$\mathbb{Q}_\ell$- VERSUS $\mathbb{F}_\ell$-COEFFICIENTS IN THE GROTHENDIECK-SERRE/TATE CONJECTURES

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In honor of Moshe Jarden's 80th birthday

ABSTRACT. We investigate the relation between the Grothendieck-Serre/Tate (G-S/T for short) conjectures with $\mathbb{Q}_\ell$- and $\mathbb{F}_\ell$-coefficients for $\ell \gg 0$ going through their ultraproduct formulations. Our main result roughly asserts that the G-S/T conjecture with $\mathbb{F}_\ell$-coefficients for $\ell \gg 0$ always implies the G-S/T conjecture with $\mathbb{Q}_\ell$-coefficients for $\ell \gg 0$ and that the converse implication holds at least in characteristic $p > 0$. In characteristic $p > 0$, this completes partly the motivic picture predicting that the G-S/T conjecture should be independent of the field of coefficients. As a concrete application of our result, we obtain that over an arbitrary finitely generated fields of characteristic $p > 0$, the Tate conjecture with $\mathbb{Q}_\ell$-coefficients for divisors and some $\ell \neq p$ is equivalent to the finiteness of the Galois-fixed part of the prime-to-p torsion subgroup of the geometric Brauer group. This generalizes a well-known theorem of Tate over finite fields.

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1. INTRODUCTION

Let K be a field of characteristic $p \geq 0$. Fix an algebraic closure $\overline{K}$; write $\pi_1(K) := \pi_1(\mathrm{Spec}(K), \mathrm{Spec}(\overline{K}))(= \mathrm{Aut}(\overline{K}/K))$ for the absolute Galois group of K. A variety over K (or a K-variety) means a scheme separated and of finite type over K. Let $\mathrm{SmP}(K)$ denote the symmetric monoidal category of smooth projective varieties over K.

1.1. Conjectures for realization functors.

For $X \in \mathrm{SmP}(K)$, let $CH^w(X)$ denote the Chow group of codimension w cycles (modulo rational equivalence) and $CH(X) := \oplus_{w\geq 0} CH^w(X)$ the $\mathbb{Z}$-graded Chow ring.

Let $CHM(K)$ denote the category of Chow motives over K with $\mathbb{Q}$-coefficients and $\mathrm{SmP}(K)^{op} \to CHM(K)$ the canonical functor [A04, 4.1.3]; fix a Weil cohomology $\mathrm{H} : CHM(K) \otimes C_H \to \mathcal{T}_H$ with field of coefficients C_H and enriched Tannakian target category $\mathcal{T}_H$ - See [A04, 3.3, 4.2.5, 7.1]. For $X \in \mathrm{SmP}(K)$, let $G_H(X)$ denote the Tannakian group of the Tannakian subcategory $\langle \mathrm{H}(X) \rangle$ generated by $\mathrm{H}(X)$ in $\mathcal{T}_H$. The following unifying conjecture is at the heart of the philosophy of pure motives.

1.1.1. **Conjecture.** *For every $X \in \mathrm{SmP}(K)$,*
(1) (Semisimplicity) $\mathrm{H}(X)$ *is semisimple - equivalently $G_H(X)$ is a reductive algebraic group over C_H;*
(2) (Fullness) *The image of the cycle class map $[-]_H : CH(X) \otimes C_H \to \oplus_{w\geq 0} \mathrm{H}^{2w}(X)(w)$ is the subspace of $G_H(X)$-invariant classes.*

The most standard avatars of Conjecture 1.1.1 are (for $K = \mathbb{C}$) the Hodge conjecture ([H52], [A04, 7.2]) for singular cohomology with enriched Tannakian target category the category of $\mathbb{Q}$-Hodge structures (so that $C_H = \mathbb{Q}$) and (for K finitely generated over its prime field) the Grothendieck-Serre/Tate (G-S/T for short) conjecture ([T65], [A04, 7.3]) for ℓ-adic cohomology ($\ell \neq p$) with enriched Tannakian target category the category of finite-dimensional $\mathbb{Q}_\ell$-vector spaces endowed with a continuous action of $\pi_1(K)$ (so that $C_H = \mathbb{Q}_\ell$). The fullness part of Conjecture 1.1.1 for H implies the standard conjecture of Lefschetz type [A04, 5.2.4] for H. If $p = 0$ this is already enough to imply all the standard conjectures for H [A04, 5.4.2.2]. If $p > 0$, combined with the semisimplicity part of Conjecture 1.1.1 for H, this also implies all the standard conjectures for H (except possibly the standard conjecture of Hodge type) [A04, 7.1.1.1]. In particular, Conjecture 1.1.1 for H implies that numerical and H-homological equivalences coincide so that, after modifying the commutativity constraint, the category of numerical motives becomes a semisimple Tannakian category over $\mathbb{Q}$. Let Q_X be any finite field extension of $\mathbb{Q}$ neutralizing the

Tannakian subcategory $\langle X\rangle$ generated by the numerical motive X in the category of numerical motives (with modified commutativity constraint) [DM82, Rem. 3.10], let $H : \langle X \otimes Q_X\rangle \to Vect_{Q_X}$ be a fiber functor and let $G(X)$ denote the corresponding Tannakian group; this is a reductive group over Q_X acting faithfully on the finite-dimensional Q_X-vector space $H(X)$. Assume Conjecture 1.1.1 holds for another Weil cohomology $\mathrm{H}' : CHM(K) \otimes C_{H'} \to \mathcal{T}_{H'}$. Then the general formalism of Tannakian categories implies the following.

1.1.2. **Conjecture.** *For every* $X \in \mathrm{SmP}(K)$ *and embedding of* Q_X *in* $\overline{C}_{H'}$*, one has* $G(X) \times_{Q_X} \overline{C}_{H'} \simeq G_{H'}(X) \times_{C_{H'}} \overline{C}_{H'}$ *acting on* $H(X) \otimes_{Q_X} \overline{C}_{H'} \simeq \mathrm{H}'(X) \otimes_{C_{H'}} \overline{C}_{H'}$.

When K has characteristic 0, one expects $Q_X = \mathbb{Q}$ and the isomorphisms of Conjecture 1.1.2 to hold over $C_{H'}$. When K has characteristic $p > 0$, as Serre noticed, this cannot always hold [Gr68, §1.7].

1.2. **Realization functors arising from étale cohomology.** Let $\mathcal{L}$ denote the set of all primes $\neq p$ and let $\mathcal{U}$ denote the set of all non-principal ultrafilters on $\mathcal{L}$. For $\ell \in \mathcal{L}$ let $\mathbb{F}_\ell$ denote the finite field with ℓ elements and $\mathbb{Q}_\ell$ the completion of $\mathbb{Q}$ at ℓ. For $\mathfrak{u} \in \mathcal{U}$ let $\mathbb{Q}_\mathfrak{u}$ (resp. $\mathbb{Q}_\mathfrak{u}$) denote the residue field of the maximal ideal of $\underline{\mathbb{F}} := \prod_{\ell\in\mathcal{L}} \mathbb{F}_\ell$ (resp. $\underline{\mathbb{Q}} := \prod_{\ell\in\mathcal{L}} \mathbb{Q}_\ell$) corresponding to $\mathfrak{u}$ (See Section 8 for details about ultraproducts).

The G-S/T conjecture is the incarnation of Conjecture 1.1.1 for the Weil cohomologies derived from étale cohomology.

1.2.1. These are built from the following cohomology groups:

- For every $\ell \in \mathcal{L}$, $\mathbb{Q}_\ell$-cohomology $\mathrm{H}^w(X_{\overline{K}}, \mathbb{Q}_\ell) := (\varprojlim_n \mathrm{H}^w(X_{\overline{K}}, \mathbb{Z}/\ell^n)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$;
- For every $\mathfrak{u} \in \mathcal{U}$, $\mathbb{Q}_\mathfrak{u}$-cohomology $\mathrm{H}^w(X_{\overline{K}}, \mathbb{Q}_\mathfrak{u}) := (\prod_{\ell\in\mathcal{L}} \mathrm{H}^w(X_{\overline{K}}, \mathbb{F}_\ell) \otimes_{\underline{\mathbb{F}}} \mathbb{Q}_\mathfrak{u}$;
  $\mathbb{Q}_\mathfrak{u}$-cohomology $\mathrm{H}^w(X_{\overline{K}}, \mathbb{Q}_\mathfrak{u}) := (\prod_{\ell\in\mathcal{L}} \mathrm{H}^w(X_{\overline{K}}, \mathbb{Q}_\ell)) \otimes_{\underline{\mathbb{Q}}} \mathbb{Q}_\mathfrak{u}$.

The following diagram summarizes the relation between the various coefficients:

$$\begin{array}{ccccccccc}
\mathbb{Q}_\ell & \twoheadleftarrow & \underline{\mathbb{Q}} & \longleftarrow & \widehat{\mathbb{Z}} & \twoheadrightarrow & \underline{\mathbb{F}} & \twoheadrightarrow & \mathbb{F}_\ell \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathbb{Q}_\mathfrak{u} & \longleftarrow & \widehat{\mathbb{Z}} \otimes \mathbb{Q} & \twoheadrightarrow & \mathbb{Q}_\mathfrak{u} & &
\end{array}$$

From now on, assume the base field K is finitely generated. Let C denote any of $\mathbb{Q}_\ell$, $\mathbb{Q}_\mathfrak{u}$, $\mathbb{Q}_\mathfrak{u}$ and write $\mathrm{H}_C(X) := \mathrm{H}(X_{\overline{K}}, C)$. The Tannakian target category $\mathcal{T}_{H_C}$ is the category of finite-dimensional continuous C-representations of $\pi_1(K)$ (as usual, $\mathbb{F}_\ell$ is equipped with the discrete topology, $\mathbb{Q}_\ell$ with the ℓ-adic topology, $\underline{\mathbb{F}}$, $\underline{\mathbb{Q}}$ with the product topology and $\mathbb{Q}_\mathfrak{u}$, $\mathbb{Q}_\mathfrak{u}$ with the quotient topology of the product topology on $\underline{\mathbb{F}}$, $\underline{\mathbb{Q}}$). For $X \in \mathrm{SmP}(K)$ the group $G_{H_C}(X)$ is the Zariski-closure of the image of $\pi_1(K)$ acting on $\mathrm{H}_C(X)$.

1.2.2. *The G-S/T conjecture.* For an integer $w \geq 0$ and $X \in \mathrm{SmP}(K)$, consider the following assertions[1].

(S, C, $\frac{w}{2}$, X)	The action of $\pi_1(K)$ on $\mathrm{H}^w(X_{\overline{K}}, C)$ is semisimple.
(wS, C, w, X)	The inclusion $\mathrm{H}^{2w}(X_{\overline{K}}, C(w))^{\pi_1(K)} \hookrightarrow \mathrm{H}^{2w}(X_{\overline{K}}, C(w))$ splits $\pi_1(K)$-equivariantly.
(wS', C, w, X)	The canonical morphism $c_w : \mathrm{H}^{2w}(X_{\overline{K}}, C(w))^{\pi_1(K)} \to \mathrm{H}^{2w}(X_{\overline{K}}, C(w))_{\pi_1(K)}$ induced by the identity is an isomorphism.
(F, C, w, X)	The cycle map $[-] : CH^w(X) \otimes C \to \mathrm{H}^{2w}(X_{\overline{K}}, C(w))^{\pi_1(K)}$ is surjective.
(sF, C, w, X)	The cycle map $[-] : CH^w(X_{\overline{K}}) \otimes C \to \varinjlim_{K'/K \text{ finite}} \mathrm{H}^{2w}(X_{\overline{K}}, C(w))^{\pi_1(K')}$ is surjective.

Apart from sF, the above assertions also make sense with C replaced by $\mathbb{F}_\ell$, $\ell \in \mathcal{L}$; we will use the corresponding notation.

[1] S stands for 'semisimplicity', wS for 'weak semisimplicity', F for 'Fullness' and sF for 'stabilized Fullness'.

With this notation, the classical ([T66], where it is only formulated for $C = \mathbb{Q}_\ell$) G-S/T conjecture (= Conjecture 1.1.1) for C asserts that (S, C, $\frac{w}{2}$, X) and (F, C, w, X) hold for every $X \in \mathrm{SmP}(K)$ and integer $w \geq 0$.

1.2.3. *Known results.* The G-S/T conjecture is widely open. If $p > 0$ and K is finite (resp. $p > 0$, resp. $p = 0$), Tate [T66] (resp. Zarhin [Z75], [Z77], Mori [Mo77], resp. Faltings [FW84]) proved (S, $\mathbb{F}_\ell$, $\frac{1}{2}$, X), $\ell \gg 0$ and (S, $\mathbb{Q}_\ell$, $\frac{1}{2}$, X) for X arbitrary and (F, $\mathbb{F}_\ell$, 1, X), (S, $\mathbb{F}_\ell$, $\frac{w}{2}$, X), $\ell \gg 0$ and (F, $\mathbb{Q}_\ell$, 1, X), (S, $\mathbb{Q}_\ell$, $\frac{w}{2}$, X) for X an abelian variety. Their proofs for $\mathbb{F}_\ell$, $\ell \gg 0$ mimic their proofs for $\mathbb{Q}_\ell$; they do not deduce one of the statements from the other.

By works of several authors ([N83], [NO85], [Ma14], [Cha13], [MP15], [KMP16], [MP20], [I18]), (F, $\mathbb{Q}_\ell$, w, X), (S, $\mathbb{Q}_\ell$, $\frac{w}{2}$, X) are now established for X a K3 surface. For K3 surfaces, (F, $\mathbb{F}_\ell$, w, X), $\ell \gg 0$ and (S, $\mathbb{F}_\ell$, $\frac{w}{2}$, X), $\ell \gg 0$ hold as well. This is due to Skorobogatov-Zarhin if $p \geq 3$ ([SkZ15]), Ito if $p = 2$ [I18] and Skorobogatov-Zarhin ([SkZ08]). To our knowledge, these are the only instances where (F, $\mathbb{F}_\ell$, w, X), $\ell \gg 0$ and (S, $\mathbb{F}_\ell$, $\frac{w}{2}$, X), $\ell \gg 0$ are deduced directly from (F, $\mathbb{Q}_\ell$, w, X), $\ell \gg 0$ and (S, $\mathbb{Q}_\ell$, $\frac{w}{2}$, X) (and not by mimicking or adjusting the proof for $\mathbb{Q}_\ell$-coefficients to $\mathbb{F}_\ell$-coefficients). The arguments of these authors, however, rely on specific features of K3 surfaces, in particular the Kuga-Satake construction[2].

