}(X)$. Parametrize a neighborhood of $[\alpha] \in \mathscr{C}_0(X)$ by an open holomorphic embedding $\varphi : U \to \mathscr{C}_0(X)$ from a neighborhood U of 0 in $T_{[\alpha]}(\mathscr{C}_0(X))$ by $\varphi(\xi) = \alpha + \xi + O(\|\xi\|^2)$, where $T_{[\alpha]}(\mathscr{C}_0(X)) \cong P_\alpha/\mathbb{C}\alpha$ is identified with a complementary linear subspace $H_\alpha \subset P_\alpha$ of $\mathbb{C}\alpha$. Consider $F : \mathbb{C} \times U \to X$ defined by $F(s, \xi) = s\varphi(\xi) - \alpha = s(\alpha + \xi + O(\|\xi\|^2)) - \alpha$. Thus $F(0, 0) = -\alpha$ and $F(1, 0) = 0$, so that F maps a neighborhood of $(1, 0)$ in $\mathbb{C} \times H_\alpha$ biholomorphically onto a neighborhood of 0 in $\mathcal{V}(-\alpha)$, $dF(1, 0)\big(a\frac{\partial}{\partial s} + \xi\big) = a\alpha + \xi$. Noting that for any $\eta \in \mathbb{C}^n$ the Euclidean translation $T_\eta(z) := z + \eta$ on $\mathbb{C}^n$ extends to $\Phi_\eta \in \mathrm{Aut}(X)$, for $t \geq 1$ we define $F_t : \mathbb{C} \times tU \to X$ by

$$F_t(s, \xi) := t(1 + \tfrac{s-1}{t})\big(\varphi(\tfrac{\xi}{t})\big) - t\alpha = (s - 1 + t)\Big(\alpha + \tfrac{\xi}{t} + O\big(\|\tfrac{\xi}{t}\|^2\big)\Big) - t\alpha\,,$$

so that $F_1 \equiv F$, $F_t(1 - t, 0) = -t\alpha$, $F_t(1, 0) = 0$ and $dF_t(1, 0)\big(a\frac{\partial}{\partial s} + \xi\big) = a\alpha + \xi$. F_t maps $\Delta(1; t) \times tU$ biholomorphically onto a neighborhood of 0 in $\mathcal{V}(-t\alpha)$,

$$F_t(s, \xi) = (s - 1)\alpha + \tfrac{s-1+t}{t}\,\xi + \tfrac{s-1+t}{t^2}\,O(\|\xi\|^2)\,.$$

As $t \to \infty$ the holomorphic mappings F_t converge uniformly on compact subsets of $\mathbb{C} \times H_\alpha$ to $G : \mathbb{C} \times H_\alpha \to \mathbb{C}^n$ given by $G(s, \xi) = (s-1)\alpha + \xi$. On the other hand, on X the cones of minimal rational curves $\mathcal{V}(-t\alpha) \subset X$ converge as subvarieties to $\mathcal{V}(\infty_{\alpha 0})$, where $\infty_{\alpha 0}$ is the point at infinity of the projective line $\ell(\alpha, 0) \subset X$ containing the points α and 0 on $\mathbb{C}^n \subset X$. Thus $\mathcal{V}(\infty_{\alpha 0}) \cap \mathbb{C}^n \subset \mathbb{C}^n$ is the linear subspace $P_\alpha = \mathbb{C}\alpha \oplus H_\alpha$. Moreover for any $x \in P_\alpha$ we have $\mathscr{C}_x(\mathcal{V}(\infty_{\alpha 0})) = \mathbb{P}(P_\alpha) \cap \mathscr{C}_x(X)$, which forms a constant family in the Harish-Chandra coordinates, proving Lemma 5.1. □

By the proof of Theorem 2.2 (which is Mok-Zhang [MZ17, Main Theorem 2]), to find $Z \supset S$ projective, $\dim(Z) = \dim(S)$, it suffices that S is linearly saturated. By Proposition 4.1 we have

Theorem 5.1 *In the exceptional cases in Theorem 4.1 of hyperquadrics $X = Q^n$, $n \geq 4$, and of Grassmannians $X = G(2,q)$, $q \geq 2$ where $(\mathscr{S}_{[\alpha]}, \mathscr{C}_0(X))$ fails to be nondegenerate for substructures, $(\mathscr{S}_{[\alpha]}, \mathscr{C}_0(X))$ remains weakly nondegenerate for mappings. In particular, if $S \subset X$ inherits an intrinsically flat sub-VMRT structure modeled on $\big(\mathscr{S}_{[\alpha]} \subset \mathscr{C}_0(X)\big)$, then S is linearly saturated and there exists a subvariety $Z \subset X$ such that $S \subset Z$ and $\dim(Z) = \dim(S)$.*

Remark As for Theorem 4.1, by the process of reconstructing S by adjunction of minimal rational curves it can be established that $Z = \gamma(\mathcal{V})$ for some $\gamma \in \mathrm{Aut}(X)$.

