A Filon-Clenshaw-Curtis-Smolyak rule for multi-dimensional oscillatory integrals with application to a UQ problem for the Helmholtz equation

Zhizhang Wu[†], Ivan G. Graham[*], Dingjiong Ma[†], Zhiwen Zhang[†]

[†] Department of Mathematics, The University of Hong Kong,
Pokfulam Road, Hong Kong SAR, China.
wuzz@hku.hk, martin35@hku.hk, zhangzw@hku.hk

[*] Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK.
I.G.Graham@bath.ac.uk

August 23, 2022

Abstract

In this paper, we combine the Smolyak technique for multi-dimensional interpolation with the Filon-Clenshaw-Curtis (FCC) rule for one-dimensional oscillatory integration, to obtain a new Filon-Clenshaw-Curtis-Smolyak (FCCS) rule for oscillatory integrals with linear phase over the $d-$dimensional cube $[-1,1]^d$. By combining stability and convergence estimates for the FCC rule with error estimates for the Smolyak interpolation operator, we obtain an error estimate for the FCCS rule, consisting of the product of a Smolyak-type error estimate multiplied by a term that decreases with $\mathcal{O}(k^{-\tilde{d}})$, where k is the wavenumber and $\tilde{d}$ is the number of oscillatory dimensions. If all dimensions are oscillatory, a higher negative power of k appears in the estimate. As an application, we consider the forward problem of uncertainty quantification (UQ) for a one-space-dimensional Helmholtz problem with wavenumber k and a random heterogeneous refractive index, depending in an affine way on d i.i.d. uniform random variables. After applying a classical hybrid numerical-asymptotic approximation, expectations of functionals of the solution of this problem can be formulated as a sum of oscillatory integrals over $[-1,1]^d$, which we compute using the FCCS rule. We give numerical results for the FCCS rule which illustrate its theoretical properties and show that the accuracy of the UQ algorithm improves when both k and the order of the FCCS rule increase. We also give results for both the quadrature and UQ problems when the underlying FCCS rule uses a dimension-adaptive Smolyak interpolation. These show increasing accuracy for the UQ problem as both the adaptive tolerance decreases and k increases, requiring very modest increase in work as the stochastic dimension increases, for a case when the affine expansion in random variables decays quickly.

AMS subject classification: 35C20, 42B20, 65D30, 65D32, 65D40.

Keywords: oscillatory integrals, high dimension, Helmholtz equation, uncertainty quantification, hybrid numerical asymptotic methods.

1 Introduction

In this paper, we formulate and analyse a novel numerical method for computing the multi-dimensional oscillatory integral

$$\mathcal{I}^{k,d,\mathbf{a}} f := \int_{[-1,1]^d} f(\boldsymbol{y}) \exp(\mathrm{i}k\mathbf{a}\cdot\boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \tag{1.1}$$

where $k > 0$ is a parameter, which may be large, and $\mathbf{a} = (a_1, ..., a_d) \in \mathbb{R}^d$ is a fixed vector. As an application of this, we solve an uncertainty quantification problem for the Helmholtz equation (modelling frequency-domain linear waves), via a hybrid numerical asymptotic method, yielding increasing accuracy as the frequency increases.

Background The computation of oscillatory integrals is a classical problem in applied mathematics (e.g., [33]), which has enjoyed considerable recent interest. By combining numerical and asymptotic techniques, research has focussed on providing methods which work well for moderate k, but remain accurate (or even improve in accuracy) as the parameter k (proportional to frequency) gets large, While there has been strong interest in this topic the in recent past (partly driven by applications in high-frequency scattering e.g., [4, 15, 14]), most methods proposed in this context are appropriate only for relatively low-dimensional oscillatory integration problems.

On the other hand there is a considerable literature on multi-dimensional integration for the non-oscillatory version of (1.1), where k is small - here we mention just [12, 3, 7, 25, 35] as exemplars of the huge literature on this topic. However, since the accuracy of these rules depends on the derivatives of the integrand, their direct application to the whole integrand in (1.1) will incur an error which in general will blow up strongly with increasing k.

Quite a large portion of research on oscillatory integration in the low-dimensioanl case (mostly $d = 1$) is concerned with Filon-type methods. In [18, 19, 20, 22, 34] the analysis concentrates on accelerating the convergence as $k \to \infty$. The basic 1D method central to the current paper is [9], which proves stability and superalgebraic convergence for the Filon Clenshaw-Curtis rule, with a negative power of k as $k \to \infty$, and gives an error estimate which is explicit in the regularity required of f. The range of application of this approach was considerably extended in recent work [23]. Extensions of Filon methods to *hp* approximation and the handling of nonlinear phase functions (again in 1D) are given in [24], [8] and [23].

To extend the approach of [9] to the higher dimensional case, the factor $f(\boldsymbol{y})$ in (1.1) should be approximated by some linear combination of simple basis functions, with coefficients which can be computed easily from f, and then this approximation should be integrated analytically against the oscillatory factor $\exp(\mathrm{i}k\mathbf{a} \cdot \boldsymbol{y})$. There are relatively few papers on the higher dimensional oscillatory case. Exceptions are [21, 17] which include discussion of generalization of a Filon-type method to problem (1.1), making use of function values and derivatives at vertices of the boundary and proving increasing accuracy as $k \to \infty$, but without explicit error estimates showing how the error depends on the number of function evaluations, the regularity of f or the dimension d.

Overview of Algorithm Our method for (1.1) essentially extends the 1D 'Filon-Clenshaw-Curtis' approach to the multidimensional case by applying Smolyak-type interpolation to the non-oscillatory part of the integrand in (1.1). Since we shall allow $\mathbf{a}$ to have entries of either sign and possibly small, we introduce the following notation in order to identify the oscillatory and non-oscillatory dimensions in (1.1).

Notation 1.1. *For $a \in \mathbb{R}$ and $k > 0$ we define*

$$\widetilde{a} = \begin{cases} a & \text{if} \quad k|a| \geq 1 \\ 0 & \text{otherwise} \end{cases}$$

and we set $\widehat{a} = a - \widetilde{a}$. (We note that $\widetilde{a}$ and $\widehat{a}$ depend on k as well as a.)

Applying Notation 1.1 to each component of the vector $\mathbf{a}$ in (1.1), we obtain the decomposition $\mathbf{a} = \widetilde{\mathbf{a}} + \widehat{\mathbf{a}}$, and we see that $\widehat{f}(\boldsymbol{y}) := f(\boldsymbol{y}) \exp(\mathrm{i}k\widehat{\mathbf{a}} \cdot \boldsymbol{y})$ is the non-oscillatory part of the integrand in (1.1), and we then rewrite (1.1) as

$$\mathcal{I}^{k,d,\mathbf{a}} f := \int_{[-1,1]^d} \widehat{f}(\boldsymbol{y}) \exp(\mathrm{i}k\widetilde{\mathbf{a}} \cdot \boldsymbol{y}) \mathrm{d}\boldsymbol{y}, \tag{1.2}$$

Our quadrature rule for (1.2) (and hence (1.1)) is then defined by replacing the factor $\widehat{f}$ by its classical Smolyak polynomial interpolant $\mathcal{Q}^{r,d}\widehat{f}$ of maximum level r (formally defined in (3.3)). This sparse grid interpolant employs separable polynomial interpolation at points on sparse grids generated by a nested sequence of 1-d grids. Here we use, at level ℓ, the points:

$$\{0\} \text{ for } \ell = 1, \quad \text{and} \quad \left\{ t_{j,\ell} := \cos\left(\frac{j\pi}{n_\ell}\right) \right\}_{j=0}^{n_\ell} \quad \text{for } \ell \geq 2, \quad \text{where } n_\ell = 2^{\ell-1}, \tag{1.3}$$

i.e. the mid-point rule is used at level $\ell = 1$ and $2^{\ell-1}+1$ Clenshaw-Curtis points are used at level $\ell \geq 2$. Using this, we approximate (1.2) by

$$\mathcal{I}^{k,d,\mathbf{a},r} f := \int_{[-1,1]^d} (\mathcal{Q}^{r,d}\widehat{f})(\boldsymbol{y}) \exp(\mathrm{i}k\widetilde{\mathbf{a}} \cdot \boldsymbol{y}) \mathrm{d}\boldsymbol{y}. \tag{1.4}$$

This is a d dimensional version of the 1D Filon-Clenshaw-Curtis (FCC) rule from [9].

Since, in each dimension the interpolant on the Clenshaw-Curtis grid can be written as a linear combination of Chebyshev polynomials of the first kind of degree n (here denoted T_n), the integral (1.4) can be computed exactly, provided the $k-$dependent 'weights'

$$W_n(ka_j) := \int_{-1}^{1} T_n(y) \exp(\mathrm{i}ka_j y) dy, \quad n = 0, \ldots, n_\ell, \quad \ell \geq 2, \quad j = 1, \ldots d \tag{1.5}$$

are known. A stable algorithm for computing these weights (for any k, a_j and n), and its analysis, are given in [9]. (The weight for the case $\ell = 1$ is trivial to compute.) Since the cost of computing the weights for an $M+1$ point rule in 1-d and with a fixed choice of k and a_j has complexity $\mathcal{O}(M \log M)$ (see [9, Remark 5.4]), and since the weights for each dimension can be computed independently, the cost of computing the weights (1.5) (on a serial computer) grows with $\mathcal{O}(d\, 2^{r-1} \log(2^{r-1}))$ as dimension d or the maximum level r increases. Weight computation in each dimension could be done in parallel.

Main results of the paper In order to prove an error estimate, we assume that

$$f \in \mathcal{W}^{p,d} := \left\{ f : [-1,1]^d \to \mathbb{R} : \frac{\partial^{|\mathbf{s}|} f}{\partial \boldsymbol{y}^{\mathbf{s}}} \in C([-1,1]^d), \text{ for all } |\mathbf{s}|_\infty \leq p \right\} \tag{1.6}$$

for some positive integer p, where $\|\cdot\|_\infty$ denotes the uniform norm over $[-1,1]$, $\mathbf{s} = (s_1, ..., s_d) \in \mathbb{N}_0^d$ are the multi-indices of order $|\mathbf{s}| = s_1 + \cdots + s_d$,

$$\frac{\partial^{|\mathbf{s}|} f}{\partial \boldsymbol{y}^{\mathbf{s}}} = \frac{\partial^{|\mathbf{s}|} f}{\partial y_1^{s_1} \cdots \partial y_d^{s_d}},$$

and $|\mathbf{s}|_\infty = \max\limits_{1 \leq i \leq d} s_i$. We introduce the norm on $\mathcal{W}^{p,d}$:

$$\|f\|_{\mathcal{W}^{p,d}} := \max_{\mathbf{s} \in \mathbb{N}_0^d : |\mathbf{s}|_\infty \leq p} \left\| \frac{\partial^{|\mathbf{s}|} f}{\partial \boldsymbol{y}^{\mathbf{s}}} \right\|_{\infty, [-1,1]^d}. \tag{1.7}$$

We note that $\mathcal{W}^{p,1}$ is just the usual space $C^p[-1,1]$ with norm given by

$$\|f\|_{\mathcal{W}^{p,1}} = \max\{\|f^{(j)}\|_{\infty,[-1,1]}, \quad j = 0, \ldots, p\}.$$

By combining the properties of the Smolyak algorithm and the FCC rule, together with the regularity assumption (1.6), we prove in §4 the following error estimates.

Theorem 1.2. *For each $p \geq 1$ and $d \geq 1$ there is a constant $C_{d,p} > 0$ such that, for all $\mathbf{a} \in \mathbb{R}^d$, $k > 0$, and r sufficiently large, we have*

$$|\mathcal{I}^{k,d,\mathbf{a}} f - \mathcal{I}^{k,d,\mathbf{a},r} f| \\ \leq C_{d,p} \left(\prod_{\substack{j \in \{1,\ldots,d\} \\ k|a_j| \geq 1}} k|a_j| \right)^{-1} \left(\log^{d-1} N(r,d) \right)^p \left(\frac{1}{N(r,d)} \right)^{p-1} \|f\|_{\mathcal{W}^{p,d}}, \tag{1.8}$$

where $N(r,d)$ is the number of point evaluations of f used in the quadrature rule (1.4).

While this result is explicit in k and $\mathbf{a}$, we can obtain the following better estimate, for sufficiently large k, although this is implicit in its dependence on $\mathbf{a}$ and requires more regularity on f.

Theorem 1.3. *Suppose $\mathbf{a} \in \mathbb{R}^d$ with $a_j \neq 0$ for each j. Then, for each $p \geq 1$ and $d \geq 1$, there is a constant $C_{d,p,\mathbf{a}} > 0$ such that, for $r \geq d+1$, we have*

$$|\mathcal{I}^{k,d,\mathbf{a}} f - \mathcal{I}^{k,d,\mathbf{a},r} f| \;\leq\; C_{d,p,\mathbf{a}}\, k^{-(d+1)} \left(\log^{d-1} N(r,d) \right)^p \left(\frac{1}{N(r,d)} \right)^{p-1} \|f\|_{\mathcal{W}^{p+3,d}}, \tag{1.9}$$

The third and fourth terms on the right-hand sides of (1.8) and (1.9) are standard in the analysis of Smolyak-type approximation methods, having a power of $\log N$ which grows with dimension. While this is good for moderate (but not very high) dimension d, our estimates (1.8), (1.9) also contain additional factors containing potentially high negative powers of k, giving a substantial advantage over traditional tensor product rules for these integrals: If a $d-$dimensional tensor product of the standard one-dimensional (non-Filon) Clenshaw-Curtis rule were used to approximate (1.1) then (for fixed p and d), the standard error estimate would give $\mathcal{O}(k^p N^{-p/d})$ as $k, N \to \infty$ (where N is the total number of points used - see Example 3.1 for a precise statement). This is vastly inferior to (1.8), (1.9) when either d or k is large.

The estimate (1.8) can be seen as a generalisation of the concept of *universality* discussed in [27, eqn. (7)] to the case of oscillatory integrals. In (1.8), one additional bounded mixed derivative is needed (compared with the non-oscillatory case) in order to obtain the advantageous negative power of k in the second term on the right-hand side of (1.8). This implies that, in general, the absolute error of the FCCS rule decays with at least the same order in k as the integral itself as $k \to \infty$. The estimate (1.9) essentially shows that, under certain conditions, the relative error of the FCCS rule actually decays with repect to k.

In proving (1.8), (1.9) we make no special assumption concerning the decay of the derivatives of f with respect to increasing dimension. Numerical experiments for the non-oscillatory case (e.g., [13, 26]) have shown that if a suitable decay rate is present, then dimension-adaptive algorithms will give better results as d increases. Theory underpinning the idea of the dimension-adaptive algorithms is given in [5, 30]. In §6.1 we give computations using both a standard and a dimension-adaptive method (the latter adapted from the algorithm in [26]) for the oscillatory quadrature problem. These demonstrate the theoretical properties of the former in general and the efficiency of the latter in cases where the importance of the dimensions is decaying.

As an application of our FCCS rule, we consider the UQ problem for the one-space-dimensional Helmholtz boundary-value problem:

$$Lu(x) := u''(x) + k^2 n^2(x) u(x) = F(x), \quad 0 < x < 1 \tag{1.10}$$

$$B_L u(x) := u(0) = u_L, \tag{1.11}$$

$$B_R u(x) := u'(1) - ikn_\infty u(1) = 0, \tag{1.12}$$

where k is the wave number, u_L and $n_\infty > 0$ are constants, F is a smooth funtion and n^2 is the (generally variable) refractive index, here also assumed sufficiently smooth. The boundary conditions model a sound-soft scattering boundary at $x = 0$ and a simple absorbing boundary condition at $x = 1$. In our UQ problem,

n is assumed to be a random field of the form

$$n(x, \boldsymbol{y}) = n_0(x) + \sum_{j=1}^{d} n_j(x) y_j, \tag{1.13}$$

where $\boldsymbol{y} \in [-1, 1]^d$ are uniform i.i.d. random variables and the quantity of interest (QoI) is a linear functional (with respect to the x variable) of the solution u. Although 1-d in space, this problem still has some considerable difficulty for large k, because the solution $u(x, \boldsymbol{y})$ suffers oscillations with respect not only to the spatial variable x but also the random variable $\boldsymbol{y}$ as $k \to \infty$, and the latter phenomenon poses considerable difficulty for UQ algorithms at high wavenumber. The structure of these oscillations is explained in more detail in §5.3.