Eventually, formal arguments allow to deduce a few other cases from the above ones - See *e.g.* [T94, Thm. 5.2].

1.3. When $p = 0$ and K is embedded into $\mathbb{C}$, the existence of comparison isomorphisms between étale and singular cohomologies (See *e.g.* [A04, 3.4.2]) implies that $\mathrm{H}_{\mathbb{Q}_\dagger}$-homological equivalence is independent of $\dagger \in \mathcal{L} \cup \mathcal{U}$, which ensures that Conjecture 1.1.2 for the $\mathrm{H}_{\mathbb{Q}_\dagger}$, $\dagger \in \mathcal{L} \cup \mathcal{U}$ and Conjecture 1.1.1 for one single $\mathrm{H}_{\mathbb{Q}_\dagger}$, $\dagger \in \mathcal{L} \cup \mathcal{U}$ imply Conjecture 1.1.1 for every $\mathrm{H}_{\mathbb{Q}_\dagger}$, $\dagger \in \mathcal{L} \cup \mathcal{U}$. But, unfortunately, very little is known about Conjecture 1.1.2 when $p = 0$. In contrast, when $p > 0$, and modulo the semisimplicity part of Conjecture 1.1.1, Conjecture 1.1.2 essentially boils down to the Langlands correspondence [L02], [Chi04], [CZ21]. However, in this case, the lack of comparison isomorphisms between the $\mathrm{H}_{\mathbb{Q}_\dagger}$, $\dagger \in \mathcal{L} \cup \mathcal{U}$ makes it unclear whether Conjecture 1.1.1 for one single $\mathrm{H}_{\mathbb{Q}_\dagger}$, $\dagger \in \mathcal{L} \cup \mathcal{U}$ implies Conjecture 1.1.1 for every $\mathrm{H}_{\mathbb{Q}_\dagger}$, $\dagger \in \mathcal{L} \cup \mathcal{U}$.

Let $\mathfrak{u} \in \mathcal{U}$. The aim of this note is to study a related but easier version of the above problem, namely to relate Conjecture 1.1.1 (in our case, the G-S/T conjecture) for $\mathrm{H}_{\mathbb{Q}_\ell}$, $\ell \in S$ for some $S \in \mathfrak{u}$, for $\mathrm{H}_{\mathbb{Q}_\mathfrak{u}}$ and for $\mathrm{H}_{\mathbb{Q}\mathfrak{u}}$. One motivation is to give conceptual and completely general (*i.e.* working for arbitrary smooth projective varieties) proofs of results like the above mentioned results of Skorobogatov-Zarhin and Ito for K3 surfaces. Another motivation is that we may hope that some new cases of the G-S/T conjecture could be proved more easily for $\mathbb{Q}_\mathfrak{u}$-coefficients and then transferred to $\mathbb{Q}\mathfrak{u}$- hence $\mathbb{Q}_\ell$-coefficients.

Assume $p > 0$. Let C denote any of $\mathbb{Q}_\ell$, $\mathbb{Q}_\mathfrak{u}$ or $\mathbb{Q}\mathfrak{u}$. For any integer $w \geq 0$, v and $X \in \mathrm{SmP}(K)$, let $G_C(X)$ denote the Zariski closure of the image of $\pi_1(K)$ acting on $\mathrm{H}^w(X_{\overline{K}}, C(v))$. Before considering Conjecture 1.1.1, we prove the following variant of Conjecture 1.1.2 for the group of connected components.

1.3.1. **Theorem.** *For every $X \in \mathrm{SmP}(K)$ the kernel of the canonical map $\pi_1(K) \to \pi_0(G_C(X))$ is independent of $C = \mathbb{Q}_\ell$, $\mathbb{Q}_\mathfrak{u}$, $\mathbb{Q}\mathfrak{u}$.*

For $C = \mathbb{Q}_\ell$, Theorem 1.3.1 is due to Serre [S00, p. 15 sqq] but Serre's arguments do not transfer as they are to $C = \mathbb{Q}_\mathfrak{u}$ or $\mathbb{Q}\mathfrak{u}$. Instead, we follow the argument of [LaP95, Prop. 2.2] and give a uniform proof of Theorem 1.3.1 (which for $C = \mathbb{Q}_\mathfrak{u}$, relies on the results of [CHT17]).

When $G_C(X)$ is connected for one of (equivalently every) $C = \mathbb{Q}_\ell$, $\mathbb{Q}_\mathfrak{u}$, $\mathbb{Q}\mathfrak{u}$, one says that X has *connected monodromy in degrees* (w, v). Under the connected monodromy assumption in degrees $(2w, w)$,

[2]The restriction $p \geq 3$ in [SkZ15] is related to the fact that the Kuga-Satake construction was not available for $p = 2$ at the time of [SkZ15]. This missing ingredient was developed by Kim and Madapusi Pera in [KMP16]. Building on [KMP16] and the method of [SkZ15], Ito extended Skorobogatov-Zarhin's result to the $p = 2$ case.

$\mathrm{H}^{2w}(X_{\overline{K}}, C(w))^{\pi_1(K)} = \mathrm{H}^{2w}(X_{\overline{K}}, C(w))^{\pi_1(K')}$ for every finite field extension K'/K and the G-S/T conjecture for X and $X' := X \times_K K'$ become equivalent - See Lemma 4.2.

Our second main result is the following statements.

1.3.2. **Proposition.** *For every $X \in \mathrm{SmP}(K)$, equidimensional of dimension d, and $\mathfrak{u} \in \mathcal{U}$, the following hold.*
(1) (F, $\mathbb{Q}_\mathfrak{u}$, d, X^2) + (S, $\mathbb{Q}_\mathfrak{u}$, $\frac{w}{2}$, X) $\Longrightarrow$ (S, $\mathbb{Q}_\mathfrak{u}$, $\frac{w}{2}$, X);
(2) (F, $\mathbb{Q}_\mathfrak{u}$, d, X^2) + (F, $\mathbb{Q}_\mathfrak{u}$, w, X) + (wS, $\mathbb{Q}_\mathfrak{u}$, w, X) $\Longrightarrow$ (wS, $\mathbb{Q}_\mathfrak{u}$, w, X).

1.3.3. **Theorem.** *Assume $p > 0$. For every $X \in \mathrm{SmP}(K)$, equidimensional of dimension d, and $\mathfrak{u} \in \mathcal{U}$, the following hold.*
(1) (wS, $\mathbb{Q}_\mathfrak{u}$, w, X) $\Longrightarrow$ (wS, $\mathbb{Q}_\mathfrak{u}$, w, X);
(2) (S, $\mathbb{Q}_\mathfrak{u}$, $\frac{w}{2}$, X) $\Longrightarrow$ (S, $\mathbb{Q}_\mathfrak{u}$, $\frac{w}{2}$, X);
(3) (F, $\mathbb{Q}_\mathfrak{u}$, i, X), $i = w, d-w$ + (wS, $\mathbb{Q}_\mathfrak{u}$, w, X) $\Longrightarrow$ (F, $\mathbb{Q}_\mathfrak{u}$, i, X), $i = w, d-w$ (+ (wS, $\mathbb{Q}_\mathfrak{u}$, w, X)).

Proposition 1.3.2 and Theorem 1.3.3 imply formally (See Lemma 4.1) the following.

1.3.4. **Corollary.** *For every $X \in \mathrm{SmP}(K)$, equidimensional of dimension d,*
(1) (F, $\mathbb{F}_\ell$, d, X^2) + (S, $\mathbb{F}_\ell$, $\frac{w}{2}$, X), $\ell \gg 0 \Longrightarrow$ (S, $\mathbb{Q}_\ell$, $\frac{w}{2}$, X), $\ell \gg 0$;
(2) (F, $\mathbb{F}_\ell$, d, X^2) + (F, $\mathbb{F}_\ell$, w, X) + (wS, $\mathbb{F}_\ell$, w, X), $\ell \gg 0 \Longrightarrow$ (wS, $\mathbb{Q}_\ell$, w, X), $\ell \gg 0$.
Assume $p > 0$. Then
(3) (wS, $\mathbb{Q}_\ell$, w, X), $\ell \gg 0 \Longrightarrow$ (wS, $\mathbb{F}_\ell$, w, X), $\ell \gg 0$;
(4) (S, $\mathbb{Q}_\ell$, $\frac{w}{2}$, X), $\ell \gg 0 \Longrightarrow$ (S, $\mathbb{F}_\ell$, $\frac{w}{2}$, X), $\ell \gg 0$;
(5) (F, $\mathbb{Q}_\ell$, i, X), $i = w, d-w$ + (wS, $\mathbb{Q}_\ell$, w, X), $\ell \gg 0 \Longrightarrow$ (F, $\mathbb{F}_\ell$, i, X), $i = w, d-w$ (+ (wS, $\mathbb{F}_\ell$, w, X)), $\ell \gg 0$.

1.3.5. For divisors, Theorem 1.3.3, Corollary 1.3.4 (3)-(5) yield [T94, Prop. 5.1] that for every $X \in \mathrm{SmP}(K)$,

(1) (F, $\mathbb{Q}_\mathfrak{u}$, 1, X) $\Longrightarrow$ (F, $\mathbb{Q}_\mathfrak{u}$, 1, X) + (wS, $\mathbb{Q}_\mathfrak{u}$, 1, X);
(2) (F, $\mathbb{Q}_\ell$, 1, X), $\ell \gg 0 \Longrightarrow$ (F, $\mathbb{F}_\ell$, 1, X) + (wS, $\mathbb{F}_\ell$, 1, X), $\ell \gg 0$.

In particular, for X an abelian variety or a K3 surface one can directly deduce (F, $\mathbb{F}_\ell$, 1, X) + (wS, $\mathbb{F}_\ell$, 1, X), $\ell \gg 0$ from (F, $\mathbb{Q}_\ell$, 1, X) (See Subsection 1.2.3) without resorting to any specific arithmetico-geometric features of X as in [SkZ15] or [I18].

1.3.6. **Remark.** The implication (F, $\mathbb{F}_\ell$, w, X) $\Longrightarrow$ (F, $\mathbb{Q}_\ell$, w, X) always holds for $\ell \gg 0$ (hence the implication (F, $\mathbb{Q}_\mathfrak{u}$, w, X) $\Longrightarrow$ (F, $\mathbb{Q}_\mathfrak{u}$, w, X)). This follows from Nakayama's lemma and the fact that $\mathrm{H}^{2w}(X_{\overline{K}}, \mathbb{Z}_\ell)$ is torsion-free for $\ell \gg 0$ ([G83] - See Fact 2.2). More precisely, we have the commutative diagram

$$\begin{array}{ccc} CH^w(X) & \longrightarrow & \mathrm{H}^{2w}(X_{\overline{K}}, \mathbb{Z}_\ell(w))^{\pi_1(K)} \\ \downarrow & \searrow & \downarrow \\ \mathrm{H}^{2w}(X_{\overline{K}}, \mathbb{F}_\ell(w))^{\pi_1(K)} & \longleftarrow & \mathrm{H}^{2w}(X_{\overline{K}}, \mathbb{Z}_\ell(w))^{\pi_1(K)} \otimes \mathbb{F}_\ell, \end{array}$$

where the bottom arrow is injective for $\ell \gg 0$. So if the left vertical arrow is surjective, the bottom arrow is an isomorphism hence the diagonal arrow is surjective.

1.4. **Divisors and finiteness of Brauer groups.** Let $X \in \mathrm{SmP}(K)$ with connected monodromy in degrees $(2, 1)$. Then (F, $\mathbb{Q}_\ell$, 1, X) is equivalent to the finiteness of the ℓ-primary $\pi_1(K)$-invariant part $\mathrm{Br}(X_{\overline{K}})^{\pi_1(K)}[\ell^\infty]$ of the Brauer group of $X_{\overline{K}}$ (*e.g.* [CCh20, Prop. 2.1.1] and the references therein). One has the following strengthening.

Corollary. *Assume $p > 0$. Then for every $X \in \mathrm{SmP}(K)$ the following assertions are equivalent*

(1) (F, $\mathbb{Q}_\ell$, 1, X), *for some $\ell \neq p$;*
(2) (F, $\mathbb{Q}_\ell$, 1, X), *for every $\ell \neq p$;*

(3) $\mathrm{Br}(X_{\overline{K}})^{\pi_1(K)}[p']$ *is finite,*

(where $\mathrm{Br}(X_{\overline{K}})^{\pi_1(K)}[p']$ denotes the prime-to-p part of $\mathrm{Br}(X_{\overline{K}})^{\pi_1(K)}$).

When K is finite, Corollary 1.4 was proved by Tate [T94, Prop. 4.3]. In this setting, it is even known that (F, $\mathbb{Q}_\ell$, 1, X) is independent of $\ell \neq p$ and implies that $\mathrm{Br}(X)$ is finite (See the references in the proof of [T94, Prop. 4.3]). That the equivalence (1) $\Leftrightarrow$ (2) holds in general was pointed out to us by Yanshuai Qin. This is essentially the same argument as in the finite field case and relies on [T94, Prop. 2.9]. Though it is well-known to experts (see *e.g.* [P15, §7] or [Q20, Cor. 1.7]), for completeness we briefly recall the proof in Subsection 7.1. The delicate implication is (2) $\Rightarrow$ (3), which requires Corollary 1.3.4 (3). Establishing (3) when X is a K3 surface was the main motivation of Skorobogatov-Zarhin and Ito in [SkZ15], [I18].