§6 Examples of non-standard sub-VMRT structures modeled on full cones of rational curves

In this section we give an elementary construction of examples of *transcendental* sub-VMRT structures modeled on the full cone $\mathcal{V}$ of minimal rational curves in the case where the ambient manifold X is a hyperquadric Q^n for dimension ≥ 4. Fix $n \geq 4$. Let $\mathbb{C}^n \subset Q^n$ be a Harish-Chandra coordinate chart on which the holomorphic conformal structure on $\mathbb{C}^n$ with Euclidean coordinates $(z_1, \cdots, z_n)$ is given by the class of holomorphic nondegenerate quadratic forms $\lambda(z)((dz^1 \otimes dz^1) + \cdots + (dz^n \otimes dz^n))$, where λ is any nowhere zero holomorphic function on $\mathbb{C}^n$. We are going to write down explicit examples of transcendental hypersurfaces $S \subset \mathbb{C}^n \subset Q^n$ such that S inherits a sub-VMRT structure modeled on $(\mathscr{S}_{[\alpha]}(Q^n) \subset \mathscr{C}_0(Q^n))$. Here $[\alpha] \in \mathscr{C}_0(Q^n) \cong Q^{n-2} \subset \mathbb{P}T_0(Q^n) \cong \mathbb{P}^{n-1}$, and $\mathscr{S}_{[\alpha]}(Q^n) = \mathscr{C}_0(Q^n) \cap \mathbb{P}(P_\alpha)$ is the singular hyperplane section with the isolated singularity at $[\alpha]$. We have

Proposition 6.1. *Let n, m be positive integers such that $n \geq 4$ and $2 \leq m \leq n-2$. Let $A(z_1, \cdots, z_m) = a_1 z_1 + \cdots + a_m z_m$ be a linear function in $(z_1, \cdots, z_m)$ such that $(a_1, \cdots, a_m) \neq 0$ and $a_1^2 + \cdots + a_m^2 = 0$. Let $(b_{m+1}, \cdots, b_n) \neq 0$ be such that $b_{m+1}^2 + \cdots + b_n^2 = 0$. Let $f : \mathbb{C}^n \to \mathbb{C}$ be defined by $f(z_1, \cdots, z_n) = e^{A(z_1, \cdots, z_m)} + b_{m+1} z_{m+1} + \cdots + b_n z_n$. Then, for $c \in \mathbb{C}$ the level set $S = \{f(z) = c\}$ is a transcendental smooth hypersurface on $\mathbb{C}^n \subset Q^n$ such that, defining $\mathscr{C}(S) := \mathbb{P}T(S) \cap \mathscr{C}(Q^n)$ and writing $\varpi : \mathscr{C}(S) \to S$ for the canonical projection, the latter is a sub-VMRT structure modeled on $\big(\mathscr{S}_{[\alpha]}(Q^n) \subset \mathscr{C}_0(Q^n)\big)$ which is neither linearly saturated nor intrinsically flat.*