However (because we are in 1d in space), for each fixed choice of $\boldsymbol{y}$, the resulting deterministic Helmholtz problem can be solved (with accuracy up to any negative order in k) using a hybrid numerical asymptotic method, originally proposed by Aziz, Kellogg and Stephens [1], in which the work involved is independent of k. This hybrid method also provides us with an asymptotic ansatz for the solution of the random Helmholtz problem, and (it turns out that) the expected value of the QoI then can be expressed as a sum of oscillatory integrals of the form (1.1) with respect to the random variables, which we can compute with $k-$independent accuracy using our FCCS rule.

In §6.2 we give numerical results for the Helmholtz UQ problem, using the hybrid numerical-asymptotic approximation for the Helmholtz problem and comparing the standard and dimension-adaptive methods for the oscillatory integrals. In the dimension-adaptive case, for an example where the expansion (1.13) decays exponentially, we observe increasing accuracy of the results as both the adaptive tolerance decreases and the wave number k increases, with very modest growth in complexity with increasing dimension.

This is significant, since it is known that for Helmholtz problems in any space dimension, derivatives with respect to the random parameters $\boldsymbol{y}$ for the Helmholtz problem blow up with wavenumber k, thus enforcing strong constraints on UQ methods in general at high wavenumber. For example, in [11], conditions ensuring convergence of first order randomized Quasi-Monte Carlo methods for a Helmholtz problem in any space dimension were studied. There, to ensure a dimension-independent optimal error estimate, one requires $\sum_{j=1}^{\infty} \|n_j\|_{W^{1,\infty}}^{2/3} = \mathcal{O}(k^{-2/3})$, (i.e., the amplitude of the allowed randomness must decrease as k increases). Strong constraints on the allowed amplitude of the randomness also appear in the multi-modes method in [10]. We impose no such constraint in our computations. Quasi-Monte Carlo methods for random Helmholtz problems in 2d were studied in [28, Chapter 4], where it was observed (for moderate wavenumbers) that the number of quadrature points needed to ensure a bounded error as k increased apparently grew exponentially in k.

The blow-up as k increases of the derivatives with respect to $\boldsymbol{y}$ of the Helmholtz solution is directly related to the width of the region in $\mathbb{C}^d$ in which (the complex extension of) u is holomorphic. This region of holomorphy is analysed in detail in the recent paper [31] for trapping and non-trapping geometries in any dimension and with general random perturbations of a deterministic base problem. In particular it is shown there that the estimates in [11] (for a non-trapping case and the expansion (1.13)) are sharp.

The remainder of the paper is organized as follows. In §2, we recall the Filon-Clenshaw-Curtis (FCC) rule for the 1d oscillatory integral and give some basic theory for it . In §3, we combine the 1d quadrature with the Smolyak algorithm to obtain our new FCCS rule for the multi-dimensional oscillatory integrals. In §4, we give the error analysis for the FCCS rule, proving Theorems 1.2 and 1.3. The application to the UQ problem for the Helmholtz equation is given in §5. In §6, we present numerical examples to demonstrate the performance of the FCCS rule and its application to the UQ problem.

Recalling Notation 1.1, the following proposition is then easy to prove using the multidimensional Leibniz rule.

Proposition 1.4. *For each $p \geq 0$, there exists a constant K_p, depending on p but independent of k such that*

$$\|\widehat{f}\|_{\mathcal{W}^{p,d}} \leq K_p \|f\|_{\mathcal{W}^{p,d}}$$

2 The FCC quadrature rule for 1D problems

In this section, we briefly review from [9] the Filon-Clenshaw-Curtis (FCC) rule for approximating the one-dimensional integral

$$I^{\omega} g := \int_{-1}^{1} g(y) \exp(\mathrm{i}\omega y) dy, \tag{2.1}$$

for $\omega \in \mathbb{R}$, where $g \in C^p[-1,1]$ for some integer $p \geq 1$.

In the oscillatory case ($|\omega| \geq 1$), the FCC quadrature rule is obtained by replacing the factor g in (2.1) by its degree N polynomial interpolant at the Clenshaw-Curtis points $\cos(j\pi/N)$, $j = 0, ..., N$ for $N \geq 1$ (extrema of the Chebyshev polynomial of the first kind $T_N(y) := \cos\big(N \arccos(y)\big)$). The interpolant is expressed in terms of the basis $\{T_n : n = 0, \ldots, N\}$, and the products of these basis functions with the oscillatory function $\exp(i\omega y)$ are integrated exactly to obtain the quadrature weights.

Starting with the nested set of quadrature points (1.3), the FCC approximation to (2.1) is then

$$I^{\omega,\ell}_{\mathrm{FCC}} g := \int_{-1}^{1} (Q^{\ell} g)(y) \exp(\mathrm{i}\omega y) dy, \tag{2.2}$$

where $(Q^1 g) = g(0)$ and, for $\ell \geq 2$, $Q^{\ell} g$ is the polynomial of degree n_ℓ satisfying

$$(Q^{\ell} g)(t_{j,\ell}) = g(t_{j,\ell}), \quad j = 0, \ldots, n_\ell. \tag{2.3}$$

It is a classical result that, for $\ell \geq 2$, $Q^{\ell} g$ can be written as

$$(Q^{\ell} g)(y) = {\sum_{n=0}^{n_\ell}}'' a_{n,\ell}(g) T_n(y), \tag{2.4}$$

where the notation $\sum''$ means that the first and the last terms in the sum are to be halved, and the coefficients $a_{n,\ell}(g)$ are given by the discrete cosine transform:

$$a_{n,\ell}(g) \;=\; \frac{2}{n_\ell} {\sum_{j=0}^{n_\ell}}'' \cos\left(\frac{jn\pi}{n_\ell}\right) g(t_{j,\ell}), \quad n = 0, \ldots, n_\ell. \tag{2.5}$$

Substituting (2.4) into (2.2) for $\ell \geq 2$, we obtain the quadrature rule

$$I^{\omega,\ell}_{\mathrm{FCC}}\, g := \begin{cases} W_0(\omega) g(0), & \ell = 1, \\ {\displaystyle\sum_{n=0}^{n_\ell}}'' W_n(\omega) a_{n,\ell}(g), & \ell \geq 2, \end{cases} \tag{2.6}$$

where the weights

$$W_n(\omega) = \int_{-1}^{1} T_n(y) \exp(\mathrm{i}\omega y)\, dy, \quad n = 0, \ldots, n_\ell \tag{2.7}$$

have to be computed. An algorithm for computing $W_n(\omega)$ for $\ell \geq 2$ is given and shown to be stable for all n_ℓ and ω in [9].

In the case where $\omega = 0$, the weights are known analytically:

$$W_n(0) = \int_{-1}^{1} T_n(y)dy = \begin{cases} 0, & n \text{ is odd}, \\ \dfrac{2}{1-n^2}, & n \text{ is even}, \end{cases} \tag{2.8}$$

and these provide us with the classical standard Clenshaw-Curtis (CC) rule [6]:

$$\int_{-1}^{1} g(y)\mathrm{d}y \;\approx I_{\mathrm{CC}}^{\ell} g \;:=\; \int_{-1}^{1} Q^{\ell} g \;=\; \begin{cases} 2g(0), & \ell = 1, \\ {\sum_{n=0}^{n_\ell}}'' W_n(0) a_{n,\ell}(g), & \ell \geq 2. \end{cases} \tag{2.9}$$

When $|\omega| < 1$ the integral (2.1) is considered to be non-oscillatory, and can be approximated directly with the CC rule. Hence our 1D quadrature method is:

Definition 2.1. *For $\omega \in \mathbb{R}$ and $\ell \geq 1$, we define the approximation $I^{\omega,\ell} g$ to the integral (2.1) by*

$$I^{\omega,\ell} g = \begin{cases} I_{FCC}^{\omega,\ell} g, & \text{when} \quad |\omega| \geq 1, \\ \\ I_{CC}^{\ell}\left(g(\cdot)\exp(\mathrm{i}\omega\cdot)\right), & \text{when} \quad |\omega| < 1. \end{cases} \tag{2.10}$$

Remark 2.2. *Given the values $\{g(t_{j,\ell}) : j = 1, \dots 2^{\ell-1}\}$, the quadrature rule $I^{\omega,\ell} g$ can be computed with complexity $\mathcal{O}(n_\ell \log n_\ell)$, using FFT (see, e.g., [9, Remarks 2.1, 5.4]).*

The following simple proposition uses Notation 1.1 to give a unified expression for (2.10) and will be useful later.

Proposition 2.3. *Let $g \in C[-1,1]$ and $\ell \geq 1$. Then the rule defined in (2.10) can be written*

$$I^{ka,\ell} g = \int_{-1}^{1} (Q^{\ell}\widehat{g})(y)\, \exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y, \quad \text{where} \quad \widehat{g}(y) = g(y)\exp(\mathrm{i}k\widehat{a}y). \tag{2.11}$$

The following error estimate is a minor variation on the result in [9, Corollary 2.3].

Theorem 2.4. *Define $\eta(1) = 0$, $\eta(2) = 3$. Then, for $p \in \mathbb{N}$, $p > 1$, and $s \in \{1, 2\}$, there exists a constant C_p such that, for all $\ell \geq 2$ and $g \in \mathcal{W}^{p+\eta(s),1}$, the quadrature rule (2.10) has the error estimate:*

$$\left| I^{\omega} g - I^{\omega,\ell} g \right| \;\leq\; C_p \, \min\left\{1, |\omega|^{-s}\right\} \left(\frac{1}{n_\ell}\right)^{p-1} \|g\|_{\mathcal{W}^{p+\eta(s),1}}, \tag{2.12}$$

for all $\omega \in \mathbb{R}$, where $\min\left\{1, |\omega|^{-s}\right\} := 1$, if $\omega = 0$. Moreover for $\ell = 1$ and any $p > 1$ we have

$$\left| I^{\omega} g - I^{\omega,\ell} g \right| \;\leq\; C_p \, \min\left\{1, |\omega|^{-1}\right\} \left(\frac{1}{n_\ell}\right)^{p-1} \|g\|_{\mathcal{W}^{p,1}}. \tag{2.13}$$

Proof. When $\ell \geq 2$, we have $n_\ell \geq 3$. A slight variation of [9, Theorem 2.2] then shows that there exists a constant $C > 0$ such that, for all $p > 1$, the estimate

$$\left| I^{\omega} g - I^{\omega,\ell} g \right| \;\leq\; C|\omega|^{-s} \left(\frac{1}{n_\ell}\right)^{p-1} \|g_c\|_{H^{p+\eta(s)}}, \tag{2.14}$$

holds, for $|\omega| \geq 1$, where $g_c(\theta) := g(\cos\theta)$ is the cosine transform of g and $\|\cdot\|_{H^p}$ is the usual univariate Sobolev norm of order p. The estimate in [8] is only stated for $\omega \geq 1$, but the case $\omega \leq -1$ can be trivially obtained from this by replacing y by $-y$ and ω by $-\omega$ in (2.1).

For $|\omega| < 1$, we proceed by estimating the error in the classical Clenshaw-Curtis rule applied to $\widetilde{g}(y) := g(y)\exp(\mathrm{i}\omega y)$ by

$$\begin{aligned}\left|I^{\omega}g - I^{\omega,\ell}g\right| &= \left|\int_{-1}^{1}(I-Q^{\ell})\widetilde{g}\right| \le \sqrt{2}\|(I-Q^{\ell})\widetilde{g}\|_{L^2[-1,1]} \\ &\le \sqrt{2}\left(\int_0^{\pi}|((I-Q^{\ell})\widetilde{g})_c(\theta)|^2\mathrm{d}\theta\right)^{1/2} \\ &\le C\left(\frac{1}{n_\ell}\right)^{p}\|\widetilde{g}_c\|_{H^p}, \end{aligned} \tag{2.15}$$

which can then be bounded in the required form (2.12). (In the last step, we used [9, eq (15)]).

The constants C appearing in (2.14) and (2.15) are independent of ℓ and ω as well as p, but to complete the proof we need to estimate the Sobolev norms on the right-hand sides of (2.14) and (2.15) in terms of the $\mathcal{W}^{p,1}$ norm of g. This is where the p-dependence appears. Since all the derivatives of $\cos\theta$ are bounded above by 1, the Faa di Bruno formula (e.g. [16])) readily yields

$$\left|\left(\frac{\mathrm{d}}{\mathrm{d}\theta}\right)^{p}(g_c(\theta))\right| \;\le\; \sum_{\mathcal{P}}|g^{(|\mathcal{P}|)}(\cos\theta)|, \quad \theta\in[-\pi,\pi],$$

where the sum is over all partitions $\mathcal{P}$ of the set $\{1,\dots,p\}$, with $|\mathcal{P}|$ denoting the number of subsets in $\mathcal{P}$. Since $|\mathcal{P}| \le p$ and the number of all such partitions is B_p (the pth *Bell number*) it follows that $\|g_c\|_{H^p} \le CB_p\|g\|_{\mathcal{W}^{p,1}}$. Recalling Proposition 1.4, $\|\widetilde{g}_c\|_{H^p}$ satisfies the same estimate, thus completing the proof of (2.12).

To obtain (2.13) for $|\omega| \ge 1$, we integrate by parts to obtain

$$\begin{aligned} I^{\omega}g - I^{\omega,1}g &= \int_{-1}^{1}(g(y)-g(0))\exp(\mathrm{i}\omega y)\mathrm{d}y \\ &= \frac{1}{\mathrm{i}\omega}\left((g(y)-g(0))\left[\exp(\mathrm{i}\omega y)\right]_{-1}^{1} - \int_{-1}^{1}g'(y)\exp(\mathrm{i}\omega y)\mathrm{d}y\right), \end{aligned}$$

from which the required estimate follows, since $n_1 = 1$. When $|\omega| < 1$, the proof is trivial. □

Remark 2.5. *In the proof above, the constant C_p in (2.12) grows quickly with p, in fact with the order of the growth of the Bell number B_p. This is the price we pay for an estimate to be uniform in ℓ. (Uniformity of the estimate with respect to ℓ is required in the proof of Theorem 4.2, because in the Smolyak construction, low order approximations in some dimensions are combined with high order in others, so we need estimates for all orders.) If (2.12) were only required to hold for ℓ sufficiently large (relative to p), then the constant C_p can be bounded independently of p (see, for example, [8, Remark 2.4]).*

3 An FCCS rule for multi-dimensional integrals

The direct application of the tensor product version of a conventional 1D rule to the multi-dimensional problem (1.1) will give very poor results as d or k increases, first because of the high oscillation and second because of the curse of dimensionality. The difficulty is illustrated by the following simple example.

Example 3.1. *Suppose the integral (1.1) is approximated by the tensor product of the 1d Clenshaw-Curtis rule (2.9), using $n+1$ points in each coordinate direction, so that the integrand is evaluated at $N := (n+1)^d$ points. Then, in the special case $f(\boldsymbol{y}) = 1$ and $\mathbf{a} = (1,0,0,\dots,0)^{\top}$, the error is*

$$\int_{[-1,1]^{d-1}}\int_{-1}^{1}(I-P^n)\Big(\exp(\mathrm{i}k\cdot)\Big)(y_1)\,\mathrm{d}y_1\mathrm{d}y_2\dots\mathrm{d}y_d \;=\; 2^{d-1}\int_{-1}^{1}(I-P^n)\Big(\exp(\mathrm{i}k\cdot)\Big)(y_1)\,\mathrm{d}y_1,$$

where P^n denotes the polynomial interpolant at $n+1$ Clenshaw-Curtis points in 1d (i.e. the operator Q^ℓ in (2.4), with n_ℓ replaced by n). Using (2.15), the error estimate for this is $k^p\, n^{-p} = k^p\, \mathcal{O}(N^{-p/d})$ for any p.

We alleviate problem of growth with respect to k by adopting the Filon approach described above. Then, to reduce the effect of dimension (encapsulated in the $N^{-p/d}$ estimate), we approximate (1.1) by replacing $\widehat{f}$ in (1.2) by its Smolyak interpolant $\mathcal{Q}^{r,d}\widehat{f}$ defined as follows.