1.5. The proof of Theorem 1.3.1 is carried out in Section 3, the proof of Proposition 1.3.2 in Section 5 and the proof of Theorem 1.3.3 in Section 6. The proof of Proposition 1.3.2 is formal; this is why it also holds for $p = 0$. The proofs of Theorem 1.3.1 and Theorem 1.3.3 rely on deeper arithmetico-geometric inputs which, for the convenience of the reader, are summarized in Section 2; the assumption that $p > 0$ is crucial. Eventually, the proof of Corollary 1.4 is carried out in Section 7. In Section 8, we gathered basic properties of ultraproducts of fields.

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2. Étale cohomology

Let K be a finitely generated field of characteristic $p \geq 0$ and let $X \in \mathrm{SmP}(K)$. Let k denote the algebraic closure of the prime field of K in K.

2.1. **Convention.** In several places, we will fix a smooth projective model $f : \mathcal{X} \to \mathcal{S}$ of $X \to \mathrm{Spec}(K)$ with $\mathcal{S}$ a smooth separated and geometrically connected scheme over k with generic point η and set of closed points $|\mathcal{S}|$. In particular, for every geometric point $\overline{s}$ over a point $s \in \mathcal{S}$, locally constant constructible $\mathbb{Z}_\ell$-sheaf $\mathcal{F}$ ($\ell \neq p$) and up to choosing étale paths from $\overline{s}$ to $\overline{\eta}$, one gets canonical equivariant isomorphisms

$$\begin{array}{ccccccc} & (R^*f_*\mathcal{F}(v))_{\overline{s}} & \xrightarrow{\simeq} & (R^*f_*\mathcal{F}(v))_{\overline{\eta}} & = & \mathrm{H}^w(\mathcal{X}_{\overline{\eta}}, \mathcal{F}) = \mathrm{H}^w(X_{\overline{K}}, \mathcal{F}) & \\ \nearrow & \circlearrowleft & & \circlearrowleft & & \circlearrowleft & \\ \pi_1(s, \overline{s}) \to & \pi_1(\mathcal{S}, \overline{s}) & \xrightarrow{\simeq} & \pi_1(\mathcal{S}, \overline{\eta}) & \twoheadleftarrow & \pi_1(\eta, \overline{\eta}) = \pi_1(K). & \end{array}$$

When $p > 0$ (so that k is a finite field) and $s \in |\mathcal{S}|$, let $\varphi_s \in \pi_1(s)$ denote the geometric Frobenius, which we identify with its image (well-defined up to conjugacy if we ignore base points, which we will do most of the time) in $\pi_1(\mathcal{S}, \overline{s}) \tilde{\to} \pi_1(\mathcal{S}, \overline{\eta})$.

Assume $p > 0$. Fix integers $w \geq 0$, v. The following are consequences of the theory of Frobenius weights developed by Deligne in [D80].

2.2. **Fact.**
(1) ([G83]) *The $\mathbb{Z}_\ell$-local systems $R^w f_*\mathbb{Z}_\ell(v)$ are torsion-free (of finite constant rank) for $\ell (\neq p) \gg 0$. In particular, for every geometric point $\overline{s}$ on $\mathcal{S}$, $(R^*f_*\mathbb{Z}_\ell(v))_{\overline{s}} \otimes \mathbb{F}_\ell \tilde{\to} (R^w f_*\mathbb{F}_\ell(v))_{\overline{s}}$, $\ell (\neq p) \gg 0$;*
(2) ([CHT17, Thm. 1.3]) $\mathrm{H}^0(\mathcal{S}_{\overline{k}}, R^w f_*\mathbb{Z}_\ell(v)) \otimes \mathbb{F}_\ell \tilde{\to} \mathrm{H}^0(\mathcal{S}_{\overline{k}}, R^w f_*\mathbb{F}_\ell(v))$, $\ell (\neq p) \gg 0$.

2.3. **Fact.**
(1) ([D80, 3.4.1 (iii)]) $R^w f_*\mathbb{Q}_\ell(v)|_{\mathcal{S}_{\overline{k}}}$ *is a semisimple $\mathbb{Q}_\ell$-local system on $\mathcal{S}_{\overline{k}}$, $\ell \neq p$;*
(2) ([CHT17, Thm. 1.1]) $R^w f_*\mathbb{F}_\ell(v)|_{\mathcal{S}_{\overline{k}}}$ *is a semisimple $\mathbb{F}_\ell$-local system on $\mathcal{S}_{\overline{k}}$ for $\ell (\neq p) \gg 0$.*

2.4. **Fact.**
(1) ([D80, Cor. 3.2.9]) *For every closed point $s \in |\mathcal{S}|$ the characteristic polynomial $P_s := \det(IdT - \varphi_s|(R^w f_* \mathbb{Q}_\ell(v))_{\bar{s}})$ of the geometric Frobenius $\varphi_s \in \pi_1(s)$ is in $\mathbb{Z}[1/p][T]$, independent of $\ell (\neq p)$;*
(2) (*e.g.* [LaP95, (proof of) Prop. 2.1]) *The characteristic polynomial $P := \det(IdT - \varphi|\mathrm{H}^0(\mathcal{S}_{\bar{k}}, R^w f_* \mathbb{Q}_\ell(v)))$ of the geometric Frobenius $\varphi \in \pi_1(k)$ is in $\mathbb{Z}[1/p][T]$ and independent of ℓ.*

From Fact 2.2 (1), Fact 2.4 (1) implies that, for $\ell(\neq p) \gg 0$, the reduction modulo ℓ of $P_s \in \mathbb{Z}[1/p][T]$ coincides with the characteristic polynomial $P_{s,\mathbb{F}_\ell} := \det(IdT - \varphi_s|(R^w f_* \mathbb{F}_\ell(v))_{\bar{s}}) \in \mathbb{F}_\ell[T]$. In turn, this implies that $P_s \in \mathbb{Z}[1/p][T]$ coincides with the characteristic polynomial $\det(IdT - \varphi_s|\mathrm{H}^w(\mathcal{X}_{\bar{s}}, \mathbb{Q}_\mathfrak{u}(v)))$, $\mathfrak{u} \in \mathcal{U}$. (It also directly follows from Fact 2.2 (1) that $P_s \in \mathbb{Z}[1/p][T]$ coincides with the characteristic polynomial $\det(IdT - \varphi_s|\mathrm{H}^w(\mathcal{X}_{\bar{s}}, \mathbb{Q}_\mathfrak{u}(v)))$, $\mathfrak{u} \in \mathcal{U}$).

From Fact 2.2 (2), Fact 2.4 (2) implies that, for $\ell(\neq p) \gg 0$, the reduction modulo ℓ of $P \in \mathbb{Z}[1/p][T]$ coincides with the characteristic polynomial $P_{\mathbb{F}_\ell} := \det(IdT - \varphi|\mathrm{H}^0(\mathcal{S}_{\bar{k}}, R^w f_* \mathbb{F}_\ell(v))) \in \mathbb{F}_\ell[T]$. In particular, if $\delta_{\mathbb{Q}_\ell}(1)$ (resp. $\delta_{\mathbb{F}_\ell}(1)$) denotes the multiplicity of 1 as a root of P (resp. $P_{\mathbb{F}_\ell}$), $\delta_{\mathbb{Q}_\ell}(1)$ is independent of ℓ and one $\delta_{\mathbb{Q}_\ell}(1) = \delta_{\mathbb{F}_\ell}(1)$ for $\ell(\neq p) \gg 0$.

2.5. Let $\overline{\Pi}$ (resp. Π) denote the image of $\pi_1(\mathcal{S}_{\bar{k}})$ (resp. $\pi_1(\mathcal{S})$) acting on $\prod_{\ell \in \mathcal{L}} (R^w f_* \mathbb{F}_\ell(v))_{\bar{s}}$. Then,

Fact. ([CT19, §3.1]) $\overline{\Pi}$ *(hence Π) is a topologically finitely generated profinite group.*

Let $\Pi_{\mathbb{Q}_\mathfrak{u}}$ denote the image of $\pi_1(\mathcal{S})$ acting on $\mathrm{H}^w(\mathcal{X}_{\bar{\eta}}, \mathbb{Q}_\mathfrak{u}(v)))$. Fact 2.5 has the following (non-trivial!) consequence

Corollary. *For every finite index subgroup $\Pi'_{\mathbb{Q}_\mathfrak{u}} \subset \Pi_{\mathbb{Q}_\mathfrak{u}}$ there exists a connected étale cover $\mathcal{S}' \to \mathcal{S}$ such that $\Pi'_{\mathbb{Q}_\mathfrak{u}}$ coincides with the image of $\pi_1(\mathcal{S}')$ acting on $\mathrm{H}^w(\mathcal{X}_{\bar{\eta}}, \mathbb{Q}_\mathfrak{u}(v)))$.*

Proof. From Fact 2.5, Π is a topologically finitely generated profinite group. As the inverse image $\Pi' \subset \Pi$ of $\Pi'_{\mathbb{Q}_\mathfrak{u}}$ in Π is again of finite index it follows from [NS07a, Thm. 1.1] (which relies on [NS07b]) that Π' is automatically open in Π hence corresponds to a connected étale cover $\mathcal{S}' \to \mathcal{S}$. □

The fact that $\overline{\Pi}$ is topologically finitely generated also ensures (Lemma 8.4.2)

$$\mathrm{H}^w(\mathcal{X}_{\bar{\eta}}, \mathbb{Q}_\mathfrak{u}(v))^{\pi_1(\mathcal{S}_{\bar{k}})} = (\prod_{\ell \in \mathcal{L}} \mathrm{H}^w(\mathcal{X}_{\bar{\eta}}, \mathbb{F}_\ell(v))^{\pi_1(\mathcal{S}_{\bar{k}})}) \otimes \mathbb{Q}_\mathfrak{u} = (\prod_{\ell \in \mathcal{L}} \mathrm{H}^0(\mathcal{S}_{\bar{\eta}}, R^w f_* \mathbb{F}_\ell(v)) \otimes \mathbb{Q}_\mathfrak{u}$$

so that, from Fact 2.2 (2) and Fact 2.4 (2), $P \in \mathbb{Q}[T]$ coincides with the characteristic polynomial $\det(IdT - \varphi|\mathrm{H}^w(\mathcal{X}_{\bar{s}}, \mathbb{Q}_\mathfrak{u}(v))^{\pi_1(\mathcal{S}_{\bar{k}})})$, $\mathfrak{u} \in \mathcal{U}$.

(From Fact 2.4 (2) and [B96, 6.3.1, 6.3.2], similar results hold for $\mathbb{Q}_\mathfrak{u}$-coefficients).

3. Proof of Theorem 1.3.1

Let K be a finitely generated field of characteristic $p > 0$ and let $X \in \mathrm{SmP}(K)$. We retain the notation of 2.1. For $C = \mathbb{Q}_\mathfrak{u}, \mathbb{Q}_\ell, \mathbb{Q}_\mathfrak{u}$, set $\mathrm{H}_C := \mathrm{H}^w(X_{\overline{K}}, C(v))$ and let $G_C \subset \mathrm{GL}(\mathrm{H}_C)$ denote the Zariski-closure of the image Π_C of $\pi_1(K)$ acting on H_C.

Let C_1, C_2 be any fields of the form $\mathbb{Q}_\ell, \mathbb{Q}_\mathfrak{u}$ or $\mathbb{Q}_\mathfrak{u}$. Since $\pi_1(\mathcal{S})$-semisimplification does not change the kernel of $\pi_1(\mathcal{S}) \twoheadrightarrow \pi_0(G_{C_i})$, one may assume H_{C_i} is a semisimple Π_{C_i}-module. Note also that $\pi_1(\mathcal{S})$-semisimplification does not affect the action of $\pi_1(\mathcal{S}_{\bar{k}})$ on H_{C_i} by Fact 2.3 (and Lemma 8.4.5 if $C_i = \mathbb{Q}_\mathfrak{u}$ or $\mathbb{Q}_\mathfrak{u}$). As $\pi_1(\mathcal{S})$ acts on H_{C_i} through a topologically finitely generated quotient, the kernel of $\pi_1(\mathcal{S}) \twoheadrightarrow \pi_0(G_{C_1})$ is an (a normal) *open* subgroup of $\pi_1(\mathcal{S})$ ([NS07a], [NS07b]) so that, up to replacing $\mathcal{S}$ by the corresponding étale (Galois) cover, one may assume G_{C_1} is connected that is, equivalently ([D82, Prop. 3.1 (a), (c)]), for every finite index subgroup $U \subset \pi_1(\mathcal{S})$ and integers $m, n \geq 0$, $dim((H_{C_1}^{\otimes m} \otimes H_{C_1}^{\vee \otimes n})^U) = dim((H_{C_1}^{\otimes m} \otimes H_{C_1}^{\vee \otimes n})^{\pi_1(\mathcal{S})})$. One has to show that this implies that for every finite index subgroup $U \subset \pi_1(\mathcal{S})$ and integers $m, n \geq 0$, $dim((H_{C_2}^{\otimes m} \otimes H_{C_2}^{\vee \otimes n})^U) = dim((H_{C_2}^{\otimes m} \otimes H_{C_2}^{\vee \otimes n})^{\pi_1(\mathcal{S})})$ [LaP95, Lemma 2.3]. Again, since $\pi_1(\mathcal{S})$ acts on H_{C_i} through a topologically finitely generated quotient, one may restrict to *open* subgroups $U \subset \pi_1(\mathcal{S})$. That is, equivalently, one has to show that for every connected étale cover $\mathcal{S}' \to \mathcal{S}$ and integers $m, n \geq 0$, $dim((H_{C_2}^{\otimes m} \otimes H_{C_2}^{\vee \otimes n})^{\pi_1(\mathcal{S}')}) = dim((H_{C_2}^{\otimes m} \otimes H_{C_2}^{\vee \otimes n})^{\pi_1(\mathcal{S})})$. But recall that $H_{C_i} = \mathrm{H}^w(\mathcal{X}_{\overline{\eta}}, C_i(v))$ so that, by Kunneth formula, $H_{C_i}^{\otimes m} \otimes H_{C_i}^{\vee \otimes n}$ is a direct factor of $\mathrm{H}^{mw+n(2d-w)}(\mathcal{X}_{\overline{\eta}}^{m+n}, C_i(n(d-v)))$. In other words, replacing $\mathcal{X} \to \mathcal{S}$ with the the $m+n$th fibered power $\mathcal{X}^{m+n} = \mathcal{X} \times_{\mathcal{S}} \times \cdots \times_{\mathcal{S}} \mathcal{X} \to \mathcal{S}$ (and the Tate twists $-(v)$ with $-(n(d-v))$), it is enough to show that for every connected étale cover $\mathcal{S}' \to \mathcal{S}$, $dim((H_{C_2})^{\pi_1(\mathcal{S}')}) = dim((H_{C_2})^{\pi_1(\mathcal{S})})$. But as, by assumption, $dim(H_{C_1}^{\pi_1(\mathcal{S}')}) = dim(H_{C_1}^{\pi_1(\mathcal{S})})$, it is actually enough to show that for every connected étale cover $\mathcal{S}' \to \mathcal{S}$, $dim((H_{C_2})^{\pi_1(\mathcal{S}')}) = dim((H_{C_1})^{\pi_1(\mathcal{S}')})$. Write $\mathcal{S} := \mathcal{S}'$ to simplify. As H_{C_i} is a semisimple Π_{C_i}-module (and using Lemma 8.4.2 for $C_i = \mathbb{Q}_{\mathfrak{u}}$ or $\mathbb{Q}_{\mathfrak{u}}$), $dim((H_{C_i})^{\pi_1(\mathcal{S})})$ is the multiplicity of 1 as an eigenvalue of the Frobenius $\varphi \in \pi_1(k) \simeq \pi_1(\mathcal{S})/\pi_1(\mathcal{S}_{\overline{k}})$ acting on $(H_{C_i})^{\pi_1(\mathcal{S}_{\overline{k}})}$. So the assertion follows from the last paragraph of Subsection 2.5.