Proof We may assume that $b_n = -1$, hence the level set $S = \{f(z) = c\}$ is given by $z_n = h(z_1, \cdots, z_{n-1}) = e^{A(z_1, \cdots, z_m)} + b_{m+1} z_{m+1} + \cdots + b_{n-1} z_{n-1} - c$ so that $S \subset \mathbb{C}^n$ is a transcendental smooth hypersurface. Write $q = (dz^1 \otimes dz^1) + \cdots + (dz^n \otimes dz^n)$. It induces an isomorphism $L : T^*(\mathbb{C}^n) \xrightarrow{\cong} T(\mathbb{C}^n)$ such that for any $z \in \mathbb{C}^n$, $\omega \in T_z^*(\mathbb{C}^n), \xi \in T_z(\mathbb{C}^n)$, we have $q(L(\omega), \xi) = \omega(\xi)$. Write $p = \frac{\partial}{\partial z_1} \otimes \frac{\partial}{\partial z_1} + \cdots + \frac{\partial}{\partial z_n} \otimes \frac{\partial}{\partial z_n}$. We have $df = e^{A(z_1, \cdots, z_m)}(a_1 dz^1 + \cdots + a_m dz^m) + (b_{m+1} dz^{m+1} + \cdots + b_n dz^n)$, so that $p(df(z), df(z)) = e^{A(z_1, \cdots, z_m)}(a_1^2 + \cdots + a_m^2) + (b_{m+1}^2 + \cdots + b_n^2) = 0$. Now fix c and let $x \in S = \{f(z) = c\}$. We have $T_x(S) = \mathrm{Ker}(df(x)) \subset T_x(\mathbb{C}^n)$. Thus, for $\eta := L(df(x))$ and $\xi \in T_x(\mathbb{C}^n)$ we have $q(\eta, \xi) = q(L(df(x)), \xi) = df(x)(\xi)$, and $\xi \in T_x(S)$ if and only if $q(\eta, \xi) = 0$. On the other hand, $q(\eta, \eta) = p(df(x), df(x)) = 0$, so that $\eta \in T_x(S)$, and $q|_{T_x(S)}$ is degenerate, with kernel spanned by $\eta = e^{A(z_1, \cdots, z_m)}(a_1 \frac{\partial}{\partial z_1} + \cdots + a_m \frac{\partial}{\partial z_m}) + (b_{m+1} \frac{\partial}{\partial z_{m+1}} + \cdots + b_n \frac{\partial}{\partial z_n})$. Hence, $\mathscr{C}_x(S) = \mathbb{P}T_x(S) \cap \mathscr{C}_x(X)$ is the singular hyperplane section of $\mathscr{C}_x(X)$ with the isolated singularity at $[\eta]$, and $\varpi : \mathscr{C}(S) \to S$ inherits a sub-VMRT structure modeled on $\big(\mathscr{S}_{[\alpha]}(Q^n) \subset \mathscr{C}_0(Q^n)\big)$. Consider the projective line

ℓ passing through $x = (z_1, \cdots, z_{n-1}, h(z_1, \cdots, z_{n-1}))$ such that $T_x(\ell) = \mathbb{C}\eta$. From the explicit description of S one checks readily that in general $\ell \not\subset S$, so that $S \subset Q^n$ is not linearly saturated. Alternatively, linear saturation of S would imply by the proof of Theorem 2.2 (i.e., proof of Main Theorem 2 of [MZ17]) that $\overline{S} \subset Q^n$ is projective, contradicting with the transcendence of S. Finally, again from the transcendence of $S \subset \mathbb{C}^n$ it follows by Theorem 5.1 that $\varpi : \mathscr{C}(S) \to S$ is not intrinsically flat, completing the proof of Proposition 5.1. $\square$

Remark Replacing $f(z_1, \cdots, z_n) = e^{A(z_1,\cdots,z_m)} + b_{m+1}z_{m+1} + \cdots + b_n z_n$ by $g(z_1, \cdots, z_n) = A(z_1, \cdots, z_m)^q + b_{m+1}z_{m+1} + \cdots + b_n z_n$ for any integer $q \geq 2$ by the same arguments we obtain examples of non-standard sub-VMRT structures modeled on $\big(\mathscr{S}_{[\alpha]}(Q^n) \subset \mathscr{C}_0(Q^n)\big)$ on nonsingular level sets $S = \{g(z) = c\} \subset \mathbb{C}^n$ which are affine algebraic.

§7 Concluding Remarks

Cohomological methods play an important role in earlier known approaches on various problems of rigidity concerning Schubert cycles on Hermitian symmetric spaces of the compact type. By contrast, in the approach stemming from the geometric theory of uniruled projective manifolds our perspective is to treat rigidity concerning special subvarieties as a problem in differential geometry revolving around varieties of minimal rational tangents. We note however that there are links between the cohomological and the differential-geometric methods worthy of further exploration.

As an example, in the work of Hong [Ho05], in which Schur rigidity is established for nonsingular Schubert cycles of irreducible Hermitian symmetric spaces $X = G/P$ of the compact type, the author made use of the result of Goncharov [Go87] from the theory of generalized conformal structures on integral varieties of F-structures, where the cohomology groups concerned are defined on orbits under the action (of the semisimple part) of P on the Grassmann of k-planes of $T_0(X)$. From our perspective much information is already stored in the VMRT, which is the highest weight orbit of the semisimple part of P under the isotropy action on $\mathbb{P}T_0(X)$. In a certain sense, in place of requiring vanishing results on cohomological groups defined on the orbits, we replace them by nondegeneracy conditions defined from projective geometry, introduced by Hong-Mok [HoM10] for mappings and by Mok-Zhang [MZ17] for substructures. One may say that our approach is microlocal in nature, imposing conditions at a general point of the pair consisting of a VMRT, which in the case of irreducible Hermitian symmetric cases of the compact type is an orbit in the projectivized tangent space under the isotropy representation, and a sub-VMRT (which is a linear section of the VMRT) rather than global conditions on certain projective varieties which are orbits in Grassmannians under actions derived from the isotropy representation. In this way we relax the requirement from global vanishing results to microlocal vanishing results (on kernels arising from certain quadratic forms).