Using the 1D interpolation operator Q^ℓ in (2.3), and the nested sequence of Clenshaw-Curtis grids in (1.3), we define the difference operator D^ℓ by

$$D^\ell g := (Q^\ell - Q^{\ell-1})g, \quad \text{with} \quad Q^0 g := 0. \tag{3.1}$$

To define the Smolyak interpolation operator, it is convenient to define the index set $\Lambda(q,d)$, for integers q, d with $q \geq d$ by

$$\Lambda(q,d) \;=\; \{\boldsymbol{\ell} \in \mathbb{N}^d : \mathbf{1} \leq \boldsymbol{\ell},\, |\boldsymbol{\ell}| \leq q\},$$

where $\mathbf{1} = (1,\ldots,1)^\top$ and $|\boldsymbol{\ell}| = \ell_1 + \ell_2 + \ldots + \ell_d$. By [32, p.13], the cardinality of $\Lambda(q,d)$ is given by the binomial coefficient:

$$\#\Lambda(q,d) = \begin{pmatrix} q \\ d \end{pmatrix}. \tag{3.2}$$

Smolyak's formula for interpolating the d-dimensional function $f : \mathbb{R}^d \to \mathbb{R}$ in dimension d, with maximum level $r \in \mathbb{N} := \{1,2,3,\ldots\}$, is then given (e.g., in [32, eq (10)] or [12, p.214]) by

$$(\mathcal{Q}^{r,d}\widehat{f})(\boldsymbol{y}) := \sum_{\boldsymbol{\ell}\in\Lambda(r+d-1,d)} (D^{\ell_1} \otimes \ldots \otimes D^{\ell_d})\widehat{f}(\boldsymbol{y}). \tag{3.3}$$

See [3, Prop 6] for a discussion of the interpolatory properties of $\mathcal{Q}^{r,d}$. The notation $D^{\ell_1}\otimes\ldots\otimes D^{\ell_d}$ indicates that we apply D^{ℓ_j} with respect to variable y_j, for each $j = 1,\ldots,d$. Note that, since $\boldsymbol{\ell} \in \Lambda(r+d-1,d)$ in (3.3), we have $d \leq |\boldsymbol{\ell}| \leq r+d-1$ and also $1 \leq \ell_j \leq r$ for each $j = 1,\ldots,d$.

Then, to define the FCCS rule for (1.2) (and hence (1.1)), we replace $\widehat{f}$ in (1.2) by $(\mathcal{Q}^{r,d}\widehat{f})$, thus obtaining (1.4). An alternative formula for $\mathcal{Q}^{r,d}\widehat{f}$ is obtained using the *combination technique* ([32, Lemma 1], [12, Section 4.1]). This allows the formula (3.3) to be written in terms of Q^l instead of D^l; the result is:

$$(\mathcal{Q}^{r,d}\widehat{f})(\boldsymbol{y}) := \sum_{\substack{\boldsymbol{\ell}\geq\mathbf{1} \\ r\leq|\boldsymbol{\ell}|\leq r+d-1}} (-1)^{r+d-|\boldsymbol{\ell}|-1}\binom{d-1}{|\boldsymbol{\ell}|-r}(Q^{\ell_1}\otimes\ldots\otimes Q^{\ell_d})\widehat{f}(\boldsymbol{y}). \tag{3.4}$$

A useful observation from this is (when $d = 1$),

$$Q^{r,1}\widehat{f} = Q^r\widehat{f} \tag{3.5}$$

Then, inserting (3.4) into (1.2), we obtain the following approximation of (1.1):

Proposition 3.2.

$$\mathcal{I}^{k,d,\mathbf{a},r} f = \sum_{\substack{\boldsymbol{\ell}\geq\mathbf{1} \\ r\leq|\boldsymbol{\ell}|\leq r+d-1}} (-1)^{r+d-|\boldsymbol{\ell}|-1}\binom{d-1}{|\boldsymbol{\ell}|-r}(I^{\omega_1,\ell_1}\otimes\cdots\otimes I^{\omega_d,\ell_d})f, \tag{3.6}$$

where

$$\omega_j = k a_j \quad \textit{for each} \quad j = 1,\ldots,d. \tag{3.7}$$

Proof. By (1.4) and (3.4), it is sufficient to prove that

$$\int_{[-1,1]^d} (Q^{\ell_1} \otimes ... \otimes Q^{\ell_d}) \widehat{f}(\boldsymbol{y}) \exp(\mathrm{i}k\widetilde{\mathbf{a}} \cdot \boldsymbol{y}) \mathrm{d}\boldsymbol{y} = (I^{\omega_1,\ell_1} \otimes \cdots \otimes I^{\omega_d,\ell_d}) f, \tag{3.8}$$

with ω_j as given in (3.7). The proof of (3.8) is obtained by induction on the dimension d. For $d = 1$, we have, directly from Proposition 2.3,

$$\int_{-1}^{1} (Q^{\ell_1} \widehat{f})(y) \exp(\mathrm{i}k\widetilde{a}_1 y) \mathrm{d}y = I^{ka_1,\ell_1} f = I^{\omega_1,\ell_1} f. \tag{3.9}$$

Now suppose (3.8) holds for dimension d and, considering dimension $d+1$, we introduce new notation as follows. For $\boldsymbol{y} \in [-1,1]^{d+1}$ and $\mathbf{a} \in \mathbb{R}^{d+1}$, we write $\boldsymbol{y} = (\boldsymbol{y}^*, y_{d+1})$ and $\mathbf{a} = (\mathbf{a}^*, a_{d+1})$, where $\boldsymbol{y}^* \in [-1,1]^d$ and $\mathbf{a}^* \in \mathbb{R}^d$. Moreover, for any $f \in C([-1,1]^{d+1})$ and any fixed $\boldsymbol{y}^* \in [-1,1]^d$, we write $f(\boldsymbol{y}^*, \cdot)$ to denote the univariate function $y_{d+1} \mapsto f(\boldsymbol{y}^*, y_{d+1}) = f(\boldsymbol{y})$. Using $I^{\omega,\ell}[f(\boldsymbol{y}^*, \cdot)]$ to denote the application of the quadrature rule (2.10) to $f(\boldsymbol{y}^*, \cdot)$, we also define the $d-$variate functions

$$F^{\omega,\ell}(\boldsymbol{y}^*) = I^{\omega,\ell}[f(\boldsymbol{y}^*, \cdot)], \quad \text{and } \widehat{F}^{\omega,\ell}(\boldsymbol{y}^*) = F^{\omega,\ell}(\boldsymbol{y}^*) \exp(\mathrm{i}k\widehat{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \quad \text{for all } \boldsymbol{y}^* \in [-1,1]^d. \tag{3.10}$$

Then, we have

$$\begin{aligned} &\int_{[-1,1]^{d+1}} (Q^{\ell_1} \otimes ... \otimes Q^{\ell_d} \otimes Q^{\ell_{d+1}}) \widehat{f}(\boldsymbol{y}) \exp(\mathrm{i}k\widetilde{\mathbf{a}} \cdot \boldsymbol{y}) \mathrm{d}\boldsymbol{y} \\ &\quad = \int_{[-1,1]^d} (Q^{\ell_1} \otimes ... \otimes Q^{\ell_d}) \left(\int_{-1}^{1} (Q^{\ell_{d+1}} \widehat{f}(\boldsymbol{y}^*, \cdot) \exp(\mathrm{i}k\widetilde{a}_{d+1} \cdot))(y_{d+1}) \mathrm{d}y_{d+1} \right) \exp(\mathrm{i}k\widetilde{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \mathrm{d}\boldsymbol{y}^* \end{aligned} \tag{3.11}$$

By Proposition 2.3, we have

$$\int_{-1}^{1} (Q^{\ell_{d+1}} \widehat{f}(\boldsymbol{y}^*, \cdot) \exp(\mathrm{i}k\widetilde{a}_{d+1} \cdot))(y_{d+1}) \mathrm{d}y_{d+1} = I^{\omega_{d+1},\ell_{d+1}} [f(\boldsymbol{y}^*, \cdot)] \exp(\mathrm{i}k\widehat{\mathbf{a}}^* \cdot \boldsymbol{y}^*) = \widehat{F}^{\omega_{d+1},\ell_{d+1}}(\boldsymbol{y}^*).$$

Inserting this into (3.11) and using the inductive hypothesis (i.e., that (3.8) holds), we obtain

$$\begin{aligned} &\int_{[-1,1]^{d+1}} (Q^{\ell_1} \otimes ... \otimes Q^{\ell_d} \otimes Q^{\ell_{d+1}}) \widehat{f}(\boldsymbol{y}) \exp(\mathrm{i}k\widetilde{\mathbf{a}} \cdot \boldsymbol{y}) \mathrm{d}\boldsymbol{y} \\ &\qquad = \int_{[-1,1]^d} \left((Q^{\ell_1} \otimes ... \otimes Q^{\ell_d}) \widehat{F}^{\omega_{d+1},\ell_{d+1}} \right) (\boldsymbol{y}^*) \exp(\mathrm{i}k\widetilde{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \mathrm{d}\boldsymbol{y}^* \\ &\qquad\quad = \left(I^{\omega_1,\ell_1} \otimes \ldots \otimes I^{\omega_d,\ell_d} \right) F^{\omega_{d+1},\ell_{d+1}}. \end{aligned} \tag{3.12}$$

The fact that (3.8) holds for dimension $d+1$ then follows by the defintion of $F^{\omega_{d+1},\ell_{d+1}}$ in (3.10). □

4 Error analysis of the FCCS rule

In this section, we shall provide an error estimate of the FCCS rule (3.6) for approximating (1.1) (equivalently (1.2)). Before this we need several preliminary results.

Lemma 4.1. *(i) Recall $I^\omega g$ defined in (2.1). Then, for any $\omega \in \mathbb{R}$ and $g \in \mathcal{W}^{1,1}$,*

$$|I^\omega g| \;\le\; 4 \min\{1, |\omega|^{-1}\} \, \|g\|_{\mathcal{W}^{1,1}},$$

where $\min\{1, |\omega|^{-1}\} := 1$ when $\omega = 0$.
(ii) Let $g \in \mathcal{W}^{p,1}$, with $p > 1$, let $a \in \mathbb{R}$, and define $\widetilde{a}, \widehat{a}$ as in Notation 1.1 and set $\widehat{g}(y) = g(y) \exp(k\widehat{a}y)$. Then, for all $k \ge 0$, and $\ell \ge 1$, we have

$$\left| \int_{-1}^{1} (D^\ell \widehat{g})(y) \exp(\mathrm{i}k\widetilde{a}y) \mathrm{d}y \right| \;\le\; C'_p \min\{1, |\omega|^{-1}\} \left(\frac{1}{n_{\ell-1}} \right)^{p-1} \|g\|_{\mathcal{W}^{p,1}}, \tag{4.1}$$

where $\omega = ka$, $C_p' = 2\max\{C_p, 2\}$, *with* C_p *is as in Proposition 2.4 and we have set* $n_0 = 1$.

Moreover, for $k \geq 0$ *and* $\ell \geq 3$,

$$\left|\int_{-1}^{1} (D^\ell \widehat{g})(y)\exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y\right| \;\leq\; C_p' \min\{1, |\omega|^{-2}\}\left(\frac{1}{n_{\ell-1}}\right)^{p-1} \|g\|_{\mathcal{W}^{p+3,1}}. \tag{4.2}$$

Proof. (i) For $\omega \neq 0$. we use integration by parts to obtain

$$I^\omega g \;=\; \int_{-1}^{1} g(y)\exp(\mathrm{i}\omega y)\mathrm{d}y \;=\; \frac{1}{\mathrm{i}\omega}[g(y)\exp(\mathrm{i}\omega y)]_{-1}^{1} - \frac{1}{\mathrm{i}\omega}\int_{-1}^{1} g'(y)\exp(\mathrm{i}\omega y)\mathrm{d}y.$$

Thus

$$|I^\omega g| \;\leq\; \frac{2}{|\omega|}||g||_\infty + \frac{2}{|\omega|}||g'||_\infty \;\leq\; \frac{4}{|\omega|}\,||g||_{\mathcal{W}^{1,1}}.$$

On the other hand, a direct estimate yields $|I^\omega g| \leq 2\|g\|_\infty$ for all ω, and part (i) follows.

(ii) For $\ell \geq 2$, by the definition of D^ℓ, Proposition 2.3 and Theorem 2.4 (with $s = 1$),

$$\begin{aligned}
\left|\int_{-1}^{1} (D^\ell \widehat{g})(y)\exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y\right| &= \left|\int_{-1}^{1} (Q^\ell - Q^{\ell-1})\widehat{g}(y)\exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y\right| \\
&= |I^{\omega,\ell}g - I^{\omega,\ell-1}g| \;\leq\; |I^\omega g - I^{\omega,\ell}g| + |I^\omega g - I^{\omega,\ell-1}g| \qquad (4.3)\\
&\leq C_p \min\{1, |\omega|^{-1}\}\left[\left(\frac{1}{n_\ell}\right)^{p-1} + \left(\frac{1}{n_{\ell-1}}\right)^{p-1}\right]\|g\|_{\mathcal{W}^{p,1}} \\
&= C_p \min\{1, |\omega|^{-1}\}\left[\left(\frac{1}{n_{\ell-1}}\right)^{p-1}(2^{1-p}+1)\right]\|g\|_{\mathcal{W}^{p,1}} \\
&\leq 2C_p \min\{1, |\omega|^{-1}\}\left(\frac{1}{n_{\ell-1}}\right)^{p-1}\|g\|_{\mathcal{W}^{p,1}}. \qquad (4.4)
\end{aligned}$$

For $\ell = 1$, we use Proposition 2.3 to obtain,

$$\left|\int_{-1}^{1} (D^1 \widehat{g})(y)\exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y\right| = |I^{\omega,1}g| \;=\; \begin{cases} |g(0)|\left|\int_{-1}^{1}\exp(\mathrm{i}\omega y)dy\right| & \text{for} \quad |\omega| \geq 1, \\ 2|g(0)|, & \text{for} \quad |\omega| < 1, \end{cases}$$

which gives $\left|\int_{-1}^{1}(D^1\widehat{g})(y)\exp(\mathrm{i}k\widetilde{a}y)\mathrm{d}y\right| \leq 2\min\{1, |\omega|^{-1}\}||g||_\infty$. The last two estimates yield (4.1).

To obtain (4.2) we proceed as in the case $\ell \geq 2$, but when estimating (4.3) we can use Theorem 2.4 with $s = 2$ instead of $s = 1$. □

Theorem 4.2. *Let* C_p' *be as in Lemma 4.1 (ii) and, for* $\mathbf{a} \in \mathbb{R}^d$, *set* $\omega_j = ka_j$. *Then for fixed* $p > 1$, $r \geq 1$, *and* $f \in \mathcal{W}^{p,d}$,

$$|\mathcal{I}^{k,d,\mathbf{a}}f - \mathcal{I}^{k,d,\mathbf{a},r}f| \;\leq\; \prod_{j=1}^{d}\left(C_p' \min\{1, |\omega_j|^{-1}\}\right)\begin{pmatrix} r+d-1 \\ d-1 \end{pmatrix} 2^{-(r-d)(p-1)}\|f\|_{\mathcal{W}^{p,d}}, \tag{4.5}$$

where $\min\{1, |\omega_j|^{-1}\} := 1$ *when* $\omega_j = 0$.

Proof. This is proved by induction on d. The argument follows that of [32] and [12], although these references analysed tensor versions of standard quadrature rules and not Filon rules for oscillatory integrals, as considered here.

Note that for $d = 1$, we have, by (1.1) and (2.1), $\mathcal{I}^{k,1,\mathbf{a}}f = I^{\omega_1}f$ with $\omega_1 = ka_1$. By (1.4) with $d = 1$, then (3.5) and Proposition 2.3, we have $\mathcal{I}^{k,1,\mathbf{a},r}f \;=\; I^{\omega_1,r}f$. Then applying Theorem 2.4 with $s = 1$, we have

$$|\mathcal{I}^{k,1,\mathbf{a}}f - \mathcal{I}^{k,1,\mathbf{a},r}f| \;=\; |I^{\omega_1}f - I^{\omega_1,r}f| \;\leq\; C_p \min\{1, |\omega_1|^{-1}\}\left(\frac{1}{n_r}\right)^{p-1}\|f\|_{\mathcal{W}^{p,1}}. \tag{4.6}$$

Since $C_p \le C'_p$, this yields (4.5) for $d = 1$.