4. Preliminary observations

Let $X \in \mathrm{SmP}(K)$. We begin by the following elementary observations, which follow from the formal properties of ultraproducts.

4.1. **Lemma.** *For* $(?, ??) = (\mathrm{S}, \frac{w}{2})$, (wS, w), $(\mathrm{wS'}, w)$, (F, w) *we have*
(1) *For every* $\mathfrak{u} \in \mathcal{U}$, $(?,\ \mathbb{Q}_{\mathfrak{u}},\ ??,\ X) \iff$ *The set of all* $\ell \in \mathcal{L}$ *such that* $(?,\ \mathbb{Q}_\ell,\ ??,\ X)$ *holds is in* $\mathfrak{u}$. *In particular,* $(?,\ \mathbb{Q}_\ell,\ ??,\ X)$, $\ell \gg 0 \iff (?,\ \mathbb{Q}_{\mathfrak{u}},\ ??,\ X)$ *for every ultrafilter* $\mathfrak{u} \in \mathcal{U}$;
(2) *For every* $\mathfrak{u} \in \mathcal{U}$, $(?,\ \mathbb{Q}_{\mathfrak{u}},\ ??,\ X) \iff$ *The set of all* $\ell \in \mathcal{L}$ *such that* $(?,\ \mathbb{F}_\ell,\ ??,\ X)$ *holds is in* $\mathfrak{u}$. *In particular,* $(?,\ \mathbb{F}_\ell,\ ??,\ X)$, $\ell \gg 0 \iff (?,\ \mathbb{Q}_{\mathfrak{u}},\ ??,\ X)$ *for every* $\mathfrak{u} \in \mathcal{U}$.

Proof. For ? =F, see 8.3.3 (with P the property of being surjective) and 8.4.2 (which can be applied by **??** (2)). For ? =S, see 8.4.5 (with P the property of acting semisimply). For ? =wS, see 8.4.6. For ? =wS', see 8.4.2, 8.4.1 and 8.3.3 (with P the property of being an isomorphism). □

4.2. Let $C = \mathbb{Q}_\ell, \mathbb{Q}_{\mathfrak{u}}$ or $\mathbb{Q}_{\mathfrak{u}}$ and let K'/K be a finite field extension. Write $X' := X \times_K K'$. Then,

Lemma.
(1) $(\mathrm{S},\ C,\ \frac{w}{2},\ X') \Leftrightarrow (\mathrm{S},\ C,\ \frac{w}{2},\ X)$;
(2) $(\mathrm{sF},\ C,\ w,\ X') \Leftrightarrow (\mathrm{sF},\ C,\ w,\ X)$;
(3) *If* K'/K *is Galois,* $(\mathrm{F},\ C,\ w,\ X') \Rightarrow (\mathrm{F},\ C,\ w,\ X)$.
Assume furthermore X *has connected monodromy in degrees* $(2w, w)$. *Then,*
(4) $(\mathrm{wS},\ C,\ w,\ X) \Leftrightarrow (\mathrm{wS},\ C,\ w,\ X')$;
(5) *The assertions* $(\mathrm{sF},\ C,\ w,\ X)$, $(\mathrm{sF},\ C,\ w,\ X')$, $(\mathrm{F},\ C,\ w,\ X)$, $(\mathrm{F},\ C,\ w,\ X')$ *are all equivalent.*

Proof. We show (3); the other assertions are purely group-theoretic and elementary. Let

$$\alpha \in \mathrm{H}^{2w}(X_{\overline{K}}, C(w))^{\pi_1(K)} \subset \mathrm{H}^{2w}(X_{\overline{K}}, C(w))^{\pi_1(K')}.$$

Then, from $(\mathrm{F},\ C,\ w,\ X')$, one can write $\alpha = \sum_{1 \leq i \leq r} \lambda_i [Y_i']$ with $\lambda_i \in C$ and $Y_i' \in Z^\omega(X')$ an integral cycle. But, then,

$$\alpha = \frac{1}{[K':K]} \sum_{1 \leq i \leq r} \lambda_i \sum_{\sigma \in Gal(K'/K)} \sigma[Y_i'] = \frac{1}{[K':K]} \sum_{1 \leq i \leq r} \lambda_i [\sum_{\sigma \in Gal(K'/K)} \sigma Y_i'].$$

The conclusion follows from the fact that $\sum_{\sigma \in Gal(K'/K)} \sigma Y_i'$ is in $Z^\omega(X')^{Gal(K'/K)} = Z^\omega(X)$. □

4.3. **Lemma.** *Assume* $p > 0$. *Then,*
(1) *For* $\ell \neq p$, $(\mathrm{wS},\ \mathbb{Q}_\ell,\ w,\ X) \iff (\mathrm{wS'},\ \mathbb{Q}_\ell,\ w,\ X)$;
(2) *For* $\ell \gg 0$, $(\mathrm{wS},\ \mathbb{F}_\ell,\ w,\ X) \iff (\mathrm{wS'},\ \mathbb{F}_\ell,\ w,\ X)$.

Proof. Let $C = \mathbb{Q}_\ell$ or $\mathbb{F}_\ell$. We retain the notation of 2.1. Write $\mathrm{H} := \mathrm{H}^{2w}(X, C(w))$ and $\overline{\Pi} := \pi_1(\mathcal{S}_{\overline{k}})$, $\Pi := \pi_1(\mathcal{S})$. The implication (wS', C, w, X) $\Rightarrow$ (wS, C, w, X) is straightforward since the composition of $c_w^{-1} : \mathrm{H}_\Pi \tilde{\to} \mathrm{H}^\Pi$ with the canonical projection $\mathrm{H} \twoheadrightarrow \mathrm{H}_\Pi$ provides a Π-equivariant splitting of $\mathrm{H}^\Pi \hookrightarrow \mathrm{H}$. Conversely, let $\phi \in \Pi$ such that ϕ and $\overline{\Pi}$ generate Π. As $\overline{\Pi}$ acts semisimply on H (Fact 2.3) the canonical morphism $\mathrm{H}^{\overline{\Pi}} \to \mathrm{H}_{\overline{\Pi}}$ is an isomorphism. Assume (wS, C, w, X) and consider a Π-equivariant decomposition $\mathrm{H} = \mathrm{H}^\Pi \oplus M$; in particular $M^\Pi = 0$. Then it is enough to show that $0 = M_\Pi = (M_{\overline{\Pi}})_\varphi \tilde{\leftarrow} (M^{\overline{\Pi}})_\varphi$ but this follows from the exact sequence

$$0 \to M^\Pi = (M^{\overline{\Pi}})^\varphi \to M^{\overline{\Pi}} \overset{\varphi-1}{\to} M^{\overline{\Pi}} \to (M^{\overline{\Pi}})_\varphi \to 0.$$

□

5. Proof of Proposition 1.3.2

5.1. Let $X \in \mathrm{SmP}(K)$ of dimension d. For $C = \mathbb{Q}_\ell, \mathbb{Z}_\ell, \mathbb{F}_\ell$ write $\mathrm{H}_C := \mathrm{H}^w(X_{\overline{K}}, C)$ and set $\Pi := \pi_1(K)$. To prove Proposition 1.3.2, one may freely replace $\mathcal{L}$ by a subset in $\mathfrak{u}$; in particular one may replace $\mathcal{L}$ by a cofinite subset hence assume

$$(5.1.1)\ \ \mathrm{H}_{\mathbb{Z}_\ell} \otimes \mathbb{F}_\ell = \mathrm{H}_{\mathbb{F}_\ell},\ \ \ell \in \mathcal{L}$$

(Fact 2.2 (1) if $p > 0$; if $p = 0$, this follows from comparison between singular and $\mathbb{Z}_\ell$-cohomology, using the fact that for every embedding $K \subset \mathbb{C}$, $\mathrm{H}_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module). By Künneth formula and Poincaré duality, (F, $\mathbb{Q}_\mathfrak{u}$, d, X^2) ensures that up to replacing $\mathcal{L}$ by a subset in $\mathfrak{u}$ one has

$$(5.1.2)\ \ \mathrm{End}_\Pi(\mathrm{H}_{\mathbb{Z}_\ell}) \otimes \mathbb{F}_\ell = (\mathrm{H}_{\mathbb{Z}_\ell} \otimes \mathrm{H}_{\mathbb{Z}_\ell}^\vee)^\Pi \otimes \mathbb{F}_\ell \tilde{\to} (\mathrm{H}_{\mathbb{F}_\ell} \otimes \mathrm{H}_{\mathbb{F}_\ell}^\vee)^\Pi = \mathrm{End}_\Pi(\mathrm{H}_{\mathbb{F}_\ell}),\ \ \ell \in \mathcal{L}.$$

5.2. **Proof of Proposition 1.3.2 (1).**

5.2.1. Let Q be a field and Γ a group. In this subsection, a Γ-module means a finite-dimensional Q-vector space endowed with an action of Γ by Q-linear automorphisms. For a Γ-module V, let V^{ss} denote the Γ-semisimplification of V.

Lemma. *One has* $\dim(\mathrm{End}_\Gamma(V)) \leq \dim(\mathrm{End}_\Gamma(V^{ss}))$ *and* $\dim(\mathrm{End}_\Gamma(V)) = \dim(\mathrm{End}_\Gamma(V^{ss}))$ *if and only if V is a semisimple Γ-module.*

Proof. Let $0 \to A \to V \to B \to 0$ be a short exact sequence of Γ-modules and W a Γ-module. Then

$$0 \longrightarrow \mathrm{Hom}_\Gamma(B, W) \longrightarrow \mathrm{Hom}_\Gamma(V, W) \longrightarrow \mathrm{Hom}_\Gamma(A, W)$$

and

$$0 \longrightarrow \mathrm{Hom}_\Gamma(W, A) \longrightarrow \mathrm{Hom}_\Gamma(W, V) \longrightarrow \mathrm{Hom}_\Gamma(W, B)$$

are exact and hence we obtain

$$\dim \mathrm{Hom}_\Gamma(V, W) \leq \dim \mathrm{Hom}_\Gamma(A \oplus B, W)$$

and

$$\dim \mathrm{Hom}_\Gamma(W, V) \leq \dim \mathrm{Hom}_\Gamma(W, A \oplus B).$$

By taking $W = V$ in the first inequality and $W = A \oplus B$ in the second, we obtain

$$\dim \mathrm{End}_\Gamma(V) \leq \dim \mathrm{End}_\Gamma(A \oplus B)$$

and induction implies

$$(*)\ \dim \mathrm{End}_\Gamma(V) \leq \dim \mathrm{End}_\Gamma(V^{ss}).$$

When $(*)$ is an equality, V is semisimple. Indeed, all the inequalities become equalities. Hence, the sequence

$$0 \longrightarrow \mathrm{Hom}_\Gamma(A \oplus B, A) \longrightarrow \mathrm{Hom}_\Gamma(A \oplus B, V) \longrightarrow \mathrm{Hom}_\Gamma(A \oplus B, B) \longrightarrow 0$$

and thus

$$0 \longrightarrow \mathrm{Hom}_\Gamma(B, A) \longrightarrow \mathrm{Hom}_\Gamma(B, V) \longrightarrow \mathrm{Hom}_\Gamma(B, B) \longrightarrow 0$$

are exact, implying that $0 \to A \to V \to B \to 0$ splits. □

5.2.2. From (5.1.1) and (5.1.2) $\dim(\mathrm{End}_\Pi(\mathrm{H}_{\mathbb{Q}_\ell})) = \dim(\mathrm{End}_\Pi(\mathrm{H}_{\mathbb{F}_\ell}))$. On the other hand (S, $\mathbb{Q}_\mathfrak{u}$, $\frac{w}{2}$, X) ensures that up to replacing $\mathcal{L}$ by a subset in $\mathfrak{u}$ one may assume (S, $\mathbb{F}_\ell$, $\frac{w}{2}$, X), $\ell \in \mathcal{L}$ (See Lemma 4.1 (2)). Let $T_{\mathbb{Z}_\ell} \subset \mathrm{H}^{ss}_{\mathbb{Q}_\ell}$ be any Π-stable $\mathbb{Z}_\ell$-lattice and set $T_{\mathbb{F}_\ell} := T_{\mathbb{Z}_\ell} \otimes \mathbb{F}_\ell$. Then since $T^{ss}_{\mathbb{F}_\ell}$ and $\mathrm{H}_{\mathbb{F}_\ell}$ are semisimple Π-modules with the same traces, they are isomorphic. Hence

$$\begin{aligned}\dim(\mathrm{End}_\Pi(\mathrm{H}^{ss}_{\mathbb{Q}_\ell})) \geq \dim(\mathrm{End}_\Pi(\mathrm{H}_{\mathbb{Q}_\ell})) &= \dim(\mathrm{End}_\Pi(\mathrm{H}_{\mathbb{F}_\ell}))\\ &= \dim(\mathrm{End}_\Pi(T^{ss}_{\mathbb{F}_\ell})) \geq \dim(\mathrm{End}_\Pi(T_{\mathbb{F}_\ell})) \geq \dim(\mathrm{End}_\Pi(\mathrm{H}^{ss}_{\mathbb{Q}_\ell})),\end{aligned}$$

where the first and second inequalities follow from Lemma 5.2.1 and the third inequality always holds. As a result, one obtains

$$\dim(\mathrm{End}_\Pi(\mathrm{H}_{\mathbb{Q}_\ell})) = \dim(\mathrm{End}_\Pi(\mathrm{H}^{ss}_{\mathbb{Q}_\ell})).$$

The conclusion follows from the equality case in Lemma 5.2.1.