A general form of the problem for characterizing special subvarieties of uniruled projective manifolds was formulated as the Recognition Problem in Mok [Mo16b, Problem 4.5.1]. In a nutshell the theory of sub-VMRT structure leads to an approach for resolving the Recognition Problem for special subvarieties Σ of Hermitian symmetric spaces of the compact type and more generally those of rational homogeneous spaces whenever there exists a projective line lying on the smooth locus $\mathrm{Reg}(\Sigma)$ of Σ. In cohomological approaches the Recognition Problem is a problem of integrability of certain geometric substructures. Our approach breaks the problem down into two steps, the first step being the verification of a microlocal condition implying partial integrability, more precisely the property that the support of the sub-VMRT structure is linearly saturated. The second step is that of reconstruction by a finite process of adjunction of minimal rational curves, i.e., projective lines. The current article is an an illustration of the first step of our scheme beyond smooth Schubert cycles by an examination of a very special class of singular Schubert varieties (which are nonetheless of special interest to Kähler geometry). The issue of reconstruction by an improvement of the method of adjunction of minimal rational curves will be taken up elsewhere. One advantage of our scheme is that, where applicable, it may lead to results ascertaining linear saturation when additional intrinsic conditions are imposed on the underlying complex submanifolds S of sub-VMRT structures, which is illustrated by Theorem 5.1, where linear saturation and algebraicity are proven in the cases excluded by Theorem 4.1 when S is further assumed to be intrinsically flat.

In another direction, we can study the Recognition Problem modeled on a family of sub-VMRTs which come from linear sections of Schubert cycles by the introduction of a quantitative measure of nondegeneracy for substructures, called p-nondegeneracy, $p \geq 1$, where the case $p = 1$ corresponds to the usual notion of nondegeneracy for substructures as given in Definition 2.4. In the case at hand, for sub-VMRT structures modeled on $\big(\mathscr{S}_{[\alpha]} \subset \mathscr{C}_0(X)\big)$ in the notation of Theorem 4.1, the optimal value of p is $s(X) - 1$, where $s(X)$ was listed in the second last paragraph of the proof of Theorem 4.1. Taking $[\beta] \in \mathscr{S}_{[\alpha]}$ distinct from $[\alpha]$, $s(X)$ is the minimal value of $\dim\big(\sigma_\beta(\mathbb{C}\eta \otimes T_\beta(\widetilde{\mathscr{S}_{[\alpha]}}))\big)$ as η varies over nonzero vectors in $T_\beta(\widetilde{\mathscr{C}_0}(X)) - T_\beta(\widetilde{\mathscr{S}_{[\alpha]}})$, while the minimal value of $\dim\big(\tau_\beta(\mathbb{C}\eta \otimes T_\beta(\widetilde{\mathscr{S}_{[\alpha]}}))\big)$ is equal to $s(X) - 1$. Let k be a positive integer so that $\dim(\mathscr{S}_{[\alpha]}) - k > 0$. Consider the Grassmannian $\mathrm{Gr}(n-k, T_0(X))$ of vector subspaces $V \subset T_0(X)$ of codimension k in $T_0(X), n := \dim(X)$. There is a dense Zariski open subset $\Phi_k \subset \mathrm{Gr}(n-k, T_0(X))$ such that for every linear subspace $V \subset T_0(X)$ of codimension k belonging to Φ_k, we have (a) $\mathbb{P}(V)$ intersects $\mathscr{S}_{[\alpha]}$ in pure codimension k, (b) the intersection is transversal at a general point of each irreducible component of $\mathscr{S}_{[\alpha]} \cap \mathbb{P}(V)$ and (c) $\mathscr{S}_{[\alpha]} \cap \mathbb{P}(V)$ is linearly nondegenerate in $\mathbb{P}(V)$. Note that from (b) it follows that the pair $(\mathscr{S}_{[\alpha]} \cap \mathbb{P}(V), \mathscr{C}_0(X) \cap \mathbb{P}(V))$ satisfies Condition (T) in the sense of Definition 2.5. A member of Φ_k will be called a Φ_k-general vector subspace of $T_0(X)$ of codimension k. In analogy to Mok-Zhang [MZ17, Theorem 9.1] we have the following result Theorem 7.1 on linear saturation and algebraicity for sub-VMRT structures on X modeled on the family of pairs $(\mathscr{S}_{[\alpha]} \cap \mathbb{P}(V), \mathscr{C}_0(X) \cap \mathbb{P}(V))$ for Φ_k-general vector subspaces $V \subset T_0(X)$ of codimension k.