Now suppose (4.5) holds for dimension d and consider dimension $d+1$. We adopt the notation used in the proof of Proposition 3.2, and, for $\boldsymbol{\ell} \in \mathbb{N}^{d+1}$, we write $\boldsymbol{\ell} = (\boldsymbol{\ell}^*, \ell_{d+1})$ with $\boldsymbol{\ell}^* \in \mathbb{N}^d$. Also, recalling (2.1), we introduce, for $f \in C([-1,1]^{d+1})$,

$$F^{\omega}(\boldsymbol{y}^*) := I^{\omega}[f(\boldsymbol{y}^*, \cdot)]. \tag{4.7}$$

We then estimate the error by the sum of two terms:

$$\begin{aligned} |\mathcal{I}^{k,d+1,\mathbf{a}} f - \mathcal{I}^{k,d+1,\mathbf{a},r} f| \;&\le\; |\mathcal{I}^{k,d+1,\mathbf{a}} f - \mathcal{I}^{k,d,\mathbf{a}^*,r} F^{\omega_{d+1}}| + |\mathcal{I}^{k,d,\mathbf{a}^*,r} F^{\omega_{d+1}} - \mathcal{I}^{k,d+1,\mathbf{a},r} f| \\ &=:\; |T_1| + |T_2|. \end{aligned} \tag{4.8}$$

Considering $|T_1|$ first, we use the equality $\mathcal{I}^{k,d+1,\mathbf{a}} f = \mathcal{I}^{k,d,\mathbf{a}^*} F^{\omega_{d+1}}$ and the inductive hypothesis to obtain

$$\begin{aligned} |T_1| \;&= |\left(\mathcal{I}^{k,d,\mathbf{a}^*} - \mathcal{I}^{k,d,\mathbf{a}^*,r}\right) F^{\omega_{d+1}}| \\ &\le \begin{pmatrix} r+d-1 \\ d-1 \end{pmatrix} \prod_{j=1}^{d} \left(C'_p \min\{1, |\omega_j|^{-1}\}\right) 2^{-(r-d)(p-1)} \|F^{\omega_{d+1}}\|_{\mathcal{W}^{p,d}}. \end{aligned} \tag{4.9}$$

Then, using Lemma 4.1 (i) and a little manipulation one can see that

$$\|F^{\omega_{d+1}}\|_{W^{p,d}} \;\le\; 4 \min\{1, |\omega_{d+1}|^{-1}\} \|f\|_{W^{p,d+1}} \le\; C'_p \min\{1, |\omega_{d+1}|^{-1}\} \|f\|_{W^{p,d+1}}. \tag{4.10}$$

Hence, using this in (4.9) we obtain

$$|T_1| \;\le \begin{pmatrix} r+d-1 \\ d-1 \end{pmatrix} \prod_{j=1}^{d+1} \left(C'_p \min\{1, |\omega_j|^{-1}\}\right) 2^{-(r-d)(p-1)} \|f\|_{\mathcal{W}^{p,d+1}}. \tag{4.11}$$

To estimate T_2, we first apply [32, eq. (11)], to write

$$\mathcal{Q}^{r,d+1} \;=\; \sum_{\boldsymbol{\ell} \in \Lambda(r+d,d+1)} (D^{\ell_1} \otimes \ldots \otimes D^{\ell_{d+1}}) \;=\; \sum_{\boldsymbol{\ell}^* \in \Lambda(r+d-1,d)} (D^{\ell_1^*} \otimes \ldots \otimes D^{\ell_d^*}) \otimes Q^{r+d-|\boldsymbol{\ell}^*|}.$$

Then, with notation as in (3.10) and proceeding as in (3.12), we have

$$\begin{aligned} \mathcal{I}^{k,d+1,\mathbf{a},r} f &= \int_{[-1,1]^{d+1}} (\mathcal{Q}^{r,d+1} \widehat{f})(\boldsymbol{y}) \exp(\mathrm{i} k \widetilde{\mathbf{a}} \cdot \boldsymbol{y}) \mathrm{d}\boldsymbol{y} \\ &= \sum_{\boldsymbol{\ell}^* \in \Lambda(r+d-1,d)} \int_{[-1,1]^d} (D^{\ell_1^*} \otimes \ldots \otimes D^{\ell_d^*}) \widehat{F}^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|}(\boldsymbol{y}^*) \exp(\mathrm{i} k \widetilde{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \mathrm{d}\boldsymbol{y}^*. \end{aligned} \tag{4.12}$$

Also, by (1.4) and subsequently (3.3), we have

$$\begin{aligned} \mathcal{I}^{k,d,\mathbf{a}^*,r} F^{\omega_{d+1}} &= \int_{[-1,1]^d} (\mathcal{Q}^{r,d} \widehat{F}^{\omega_{d+1}})(\boldsymbol{y}^*) \exp(\mathrm{i} k \widetilde{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \mathrm{d}\boldsymbol{y}^* \\ &= \sum_{\boldsymbol{\ell}^* \in \Lambda(r+d-1,d)} \int_{[-1,1]^d} (D^{\ell_1^*} \otimes \ldots \otimes D^{\ell_d^*}) \widehat{F}^{\omega_{d+1}}(\boldsymbol{y}^*) \exp(\mathrm{i} k \widetilde{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \mathrm{d}\boldsymbol{y}^*. \end{aligned} \tag{4.13}$$

Thus, combining (4.12) and (4.13), we obtain

$$\begin{aligned} T_2 &= \mathcal{I}^{k,d,\mathbf{a}^*,r} F^{\omega_{d+1}} - \mathcal{I}^{k,d+1,\mathbf{a},r} f \\ &= \sum_{\boldsymbol{\ell}^* \in \Lambda(r+d-1,d)} \int_{[-1,1]^d} (D^{\ell_1^*} \otimes \ldots \otimes D^{\ell_d^*}) \left(\widehat{F}^{\omega_{d+1}} - \widehat{F}^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|}\right)(\boldsymbol{y}^*) \exp(\mathrm{i} k \widetilde{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \mathrm{d}\boldsymbol{y}^*. \end{aligned} \tag{4.14}$$

Now, to estimate $|T_2|$, we consider any function $G \in \mathcal{W}^{p,d}$ and any $\mathbf{a} \in \mathbb{R}^d$, and define $\widehat{G}(\boldsymbol{y}) = G(\boldsymbol{y}) \exp(\mathrm{i}k\widehat{\mathbf{a}}.\boldsymbol{y})$, with $\widehat{\mathbf{a}}$ as defined in Notation 1.1 as in (1.2). By induction on dimension d, using Lemma 4.1 (ii) at each step, we obtain the estimate

$$\begin{aligned}\left| \int_{[-1,1]^d} (D^{\ell_1^*} \otimes \ldots \otimes D^{\ell_d^*})\, \widehat{G}(\boldsymbol{y}^*) \exp(\mathrm{i}k\widetilde{\mathbf{a}}^* \cdot \boldsymbol{y}^*) \mathrm{d}\boldsymbol{y}^* \right| \\ \leq \prod_{j=1}^{d} \left(C_p' \min\{1, |\omega_j|^{-1}\} \left(\frac{1}{n_{\ell_j^*-1}} \right)^{p-1} \right) \|G\|_{\mathcal{W}^{p,d}} \\ \leq \left[\prod_{j=1}^{d} \left(C_p' \min\{1, |\omega_j|^{-1}\} \right) \right] 2^{(2d-|\boldsymbol{\ell}^*|)(p-1)} \|G\|_{\mathcal{W}^{p,d}}, \end{aligned} \tag{4.15}$$

where, in the last step, we used the fact that

$$\prod_{j=1}^{d} \frac{1}{n_{\ell_j^*-1}} \leq \prod_{j=1}^{d} 2^{2-\ell_j^*} = 2^{2d-|\boldsymbol{\ell}^*|}.$$

Thus, to complete the estimate of (4.14) we need to estimate $\|F^{\omega_{d+1}} - F^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|}\|_{\mathcal{W}^{p,d}}$. Recalling that F^{ω} is given by (4.7) and $F^{\omega,\ell}$ is given in (3.10), we apply Theorem 2.4 with $s = 1$ to obtain, for any $j = 1, \ldots, d$, and any $q = 0, \ldots, p$,

$$\begin{aligned}\left| \partial_j^q (F^{\omega_{d+1}} - F^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|})(\boldsymbol{y}^*) \right| &= \left| \left(I^{\omega_{d+1}} - I^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|} \right) \left[(\partial_j^q f)(\boldsymbol{y}^*, \cdot) \right] \right| \\ &\leq C_p \min\{1, |\omega_{d+1}|^{-1}\}\, 2^{-(r+d-|\boldsymbol{\ell}^*|-1)(p-1)}\, \|(\partial_j^q f)(\boldsymbol{y}^*, \cdot)\|_{\mathcal{W}^{p,1}}, \end{aligned}$$

(where we used the fact that $r + d - |\boldsymbol{\ell}^*| \geq 1$ in (4.14), because $\boldsymbol{\ell}^* \in \Lambda(r + d - 1, d)$). Hence

$$\|F^{\omega_{d+1}} - F^{\omega_{d+1}, r+d-|\boldsymbol{\ell}^*|}\|_{\mathcal{W}^{p,d}} \leq C_p \min\{1, |\omega_{d+1}|^{-1}\}\, 2^{-(r+d-|\boldsymbol{\ell}^*|-1)(p-1)}\, \|f\|_{\mathcal{W}^{p,d+1}}. \tag{4.16}$$

Combining (4.14), (4.15) and (4.16), using $C_p \leq C_p'$ and then the cardinality formula (3.2), we obtain

$$\begin{aligned}|T_2| &\leq \left[\prod_{j=1}^{d+1} \left(C_p' \min\{1, |\omega_j|^{-1}\} \right) \right] \left(\sum_{\boldsymbol{\ell}^* \in \Lambda(r+d-1,d)} \left(2^{2d-|\boldsymbol{\ell}^*|} 2^{-(r+d-|\boldsymbol{\ell}^*|-1)} \right)^{p-1} \right) \|f\|_{\mathcal{W}^{p,d+1}} \\ &= \binom{r+d-1}{d} \left[\prod_{j=1}^{d+1} \left(C_p' \min\{1, |\omega_j|^{-1}\} \right) \right] 2^{-(r-d-1)(p-1)} \|f\|_{\mathcal{W}^{p,d+1}}. \end{aligned} \tag{4.17}$$

Then, combining (4.11), (4.17) and using the elementary identity $\binom{q}{d} + \binom{q}{d-1} = \binom{q+1}{d}$, with $q = r+d-1$, we have shown that the estimate (4.5) holds for dimension $d+1$. □

Remark 4.3. *(i) For those values of $\boldsymbol{\ell}^*$ satisfying $r + d - |\boldsymbol{\ell}^*| \geq 2$, an application of Theorem 2.4 shows that (4.16) holds with $|\omega_{d+1}|^{-1}$ replaced by $|\omega_{d+1}|^{-2}$ and $\mathcal{W}^{p,d+1}$ replaced by $\mathcal{W}^{p+3,d+1}$.*

(ii) If any $\ell_j^* \geq 3$ then (for that particular j), an application of Theorem 4.1 (ii), shows that $|\omega_j|^{-1}$ can be replaced by $|\omega_j|^{-2}$ and $\mathcal{W}^{p,d}$ replaced by $\mathcal{W}^{p+3,d}$ in (4.15).

While Theorem 4.2 gives an estimate for the error which is explicit in $\mathbf{a}$ and k, the following theorem gives a simpler (and higher order in k^{-1}) at the cost of a stronger regularity requirement.

Theorem 4.4. *Let $\mathbf{a} \in \mathbb{R}^d$ with $\mathbf{a}_j \neq 0$ for each j. Let $p \geq 1$ and $d \geq 1$. Then there exists a constant $C_{p,d,\mathbf{a}}$ such that, for all $r \geq d+1$ and $f \in \mathcal{W}^{p+3,d}$,*

$$|\mathcal{I}^{k,d,\mathbf{a}} f - \mathcal{I}^{k,d,\mathbf{a},r} f| \leq C_{p,d,\mathbf{a}}\ k^{-(d+1)} \binom{r+d-1}{d-1} 2^{-(r-d)(p-1)} \|f\|_{\mathcal{W}^{p+3,d}}. \tag{4.18}$$

Proof. We follow the proof of Theorem 4.2, indicating differences only briefly. This proof is less technical since we assume k is sufficiently large and we do not require explicitness with respect to $\mathbf{a}$. Throughout the proof $C_{p,d,\mathbf{a}}$ denotes a generic constant which may depend on $p, d, \mathbf{a}$, and whose value may vary from line to line.

Step 1 For the case $d = 1$, we have $r \geq 2$. Then we recall (4.6), but this time we use Theorem 2.4 with $s = 2$ to obtain

$$|\mathcal{I}^{k,1,\mathbf{a}} f - \mathcal{I}^{k,1,\mathbf{a},r} f| \;=\; |I^{\omega_1} f - I^{\omega_1,r} f| \;\leq\; C_p (k|a|)^{-2} \left(\frac{1}{n_r}\right)^{p-1} \|f\|_{\mathcal{W}^{p+3,1}},$$

which yields (4.18) for $d = 1$.

Step 2 Now assuming (4.18) holds for d we consider the corresponding result for dimension $d+1$. In this case we are assuming

$$r \geq d + 2. \tag{4.19}$$

Again we introduce the splitting (4.8).

Step 2a Analogously to (4.9) and (4.10) we obtain, via the inductive hypothesis,

$$\begin{aligned} |T_1| \;&\leq\; C_{p,d,\mathbf{a}}\, k^{-(d+1)} \begin{pmatrix} r+d-1 \\ d-1 \end{pmatrix} 2^{-(r-d)(p-1)} \|F^{\omega_{d+1}}\|_{\mathcal{W}^{p+3,d}} \\ &\leq\; C_{p,d,\mathbf{a}}\, k^{-(d+2)} \begin{pmatrix} r+d-1 \\ d-1 \end{pmatrix} 2^{-(r-d)(p-1)} \|f\|_{\mathcal{W}^{p+3,d+1}}, \end{aligned} \tag{4.20}$$

where the additional power of k comes from the estimate of $\|F^{\omega_{d+1}}\|_{\mathcal{W}^{p+3,d+1}}$ – analgous to (4.10).

Step 2b The estimate of $|T_2|$ (starting from (4.14)) is slightly more complicated. Note first that (4.14) is a sum of terms, each corresponding to a different choice of $\boldsymbol{\ell}^*$. In all cases $r + d - |\boldsymbol{\ell}^*| \geq 1$. The key estimates for each of the summands in (4.14) are given in (4.15) and (4.16). In the following discussion we will discuss only asymptotic decay as $k \to \infty$ of terms in (4.15) and (4.16), the dependence on other variables is the same as in the proof of Theorem 4.2.

If, in fact, $r + d - |\boldsymbol{\ell}^*| \geq 2$ then, via Remark 4.3 (i), we obtain

$$\|F^{\omega_{d+1}} - F^{\omega_{d+1},r+d-|\boldsymbol{\ell}^*|}\|_{\mathcal{W}^{p,d}} \;\leq\; C_{p,d,\mathbf{a}}\, k^{-2}\, 2^{-(r+d-|\boldsymbol{\ell}^*|-1)(p-1)} \|f\|_{\mathcal{W}^{p+3,d+1}},$$

i.e., one additional negative power of k compared with (4.16). In this case, the corresponding summand in (4.14) can be estimated as a product of d terms of $\mathcal{O}(k^{-1})$ (analogous to (4.15)) and one of $\mathcal{O}(k^{-2})$, yielding $\mathcal{O}(k^{-(d+2)})$ overall for that summand.

On the other hand, if $r + d - |\boldsymbol{\ell}^*| = 1$, then, by (4.19), $|\boldsymbol{\ell}^*| = r + d - 1 \geq 2d + 1$. Since also $\boldsymbol{\ell}^* \geq \mathbf{1}$, it follows that at least one ℓ_j^* must be ≥ 3. For this j, Remark 4.3 (ii) can be applied. Thus (4.15) has an estimate of $\mathcal{O}(k^{-d-1})$ (or better) and combining this with the standard $\mathcal{O}(k^{-1})$ estimate for (4.16), we again obtain $\mathcal{O}(k^{-d-2})$ overall for that summand as well.

Thus overall (4.14) has an $\mathcal{O}(k^{-(d+2)})$ estimate in the case of dimension $d + 1$. This completes the induction argument. □

Remark 4.5. *The result in Theorem 4.4 is important if one is interested in computing (1.1) to high relative accuracy for large k. This is because in general the oscillatory integral (1.1) decays with order $\mathcal{O}(k^{-d})$ as k increases, and Theorem 4.4 shows that the relative error in computing this with the FCCS rule still decays with increasing k.*

We are now ready to complete the proofs of the main theorems - Theorem 1.2 and 1.3, stated in the Introduction.

Proof of Theorems 1.2, 1.3. The proofs just require estimating the term $\binom{r+d-1}{d-1}2^{-(r-d)(p-1)}$ which appears in the estimates in Theorems 4.2 and 4.4.