5.3. **Proof of Proposition 1.3.2 (2).**

5.3.1. Given a ring R, let $Idem(R)$ and $CIdem(R)$ denote respectively the idempotents and central idempotents in R.

Let A be a $\mathbb{Z}_\ell$-algebra which, as a $\mathbb{Z}_\ell$-module, is free of finite rank. The following lemma is possibly classical (see *e.g.* [Do72, Thm. 44.3 (2)] for the surjectivity part of the assertion) but for lack of a suitable complete reference and to keep the exposition self-contained, we include a proof.

Lemma. (Lifting idempotents) *The reduction modulo-ℓ morphism $A \twoheadrightarrow A \otimes \mathbb{F}_\ell$ restricts to a surjective map $Idem(A) \twoheadrightarrow Idem(A \otimes \mathbb{F}_\ell)$ and to a bijective map $CIdem(A) \tilde{\to} CIdem(A \otimes \mathbb{F}_\ell)$.*

Proof. First, observe that for every $a, a' \in A$ such that $[a, a'] = 0$ and $a - a' \in \ell^N A$, we have $a^{\ell^n} - a'^{\ell^n} \in \ell^{N+n}A$. Indeed, write $a - a' = \ell^N b_0 \in \ell^N A$. Then, b_0 commutes with a, a' and one has

$$a^\ell - a'^\ell = \sum_{1\leq k\leq \ell} \binom{\ell}{k} \ell^{Nk} a'^{\ell-k} b_0^k = \ell^{N+1} \sum_{1\leq k\leq \ell} \binom{\ell}{k} \frac{\ell^{Nk}}{\ell^{N+1}} a'^{\ell-k} b_0^k = \ell^{N+1} b_1.$$

The conclusion follows by straightforward induction.

- Let $\epsilon \in Idem(A \otimes \mathbb{F}_\ell)$ and pick any $a \in A$ such that $\overline{a} = \epsilon$. By construction, $a^{\ell^m} - a \in \ell A$, $m \geq 0$ hence, from the preliminary observation,
$$a^{\ell^{n+p}} - a^{\ell^n} = (a^{\ell^p})^{\ell^n} - a^{\ell^n} \in \ell^{n+1} A,\ n \geq 0$$
hence $\{a^{\ell^n}\}_n$ is a Cauchy sequence. Set $e := \lim_{n\to\infty} a^{\ell^n}$. By construction, for $n \gg 0$ we have $\overline{e} = \overline{a}^{\ell^n} = \epsilon^{\ell^n} = \epsilon$. Furthermore, since $a^2 - a \in \ell A$, we get, again, $a^{2\ell^n} - a^{\ell^n} \in \ell^{n+1}A$, $n \geq 0$. Since $(-)^2 : A \to A$ is continuous, one gets $e^2 = e$. This shows $Idem(A) \twoheadrightarrow Idem(A \otimes \mathbb{F}_\ell)$.
- Let $e \in Idem(A)$ such that $\overline{e} \in CIdem(A \otimes \mathbb{F}_\ell)$. Then $\overline{e}(A \otimes \mathbb{F}_\ell)(1 - \overline{e}) = 0$ forces
$$eA(1 - e) \subset \ell A = e\ell Ae \oplus (1 - e)\ell Ae \oplus e\ell A(1 - e) \oplus (1 - e)\ell A(1 - e).$$
Multiplying by e on the left and $1-e$ on the right, one gets $eA(1-e) = \ell eA(1-e)$ hence, by Nakayama's lemma, $eA(1 - e) = 0$. Similarly $(1 - e)Ae = 0$. Hence for every $a \in A$,
$$ea = ea(e + (1 - e)) = eae = (e + (1 - e))ae = ae.$$
This shows $CIdem(A) \twoheadrightarrow CIdem(A \otimes \mathbb{F}_\ell)$. Let $e, e' \in CIdem(A)$ such that $\overline{e} = \overline{e}'$ that is, $e - e' \in \ell A$. Since $[e, e'] = 0$, the preliminary observation shows that $e - e' = e^{\ell^n} - e'^{\ell^n} \in \ell^{n+1}A$, $n \geq 0$ hence $e = e'$. This shows $CIdem(A) \tilde{\to} CIdem(A \otimes \mathbb{F}_\ell)$.

□

5.3.2. From (5.1.1) one has $\mathrm{End}_\Pi(\mathrm{H}_{\mathbb{Z}_\ell}) \otimes \mathbb{F}_\ell = \mathrm{End}_\Pi(\mathrm{H}_{\mathbb{F}_\ell})$, $\ell \in \mathcal{L}$. On the other hand, (F, $\mathbb{Q}_\mathfrak{u}$, w, X), (wS, $\mathbb{Q}_\mathfrak{u}$, w, X) ensure that up to replacing $\mathcal{L}$ by a subset in $\mathfrak{u}$, one also has (F, $\mathbb{F}_\ell$, w, X), (wS, $\mathbb{F}_\ell$, w, X), (wS' , $\mathbb{F}_\ell$, w, X), $\ell \in \mathcal{L}$ (Lemma 4.1 (2), Lemma 4.3 (2)). Using (5.1.1) one can apply Lemma 5.3.1 to $A = \mathrm{End}_\Pi(\mathrm{H}_{\mathbb{Z}_\ell})$. Write $M_0 := \mathrm{H}^{\Pi}_{\mathbb{F}_\ell}$, $M_1 := \ker(\mathrm{H}_{\mathbb{F}_\ell} \to \mathrm{H}_{\mathbb{F}_\ell \Pi})$. Then, by (wS' , $\mathbb{F}_\ell$, w, X), one has the canonical decomposition $H_{\mathbb{F}_\ell} = M_1 \oplus M_0$ as Π-modules. By definition of M_0, M_1, any element in $\mathrm{End}_\Pi(\mathrm{H}_{\mathbb{F}_\ell})$ stabilizes both M_0 and M_1 hence the elements $e_i : \mathrm{H}_{\mathbb{F}_\ell} \twoheadrightarrow M_i \hookrightarrow \mathrm{H}_{\mathbb{F}_\ell}$ (obtained by composing the canonical projection followed by the canonical injection), $i = 1, 2$ are in $CIdem(\mathrm{End}_\Pi(\mathrm{H}_{\mathbb{F}_\ell}))$. From Lemma 5.3.1, e_0, e_1 lift uniquely to $\tilde{e}_0, \tilde{e}_1 \in CIdem(\mathrm{End}_\Pi(H_{\mathbb{Z}_\ell}))$ with $Id = \tilde{e}_0 + \tilde{e}_1$. Let $\tilde{M}_{1-i} := \ker(\tilde{e}_i)$, $i = 0, 1$. Then $H_{\mathbb{Z}_\ell} = \tilde{M}_1 \oplus \tilde{M}_0$ with $\tilde{M}_i \otimes \mathbb{F}_\ell = M_i$, $i = 0, 1$. It remains to check that $\tilde{M}_0 = \mathrm{H}^{\Pi}_{\mathbb{Z}_\ell}$.

Since $\tilde{M}_0 \otimes \mathbb{F}_\ell = M_0 (= \mathrm{H}^{\Pi}_{\mathbb{F}_\ell}) = \mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \otimes \mathbb{F}_\ell$, by Nakayama's lemma, it is enough to show that $\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \subset \tilde{M}_0$. Since $\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} = (\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \cap \tilde{M}_0) \oplus (\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \cap \tilde{M}_1)$, this is equivalent to $\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \cap \tilde{M}_1 = 0$. Let $h \in \mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \cap \tilde{M}_1$. Then $h \bmod \ell \in \mathrm{H}^{\Pi}_{\mathbb{F}_\ell} \cap M_1 = 0$ that is $\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \cap \tilde{M}_1 \subset \ell H_{\mathbb{Z}_\ell}$. But as $\mathrm{H}_{\mathbb{Z}_\ell}/(H^{\Pi}_{\mathbb{Z}_\ell} \cap \tilde{M}_1) \hookrightarrow (\mathrm{H}_{\mathbb{Z}_\ell}/\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell}) \times \tilde{M}_0$ is torsion-free (equivalently, $\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \cap \tilde{M}_1 \subset \mathrm{H}_{\mathbb{Z}_\ell}$ is a $\mathbb{Z}_\ell$-direct summand), $(\ell \mathrm{H}_{Z_\ell}) \cap (\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \cap \tilde{M}_1) = \ell(\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \cap \tilde{M}_1)$. As a result, $\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \cap \tilde{M}_1 = \ell(\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \cap \tilde{M}_1)$ which, by Nakayama's lemma, forces $\mathrm{H}^{\Pi}_{\mathbb{Z}_\ell} \cap \tilde{M}_1 = 0$.

6. Proof of Theorem 1.3.3

Let K be a finitely generated field of characteristic $p > 0$ and let $X \in \mathrm{SmP}(K)$. We retain the notation of 2.1. Set $\overline{\Pi} := \pi_1(\mathcal{S}_{\overline{k}})$, $\Pi := \pi_1(\mathcal{S})$. Again, to prove Theorem 1.3.3 one may freely replace $\mathcal{L}$ by a subset in $\mathfrak{u}$; in particular one may assume $\mathrm{H}^0(\mathcal{X}_{\overline{\eta}}, R^* f_* \mathbb{Z}_\ell) \otimes \mathbb{F}_\ell \tilde{\to} \mathrm{H}^0(\mathcal{X}_{\overline{\eta}}, R^* f_* \mathbb{F}_\ell)$, $\ell \in \mathcal{L}$ (Fact 2.2 (1)). From Lemma 4.1, it is enough to show

(1') For $\ell \gg 0$, (wS, $\mathbb{Q}_\ell$, w, X) $\Longrightarrow$ (wS, $\mathbb{F}_\ell$, w, X)
(2') For $\ell \gg 0$, (S, $\mathbb{Q}_\ell$, $\frac{w}{2}$, X) $\Longrightarrow$ (S, $\mathbb{F}_\ell$, $\frac{w}{2}$, X)
(3') For $\ell \gg 0$, (F, $\mathbb{Q}_\ell$, i, X), $i = w, d-w$ + (wS, $\mathbb{Q}_\ell$, w, X) $\Longrightarrow$ (F, $\mathbb{F}_\ell$, i, X), $i = w, d-w$ + (wS, $\mathbb{F}_\ell$, w, X)

6.1. **Proof of (1').** For $C = \mathbb{Q}_\ell, \mathbb{F}_\ell$, write $\mathrm{H}_C := \mathrm{H}^{2w}(X_{\overline{K}}, C(w))$ and consider the following seemingly weak variant of (wS, C, w, X).

(wS", C, w, X) The inclusion $\mathrm{H}^{\Pi}_C \hookrightarrow \mathrm{H}^{\overline{\Pi}}_C$ splits $\pi_1(k)$-equivariantly.

Recall the definition of $\delta_C(1)$ at the end of Paragraph 2.4; by definition this is the dimension of the generalized eigenspace $\mathrm{H}^{\overline{\Pi}}_C\{1\} := \cup_{n \geq 1} \ker((Id - \varphi)^n | \mathrm{H}^{\overline{\Pi}}_C)$ attached to 1 so that

$$\text{(6.1.1)} \quad (\text{wS"}, C, w, X) \Leftrightarrow \delta_C(1) = \dim(\mathrm{H}^{\Pi}_C) \Leftrightarrow \delta_C(1) \leq \dim(\mathrm{H}^{\Pi}_C)$$

(where the last equivalence follows from the fact that $\delta_C(1) \geq \dim(\mathrm{H}^{\Pi}_C)$ always holds). One also has

6.1.2 **Lemma.** (wS, $\mathbb{Q}_\ell$, w, X) $\Leftrightarrow$ (wS", $\mathbb{Q}_\ell$, w, X) *and* (wS, $\mathbb{F}_\ell$, w, X) $\Leftrightarrow$ (wS", $\mathbb{F}_\ell$, w, X), $\ell \gg 0$.