For the formulation of the result let X be an irreducible Hermitian symmetric space of the compact type of rank ≥ 2 not biholomorphic to a Lagrangian Grassmannian. Let $0 \in X$ be a reference point and $[\alpha] \in \mathscr{C}_0(X)$ be an arbitrary point. Let k be a positive integer satisfying $k \leq s(X) - 2$. Identify X as a projective submanifold by means of the first canonical embedding $\iota : X \hookrightarrow \mathbb{P}^N$. Let $\Pi \subset \mathbb{P}^N$ be a projective linear subspace of codimension k such that $Z = X \cap \Pi$ is a smooth linear section of codimension k in X. Since $Z \subset X \subset \mathbb{P}^N$ is a smooth linear section of codimension k, each irreducible component of the set of projective lines on Z passing through a general point $x \in Z$ must be of dimension at least equal to $\dim(\mathscr{C}_0(X)) - k > s(X) - k > 2$, so that in particular Z is uniruled by projective lines. Let $E \subset Z$ be the bad locus (which is here the same as the enhanced bad locus) of Z as a projective submanifold uniruled by lines. From the deformation theory of rational curves the VMRT $\mathscr{C}_x(Z)$ of Z as a projective submanifold uniruled by lines is of dimension exactly equal to $\dim(\mathscr{C}_0(X)) - k$ for $x \in Z - E$. We have

Theorem 7.1 *Suppose $W \subset Z - E$ is a nonempty open subset in the complex topology, and $S \subset W$ is a complex submanifold of dimension $\dim(\mathscr{C}_0(X)) + 1 - k$ such that, writing $\mathscr{C}(Z) := \mathscr{C}(X) \cap \mathbb{P}T(\Pi) \cap \mathbb{P}T(Z - E)$ and $\mathscr{C}(S) := \mathscr{C}(Z) \cap \mathbb{P}T(S)$, the canonical projection $\varpi : \mathscr{C}(S) \to S$ defines a sub-VMRT structure on S such that for a general point $x \in S$, $\big(\mathscr{C}_x(S) \subset \mathscr{C}_x(Z)\big)$ is projectively equivalent to $\big(\mathscr{S}_{[\alpha]} \cap \mathbb{P}(V) \subset \mathscr{C}_0(X) \cap \mathbb{P}(V)\big)$ for some Φ_k-general vector subspace $V \subset T_0(X)$ of codimension k. Then, S is linearly saturated. Moreover, there exists a subvariety $Z \subset X$ such that $S \subset Z$ and $\dim(Z) = \dim(S)$.*

Proof By the proof of Theorem 4.1, $(\mathscr{S}_{[\alpha]}, \mathscr{C}_0(X))$ is $(s(X)-1)$-nondegenerate for substructures according to the definition of p-nondegeneracy for substructures in the preceding paragraphs. The key to the proof of Theorem 7.1 is the observation that whenever $k \leq s(X) - 2$, the pair $(\mathscr{S}_{[\alpha]} \cap \mathbb{P}(V), \mathscr{C}_0(X) \cap \mathbb{P}(V))$ remains r-nondegenerate for $r := s(X) - 1 - k \geq 1$ for a Φ_k-general vector subspace $V \subset T_0(X)$ of codimension k, which follows from Mok-Zhang [MZ17, Proposition 9.1]. By Theorem 2.1, S is linearly saturated. The last statement follows from Theorem 2.2, completing the proof of Theorem 7.1. □

In order for Theorem 7.1 to be applicable we need to have $s(X) \geq 3$, which rules out the cases of Grassmannians of rank ≤ 3, $G^{II}(5,5)$ and hyperquadrics. It is for instance applicable to the Grassmannian $X = G(n,n)$, $n \geq 4$, to prove results of linear saturation and algebraicity for germs of complex submanifolds on smooth codimension-k linear sections admitting sub-VMRT structures modeled on Φ_k-general linear sections of $\mathscr{S}_{[\alpha]}$ for $0 < k \leq n - 3$, and to $X(E_7)$ for $0 < k \leq 3$.

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