The reference [27, Lemma 4] gives an asymptotic formula (due to Müller-Gronbach) for the number of nodes $N(r,d)$ used in the quadrature rule (1.4). Using this, and a little manipulation, we obtain

$$N(r,d) \;\approx\; \frac{1}{(d-1)!\,2^d}\left(1+\frac{d-1}{r}\right)^{d-1}\left[2^r\,r^{d-1}\right],$$

where $\approx$ means that the ratio of the left-hand side and the right-hand side tends to 1 as $r\to\infty$. Hence from this it follows that, for sufficiently large r,

$$\frac{1}{(d-1)!\,2^{d-1}}\left[2^r\,r^{d-1}\right] \geq N(r,d) \;\geq\; \frac{1}{(d-1)!\,2^{d+1}}\left[2^r\,r^{d-1}\right]. \tag{4.21}$$

In particular, for sufficiently large r, we have the inequalities

$$\text{(i)}\quad 2^{-(r-d)} \;\leq\; \frac{2}{(d-1)!}\,\frac{r^{d-1}}{N(r,d)} \quad\text{and}\quad \text{(ii)}\quad 2^r \;\leq\; N(r,d). \tag{4.22}$$

Moreover, as is easily shown,

$$\lim_{r\to\infty}\frac{1}{r^{d-1}}\binom{r+d-1}{d-1} = \frac{1}{(d-1)!},$$

and so, for sufficiently large r,

$$\binom{r+d-1}{d-1} \leq \frac{2}{(d-1)!}r^{d-1}. \tag{4.23}$$

Putting together (4.22) (i) and (4.23) and then using (4.22)(ii), we obtain

$$\begin{aligned}\binom{r+d-1}{d-1}2^{-(r-d)(p-1)} &\leq \left(\frac{2}{(d-1)!}\right)^p (r^{d-1})^p \left(\frac{1}{N(r,d)}\right)^{p-1}\\ &\leq \left(\frac{2}{(d-1)!\log^{d-1}2}\right)^p (\log^{d-1}N(r,d))^p\frac{1}{N(r,d)^{p-1}}\end{aligned}$$

Combining this with Theorems 4.2, 4.4, we complete the proof of Theorems 1.2, 1.3.

Remark 4.6. *Finally we remark that the better decay rate as $k\to\infty$ obtained in Theorem 1.3 could be obtained for smaller r if, instead of the mid-point rule when $\ell=1$ (see (1.3)), we employ the two-point Clenshaw-Curtis rule. This fact is illustrated in Example 3 of §6. While this observation is useful for problems of moderate dimension d, it is less interesting for higher d because the number of nodes used in the Smolyak interpolant grows quickly with d if a rule with more than one point is employed at level 1 (see the statement after equation(4) in [3], or [27, equation (25)]).*

5 Application to a UQ problem for the Helmholtz equation

In uncertainty quantification for problems governed by PDEs, one typically wants to compute the output statistics (e.g. expectation or higher order moments) of some quantity of interest (QoI - typically a functional of the PDE solution), given the statistical properties of the random input data (e.g., coefficients) of

the PDE. The required output moments are usually written as a multi-dimensional integral, with dimension determined by the number of random parameters in the model. For problems governed by the Helmholtz equation, the solution is usually oscillatory with respect to both the physical variable(s) and the random parameters. – see, e.g., [28, Chapter 5], [11, §4], [31].

In this paper we restrict to problem (1.10) – (1.12) and we first deal with the oscillation with respect to x by applying a 'hybrid numerical-asymptotic' method in physical space. For each random parameter $\boldsymbol{y}$ this yields an expression for the solution $u(x, \boldsymbol{y})$ (increasingly accurate as k increases) which identifies the principal oscillations with respect to x, and the only parts to be computed numerically are smooth functions of x. For this Helmholtz problem, it turns out that the approximation also yields the principal oscillations with respect to $\boldsymbol{y}$ and integration with respect to $\boldsymbol{y}$ requires integration of multidimensional oscillatory integrals of the type discussed earlier in this paper. (See e.g., [4], for a general discussion of hybrid numerical-asymptotic methods in the context of Helmholtz problems.)

Later we will study the case when n is a random field, but we start here with the deterministic case in order to explain the hybrid numerical-asymptotic method.

5.1 The deterministic problem and its hybrid numerical-asymptotic solution

In this subsection we shall explain how to obtain an asymptotic approximation for the solution u of (1.10) – (1.12) which is increasingly accurate as k increases. Its implementation will involve the numerical solution of problems which are well-behaved with respect to k.

To motivate the idea we first consider problem (1.10) – (1.12) in the special case $F = 0$. Then, in the homogeneous case $n = 1$ the solution is just a linear combination of the complementary functions $\exp(\pm \mathrm{i}kx)$. When n is variable, we define $N(x) = \int_0^x n(x')\mathrm{d}x'$ and, for functions μ, ν we consider the 'approximate complementary function':

$$r := \mu\xi + \nu\xi^{-1}, \quad \text{where} \quad \xi(x) = \exp(\mathrm{i}kN(x)),$$

with μ, ν (as yet) unknown functions of x. With this, we readily find:

$$Lr = \left[\mu'' + \mathrm{i}k(2n\mu' + n'\mu)\right]\xi \; + \; \left[\nu'' - \mathrm{i}k(2n\nu' + n'\nu)\right]\xi^{-1}. \tag{5.1}$$

In order to derive to find an approximation to u which is accurate for large k, we introduce the 'approximate ray-expansion':

$$\widetilde{u}^m := \sum_{j=0}^{2m} k^{-j} r_j, \quad \text{where} \quad r_j := \mu_j\xi + \nu_j\xi^{-1}, \quad \text{for} \quad m \geq 0. \tag{5.2}$$

Then (recalling that we assumed $Lu = F = 0$), an easy calculation, using (5.1) shows

$$L(u - \widetilde{u}^m) \; = \; - \sum_{j=0}^{2m} k^{-j} \left[\mu_j'' + \mathrm{i}k(2n\mu_j' + n'\mu_j)\right]\xi \; - \; \sum_{j=0}^{2m} k^{-j} \left[\nu_j'' - \mathrm{i}k(2n\nu_j' + n'\nu_j)\right]\xi^{-1}. \tag{5.3}$$

To force $\widetilde{u}^m$ to be close to u, we choose the coefficients μ_j and ν_j to that the right-hand side of (5.3) decays rapidly with increasing k. Forcing all terms in (5.3) up to $\mathcal{O}(k^{-2m})$ to vanish leads to the system of differential equations to be satisfied by μ_j, ν_j:

$$2n\mu_j' + n'\mu_j \; = \; \mathrm{i}\mu_{j-1}'', \quad j = 0, 1, \ldots, m \tag{5.4}$$

$$2n\nu_j' + n'\nu_j \; = \; -\mathrm{i}\nu_{j-1}'', \quad j = 0, 1, \ldots, m \tag{5.5}$$

$$\text{with} \quad \mu_{-1} = \nu_{-1} = 0. \tag{5.6}$$

Then, provided we choose the boundary conditions for μ_j and ν_j so that $u - \widetilde{u}^m$ satisfies homogeneous boundary conditions and provided the regularity of μ_j, ν_j can be controlled we can ensure (at least formally) that $u - \widetilde{u}^m = \mathcal{O}(k^{-2m})$ for any choice of m.

This explains the idea in the simple case $F = 0$. Now we prove a theorem which describes the general case. To handle $F \neq 0$ we need to introduce the sequence

$$F_2 = F/n^2 \quad \text{and} \quad F_{2j+2} = -F_{2j}''/n^2, \quad \text{for} \quad j = 1, 2, \ldots. \tag{5.7}$$

Theorem 5.1. *Assume that $m \geq 1$, $F \in C^{2m}[0,1]$ and $n \in C^{2m+2}[0,1]$. Then for k sufficiently large, and for $j = 0, \ldots, 2m$, there exist unique μ_j, ν_j satisfying (5.4) – (5.6), together with the boundary conditions:*

$$B_L r_0 = u_L, \quad B_R r_0 = 0; \tag{5.8}$$

and,

$$\left.\begin{array}{ll} B_L r_{2j-1} = 0, & B_R r_{2j-1} = 0 \\ B_L r_{2j} = -F_{2j}(0), & B_R r_{2j} = -B_R F_{2j} \end{array}\right\}, \quad \text{for} \quad j = 1, \ldots, m. \tag{5.9}$$

Moreover, with the approximation to u then defined by

$$\widetilde{u}^m \;:=\; \sum_{j=0}^{2m} k^{-j} r_j \;+\; \sum_{j=1}^{m} k^{-2j} F_{2j}, \tag{5.10}$$

there exists a constant C independent of k such that, for sufficiently large k,

$$\|u - \widetilde{u}^m\|_{H^1(0,1)} \;\leq\; C k^{-2m}. \tag{5.11}$$

Proof. First, we note that, using [1, Lemma 2.2] and the regularity condition on n we can show by a simple induction that, for k sufficiently large, μ_j and ν_j are well-defined for all $j = 0, \ldots, 2m$ and also $\|\mu_{2m}''\|_{\infty,[0,1]}$ and $\|\nu_{2m}''\|_{\infty,[0,1]}$ are both bounded independently of k. Moreover the regularity conditions of n and F also ensure that F_{2j} are well defined, for $j = 1, \ldots, m+1$, and $\|F_{2m+2}\|_{\infty,[0,1]}$ is bounded independently of k.

Now, to prove the result we observe that by (5.1), and then (5.4) – (5.6),

$$\begin{aligned} L\left(\sum_{j=0}^{2m} k^{-j} r_j\right) &= \sum_{j=0}^{2m} \left(k^{-j}\mu_j'' + \mathrm{i}k^{-(j-1)}(2n\mu_j' + n'\mu_j)\right)\xi \qquad (5.12) \\ &\quad + \sum_{j=0}^{2m} \left(k^{-j}\nu_j'' - \mathrm{i}k^{-(j-1)}(2n\nu_j' + n'\nu_j)\right)\xi^{-1} \\ &= \sum_{j=0}^{2m} \left[\left(k^{-j}\mu_j'' - k^{-(j-1)}\mu_{j-1}''\right) + \left(k^{-j}\nu_j'' - k^{-(j-1)}\nu_{j-1}''\right)\right] \\ &= k^{-2m}\left(\mu_{2m}'' + \nu_{2m}''\right). \qquad (5.13) \end{aligned}$$

Moreover, using also (5.7),

$$\begin{aligned} L\left(\sum_{j=1}^{m} k^{-2j} F_{2j}\right) &= \sum_{j=1}^{m} k^{-2j}(F_{2j}'' + k^2 n^2 F_{2j}) = \; n^2 \sum_{j=1}^{m} (k^{-2j+2} F_{2j} - k^{-2j} F_{2j+2}) \\ &= n^2(F_2 - k^{-2m} F_{2m+2}) = F - k^{-2m} n^2 F_{2m+2}. \qquad (5.14) \end{aligned}$$

Combining (5.10), (5.13) and (5.14), and using $Lu = F$, we have

$$L(u - \widetilde{u}^m) = k^{-2m}(n^2 F_{2m+2} - \mu_{2m}'' - \nu_{2m}''). \tag{5.15}$$

Also, by (5.8), (5.9), we have

$$B_L\widetilde{u}^m = \sum_{j=0}^{2m} k^{-j} B_L r_j + \sum_{j=1}^{m} k^{-2j} B_L F_{2j} = u_L + \sum_{j=1}^{m} k^{-2j}(-F_{2j}(0) + F_{2j}(0)) = u_L. \tag{5.16}$$

and

$$B_R\widetilde{u}^m = \sum_{j=0}^{2m} k^{-j} B_R r_j + \sum_{j=1}^{m} k^{-2j} B_R F_{2j} = \sum_{j=1}^{m} k^{-2j}(-B_R F_{2j} + B_R F_{2j}) = 0. \tag{5.17}$$

Therefore

$$B_L(u - \widetilde{u}^m) = 0 = B_R(u - \widetilde{u}^m), \tag{5.18}$$

and it follows (see e.g., [1, Lemma 2.1]), that

$$\|u - \widetilde{u}^m\|_{H^1(0,1)} \le Ck^{-2m},$$

which leads to the required result (5.11). □

Remark 5.2. *By taking $m = 1$ in Theorem 5.1, and under the assumptions $n \in C^4[0,1]$ and $F \in C^2[0,1]$, we obtain the approximation (valid for k sufficiently large):*

$$\widetilde{u}^1 = \widetilde{\mu}\xi + \widetilde{\nu}\xi^{-1} + \widetilde{F}, \tag{5.19}$$

where

$$\xi = \exp(\mathrm{i}kN), \quad \widetilde{\mu} = \mu_0 + k^{-1}\mu_1 + k^{-2}\mu_2, \quad \widetilde{\nu} = \nu_0 + k^{-1}\nu_1 + k^{-2}\nu_2, \quad \text{and} \quad \widetilde{F} = k^{-2}F_2 = k^{-2}F/n^2. \tag{5.20}$$

With this approximation we have the error estimate $||u - \widetilde{u}^1||_{H^1(0,1)} = \mathcal{O}(k^{-2})$. We use this approximation in the numerical experiment in the following section.

5.2 Computation of $\widetilde{\mu}$ and $\widetilde{\nu}$

The computation of $\widetilde{F}$ is easy, and thus we need to compute $\widetilde{\mu}$ and $\widetilde{\nu}$ in order to obtain $\widetilde{u}^1$, which is then reduced to the computation of $\{\mu_j, \nu_j\}_{j=0}^2$. We know from §5.1 that μ_j and ν_j satisfy the ODE system (5.4) – (5.6) together with the boundary conditions (5.8) and (5.9) for $j = 0, 1, 2$. The ODE system (5.4) – (5.6) admits the following solutions:

$$\mu_j(x) = \alpha_j^1 \mu_j^G(x) + \mu_j^P(x), \quad \nu_j(x) = \alpha_j^2 \nu_j^G(x) + \nu_j^P(x), \tag{5.21}$$

where

$$\mu_j^G(x) = \nu_j^G(x) = n(x)^{-\frac{1}{2}} \tag{5.22}$$

are the general solutions and

$$\mu_j^P(x) = \frac{i}{2} n(x)^{-\frac{1}{2}} \int_0^x \mu_{j-1}''(s) n(s)^{-\frac{1}{2}} ds, \quad \nu_j^P(x) = -\frac{i}{2} n(x)^{-\frac{1}{2}} \int_0^x \nu_{j-1}''(s) n(s)^{-\frac{1}{2}} ds \tag{5.23}$$

are the particular solutions, with the coefficients α_j^1 and α_j^2 determined by the boundary conditions (5.8) and (5.9). We let

$$\mathcal{I}_j^\mu(x) = \int_0^x \mu_{j-1}''(s) n(s)^{-\frac{1}{2}} ds, \quad \mathcal{I}_j^\nu(x) = \int_0^x \nu_{j-1}''(s) n(s)^{-\frac{1}{2}} ds. \tag{5.24}$$

In general, we do not have the analytic expressions for μ_j and ν_j, since we do not always have an explicit expression for the integrals in (5.24). We sometimes need to resort to numerical integration.

Partition of $[0,1]$ Let M and L be positive integers with $H = \frac{1}{M}$ and $h = \frac{1}{LM}$, where L is assumed to be even without loss of generality. We partition $[0,1]$ into $0 = x_0 < x_1 < \ldots < x_{M-1} < x_M = 1$, where $x_m = mH$ for $m = 0, 1, \ldots, M$. For each interval $[x_m, x_{m+1}]$, we further partition it into $x_m = x_m^0 < x_m^1 < \ldots < x_m^{L-1} < x_m^L = x_{m+1}$, where $x_m^\ell = x_m + \ell h$ for $\ell = 0, 1, \ldots, L$. We aim to obtain the values of $\widetilde{\mu}$ and $\widetilde{\nu}$ at x_m for $m = 0, 1, \ldots, M$.

Computation of μ_0 and ν_0 Note that $\mu_0^P = \nu_0^P = 0$ since $\mu_{-1} = \nu_{-1} = 0$. Then $\mu_0(x)$ and $\nu_0(x)$ can be analytically obtained with the coefficients α_0^1 and α_0^2 determined by the boundary condition (5.8).

Computation of μ_1 and ν_1 Note that $\mu_0''(x)n(x)^{-\frac{1}{2}}$ and $\nu_0''(x)n(x)^{-\frac{1}{2}}$, i.e. the integrands of $\mathcal{I}_1^\mu$ and $\mathcal{I}_1^\nu$, are analytically available. We obtain the values of $\mathcal{I}_1^\mu(x_m^\ell)$ and $\mathcal{I}_1^\nu(x_m^\ell)$ for each m and ℓ by successively applying an M_G-point Gauss quadrature rule to the integration on $[x_m^\ell, x_m^{\ell+1}]$. Then we obtain the values of $\mu_1(x_m^\ell)$ and $\nu_1(x_m^\ell)$ for each $m = 0, 1, \ldots, M$ and $\ell = 0, 1, \ldots, L$, with the coefficients α_1^1 and α_1^2 determined by the boundary condition (5.9) for $j = 1$.