Proof. The implications $\Rightarrow$ are straightforward. For the converse implications, from Fact 2.3 the canonical Π-equivariant morphism $\mathrm{H}^{\overline{\Pi}}_C \to \mathrm{H}_{C\,\overline{\Pi}}$ is an isomorphism. So, setting $N := \ker(\mathrm{H}_C \to \mathrm{H}_{C\,\overline{\Pi}})$, one obtains a direct sum decomposition as Π-modules $\mathrm{H}_C = \mathrm{H}^{\overline{\Pi}}_C \oplus N$. □

6.1.3 From 6.1, it is enough to show

$$\text{(1")} \quad \text{For } \ell \gg 0,\ \delta_{\mathbb{Q}_\ell}(1) \leq \dim(\mathrm{H}^{\Pi}_{\mathbb{Q}_\ell}) \Longrightarrow \delta_{\mathbb{F}_\ell}(1) \leq \dim(\mathrm{H}^{\Pi}_{\mathbb{F}_\ell}).$$

From the last paragraph of 2.4, $\delta_{\mathbb{Q}_\ell}(1) = \delta_{\mathbb{F}_\ell}(1)$ for $\ell (\neq p) \gg 0$ so that (1") follows from

$$\dim(\mathrm{H}^{\Pi}_{\mathbb{F}_\ell}) \leq \delta_{\mathbb{F}_\ell}(1) = \delta_{\mathbb{Q}_\ell}(1) \leq \dim(\mathrm{H}^{\Pi}_{\mathbb{Q}_\ell}) \leq \dim(\mathrm{H}^{\Pi}_{\mathbb{F}_\ell}).$$

6.2. **Proof of (2').** This is proved in [CHT17, §11]. We give here a more elementary argument, which avoids Larsen-Pink's theory of regular semisimple Frobenii. For $C = \mathbb{F}_\ell, \mathbb{Q}_\ell, \mathbb{Z}_\ell$, write $\mathrm{H}_C := \mathrm{H}^w(X_{\overline{K}}, C)$. Also, let $\overline{\Pi}_\ell$ and Π_ℓ denote the image of $\overline{\Pi}$ and Π acting on $\mathrm{H}_{\mathbb{Z}_\ell}$ respectively.

We begin with the following Lemma. Recall that (S, $\mathbb{Q}_\ell$, $\frac{w}{2}$, X), (S, $\mathbb{Q}_\mathfrak{u}$, $\frac{w}{2}$, X) hence - as this holds for every $\mathfrak{u} \in \mathcal{U}$ (8.4.5 for P the property of acting semisimply) - (S, $\mathbb{F}_\ell$, $\frac{w}{2}$, X) for $\ell \gg 0$ are insensitive to finite field extensions of K (Lemma 4.2 (1)).

Lemma. *After replacing K by a finite field extension, there exists a monic polynomial $P \in \mathbb{Q}[T]$ and for every $\ell \neq p$ a semisimple element $\phi_\ell \in \Pi_\ell$ such that, for $\ell \gg 0$, Π_ℓ is generated by $\overline{\Pi}_\ell$ and ϕ_ℓ, and ϕ_ℓ has characteristic polynomial P.*

Proof. Let $\overline{\mathfrak{G}}_{\mathbb{Z}_\ell}$, $\mathfrak{G}_{\mathbb{Z}_\ell}$ denote respectively the Zariski closure of $\overline{\Pi}_\ell$, Π_ℓ in $\mathrm{GL}(\mathrm{H}_{\mathbb{Z}_\ell})$. After possibly replacing $\mathcal{S}$ by a connected étale cover, one may assume $\mathfrak{G}_{\mathbb{Q}_\ell}$ is connected for every $\ell \in \mathcal{L}$ (Theorem 1.3.1). One may also assume $\mathcal{S}$ carries a k-point $s \in \mathcal{S}(k)$. Let φ_ℓ denote the image of the geometric Frobenius φ_s acting on $\mathrm{H}_{\mathbb{Z}_\ell}$; recall that its characteristic polynomial P_s is in $\mathbb{Q}[T]$ and independent of ℓ ([D80]). Write

$\varphi_\ell = \varphi_\ell^{ss}\varphi_\ell^u$ for the multiplicative Jordan decomposition of φ_ℓ in $\mathfrak{G}_{\mathbb{Q}_\ell}$. There exists polynomials P^{ss}, P^u in $\mathbb{Q}[T]$ and independent of ℓ such that $\varphi_\ell^{ss} = P^{ss}(\varphi_\ell)$, $\varphi_\ell^u = P^u(\varphi_\ell)$. Let $\mathfrak{F}_{\mathbb{Z}_\ell}$, $\mathfrak{F}_{\mathbb{Z}_\ell}^{ss}$, $\mathfrak{F}_{\mathbb{Z}_\ell}^u$ denote the Zariski closure in $\mathfrak{G}_{\mathbb{Z}_\ell}$ of the subgroup generated by φ_ℓ, φ_ℓ^{ss} and φ_ℓ^u respectively. Then $\mathfrak{F}_{\mathbb{Z}_\ell} = \mathfrak{F}_{\mathbb{Z}_\ell}^{ss}\mathfrak{F}_{\mathbb{Z}_\ell}^u$. Since $\mathfrak{G}_{\mathbb{Q}_\ell}/\overline{\mathfrak{G}}_{\mathbb{Q}_\ell}$ is connected, abelian, reductive[3], it is a torus. Hence $\mathfrak{F}_{\mathbb{Q}_\ell}^u \subset \overline{\mathfrak{G}}_{\mathbb{Q}_\ell}$. In particular, $\varphi_\ell^u \in \overline{\mathfrak{G}}(\mathbb{Q}_\ell)$. But, actually, $\varphi_\ell^u \in \overline{\mathfrak{G}}(\mathbb{Z}_\ell)$ for $\ell \gg 0$. Indeed, $\varphi_\ell^u = P^u(\varphi_\ell)$ is in $\mathrm{End}_{\mathbb{Z}_\ell}(\mathrm{H}_{\mathbb{Z}_\ell})$ for $\ell \gg 0$ since P^u is in $\mathbb{Q}[T]$ and independent of ℓ. Also $\det(\varphi_\ell^u) = 1 \in \mathbb{Z}_\ell^\times$ shows that $\varphi_\ell^u \in \overline{\mathfrak{G}}(\mathbb{Q}_\ell) \cap \mathrm{GL}(H_{\mathbb{Z}_\ell})$. It only remains to check that $\overline{\mathfrak{G}}(\mathbb{Q}_\ell) \cap \mathrm{GL}(\mathrm{H}_{\mathbb{Z}_\ell}) = \overline{\mathfrak{G}}(\mathbb{Z}_\ell)$. The inclusion $\overline{\mathfrak{G}}(\mathbb{Q}_\ell) \cap \mathrm{GL}(H_{\mathbb{Z}_\ell}) \supset \overline{\mathfrak{G}}(\mathbb{Z}_\ell)$ is straightforward. The converse inclusion is the valuative criterion of properness for the closed immersion $\overline{\mathfrak{G}}_{\mathbb{Z}_\ell} \hookrightarrow \mathrm{GL}(\mathrm{H}_{\mathbb{Z}_\ell})$:

$$\begin{array}{ccc} \overline{\mathfrak{G}}_{\mathbb{Z}_\ell} & \hookrightarrow & \mathrm{GL}(\mathrm{H}_{\mathbb{Z}_\ell}) \\ \uparrow & \nwarrow & \uparrow \\ \mathbb{Q}_\ell & \longrightarrow & \mathbb{Z}_\ell \end{array}$$

From [CHT17, Thm. (7.3.2)], there exists an integer $N \geq 1$ independent of ℓ such that $(\varphi_\ell^u)^N \in \overline{\Pi}_\ell$. But, then, $(\varphi_\ell^{ss})^N = \varphi_\ell^N(\varphi_\ell^u)^{-N} \in \Pi_\ell$; after replacing k by its degree-N field extension, we may assume $N = 1$. Then $\phi_\ell = \varphi_\ell^{ss}$ works. □

We can now conclude the proof. The fact that ϕ_ℓ acts semisimply on $\mathrm{H}_{\mathbb{Q}_\ell}$ is equivalent to the fact that the minimal polynomial Q_ℓ of ϕ_ℓ is separable. Since P is in $\mathbb{Q}[T]$ and independent of ℓ, $Q := Q_\ell$ is in $\mathbb{Q}[T]$ and independent of ℓ as well. And since one assumes $\mathrm{H}_{\mathbb{Z}_\ell}$ is torsion free, the minimal polynomial of ϕ_ℓ acting on $\mathrm{H}_{\mathbb{F}_\ell}$ is the reduction modulo-ℓ of Q for $\ell \gg 0$; in particular, it is again separable for $\ell \gg 0$. This shows that ϕ_ℓ acts semisimply on $\mathrm{H}_{\mathbb{F}_\ell}$ for $\ell \gg 0$ hence that its image in $\mathrm{GL}(\mathrm{H}_{\mathbb{F}_\ell})$ is of prime-to-ℓ order. Thus (S, $\mathbb{F}_\ell$, $\frac{w}{2}$, X) follows from Fact 2.3 (2) and [S94a, Lem. 5(b)].

6.3. **Proof of (3').** One retains the notation of Subsection 6.1. Since we may assume $\ell \gg 0$, (F, $\mathbb{Q}_\ell$, i, X), $i = w, d - w$ + (wS, $\mathbb{Q}_\ell$, w, X) imply that the canonical morphism $Z^w(X) \otimes \mathbb{Z}_\ell \to \mathrm{H}_{\mathbb{Z}_\ell}^{\Pi}$ is surjective ([MiR04, Lem. 3.1]) and, in particular, that the morphism $Z^w(X) \otimes \mathbb{Z}_\ell \to \mathrm{H}_{\mathbb{Z}_\ell}$ has torsion-free cokernel. This and the fact that one assumes $\mathrm{H}_{\mathbb{Z}_\ell}$ is torsion free show that the images of $Z^w(X) \otimes C \to \mathrm{H}_C^{\Pi}$ for $C = \mathbb{Q}_\ell, \mathbb{F}_\ell, \mathbb{Z}_\ell$ have the same rank - say δ. As a result

$$(\mathrm{F},\ X,\ \mathbb{Q}_\ell,\ w) \Leftrightarrow \delta = \dim(\mathrm{H}_{\mathbb{Q}_\ell}^{\Pi})$$
$$(\mathrm{F},\ X,\ \mathbb{F}_\ell,\ w) \Leftrightarrow \delta = \dim(\mathrm{H}_{\mathbb{F}_\ell}^{\Pi})$$

Thus the conclusion follows from the implications:

$$\delta_{\mathbb{Q}_\ell}(1) = \dim(\mathrm{H}_{\mathbb{Q}_\ell}^{\Pi}) \overset{6.1}{\Leftrightarrow} (\mathrm{wS},\ \mathbb{Q}_\ell,\ w,\ X) \overset{(1')}{\Rightarrow} (\mathrm{wS},\ \mathbb{F}_\ell,\ w,\ X) \overset{6.1}{\Leftrightarrow} \delta_{\mathbb{F}_\ell}(1) = \dim(\mathrm{H}_{\mathbb{F}_\ell}^{\Pi}).$$

and the fact that for $\ell \gg 0$, $\delta_{\mathbb{Q}_\ell}(1) = \delta_{\mathbb{F}_\ell}(1)$ (see the last paragraph of 2.4).

7. Proof of Corollary 1.4

Assume $p > 0$ and let $X \in \mathrm{SmP}(K)$ with dimension d. Let $\mathrm{Br}(X_{\overline{K}}) := \mathrm{H}^2(X_{\overline{K}}, \mathbb{G}_m)$ denote the Brauer group of $X_{\overline{K}}$. For a prime $\ell \neq p$ and integer $n \geq 1$, let $\mathrm{Br}(X_{\overline{K}})[\ell^n] \subset Br(X_{\overline{K}})$ denote the kernel of the multiplication-by-ℓ^n map,

$$T_\ell(\mathrm{Br}(X_{\overline{K}})) := \varprojlim Br(X_{\overline{K}})[\ell^n],\quad V_\ell(\mathrm{Br}(X_{\overline{K}}))[\ell^n]) := T_\ell(\mathrm{Br}(X_{\overline{K}}))[\ell^n]) \otimes \mathbb{Q}_\ell.$$

Recall the following elementary observation.

Lemma. *For every $\ell \neq p$, $\mathrm{Br}(X_{\overline{K}})^{\pi_1(K)}[\ell^\infty]$ is finite $\Leftrightarrow V_\ell(\mathrm{Br}(X_{\overline{K}}))^{\pi_1(K)} = 0$.*

Proof. As $\mathrm{Br}(X_{\overline{K}})^{\pi_1(K)}[\ell^n]$ is finite, $n \geq 0$ one has the following equivalences

$$\begin{aligned} &\mathrm{Br}(X_{\overline{K}})^{\pi_1(K)}[\ell^\infty] \text{ is infinite} \\ &\Longleftrightarrow \mathrm{Br}(X_{\overline{K}})^{\pi_1(K)} \text{ contains an element of order exactly } \ell^n \text{ for every } n \geq 1 \\ &\overset{(1)}{\Longleftrightarrow} T_\ell(\mathrm{Br}(X_{\overline{K}}))^{\pi_1(K)} \neq 0 \\ &\overset{(2)}{\Longleftrightarrow} V_\ell(\mathrm{Br}(X_{\overline{K}}))^{\pi_1(K)} \neq 0, \end{aligned}$$

[3]This is where we use (S, $\mathbb{Q}_\ell$, $\frac{w}{2}$, X).

where $\overset{(1)}{\Longrightarrow}$ follows from the fact a projective system of non-empty finite sets is non-empty and $\overset{(2)}{\Longrightarrow}$ follows from the fact $T_\ell(\mathrm{Br}(X_{\overline{K}}))$ is torsion-free. □

7.1. **Proof of (1) ⇒ (2).** We retain, again, the notation of 2.1. Let $\rho(X)$ denote the rank of the Néron-Severi group $NS(X)$ of X. For divisors, numerical and algebraic equivalences coincide (*e.g.* [Gr71, XIII, Thm. 4.6]); in particular (F, $\mathbb{Q}_\ell$, 1, X) is equivalent to each of the assertions (a) - (d) in [T94, Prop. 2.9]. From [T94, Prop. 2.9 (a)],

$$\rho(X) = \dim(\mathrm{H}^2(X_{\overline{K}}, \mathbb{Q}_\ell(w))^{\pi_1(K)})(= \dim(\mathrm{H}^2(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_\ell(w))^{\pi_1(\mathcal{S})}))$$

while, from [T94, Prop. 2.9 (c)], (S, $\mathbb{Q}_\ell$, 1, X) holds so that $\rho(X) = \delta_{\mathbb{Q}_\ell}(1)$. As P is in $\mathbb{Z}[1/p][T]$ and independent of $\ell \neq p$ (Fact 2.4 (2)), for every other prime $\ell' \neq p$, one has

$$\rho(X) = \delta_{\mathbb{Q}_\ell}(1) = \delta_{\mathbb{Q}_{\ell'}}(1) \geq \dim(\mathrm{H}^2(\mathcal{X}_{\overline{\eta}}, \mathbb{Q}_{\ell'}(w))^{\pi_1(\mathcal{S})}) \geq \rho(X).$$

So that [T94, Prop. 2.9 (a)] holds for ℓ' as well and (F, $\mathbb{Q}_{\ell'}$, 1, X) follows from the implication (a) ⇒ (b) in [T94, Prop. 2.9].