Computation of μ_2 and ν_2 Note that

$$\left(\mu_1^P(x)\right)'' = \frac{i}{2}\left[\left(\frac{3}{4}n(x)^{-\frac{5}{2}}n'(x)^2 - \frac{1}{2}n(x)^{-\frac{3}{2}}n''(x)\right)\mathcal{I}_1^\mu(x) - \frac{3}{2}n(x)^{-2}n'(x)\mu_0''(x) + n(x)^{-1}\mu_0'''(x)\right]. \quad (5.25)$$

Hence $\mu_1''(x)n(x)^{-\frac{1}{2}}$ is available at x_m^ℓ for each $m = 0, 1, \ldots, M$ and $\ell = 0, 1, \ldots, L$, and so is $\nu_1''(x)n(x)^{-\frac{1}{2}}$. We obtain the values of $\mathcal{I}_2^\mu(x_m)$ and $\mathcal{I}_2^\nu(x_m)$ for each m by successively applying the composite Simpson's rule to the integration on $[x_m, x_{m+1}]$ using the values of $\mu_1''(x)n(x)^{-\frac{1}{2}}$ and $\nu_1''(x)n(x)^{-\frac{1}{2}}$ at x_m^ℓ for $\ell = 0, 1, \ldots, L$. The we obtain the values of $\mu_2(x_m)$ and $\nu_2(x_m)$ for each $m = 0, 1, \ldots, M$, with the coefficients α_2^1 and α_2^2 determined by the boundary condition (5.9) for $j = 2$.

5.3 The random problem

We now introduce the random model by assuming that $n(x)$ depends in an affine way on d i.i.d. random variables $\boldsymbol{y} = (y_1, ..., y_d)$ with $y_i \in U[-1, 1]$. That is, we assume

$$n(x, \boldsymbol{y}) = n_0(x) + \sum_{j=1}^{d} n_j(x)y_j, \quad (5.26)$$

where $n_j(x) \in C^4[-1, 1]$ for $j = 0, 1, \ldots, d$. We also assume that the expansion functions satisfy, for some constant $C > 0$,

$$\min_{x\in[0,1]} n_0(x) - \sum_{j=1}^{d} ||n_j||_{\infty,[0,1]} \;\geq\; C \quad (5.27)$$

This condition ensures that

$$C \;\leq\; n(x, \boldsymbol{y}) \;\leq\; \sum_{j=0}^{d} \|n_j\|_{\infty,[0,1]}, \quad \text{for all} \quad x \in [0, 1], \quad \boldsymbol{y} \in U[-1, 1]^d. \quad (5.28)$$

With the parametrization of n given in (5.26), we have

$$N(x, \boldsymbol{y}) = N_0(x) + \sum_{j=1}^{d} N_j(x)y_j, \quad \text{with} \quad N_j(x) = \int_0^x n_j(x')\mathrm{d}x', \quad j = 0, \ldots, d, \quad (5.29)$$

and hence

$$\xi(x,\boldsymbol{y}) = \exp(\mathrm{i}kN_0(x))\exp(\mathrm{i}k\mathbf{a}(x)\cdot\boldsymbol{y}), \tag{5.30}$$

where $\mathbf{a}\in C[0,1]^d$ is the real vector-valued function with components given by

$$a_j(x) = N_j(x), \quad j=1,\dots,d. \tag{5.31}$$

In UQ applications one is often interested in computing the expectation or higher moments of a Quantity of Interest, typically a functional of the solution u. Since our aim here is to provide an application of the use of the FCCS rule combined with the hybrid numerical-asymptotic we restrict attention to the computation of the the expectation of $u(x)$ at any given point $x\in[0,1]$ which we can approximate by $\mathbb{E}[\widetilde{u}^1(x)]$, with $\widetilde{u}^1$ given in (5.19), (5.20) above. To express this quantity neatly, for any smooth enough function ξ defined on $[0,1]\times[-1,1]^d$, we define the integrals

$$(\mathcal{I}^{\pm\mathbf{a}}\xi)(x) = 2^{-d}\int_{[-1,1]^d}\xi(x,\boldsymbol{y})\exp(\pm\mathrm{i}k\mathbf{a}(x)\boldsymbol{y})d\boldsymbol{y} \quad\text{and}\quad (\mathcal{I}\xi)(x) = 2^{-d}\int_{[-1,1]^d}\xi(x,\boldsymbol{y})d\boldsymbol{y}. \tag{5.32}$$

Using this notation we have

$$\begin{aligned}\mathbb{E}[\widetilde{u}^1(x)] = \quad & \exp(\mathrm{i}kN_0(x))\,(\mathcal{I}^{+\mathbf{a}}\widetilde{\mu})(x) && (5.33)\\ & + \exp(-\mathrm{i}kN_0(x))\,(\mathcal{I}^{-\mathbf{a}}\widetilde{\nu})(x) && (5.34)\\ & + (\mathcal{I}\widetilde{F})(x) && (5.35)\end{aligned}$$

The integral (5.35) is not oscillatory and can be computed using the standard Clenshaw-Curtis-Smolyak rule, but (5.33) and (5.34) need to be computed using our FCCS rule. The values of $\widetilde{\mu}(x,\boldsymbol{y})$ and $\widetilde{\nu}(x,\boldsymbol{y})$ at the multidimensional quadrature nodes are computed using the procedure described in §5.2.

At this point we remark that, although the functions μ_j,ν_j are non-oscillatory with respect to spatial variable x, the coefficients α_j^ℓ appearing in (5.21) do depend on k (via application of the boundary conditions (5.8), (5.9)), and so $\widetilde{\mu},\widetilde{\nu}$ could potentially have derivatives (with respect to the random paramneters $\boldsymbol{y}$) which depend on k. We investigate this point in §6.2 and show that, under reasonable assumptions, the application of the FCCS rule to (5.33), (5.34), where $\widetilde{\mu},\widetilde{\nu}$ are approximated by the Smolyak inrterpolation can be justified and works well in practice.

Before proceeding, we remark that the computation above can easily be extended to general linear functionals on $H^1(0,1)$. For example, to approximate the expected value of $Gu := (u,g)_{L^2(0,1)}$, with $g\in L^2(0,1)$, we compute

$$\begin{aligned}\mathbb{E}\left[\int_0^1\widetilde{u}^1(x,\cdot)\overline{g}(x)\mathrm{d}x\right] &= \int_0^1\mathbb{E}\left[\widetilde{u}^1(x,\cdot)\right]\overline{g}(x)\mathrm{d}x\\ &= \int_0^1\exp(\mathrm{i}kN_0(x))(\mathcal{I}^{+\mathbf{a}}\widetilde{\mu})(x)\overline{g}(x)\mathrm{d}x + \int_0^1\exp(-\mathrm{i}kN_0(x))(\mathcal{I}^{-\mathbf{a}}\widetilde{\nu})(x)\overline{g}(x)\mathrm{d}x + \int_0^1(\mathcal{I}\widetilde{f})(x)\overline{g}(x)\mathrm{d}x,\end{aligned} \tag{5.36}$$

so that additional 1d oscillatory integrals now appear in the first and second terms of (5.36). Similar but slightly more complicated terms appear in the computation of the general linear functional on $H^1(0,1)$: $Gu := (u',g')_{L^2(0,1)} + (u,g)_{L^2(0,1)}$, for any $g\in H^1(0,1)$.

6 Numerical Experiments

Our code for the following numerical examples was based on the Sparse Grids Matlab Kit, available at `https://sites.google.com/view/sparse-grids-kit`. See also [2, 29].

6.1 Multi-dimensional quadrature

In this subsection we illustrate the convergence properties of the Filon-Clenshaw-Curtis-Smolyak rule. The absolute and relative errors are defined respectively as:

$$e^{k,d,\mathbf{a},r}(f) = \left|\mathcal{I}^{k,d,\mathbf{a}}f - \mathcal{I}^{k,d,\mathbf{a},r}f\right| \quad \text{and} \quad E^{k,d,\mathbf{a},r}(f) = \frac{\left|\mathcal{I}^{k,d,\mathbf{a}}f - \mathcal{I}^{k,d,\mathbf{a},r}f\right|}{|\mathcal{I}^{k,d,\mathbf{a}}f|},$$

with $\mathcal{I}^{k,d,\mathbf{a}}f$ and $\mathcal{I}^{k,d,\mathbf{a},r}f$ defined in (1.2) and (3.6) respectively.

Example 1: Exactness check. Recall that by [3, Proposition 2], we have $\mathcal{Q}^{r,d}f = f$, for all

$$f \in \sum_{|\boldsymbol{\ell}|=r+d-1} \mathbb{P}(n_{\ell_1}) \otimes \mathbb{P}(n_{\ell_2}) \otimes \ldots \otimes \mathbb{P}(n_{\ell_d}), \tag{6.1}$$

where $n_1 = 0$ and $n_\ell = 2^{\ell-1}$ for $\ell \geq 2$ and $\mathbb{P}(n)$ denotes the univariate polynomials of degree n. Choosing

$$f(\boldsymbol{y}) = \prod_{j=1}^{d} y_j^2, \tag{6.2}$$

we have $\mathcal{Q}^{r,d}f = f$ when $r \geq d+1$. (This is because $\boldsymbol{\ell} := (2, 2, \ldots, 2)^\top$ satisfies $|\boldsymbol{\ell}| = 2d \leq r + d - 1$.) Results for $d = 4$, $\mathbf{a} = (1, 0, 1, 0)^\top$ are given in Table 1 and clearly indicate exactness of the approximation for $r \geq 5 = d + 1$.

$e^{k,d,\mathbf{a},r}(f)$							
r	1	2	3	4	5	6	7
$k = \pi/2$	2.59 (-2)	2.59 (-2)	2.59 (-2)	2.59 (-2)	3.47 (-17)	2.09 (-17)	3.71 (-17)
$k = 2\pi$	4.56 (-3)	4.56 (-3)	4.56 (-3)	4.56 (-3)	6.07 (-18)	1.65 (-17)	2.33 (-17)

Table 1: $e^{k,d,\mathbf{a},r}(f)$ for $d = 4$, $k = \pi/2, 2\pi$ with f as in (6.2)

In the following three examples we compute (1.1) for the case $d = 3$ with

$$f(\boldsymbol{y}) = \cos(m y_1 y_2 y_3) \tag{6.3}$$

and various choices of $\mathbf{a}$. The exact value of $\mathcal{I}^{k,d,\mathbf{a}}f$ is taken to be $\mathcal{I}^{k,d,\mathbf{a},10}f$, and this value is used to compute the errors. We repeated with exact value computed with $r = 12$ and observed no changes in the results.

Example 2 - fast convergence with increasing r. This experiment illustrates the fast decay of the error as r increases when f is smooth. The relative errors for four different values of m as r increases and $k = 101.53$ is fixed are given in Table 2. Convergence as r increases is observed in the columns corresponding to $m = 2, 4, 8$. For fixed r, the error grows roughly proportional to m. In the column headed $m = 16$ we see that convergence has hardly started yet, and higher r will be needed to see convergence. Intuition for this can be obtained from Theorem 1.3, since $\|f\|_{W^{p,d}} \sim m^p$, indicating the need for higher r before convergence commences.

	$m=2$	$m=4$	$m=8$	$m=16$
r	$E^{k,d,\mathbf{a},r}(f)$	$E^{k,d,\mathbf{a},r}(f)$	$E^{k,d,\mathbf{a},r}(f)$	$E^{k,d,\mathbf{a},r}(f)$
3	3.22 (+0)	2.67 (+0)	4.32 (+0)	2.10 (+0)
4	4.10 (-2)	1.99 (-1)	3.73 (-1)	1.37 (-1)
5	2.20 (-3)	7.13 (-2)	1.90 (-1)	1.83 (-1)
6	9.47 (-5)	2.25 (-3)	5.87 (-2)	1.62 (-1)

Table 2: Relative error as r increases when $\mathbf{a} = (1,1,1)^\top$, $k = 101.53$ and f given by (6.3)

Example 3 - asymptotic decay as $k \to \infty$, with fixed r. Again we fix $\mathbf{a} = (1,1,1)^\top$ and study the behaviour of the errors as $k \to \infty$. In general, the decay rate of the exact and approximate intergrals as $k \to \infty$ can be quite delicate. To see this, consider the following model 1D integral with integration by parts:

$$\int_{-1}^{1} \exp(\mathrm{i}ky) g(y) \mathrm{d}y = \frac{1}{\mathrm{i}k} \left[g(1) \exp(\mathrm{i}k) - g(-1) \exp(-\mathrm{i}k) \right] - \frac{1}{\mathrm{i}k} \int_{-1}^{1} \exp(\mathrm{i}ky) g'(y) \mathrm{d}y. \tag{6.4}$$

A second integration by parts shows that the second term on the right-hand side of (6.4) is $\mathcal{O}(k^{-2})$, while the first term takes the form $C(k)k^{-1}$, so is dominant in general. However in general the factor $C(k)$ can vary considerably with respect to k, leading to possibly irregular behaviour as $k \to \infty$. However by taking $k = 2\ell\pi + \pi/4$, $\ell = 2, 4, 8, \ldots, 128$, $C(k)$ turns out to be the $k-$independent constant $C(k) = ((1-\mathrm{i})g(1) + (1+\mathrm{i})g(-1))/\sqrt{2}$, thus ensuring (excluding special cases of g) a regular $O(k^{-1})$ decay for the dominant term in (6.4). We use this sequence of wavenumbers in the experiments below. With f as given in (6.3) with $m = 2$, Table 3 illustrates that $|\mathcal{I}^{k,d,\mathbf{a}} f|$ decays with $\mathcal{O}(k^{-d})$. Also, the relative error remains bounded with respect to k when $r = 3$ and decays with order at least $\mathcal{O}(k^{-1})$ for $r = 4$, as predicted by Theorem 1.3.

			$r=3$				$r=4$			
k	$\lvert\mathcal{I}^{k,d,\mathbf{a}}(f)\rvert$		$e^{k,d,\mathbf{a},r}(f)$		$E^{k,d,\mathbf{a},r}(f)$		$e^{k,d,\mathbf{a},r}(f)$		$E^{k,d,\mathbf{a},r}(f)$	
13.35	1.06(-03)		2.25(-03)		2.12(+00)		2.35(-04)		2.21(-01)	
25.92	1.04(-04)	10.21	2.66(-04)	8.44	2.56(+00)	0.83	1.88(-05)	12.47	1.81(-01)	1.22
51.05	1.12(-05)	9.30	3.24(-05)	8.21	2.90(+00)	0.88	1.28(-06)	14.76	1.14(-01)	1.59
101.32	1.28(-06)	8.73	4.00(-06)	8.11	3.12(+00)	0.93	8.22(-08)	15.52	6.42(-02)	1.78
201.85	1.52(-07)	8.39	4.96(-07)	8.06	3.26(+00)	0.96	5.20(-09)	15.79	3.41(-02)	1.88
402.91	1.86(-08)	8.21	6.18(-08)	8.03	3.33(+00)	0.98	3.27(-10)	15.91	1.76(-02)	1.94
805.03	2.29(-09)	8.10	7.71(-09)	8.01	3.36(+00)	0.99	2.05(-11)	15.96	8.94(-03)	1.97

Table 3: Errors for $\mathbf{a} = (1,1,1)^\top$, f given by (6.3) with $m = 2$, increasing k, for $r = 3$ and $r = 4$

We recall that the rule analysed in this paper uses the mid-point rule at level 1, and Clenshaw-Curtis grids thereafter (see (1.3)). By Remark 4.6, a better asymptotic decay of the relative error with respect to k – even for small r – can be obtained if we use instead the two-point Clenshaw-Curtis rule at level 1. Results using this rule for the same case as in Table 3 are given in Table 4. Here the relative error is observed to decay with $\mathcal{O}(k^{-1})$ when $r = 3$ and with $\mathcal{O}(k^{-2})$ when $r = 4$ and we observe several orders of magnitude improvement in accuracy for large k. However we recall that the number of quadrature points grows more rapidly with dimension when the level 1 rule has more than one point and so this method may

not be appropriate for higher dimensions. But if d is not too big, this variation should be useful if accurate results for very high k are required.

	$r=3$				$r=4$			
k	$e^{k,d,\mathbf{a},r}(f)$		$E^{k,d,\mathbf{a},r}(f)$		$e^{k,d,\mathbf{a},r}(f)$		$E^{k,d,\mathbf{a},r}(f)$	
13.35	6.65(-05)		6.27(-02)		2.05(-05)		1.93(-02)	
25.92	2.57(-06)	25.91	2.47(-02)	2.54	8.37(-07)	24.44	8.06(-03)	2.39
51.05	5.36(-08)	47.96	4.79(-03)	5.16	2.86(-08)	29.30	2.56(-03)	3.15
101.32	1.03(-09)	51.98	8.05(-04)	5.95	9.25(-10)	30.89	7.23(-04)	3.54
201.85	2.19(-10)	4.71	1.43(-03)	0.56	2.93(-11)	31.55	1.92(-04)	3.76
402.91	1.88(-11)	11.65	1.01(-03)	1.42	9.19(-13)	31.92	4.94(-05)	3.89
805.03	1.34(-12)	14.04	5.83(-04)	1.73	2.85(-14)	32.23	1.24(-05)	3.98

Table 4: Errors for f given in (6.3) with $m=2$, $\mathbf{a}=(1,1,1)^\top$, for $r=3$ and $r=4$ using Clenshaw-Curtis 2 point rule on level 1

Example 4 - Robustness to variation in the elements of a. Recall that integral $\mathcal{I}^{k,d,\mathbf{a}}(f)$ is only oscillatory in the jth dimension when $k|a_j| \geq 1$, and if this is not true then the standard quadrature rule (and not its Filon variant) is applied in that dimension. The standard rule is Clenshaw-Curtis when $\ell \geq 2$ and the mid-point rule when $\ell = 1$. Here we show that the algorithm proposed works stably when $k|a_j|$ passes through the value 1 or when $a_j = 0$. In this example we consider examples with $\mathbf{a} = (0.01, 1, 1)^\top$ and $\mathbf{a} = (0, 1, 1)^\top$. In the first case $\mathcal{I}^{k,d,\mathbf{a}}(f)$ is not oscillatory in the y_1 direction unless $k \geq 100$ and in the second case it is never oscillatory in the y_1 direction. Results are in Table 5. The method behaves robustly to the choice of a_1.

	$\mathbf{a}=(0.01,1,1)^\top$			$\mathbf{a}=(0,1,1)^\top$		
	$k=25.92$	$k=101.32$	$k=201.85$	$k=25.92$	$k=101.32$	$k=201.85$
$\lvert\mathcal{I}^{k,d,\mathbf{a}}(f)\rvert$:	2.30(-3)	1.68(-4)	3.96(-5)	2.30(-3)	1.70(-4)	4.38(-5)
$E^{k,d,\mathbf{a},r}(f)$:						
$r=4$	1.96(-1)	1.34(-1)	5.42(-2)	1.80(-1)	1.64(-1)	1.63(-1)
$r=5$	2.41(-2)	7.00(-3)	3.54(-3)	2.47(-2)	7.97(-3)	4.87(-3)
$r=6$	1.37(-4)	2.70(-4)	4.57(-6)	2.11(-4)	3.88(-4)	2.21(-4)
$r=7$	1.30(-5)	2.13(-5)	1.92(-5)	1.56(-5)	1.53(-5)	1.09(-5)
$r=8$	2.05(-6)	4.46(-7)	1.59(-7)	2.12(-6)	8.60(-7)	2.48(-7)

Table 5: Values of the relative error as r increases, with $d=3, m=2$ and two different choices of $\mathbf{a}$.

In the next example we illustrate the advantage arising when some dimensions of the problem are less important than others.

Example 5 - Decaying importance of dimensions Here we consider the functions

$$f(\boldsymbol{y}) = \cos(my_1y_2)\cos(my_3y_4)\cos(my_5y_6), \tag{6.5}$$

$$f(\boldsymbol{y}) = \cos(my_1y_2)\cos(0.1my_3y_4)\cos(0.01my_5y_6). \tag{6.6}$$

With $k = 16\pi + 1 \approx 51.27$ and $\mathbf{a} = (1, \ldots, 1)^\top \in \mathbb{R}^6$, the reference values of the integrals are taken to be the product of the quadrature approximations to the three 2-d integrals:

$$\mathcal{I}^{k,6,\mathbf{a}}(f) \;\approx\; \prod_{j=1}^{3} \mathcal{I}^{k,2,(a_{2j-1},a_{2j}),10}(f_j).$$

The relative errors for each choice of f are given in Table 6, illustrating the substantial benefit of the decay in importance of the dimensions in case (6.6) compared with case (6.5).

	$m=1$	$m=2$	$m=3$	$m=4$
r	$E^{k,d,\mathbf{a},r}(f)$	$E^{k,d,\mathbf{a},r}(f)$	$E^{k,d,\mathbf{a},r}(f)$	$E^{k,d,\mathbf{a},r}(f)$
6	7.92 (-1)	3.14 (+1)	7.78 (+0)	1.74 (+1)
7	8.51 (-3)	1.49 (+0)	1.00 (+0)	6.15 (+0)
8	4.47 (-5)	8.51 (-2)	1.68 (-1)	2.52 (+0)
9	3.21 (-6)	3.62 (-4)	7.35 (-3)	3.71 (-1)
6	3.52 (-7)	2.27 (-5)	4.27 (-4)	6.51 (-3)
7	7.84 (-9)	1.93 (-6)	1.67 (-5)	1.56 (-4)
8	6.74 (-10)	1.35 (-7)	6.33 (-7)	6.04 (-6)
9	2.61 (-12)	8.76 (-10)	3.23 (-8)	1.18 (-6)

Table 6: Relative error as r increases when $d=6$, $\mathbf{a}=(1,\ldots,1)^\top$, $k=51.27$ and f given by (6.5) (top panel) and (6.6) (bottom panel)

Example 6 – Dimension-adaptive methods. It is well-known (e.g., [13, 26]) (and the previous example shows) that if the dimensions can be ordered so that higher dimensions become less and less important than lower dimensions, then dimension-adaptive tensor product methods will be more efficient than standard procedures. This observation is relevant to the UQ problem considered in the next subsection. To illustrate this here, we consider the integral

$$\mathcal{I}(x) := \int_{[-1,1]^d} n^{-1/2}(x,\boldsymbol{y}) \exp(\mathrm{i}k\mathbf{a}(x).\boldsymbol{y})\mathrm{d}y, \tag{6.7}$$

where n given in (5.26), with

$$n_0(x)=1 \quad \text{and} \quad n_j(x)=\exp(-j)\sin(j\pi x), \quad \text{for} \quad x\in[0,1], \tag{6.8}$$

and so, by (5.31),

$$a_j(x) \;=\; N_j(x) \;=\; \int_0^x n_j(x')\mathrm{d}x' \;=\; \exp(-j)\int_0^x \sin(j\pi x')\mathrm{d}x' = \frac{1}{j\pi}\exp(-j)\,(1-\cos(j\pi x)).$$

Since the functions μ_0,ν_0 (which constitute the principal parts of $\widetilde{\mu},\widetilde{\nu}$ in (5.20)) are $\boldsymbol{y}$-dependent multiples of $n^{-1/2}$, the computation of $\mathcal{I}(x)$ is a good test for the UQ computation considered in the following subsection.

In this example we choose $k=101.53$ and $x=1/2$ and we compare the performance of the 'standard' FCCS rule (i.e., the rule analysed above) with an adaptive version where the approximation $\mathcal{Q}^{r,d}\widehat{f}$ in (1.4) is replaced by a dimension-adaptive procedure.

Our algorithm is implemented using the Sparse Grids Matlab kit [26, 29] and uses an adaptive procedure motivated by discussions in [13, 26]. More precisely, since $|\mathcal{I}(x)|$ decreases as k increases, we aim to compute an approximation $\mathcal{I}^\tau(x)$ of $\mathcal{I}(x)$ so that the relative error estimate $|\mathcal{I}(x)-\mathcal{I}^\tau(x)|/|\mathcal{I}(x)|\le\tau$ holds. Since this cannot be done exactly, it is done approximately by computing local relative 'profit indicators', measuring the benefit of adding more sparse grid points to the existing approximation, defining the global profit indicator by taking the maximum of these, and iterating till the global profit indicator is below the tolerance.

To compute errors, a reference value for $\mathcal{I}(x)$ is computed by 'brute force', using the tensor product Gauss-Legendre rule with 25 Gauss points in each of the d dimensions, allowing us to compute errors.

Results are given in Tables 7 – 9. The tables show the substantial advantage of the adaptive method in terms the number of function evaluations over the standard method when the dimensions have decreasing importance, a situation often encountered in UQ applications.

	adaptive	standard		
	$\tau = 10^{-4}$	$r = 4$	$r = 5$	$r = 6$
relative error	1.15 (-7)	8.37 (-6)	1.34 (-7)	7.21 (-10)
number of function evaluations	53	137	401	1105

Table 7: Comparison of the dimension-adaptive and standard FCCS rule for $\mathcal{I}(1/2)$, $d = 4$

	adaptive	standard		
	$\tau = 10^{-6}$	$r = 4$	$r = 5$	$r = 6$
relative error	9.33 (-8)	8.46 (-6)	1.41 (-7)	8.64 (-10)
number of function evaluations	129	389	1457	4865

Table 8: Comparison of the dimension-adaptive and standard FCCS rule for $\mathcal{I}(1/2)$, $d = 6$

	adaptive	standard		
	$\tau = 10^{-6}$	$r = 4$	$r = 5$	$r = 6$
relative error	1.17 (-7)	8.46 (-6)	1.41 (-7)	7.85 (-10)
number of function evaluations	151	849	3937	15713

Table 9: Comparison of the dimension-adaptive and standard FCCS rules for $\mathcal{I}(1/2)$, $d = 8$

6.2 UQ problem for the Helmholtz equation

In this subsection we consider the Helmholtz problem (1.10) – (1.12) with random $n = n(x, \boldsymbol{y})$ given by (5.26), but with F a function of x only, so that the solution $u = u(x, \boldsymbol{y})$ depends on x and $\boldsymbol{y}$ (and also on the frequency k). Then formulae (5.33) – (5.34) show that $\mathbb{E}[u(x)]$ can be written as a sum of oscillatory integrals with kernels given in equation (5.20). These integrals are (formally) in a form suitable for approximation by our FCCS rule, but in order to predict more precisely how well this will work, some further analysis is needed to investigate the regularity of $\widetilde{\mu}$ and $\widetilde{\nu}$ with respect to $\boldsymbol{y}$. Since the principal components of $\widetilde{\mu}$, $\widetilde{\nu}$ (i.e. those components which are $\mathcal{O}(1)$ as $k \to \infty$) are μ_0, ν_0 respectively, we restrict the discussion here to the analysis of the regularity of μ_0, ν_0 with respect to $\boldsymbol{y}$. In the following discussion we make the simplifying assumptions that (in the random problem), the Dirichlet data $u_L = u(0)$ is a constant independent of $\boldsymbol{y}$ and that $n_\infty = n(1, \boldsymbol{y})$ is a positive constant for all $\boldsymbol{y} \in [-1, 1]^d$. Then it can be shown, after some algebra, that

$$\mu_0(x, \boldsymbol{y}) = \alpha_0^1(\boldsymbol{y}) n(x, \boldsymbol{y})^{-1/2}, \quad \text{and} \quad \nu_0(x, \boldsymbol{y}) = \alpha_0^2(\boldsymbol{y}) n(x, \boldsymbol{y})^{-1/2}, \tag{6.9}$$

where the functions α_0^j are given by

$$\alpha_0^2(\boldsymbol{y}) = \frac{u_L \sqrt{n(0, \boldsymbol{y})} n'(1, \boldsymbol{y})}{2\mathrm{i}} \frac{\exp(\mathrm{i}kN(1, \boldsymbol{y}))}{n'(1, \boldsymbol{y}) \sin(kN(1, \boldsymbol{y})) - 2kn_\infty^2 \exp(-\mathrm{i}kN(1, \boldsymbol{y}))} \tag{6.10}$$

and

$$\alpha_0^1(\boldsymbol{y}) = u_L \sqrt{n(0, \boldsymbol{y})} - \alpha_0^2(\boldsymbol{y}). \tag{6.11}$$

From this we see that:

(i) If $n'(1,\boldsymbol{y}) = 0$ (i.e., $n(x,\boldsymbol{y})$ is a constant function of x near $x = 1$ for all $\boldsymbol{y}$), then $\alpha_0^2(\boldsymbol{y}) = 0$, $\alpha_0^1(\boldsymbol{y}) = u_L\sqrt{n(0,\boldsymbol{y})}$, and there is no $k-$dependent oscillation with respect to $\boldsymbol{y}$ in $\alpha_0^1(\boldsymbol{y})$. The component r_0 in the expansion (5.10) corresponds to a wave moving from left to right across the domain;

(ii) If $n'(1,\boldsymbol{y}) \neq 0$ then, while α_0^1 and α_0^2 are both potentially have $k-$dependent oscillations with respect to $\boldsymbol{y}$, the *amplitude* of their oscillatory components decays with $\mathcal{O}(1/k)$ as k increases.

These facts allow us to apply the FCCS rule directly to the integrals (5.33) and (5.34) without any further splitting of their kernels. We do this in the following example for a case where $n'(1,\boldsymbol{y})$ is not the zero function, and observe good results.

In the following two examples we consider computing $\mathbb{E}[u(1)]$ for the problem (1.10) – (1.12) with $u_L = 1, n_\infty = 1, F(x) = x$, with n given by (5.26), and (6.8). So in this case $n(1,\boldsymbol{y}) = n_\infty = 1$ for all $\boldsymbol{y}$ and formulae (6.10), (6.11) hold.

Using the asymptotic approximation described in §5.1, this can be approximated by $\mathbb{E}[\widetilde{u}^1(1)]$, where $\widetilde{u}^1$ is defined in Remark 5.2. The formulae in (5.33), (5.34) and (5.35) show that $\mathbb{E}[\widetilde{u}^1(1)]$ can be written as a sum of three multidimensional integrals, the first two of which are oscillatory.

The functions $\widetilde{\mu}$ and $\widetilde{\nu}$ appearing in (5.33) and (5.34) are obtained using the formulae (5.19), (5.20), (5.21), requiring the solution of a system of ODEs that are non-oscillatory with respect to x. To solve these we use the method described in §5.2 to do this with parameters chosen as $M = 1, L = 1024$ and $M_G = 10$. These parameters are chosen to give very accurate values of $\widetilde{\mu}$ and $\widetilde{\nu}$ and it is not the purpose of this paper to investigate the most efficient choice of these parameters, since this question is not related to our main task here, namely to find methods which are efficient in terms of k and d dependence.

We then study the performance of both the standard and the adaptive methods for approximating the integrals appearing in (5.33), (5.34) and (5.35).

Example 7 - The standard FCCS method. For $r \geq 1$, we denote by $\mathbb{E}^{k,r}[\widetilde{u}^1(x)]$ the approximation of $\mathbb{E}[\widetilde{u}^1(x)]$ obtained by applying the FCCS rule to (5.33), (5.34) and (5.35) with maximum level r. Since (5.35) is not oscillatory, the FCCS rule just corresponds to a standard sparse grid quadrature on the hierarchy of grids (1.3).

r	$k = 8$	$k = 16$	$k = 32$	$k = 64$
5	5.86 (-3)	1.18 (-4)	8.84 (-4)	8.11 (-4)
6	5.83 (-3)	2.49 (-5)	5.13 (-5)	3.16 (-4)
7	5.83 (-3)	2.79 (-5)	7.62 (-6)	1.11 (-4)
8	5.83 (-3)	2.80 (-5)	6.53 (-6)	2.83 (-6)
9	5.83 (-3)	2.80 (-5)	6.49 (-6)	2.82 (-6)
10	5.83 (-3)	2.80 (-5)	6.49 (-6)	2.33 (-6)
11	5.83 (-3)	2.80 (-5)	6.49 (-6)	2.31 (-6)
12	5.83 (-3)	2.80 (-5)	6.49 (-6)	2.31 (-6)

Table 10: $|\mathbb{E}^{k,r}[\widetilde{u}^1(1)] - \mathbb{E}[u(1)]|$ for $d = 4$ as r and k vary

We first consider $d = 4$. A reference value is computed by applying the continuous piecewise linear finite element method to the full $k-$dependent boundary-value problem (1.10) – (1.12) with spatial mesh size $h = (2^{14} + 1)^{-1}$. This is done for sample points $\boldsymbol{y}$ chosen on the grid formed as the tensor product of the 1d Gauss-Legendre rule with 50 Gauss points in each of the d dimensions. This is an expensive

method, but it provides a very accurate $\mathbb{E}[u(1)]$, and is done only once to allow us to study errors. The absolute error $|\mathbb{E}^{k,r}[\widetilde{u}^1(1)] - \mathbb{E}[u(1)]|$ is shown in Table 10. Recall that the method we are studying has an error with respect to k (due to the asymptotic approximation) and with respect to r (from the sparse grid approximation). Hence we see convergence as both k and r increase by reading diagonally across the table, e.g., starting from $r = 7$ and $k = 8$ we see the sequence: $5.83(-3)$, $2.80(-5)$, $6.49(-6)$, $2.33(-6)$, and similarly for other diagonals.