7.2. **Proof of (2) ⇒ (3).** From [T94, Prop. (5.1)] and Lemma 4.3, for every $\ell \neq p$, (F, $\mathbb{Q}_\ell$, 1, X) implies (wS, $\mathbb{Q}_\ell$, 1, X). Whence, in particular, split short exact sequences of $\pi_1(K)$-modules

$$0 \to \mathrm{H}^2(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{\pi_1(K)} \to \mathrm{H}^2(X_{\overline{K}}, \mathbb{Q}_\ell(1)) \to V_\ell(\mathrm{Br}(X_{\overline{K}})) \to 0,\ \ell \neq p,$$

which shows $V_\ell(\mathrm{Br}(X_{\overline{K}}))^{\pi_1(K)} = 0$, $\ell \neq p$. From the above preliminary Lemma, this is equivalent to the finiteness of $\mathrm{Br}(X_{\overline{K}})^{\pi_1(K)}[\ell^\infty]$ for $\ell \neq p$. So, to prove (3), it is enough to show

$$(7.2.1)\ \ \mathrm{Br}(X_{\overline{K}})^{\pi_1(K)}[\ell] = 0, \quad \ell \gg 0.$$

From [T94, Prop. (5.1)] and 6 (3'), (2) implies (F, $\mathbb{F}_\ell$, 1, X) for $\ell \gg 0$ whence the short exact sequences

$$(7.2.2)\ \ 0 \to \mathrm{H}^2(X_{\overline{K}}, \mathbb{F}_\ell(1))^G \to \mathrm{H}^2(X_{\overline{K}}, \mathbb{F}_\ell(1)) \to \mathrm{Br}(X_{\overline{K}})[\ell] \to 0, \quad \ell \gg 0.$$

On the other hand, from [T94, Prop. (5.1)], (2) also implies (wS', $\mathbb{Q}_\ell$, 1, X) hence, by Lemma 4.3 (1), (wS, $\mathbb{Q}_\ell$, 1, X) which, in turn, by 6 (1'), implies (wS, $\mathbb{F}_\ell$, 1, X) for $\ell \gg 0$. This shows (7.2.2) splits $\pi_1(K)$-equivariantly for $\ell \gg 0$, whence (7.2.1).

7.3. **Proof of (3) ⇒ (2).** From the above preliminary observation the finiteness of $\mathrm{Br}(X_{\overline{K}})^{\pi_1(K)}[p']$ implies $V_\ell(\mathrm{Br}(X_{\overline{K}}))^{\pi_1(K)} = 0$, $\ell \neq p$ so that taking $\pi_1(K)$-invariants in the short exact sequences

$$0 \to NS(X_{\overline{K}}) \otimes \mathbb{Q}_\ell \to \mathrm{H}^2(X_{\overline{K}}, \mathbb{Q}_\ell(1)) \to V_\ell(\mathrm{Br}(X_{\overline{K}})) \to 0,\ \ell \neq p$$

(where $NS(X_{\overline{K}})$ denotes the Néron-Severi group of $X_{\overline{K}}$) one gets

$$(NS(X_{\overline{K}}) \otimes \mathbb{Q}_\ell)^{\pi_1(K)} \tilde{\to} \mathrm{H}^2(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{\pi_1(K)}.$$

On the other hand, let $K^{perf} := K^{\pi_1(K)}$ denote the perfect closure of K and write $X^{perf} := X \times_K K^{perf}$. Then

$$(CH^1(X_{\overline{K}}) \otimes \mathbb{Q}_\ell)^{\pi_1(K)} = CH^1(X^{perf}) \otimes \mathbb{Q}_\ell \tilde{\leftarrow} CH^1(X) \otimes \mathbb{Q}_\ell$$

(note that, in general, the cokernel of $CH^1(X) \to CH^1(X^{perf})$ is of p-primary torsion). Since $\pi_1(K)$ acts through a finite quotient - hence semisimply - on every finite-dimensional $\mathbb{Q}_\ell$-vector subspace of $CH^1(X_{\overline{K}}) \otimes \mathbb{Q}_\ell$, the morphism $(CH^1(X_{\overline{K}}) \otimes \mathbb{Q}_\ell)^{\pi_1(K)} \to (NS(X_{\overline{K}}) \otimes \mathbb{Q}_\ell)^{\pi_1(K)}$ is surjective, which yields the surjectivity of

$$[-] : CH^1(X) \otimes \mathbb{Q}_\ell \tilde{\to} (CH^1(X_{\overline{K}}) \otimes \mathbb{Q}_\ell)^{\pi_1(K)} \twoheadrightarrow (NS(X_{\overline{K}}) \otimes \mathbb{Q}_\ell)^{\pi_1(K)} \tilde{\to} \mathrm{H}^2(X_{\overline{K}}, \mathbb{Q}_\ell(1))^{\pi_1(K)}, \quad \ell \neq p$$

8. Appendix: Basic properties of ultraproducts of fields

Let $\mathcal{L}$ be an infinite set. For a subset $S \subset \mathcal{L}$, write $\mathbf{1}_S : \mathcal{L} \to \{0,1\}$ for the characteristic function of S.

8.1. A filter on $\mathcal{L}$ is a family $\mathfrak{f}$ of subsets of $\mathcal{L}$ such that
(1) $A, B \in \mathfrak{f} \Rightarrow A \cap B \in \mathfrak{f}$;
(2) $A \in \mathfrak{f}$, $A \subset B \subset \mathcal{L} \Rightarrow B \in \mathfrak{f}$;
(3) $\emptyset \notin \mathfrak{f}$

8.1.1. An ultrafilter is a filter $\mathfrak{u}$ which is maximal for $\subset$ among all filters that is such that for every filter $\mathfrak{f}$ on $\mathcal{L}$, $\mathfrak{u} \subset \mathfrak{f} \Rightarrow \mathfrak{u} = \mathfrak{f}$. A filter $\mathfrak{u}$ on $\mathcal{L}$ is an ultrafilter if and only if for every $S \subset \mathcal{L}$ either $S \in \mathfrak{u}$ or $\mathcal{L} \setminus S \in \mathfrak{u}$.

8.1.2. An ultrafilter $\mathfrak{u}$ on $\mathcal{L}$ is either principal that is of the form $\mathfrak{u}_\ell := \{S \subset \mathcal{L} \mid \ell \in S\}$ for some $\ell \in \mathcal{L}$ or contains the filter $\mathfrak{f}^\# := \{S \subset \mathcal{L} \mid |\mathcal{L} \setminus S| < +\infty\}$ of cofinite subsets, and $\mathfrak{f}^\#$ is the intersection of all non-principal ultrafilters on $\mathcal{L}$.

8.1.3. For every $\ell \in \mathcal{L}$ fix a field F_ℓ and write $\underline{F} := \prod_{\ell\in\mathcal{L}} F_\ell$; note that $\underline{F}$ is 0-dimensional. For every $S \subset \mathcal{L}$, write $e_S := (1 - \mathbf{1}_S(\ell))_\ell \in \underline{F}$ for the corresponding idempotent. Filters on $\mathcal{L}$ are in bijection with the ideals of $\underline{F}$

$$\begin{array}{ccc} \text{Filters on } \mathcal{L} & \longleftrightarrow & \text{Ideals of } \underline{F} \\ \mathfrak{f} & \longmapsto & \langle e_S \mid S \in \mathfrak{f}\rangle \\ \{S \in \mathcal{P}(\mathcal{L}) \mid e_S \in \mathfrak{i}\} & \longleftarrow & \mathfrak{i} \end{array}$$

This bijection restricts to a bijection between ultrafilters on $\mathcal{L}$ and the prime (equivalently maximal) spectrum of $\underline{F}$

$$\begin{array}{ccc} \text{Ultrafilters on } \mathcal{L} & \longleftrightarrow & \mathrm{Spec}(\underline{F}) \\ \mathfrak{u} & \longmapsto & \mathfrak{m}_\mathfrak{u} := \langle e_S \mid S \in \mathfrak{u}\rangle \\ \mathfrak{u}_\mathfrak{m} := \{S \in \mathcal{P}(\mathcal{L}) \mid e_S \in \mathfrak{m}\} & \longleftarrow & \mathfrak{m} \end{array}$$

and, *via* this bijection, principal (prime) ideals corresponds to principal ultrafilters. In particular, 8.1.2 shows that the intersection of all non-principal maximal ideals in $\underline{F}$ is the ideal $\oplus_{\ell\in\mathcal{L}} F_\ell \subset \underline{F}$.

Let $\mathcal{U}$ denote the set of all *non-principal* ultrafilters on $\mathcal{L}$.

8.1.4. For $\mathfrak{u} \in \mathcal{U}$ and an $\underline{F}$-module $\underline{M}$, write

$$M_\mathfrak{u} := \underline{M}/\mathfrak{m}_\mathfrak{u}\underline{M} = \varinjlim_{S\in\mathfrak{u}} (1 - e_S)\underline{M}$$

(direct limit by reverse inclusion). For $\ell \in \mathcal{L}$, write $M_\ell := M_{\mathfrak{u}_\ell}$ for its 'ℓth component'.

Since for every $S \in \mathfrak{u}$ the projection $p_S : \underline{F} = e_S\underline{F} \times (1 - e_S)\underline{F} \twoheadrightarrow \underline{F}/e_S\underline{F} = (1 - e_S)\underline{F}$ is flat and $\underline{F} \to F_\mathfrak{u}$ is the direct limit of the $p_S : \underline{F} \twoheadrightarrow \underline{F}/e_S\underline{F}$, one gets the following.

Lemma. *For every ultrafilter $\mathfrak{u}$ on $\mathcal{L}$, the morphism $\underline{F} \to F_\mathfrak{u}$ is flat.*

8.2. A finitely generated $\underline{F}$-module $\underline{M}$ is the direct product $\underline{M} = \prod_{\ell\in\mathcal{L}} M_\ell$ of its ℓth components if and only if it is finitely presented. Write $\mathrm{Mod}_{/\underline{F}}$ for the full subcategory of the category of $\underline{F}$-modules whose objects are direct products $\underline{M} = \prod_{\ell\in\mathcal{L}} M_\ell$ of their components. One easily checks that $\mathrm{Mod}_{/\underline{F}}$ is an abelian category. For $\underline{M} \in \mathrm{Mod}_{/\underline{F}}$, one has

$$\underline{M} \text{ is finitely generated } \Leftrightarrow\ \underline{M} \text{ is finitely presented} \Leftrightarrow \sup_{\ell\in\mathcal{L}}\mathrm{dim}_{F_\ell}(M_\ell) < +\infty \Leftrightarrow \sup_{\mathfrak{u}\in\mathcal{U}}\mathrm{dim}_{F_\mathfrak{u}}(M_\mathfrak{u}) < +\infty$$

In particular, for $\underline{M} \in \mathrm{Mod}_{/\underline{F}}$ finitely generated and
- (8.2.1) for $\underline{N} \subset \underline{M}$ an $\underline{F}$-submodule, one has

$$\underline{N} \in \mathrm{Mod}_{/\underline{F}} \Leftrightarrow \underline{N} \text{ is finitely generated} \Leftrightarrow \underline{N} \text{ is finitely presented}$$

- (8.2.2) for every $\underline{F}$-module $\underline{N}$ and $\mathfrak{u} \in \mathcal{U}$, the canonical morphism

$$\mathrm{Hom}_{\underline{F}}(\underline{M}, \underline{N}) \otimes_{\underline{F}} F_\mathfrak{u} \to \mathrm{Hom}_{F_\mathfrak{u}}(M_\mathfrak{u}, N_\mathfrak{u})$$

  is an isomorphism ([Bo85, Chap. I, §2.10, Prop. 11], using 8.1.4).

8.2.1. The full subcategory of finitely generated $\underline{F}$-modules in $\mathrm{Mod}_{/\underline{F}}$ is an abelian subcategory of $\mathrm{Mod}_{/\underline{F}}$, stable under taking internal Hom and tensor products: for finitely generated $\underline{M}, \underline{N} \in \mathrm{Mod}_{/\underline{F}}$, the canonical morphisms $\mathrm{Hom}_{\underline{F}}(\underline{M}, \underline{N}) \to \prod_{\ell\in\mathcal{L}} \mathrm{Hom}_{F_\ell}(M_\ell, N_\ell)$ and $\underline{M} \otimes_{\underline{F}} \underline{N} \to \prod_{\ell\in\mathcal{L}} M_\ell \otimes_{F_\ell} N_\ell$ are isomorphisms.