Since, for $r = 12$ there is not much error in the oscillatory integrals, we see steady decay of the error (between $\mathcal{O}(k^{-1})$ amd $\mathcal{O}(k^{-2})$) as k increases. For small r, on the other hand, the rows of Table 10 do not exhibit steady decay with respect to k due to the error in the oscillatory integrals.

The computation of the exact reference value used in Table 10 is costly and not feasible for higher dimensions or wavenumbers. Instead, in Table 11 we study the error proxy

$$|\mathbb{E}^{k,r}[\widetilde{u}^1(1)] - \mathbb{E}^{k,r+4}[\widetilde{u}^1(1)]| \tag{6.12}$$

for $d = 4$ and higher values of k. Table 10 tells us that we should use this proxy cautiously, since (for example) computing this quantity for the column corresponding to $k = 8$ will give values uniformly of order 10^{-6} where the true error is much larger. However reading Table 11 diagonally we still see quite convincing convergence of this proxy to zero as k, r both increase simultaneously, although the convergence is not always monotonic.

r	$k = 32$	$k = 64$	$k = 128$
4	2.17 (-3)	5.35 (-4)	4.04 (-5)
5	8.77 (-4)	8.09 (-4)	5.43 (-5)
6	4.48 (-5)	3.19 (-4)	1.02 (-4)
7	2.22 (-6)	1.13 (-4)	5.19 (-5)
8	1.30 (-7)	1.50 (-6)	5.84 (-5)

Table 11: Values of the 'Error proxy' (6.12) for $d = 4$ for various r and k

In Table 11 we study the 'error proxy':

$$|\mathbb{E}^{k,r}[\widetilde{u}^1(1)] - \mathbb{E}^{k,r+2}[\widetilde{u}^1(1)]| \tag{6.13}$$

for the case $d = 6$, and observe a similar diagonal behaviour.

r	$k = 32$	$k = 64$	$k = 128$
4	2.21 (-3)	2.05 (-4)	1.25 (-4)
5	8.89 (-4)	9.17 (-4)	4.03 (-5)
6	4.20 (-5)	3.31 (-4)	1.54 (-4)
7	2.12 (-6)	1.08 (-4)	5.13 (-5)
8	1.20 (-7)	1.71 (-6)	5.63 (-5)

Table 12: Values of the 'Error proxy' (6.13) for $d = 6$ for various r and k

Example 8 - The dimension adaptive algorithm. Finally we consider the dimension adaptive method for the UQ problem. In this case, for any given k, $\mathbb{E}[u(1)]$ is computed by approximating the three integrals (5.33), (5.34) and (5.35), using the dimension adaptive method. To do this we introduce a tolerance parameter τ. Since the formulae (6.10), (6.11) show that the amplitude of ν_0 appearing in the integral (5.34) decays with $\mathcal{O}(1/k)$, and (recall Example 6 above), the adaptive method aims to control

the relative error (and not the absolute error) in the approximate integral, we use a smaller tolerance τ for integrals (5.33) and (5.35) and larger tolerance $k\tau$ for integral (5.34), and the resulting approximation of $\mathbb{E}[\widetilde{u}^1(x)]$ is denoted by $\mathbb{E}^{k,\tau}[\widetilde{u}^1(x)]$. For $d = 6, 8, 10$, we display values of the error proxy:

$$|\mathbb{E}^{k,\tau}[\widetilde{u}^1(1)] - \mathbb{E}^{k,\tau/4}[\widetilde{u}^1(1)]|. \tag{6.14}$$

We also let $N_{\widetilde{\mu}}, N_{\widetilde{\nu}}, N_{\widetilde{f}}$ denote, respectively, the number of grid points in the adaptive sparse grids used for computing integrals (5.33), (5.34) and (5.35) and set $N_{tot} = N_{\widetilde{\mu}} + N_{\widetilde{\nu}} + N_{\widetilde{f}}$.

τ	$k = 32$	$k = 64$	$k = 128$	$k = 256$
0.01	3.84 (-5)	1.13 (-4)	2.77 (-5)	2.78 (-7)
0.005	6.37 (-5)	1.86 (-4)	2.57 (-5)	7.76 (-6)
0.0025	7.01 (-5)	5.98 (-5)	3.12 (-5)	3.19 (-7)
0.00125	2.64 (-4)	1.81 (-6)	6.63 (-5)	7.19 (-6)
0.01	(21, 27, 13, 61)	(21, 75, 13, 109)	(13, 15, 13, 41)	(13, 13, 13, 39)
0.005	(49, 43, 13, 105)	(21, 75, 13, 109)	(13, 21, 13, 47)	(21, 13, 13, 47)
0.0025	(53, 53, 13, 119)	(81, 75, 13, 169)	(13, 149, 13, 175)	(21, 15, 13, 49)
0.00125	(53, 77, 13, 143)	(141, 81, 13, 235)	(21, 149, 13, 183)	(39, 15, 13, 67)
0.000625	(53, 77, 13, 143)	(149, 141, 13, 303)	(31, 157, 13, 201)	(55, 15, 13, 83)
0.0003125	(91, 77, 13, 181)	(219, 141, 13, 373)	(167, 277, 13, 457)	(55, 45, 13, 113)

Table 13: Results of the dimension adaptive method for $d = 6$ for various τ and k. Top panel: Values of the 'Error proxy' (6.14). Bottom panel: $(N_{\widetilde{\mu}}, N_{\widetilde{\nu}}, N_{\widetilde{f}}, N_{tot})$

τ	$k = 32$	$k = 64$	$k = 128$	$k = 256$
0.01	3.84 (-5)	1.13 (-4)	2.77 (-5)	2.78 (-7)
0.005	6.37 (-5)	1.86 (-4)	2.57 (-5)	7.76 (-6)
0.0025	7.01 (-5)	5.98 (-5)	3.12 (-5)	3.19 (-7)
0.00125	2.64 (-4)	1.81 (-6)	6.63 (-5)	7.19 (-6)
0.01	(25, 31, 17, 73)	(25, 79, 17, 121)	(17, 19, 17, 53)	(17, 17, 17, 51)
0.005	(53, 47, 17, 117)	(25, 79, 17, 121)	(17, 25, 17, 59)	(25, 17, 17, 59)
0.0025	(57, 57, 17, 131)	(85, 79, 17, 181)	(17, 153, 17, 187)	(25, 19, 17, 61)
0.00125	(57, 81, 17, 155)	(145, 85, 17, 247)	(25, 153, 17, 195)	(43, 19, 17, 79)
0.000625	(57, 81, 17, 155)	(153, 145, 17, 315)	(35, 161, 17, 213)	(59, 19, 17, 95)
0.0003125	(95, 81, 17, 193)	(223, 145, 17, 385)	(171, 281, 17, 469)	(59, 49, 17, 125)

Table 14: Results of the dimension adaptive method for $d = 8$ for various τ and k. Top panel: Values of the 'Error proxy' (6.14). Bottom panel: $(N_{\widetilde{\mu}}, N_{\widetilde{\nu}}, N_{\widetilde{f}}, N_{tot})$

τ	$k=32$	$k=64$	$k=128$	$k=256$
0.01	3.84 (-5)	1.13 (-4)	2.77 (-5)	2.78 (-7)
0.005	6.37 (-5)	1.86 (-4)	2.57 (-5)	7.76 (-6)
0.0025	7.01 (-5)	5.98 (-5)	3.12 (-5)	3.19 (-7)
0.00125	2.64 (-4)	1.81 (-6)	6.63 (-5)	7.19 (-6)
0.01	(29, 35, 21, 85)	(29, 83, 21, 133)	(21, 23, 21, 65)	(21, 21, 21, 63)
0.005	(57, 51, 21, 129)	(29, 83, 21, 133)	(21, 29, 21, 71)	(29, 21, 21, 71)
0.0025	(61, 61, 21, 143)	(89, 83, 21, 193)	(21, 157, 21, 199)	(29, 23, 21, 73)
0.00125	(61, 85, 21, 167)	(149, 89, 21, 259)	(29, 157, 21, 207)	(47, 23, 21, 91)
0.000625	(61, 85, 21, 167)	(157, 149, 21, 327)	(39, 165, 21, 225)	(63, 23, 21, 107)
0.0003125	(99, 85, 21, 205)	(227, 149, 21, 397)	(175, 285, 21, 481)	(63, 53, 21, 137)

Table 15: Results of the dimension adaptive method for $d = 10$ for various τ and k. Top panel: Values of the 'Error proxy' (6.14). Bottom panel: $(N_{\widetilde{\mu}}, N_{\widetilde{\nu}}, N_{\widetilde{f}}, N_{tot})$

In Tables 13 – 15 we again observe diagonal convergence as $\tau \to 0$ and $k \to \infty$. For fixed k and τ we see only very modest growth in the amount of work as the dimension increases. In fact for $\tau = 0.00125$ and any fixed k a linear least squares fit on the data here suggests the number of function evaluations grows at most like $\mathcal{O}(d^{0.1})$ as d increases. For fixed τ and d, we see some initial growth of the work as k increases, but this seems to reduce substantially as k gets higher.

Acknowledgement

The research of I.G. Graham is supported by UK EPSRC grant EP/S003975/1. The research of Z. Zhang is supported by Hong Kong RGC grants (Projects 17300318 and 17307921), National Natural Science Foundation of China (Project 12171406), and Seed Funding for Strategic Interdisciplinary Research Scheme 2021/22 (HKU). The computations were performed using research computing facilities offered by Information Technology Services, the University of Hong Kong.

References

[1] A. K. Aziz, R. B. Kellogg, and A. B. Stephens, *A two point boundary value problem with a rapidly oscillating solution*, Numer. Math., 53 (1988), pp. 107–121.

[2] J. Bäck, F. Nobile, L. Tamellini, and R. Tempone, *Stochastic spectral Galerkin and collocation methods for PDEs with random coefficients: a numerical comparison*, in Spectral and High Order Methods for Partial Differential Equations, J. Hesthaven and E. Ronquist, eds., vol. 76 of Lecture Notes in Computational Science and Engineering, Springer, 2011, pp. 43–62. Selected papers from the ICOSAHOM '09 conference, June 22-26, Trondheim, Norway.

[3] V. Barthelmann, E. Novak, and K. Ritter, *High dimensional polynomial interpolation on sparse grids*, Adv. Comput. Math., 12 (2000), pp. 273–288.

[4] S. N. Chandler-Wilde, I. G. Graham, S. Langdon, and E. A. Spence, *Numerical-asymptotic boundary integral methods in high-frequency acoustic scattering*, Acta Numer., 21 (2012), pp. 89–305.

[5] A. Chkifa, A. Cohen, and C. Schwab, *High-dimensional adaptive sparse polynomial interpolation and applications to parametric PDEs*, Found. Comput. Math., 14 (2014), pp. 601–633.

[6] C. W. Clenshaw and A. R. Curtis, *A method for numerical integration on an automatic computer*, Numer. Math., 2 (1960), pp. 197–205.

[7] J. Dick, F. Y. Kuo, and I. H. Sloan, *High-dimensional integration: the quasi-Monte Carlo way*, Acta Numer., 22 (2013), pp. 133–288.

[8] V. Domínguez, I. G. Graham, and T. Kim, *Filon-Clenshaw-Curtis rules for highly-oscillatory integrals with algebraic singularities and stationary points*, SIAM J. Numer. Anal., 51 (2013), pp. 1542–1566.

[9] V. Domínguez, I. G. Graham, and V. P. Smyshlyaev, *Stability and error estimates for Filon-Clenshaw-Curtis rules for highly oscillatory integrals*, IMA J. Numer. Anal., 31 (2011), pp. 1253–1280.

[10] X. Feng, J. Lin, and C. Lorton, *An efficient numerical method for acoustic wave scattering in random media*, SIAM/ASA J. Uncertainty Quantification, 3 (2015), pp. 790–822.

[11] M. Ganesh, F. Kuo, and I. Sloan, *Quasi-Monte Carlo finite element analysis for wave propagation inheterogeneous random media*, SIAM/ASA J Uncertainty Quantification, 9 (2021), pp. 106–134.

[12] T. Gerstner and M. Griebel, *Numerical integration using sparse grids*, Numer. Algorithms, 18 (1998), pp. 209–232.

[13] ——, *Dimension-adaptive tensor-product quadrature*, Computing, 71 (2003), pp. 65–87.

[14] A. Gibbs, D. P. Hewett, D. Huybrechs, and E. Parolin, *Fast hybrid numerical-asymptotic boundary element methods for high frequency screen and aperture problems based on least-squares collocation*, SN Partial Differ. Equat. Appl., 1 (2020), pp. 1–26.

[15] S. P. Groth, D. P. Hewett, and S. Langdon, *A hybrid numerical–asymptotic boundary element method for high frequency scattering by penetrable convex polygons*, Wave Motion, 78 (2018), pp. 32–53.

[16] M. Hardy, *Combinatorics of partial derivatives*, Electron. J. Combin., 13 (2006), pp. 1–13.

[17] D. Huybrechs and S. Vandewalle, *The construction of cubature rules for multivariate highly oscillatory integrals*, Math. Comp., 76 (2007), pp. 1955–1980.

[18] A. Iserles, *On the numerical quadrature of highly-oscillating integrals I: Fourier transforms*, IMA J. Numer. Anal., 24 (2004), pp. 365–391.

[19] ——, *On the numerical quadrature of highly-oscillating integrals II: Irregular oscillators*, IMA J. Numer. Anal., 25 (2005), pp. 25–44.

[20] A. Iserles and S. Nørsett, *On quadrature methods for highly oscillatory integrals and their implementation*, BIT Numer. Math., 44 (2004), pp. 755–772.

[21] ——, *Quadrature methods for multivariate highly oscillatory integrals using derivatives*, Math. Comp., 75 (2006), pp. 1233–1258.

[22] A. Iserles, S. Nørsett, and S. Olver, *Highly oscillatory quadrature: The story so far*, in Numerical mathematics and advanced applications, Springer, 2006, pp. 97–118.

[23] G. Maierhofer, A. Iserles, and N. Peake, *An extended Filon–Clenshaw–Curtis method for high-frequency wave scattering problems in two dimensions*, arXiv preprint arXiv:2006.05551, (2020).

[24] J. M. Melenk, *On the convergence of Filon quadrature*, J. Comput. Appl. Math., 234 (2010), pp. 1692–1701.

[25] F. Nobile, L. Tamellini, and R. Tempone, *Convergence of quasi-optimal sparse-grid approximation of Hilbert-space-valued functions: application to random elliptic PDEs*, Numer. Math., 134 (2016), pp. 343–388.

[26] F. Nobile, L. Tamellini, F. Tesei, and R. Tempone, *An adaptive sparse grid algorithm for elliptic PDEs with lognormal diffusion coefficient*, in Sparse Grids and Applications-Stuttgart 2014, Springer, 2016, pp. 191–220.

[27] E. Novak and K. Ritter, *Simple cubature formulas with high polynomial exactness*, Constr. Approx., 15 (1999), pp. 499–522.

[28] O. Pembery, *The Helmholtz equation in heterogeneous and random media: analysis and numerics*, PhD thesis, University of Bath, 2020.

[29] C. Piazzola and L. Tamellini, *The Sparse Grids Matlab kit–a Matlab implementation of sparse grids for high-dimensional function approximation and uncertainty quantification*, arXiv preprint arXiv:2203.09314, (2022).

[30] C. Schillings and C. Schwab, *Sparse, adaptive Smolyak quadratures for Bayesian inverse problems*, Inverse Probl., 29 (2013), p. 065011.

[31] E. A. Spence and J.Wunsch, *Wavenumber-explicit parametric holomorphy of Helmholtz solutions in the context of uncertainty quantification*, ArXiv preprint: arXiv:2203.10270, (2022).

[32] G. W. Wasilkowski and H. Wozniakowski, *Explicit cost bounds of algorithms for multivariate tensor product problems*, J. Complexity, 11 (1995), pp. 1–56.

[33] R. Wong, *Asymptotic approximations of integrals*, Classics in Applied Mathematics 24, SIAM, 2001.

[34] S. Xiang, *Efficient Filon-type methods for* $\int_a^b f(x)e^{i\omega g(x)}dx$, Numer. Math., 105 (2007), pp. 633–658.

[35] J. Zech and C. Schwab, *Convergence rates of high dimensional Smolyak quadrature*, ESAIM: Math. Model. Numer. Anal., 54 (2020), pp. 1259–1307.