8.3. For every $\mathfrak{u} \in \mathcal{U}$,

8.3.1. **Lemma.** *Let $\underline{M} \in \mathrm{Mod}_{/\underline{F}}$ be finitely generated and let $N_\bullet : N_0 = M_\mathfrak{u} \supset N_1 \supset \cdots \supset N_r \supset N_{r+1} = 0$ be a finite filtration by $F_\mathfrak{u}$-submodules. Then there exists a filtration $\underline{N}_\bullet : \underline{N}_0 = \underline{M} \supset \underline{N}_1 \supset \cdots \supset \underline{N}_r \supset \underline{N}_{r+1} = 0$ in $\mathrm{Mod}_{/\underline{F}}$ such that $N_{\bullet,\mathfrak{u}} = N_\bullet$.*

Proof. One may assume $r = 1$; write $N := N_1$. Fix an $F_\mathfrak{u}$-basis $n_1, \ldots, n_r$ of N and lift it to a family $\underline{n}_1, \ldots, \underline{n}_r \in \underline{M}$. Then the $\underline{F}$-submodule $\underline{N} = \sum_{1\le i\le r} \underline{F}\, \underline{n}_i \subset \underline{M}$ is in $\mathrm{Mod}_{/\underline{F}}$ by (8.2.1). □

8.3.2. **Lemma.** *Let $\underline{M} \in \mathrm{Mod}_{/\underline{F}}$ and consider the following properties.*
- *(8.3.2.1) $M_{\mathfrak{u}} = 0$;*
- *(8.3.2.2) The set of $\ell \in \mathcal{L}$ such that $M_\ell = 0$ is in $\mathfrak{u}$.*

Then (8.3.2.2) $\Rightarrow$ (8.3.2.1). If $\underline{M}$ is finitely generated, (8.3.2.1) $\Rightarrow$ (8.3.2.2).

Proof. (8.3.2.2) $\Rightarrow$ (8.3.2.1) is straightforward. Conversely, if $\mathfrak{m}_{\mathfrak{u}}\underline{M} = \underline{M}$ and $\underline{M}$ is finitely generated with $\underline{F}$-generators $\underline{m}_1, \dots, \underline{m}_r$ then for every $i = 1, \dots, r$, there exists $S_i \in \mathfrak{u}$ such that $\underline{m}_i \in e_{S_i}\underline{M}$ hence $\underline{M} = e_S\underline{M}$ with $S = S_1 \cap \cdots \cap S_r \in \mathfrak{u}$. □

8.3.3. **Lemma.** *Let $\underline{\phi} : \underline{M} \to \underline{N}$ be a morphism in $\mathrm{Mod}_{/\underline{F}}$ and consider the following properties.*
- *(8.3.3.1) $\phi_{\mathfrak{u}} : M_{\mathfrak{u}} \to N_{\mathfrak{u}}$ has P;*
- *(8.3.3.2) The set S of all $\ell \in \mathcal{L}$ such that $\phi_\ell : M_\ell \to N_\ell$ has P is in $\mathfrak{u}$,*

where P is one of the properties of being injective, surjective, an isomorphism. Then (8.3.3.2) $\Rightarrow$ (8.3.3.1). If the conditions below are satisfied, (8.3.3.1) $\Rightarrow$ (8.3.3.2).

P	*Condition*
Surjective	*$\underline{\phi}$ has finitely generated cokernel*
Injective	*$\underline{\phi}$ has finitely generated kernel*
Isomorphism	*$\underline{\phi}$ has finitely generated kernel and cokernel*

Proof. By right-exactness (resp. left-exactness - 8.1.4) of $-\otimes_{\underline{F}}\mathbb{Q}_{\mathfrak{u}}$, $\mathrm{coker}(\underline{\phi})_{\mathfrak{u}} = \mathrm{coker}(\phi_{\mathfrak{u}})$ (resp. $\ker(\underline{\phi})_{\mathfrak{u}} = \ker(\phi_{\mathfrak{u}})$). So the conclusion follows from 8.3.2. □

8.4. Let $\underline{M} \in \mathrm{Mod}_{/\underline{F}}$ and Π be a group acting on $\underline{M}$. For every $\mathfrak{u} \in \mathcal{U}$,

8.4.1. **Lemma.** $(M_{\mathfrak{u}})_\Pi = (\underline{M}_\Pi)_{\mathfrak{u}}$.

Proof. This follows from the exact sequence $\underline{M}^{\oplus \Pi} \stackrel{\sum_{\pi\in\Pi}(Id-\pi)}{\longrightarrow} \underline{M} \to \underline{M}_\Pi \to 0$, right-exactness of $-\otimes_{\underline{F}} F_{\mathfrak{u}}$ and the fact that tensor products commute with direct sums. □

From now on, assume furthermore that $\underline{M} \in \mathrm{Mod}_{/\underline{F}}$ is finitely generated, that for every $\ell \in \mathcal{L}$, F_ℓ is a Hausdorff topological field, that Π is a topological group which acts continuously on $\underline{M}$ for $\underline{M}$ equipped with the product topology of the topologies of the M_ℓ (recall M_ℓ is a finitely generated F_ℓ-module) and that Π is topologically finitely generated with topological generators $\pi_1, \dots, \pi_s$. Let $\Pi^\circ \subset \Pi$ denote the abstract group generated by $\pi_1, \dots, \pi_s$.

8.4.2. **Lemma.** $(M_{\mathfrak{u}})^\Pi = (\underline{M}^\Pi)_{\mathfrak{u}}$.

Proof. The exact sequence $0 \to \underline{M}^\Pi \to \underline{M} \stackrel{(Id-\pi_1,\dots,Id-\pi_s)}{\longrightarrow} \underline{M}^s$, 8.1.4 and the fact that tensor products commute with finite direct products (=direct sums) yield $(\underline{M}^\Pi)_{\mathfrak{u}} = (M_{\mathfrak{u}})^{\Pi^\circ}$. So the assertion follows from the obvious inclusions $(M_{\mathfrak{u}})^\Pi \supset (\underline{M}^\Pi)_{\mathfrak{u}} = (M_{\mathfrak{u}})^{\Pi^\circ} \supset (M_{\mathfrak{u}})^\Pi$. □

In particular, if $\underline{N} \in \mathrm{Mod}_{/\underline{F}}$ is also finitely generated and equipped with a continuous action of Π, (8.2.2) and 8.4.2 yield

$$(8.4.2.1)\quad \mathrm{Hom}_\Pi(M_{\mathfrak{u}}, N_{\mathfrak{u}}) = \mathrm{Hom}_\Pi(\underline{M}, \underline{N})_{\mathfrak{u}}$$

8.4.3. **Lemma.** *For every finite filtration $\underline{N}_\bullet : \underline{N}_0 = \underline{M} \supset \underline{N}_1 \supset \cdots \supset \underline{N}_r \supset \underline{N}_{r+1} = 0$ in $\mathrm{Mod}_{/\underline{F}}$, map $\sigma : \{0, \dots, r+1\} \to \{0, \dots, r+1\}$ and subset $X \subset \underline{F}[\Pi]$, consider the following assertions.*
- *(8.4.3.1) $XN_{i,\mathfrak{u}} \subset N_{\sigma(i),\mathfrak{u}}$, $i = 0, \dots, r+1$;*
- *(8.4.3.2) The set of all $\ell \in \mathcal{L}$ such that $XN_{\ell,i} \subset N_{\ell,\sigma(i)}$, $i = 0, \dots, r+1$ is in $\mathfrak{u}$.*

Then (8.4.3.2) $\Rightarrow$ (8.4.3.1). If X is finite (8.4.3.1) $\Rightarrow$ (8.4.3.2).

Proof. (8.4.3.2) $\Rightarrow$ (8.4.3.1) is straightforward. For (8.4.3.1) $\Rightarrow$ (8.4.3.2), write $X = \{x_1, \dots, x_t\}$. Then for every $i = 0, \dots, r+1$, one has $XN_{\ell,i} \subset N_{\ell,\sigma(i)}$ if and only if $(x_1, \dots, x_t)(N_{\ell,i}) \subset N^t_{\ell,\sigma(i)} \subset M^t_\ell$. Let $\underline{n}_{i,1}, \dots, \underline{n}_{i,t_i}$ be a set of $\underline{F}$-generators for $\underline{N}_i$ (8.2.1). By (8.4.3.1), for every $1 \leq j \leq t_i$, there exists $S_{i,j} \in \mathfrak{u}$ such that $(x_1, \dots, x_t)(\underline{n}_j) \in \underline{N}^t_{\sigma(i)} + e_{S_{i,j}}\underline{M}^t$. Hence $(x_1, \dots, x_t)(\underline{N}_i) \subset \underline{N}^t_{\sigma(i)} + \sum_{1\leq j\leq t_i} e_{S_{i,j}}\underline{M}^t \subset \underline{N}^t_{\sigma(i)} + e_{S_i}\underline{M}^t$ with $S_i = S_{i,1} \cap \cdots \cap S_{i,t_i} \in \mathfrak{u}$. The set of $\ell \in \mathcal{L}$ satisfying (8.4.3.2) then contains $S_0 \cap \cdots \cap S_{r+1} \in \mathfrak{u}$ □

In particular,
- (8.4.3.3) ($\sigma = Id$, $X = \{\pi_1, \dots, \pi_s\}$) $N_{\bullet,\mathfrak{u}}$ is Π-stable if and only if the set of all $\ell \in \mathcal{L}$ such that $N_{\bullet,\ell}$ is Π-stable is in $\mathfrak{u}$.

- (8.4.3.4) ($\sigma(i) = i+1$, $X = \{1-\pi_1, \dots, 1-\pi_s\}$ - See 8.3.1) Π acts unipotently on $M_{\mathfrak{u}}$ if and only if the set of all $\ell \in \mathcal{L}$ such that Π acts unipotently on M_ℓ is in $\mathfrak{u}$.

8.4.4. 8.3.1 and 8.4.3 imply that, every Π-submodule $N \subset M_{\mathfrak{u}}$ (hence resp. every Π-quotient $M_{\mathfrak{u}} \twoheadrightarrow N$) lifts to a Π-submodule $\underline{N} \subset \underline{M}$ (resp. a Π-quotient $\underline{M} \twoheadrightarrow \underline{N}$) in $\mathrm{Mod}_{/\underline{F}}$. From this, one immediately deduces that any Π-module N in the Tannakian category generated by the Π-module $M_{\mathfrak{u}}$ lifts to some $\underline{N}$ in $\mathrm{Mod}_{/\underline{F}}$ which is a Π-subquotient of a Π-module of the form $\oplus_{(m,n)\in\mathbb{Z}^2_{\geq 0}}(\underline{M}^{\otimes m} \otimes \check{\underline{M}}^{\otimes n})^{\oplus \mu(m,n)}$ for some function $\mu : \mathbb{Z}^2_{\geq 0} \to \mathbb{Z}_{\geq 0}$ with finite support.

8.4.5. **Lemma.** *The following assertions are equivalent.*
- *(8.4.5.1)* Π *acting on* $M_{\mathfrak{u}}$ *has* P*;*
- *(8.4.5.2) The set* S *of all* $\ell \in \mathcal{L}$ *such that* Π *acting on* M_ℓ *has* P *is in* $\mathfrak{u}$*,*

where P *is one of the properties of acting irreducibly or semisimply.*

Proof. The assertion for P the property of acting irreducibly follows from 8.3.2 and 8.4.4. Let P be the property of acting semisimply and assume (8.4.5.2). Let $N \subset M_{\mathfrak{u}}$ be a Π-submodule. By 8.4.4, N lifts to an $\underline{F}$-submodule $\underline{N}$ in $\mathrm{Mod}_{/\underline{F}}$ which is Π-stable. As $S \in \mathfrak{u}$, one may take $N_\ell = 0$ for $\ell \in \mathcal{L} \setminus S$. By (8.4.5.2), the projection $\underline{M} \twoheadrightarrow \underline{M}/\underline{N}$ splits Π-equivariantly. The conclusion follows by applying $-\otimes_{\underline{F}} F_{\mathfrak{u}}$. Conversely, assume (8.4.5.1). If $S \notin \mathfrak{u}$ then $\mathcal{L} \setminus S \in \mathfrak{u}$ and for every $\ell \in \mathcal{L} \setminus S$ there exists a Π-submodule $N_\ell \subset M_\ell$ such that

$$Q_\ell := \mathrm{coker}(\mathrm{Hom}_\Pi(M_\ell/N_\ell, M_\ell) \stackrel{p_\ell\circ -}{\to} \mathrm{Hom}_\Pi(M_\ell/N_\ell, M_\ell/N_\ell))$$

is non-zero, where $p_\ell : M_\ell \to M_\ell/N_\ell$ is the canonical quotient morphism. In particular, $Q_{\mathfrak{u}} \neq 0$, where $\underline{Q} := \prod_{\ell\in\mathcal{L}} Q_\ell$. Write also $\underline{N} := \prod_{\ell\in\mathcal{L}\setminus S} N_\ell$ and let $\underline{p} : \underline{M} \to \underline{M}/\underline{N}$ denote the canonical quotient morphism. By right-exactness of $-\otimes_{\underline{F}} F_{\mathfrak{u}}$, one obtains an exact sequence

$$\mathrm{Hom}_\Pi(\underline{M}/\underline{N}, \underline{M})_{\mathfrak{u}} \stackrel{p\circ -}{\to} \mathrm{Hom}_\Pi(\underline{M}/\underline{N}, \underline{M}/\underline{N})_{\mathfrak{u}} \to Q_{\mathfrak{u}} \to 0,$$

which, by (8.4.2.1), identifies with

$$\mathrm{Hom}_\Pi(M_{\mathfrak{u}}/N_{\mathfrak{u}}, M_{\mathfrak{u}}) \stackrel{p\circ -}{\to} \mathrm{Hom}_\Pi(M_{\mathfrak{u}}/N_{\mathfrak{u}}, M_{\mathfrak{u}}/N_{\mathfrak{u}}) \to Q_{\mathfrak{u}} \to 0,$$

contradicting the fact that the morphism of Π-modules $N_{\mathfrak{u}} \hookrightarrow M_{\mathfrak{u}}$ splits Π-equivariantly by (8.4.5.1). □

The same arguments show the following.

8.4.6. **Lemma.** *Let* $\underline{N} \subset \underline{M}$ *be an* $\underline{F}$*-submodule in* $\mathrm{Mod}_{/\underline{F}}$ *which is* Π*-stable. The following assertions are equivalent.*
- *(8.4.6.1) The inclusion* $N_{\mathfrak{u}} \hookrightarrow M_{\mathfrak{u}}$ *splits* Π*-equivariantly;*
- *(8.4.6.2) The set of all* $\ell \in \mathcal{L}$ *such that the inclusion* $N_\ell \hookrightarrow M_\ell$ *splits* Π*-equivariantly is in* $\mathfrak{u}$*.*

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