Conjecture. 1.1.1. by H( field of coefficients C K )
The most standard avatars of Conjecture 1.1.1 are (for S/T for short) conjecture ([T65], [A04, 7.3]) for
ℓ C (so that for singular cohomology with enriched Tannakian target category the category of
Q (1) (Semisimplicity) H( X CHM
Aut( Conjectures for realization functors.
1.1. Q semisimple Tannakian category over so that, after modifying the commutativity constraint, the category of numerical motives becomes a
In particular, Conjecture 1.1.1 for H implies that numerical and H-homological equivalences coincide
all the standard conjectures for H (except possibly the standard conjecture of Hodge type) [A04, 7.1.1].
H [A04, 5.4.2.2]. If p > 0, combined with the semisimplicity part of Conjecture 1.1.1 for H, this also implies
P Lefschetz type [A04, 5.2.4] for H. If π Q target category the category of finite-dimensional
Q cycles (modulo rational equivalence) and

\[ CH^w(X) := \oplus_{w \geq 0} CH^w(X) \]  the Z-graded Chow ring.

Let CHM(K) denote the category of Chow motives over K with Q-coefficients and SmP(K)^op →
C HM(K) the canonical functor [A04, 4.1.3]; fix a Weil cohomology H : CHM(K) ⊗ C_H → T_H with
field of coefficients C_H and enriched Tannakian target category T_H - See [A04, 3.3, 4.2.5, 7.1]. For
X ∈ SmP(K), let G_H(X) denote the Tannakian group of the Tannakian subcategory (H(X)) generated
by H(X) in T_H. The following unifying conjecture is at the heart of the philosophy of pure motives.
1.1.1. Conjecture. For every X ∈ SmP(K),
(1) (Semisimplicity) H(X) is semisimple - equivalently G_H(X) is a reductive algebraic group over C_H;
(2) (Fullness) The image of the cycle class map [-]_H : CH(X) ⊗ C_H → \oplus_{w \geq 0} H^{2w}(X)(w) is the subspace
of G_H(X)-invariant classes.

The most standard avatars of Conjecture 1.1.1 are (for K = C) the Hodge conjecture ( [H52], [A04, 7.2])
for singular cohomology with enriched Tannakian target category the category of Q-Hodge structures
(so that C_H = Q) and (for K finitely generated over its prime field) the Grothendieck-Serre/Tate (G-S/T for short) conjecture ( [T65], [A04, 7.3]) for ℓ-adic cohomology (ℓ ≠ p) with enriched Tannakian
target category the category of finite-dimensional Q_ℓ-vector spaces endowed with a continuous action of
π_1(K) (so that C_H = Q_ℓ). The fullness part of Conjecture 1.1.1 for H implies the standard conjecture of Lefschetz type [A04, 5.2.4] for H. If p = 0 this is already enough to imply all the standard conjectures for H [A04, 5.4.2.2]. If p > 0, combined with the semisimplicity part of Conjecture 1.1.1 for H, this also implies
all the standard conjectures for H (except possibly the standard conjecture of Hodge type) [A04, 7.1.1.1].
In particular, Conjecture 1.1.1 for H implies that numerical and H-homological equivalences coincide
so that, after modifying the commutativity constraint, the category of numerical motives becomes a
semisimple Tannakian category over Q. Let Q_X be any finite field extension of Q neutralizing the
Tannakian subcategory $(X)$ generated by the numerical motive $X$ in the category of numerical motives (with modified commutativity constraint) [DM82, Rem. 3.10], let $H : (X \otimes Q_X) \to Vect_{Q_X}$ be a fiber functor and let $G(X)$ denote the corresponding Tannakian group; this is a reductive group over $Q_X$ acting faithfully on the finite-dimensional $Q_X$-vector space $H(X)$. Assume Conjecture 1.1.1 holds for another Weil cohomology $H' : CHM(K) \otimes C_{H'} \to T_{H'}$. Then the general formalism of Tannakian categories implies the following.

1.2.2. Conjecture. For every $X \in \text{SmP}(K)$ and embedding of $Q_X$ in $C_{H'}$, one has $G(X) \times_{Q_X} C_{H'} \simeq G_{H'}(X) \times_{C_{H'}} C_{H'}$ acting on $H(X) \otimes_{Q_X} C_{H'} \simeq H'(X) \otimes_{C_{H'}} C_{H'}$.

When $K$ has characteristic 0, one expects $Q_X = Q$ and the isomorphisms of Conjecture 1.1.2 to hold over $C_{H'}$. When $K$ has characteristic $p > 0$, as Serre noticed, this cannot always hold [Gr68, §1.7].

1.2. Realization functors arising from étale cohomology. Let $\mathcal{L}$ denote the set of all primes $\neq p$ and let $\mathcal{U}$ denote the set of all non-principal ultrafilters on $\mathcal{L}$. For $\ell \in \mathcal{L}$ let $\mathbb{F}_\ell$ denote the finite field with $\ell$ elements and $\mathcal{Q}_\ell$ the completion of $\mathbb{Q}$ at $\ell$. For $u \in \mathcal{U}$ let $\mathcal{Q}_u$ (resp. $\mathcal{Q}_a$) denote the residue field of the maximal ideal of $\mathbb{F} := \prod_{\ell \in \mathcal{L}} \mathbb{F}_\ell$ (resp. $\mathcal{Q} := \prod_{\ell \in \mathcal{L}} \mathcal{Q}_\ell$) corresponding to $u$ (See Section 8 for details about ultraproducts).

The G-S/T conjecture is the incarnation of Conjecture 1.1.1 for the Weil cohomologies derived from étale cohomology.

1.2.1. These are built from the following cohomology groups:

- For every $\ell \in \mathcal{L}$, $\mathcal{Q}_\ell$-cohomology $H^w(X, \mathcal{Q}_\ell) := (\lim_{\to} H^w(X, \mathbb{Z}/\ell^n)) \otimes_{\mathbb{Z}_\ell} \mathcal{Q}_\ell$;
- For every $u \in \mathcal{U}$, $\mathcal{Q}_u$-cohomology $H^w(X, \mathcal{Q}_u) := (\prod_{\ell \in \mathcal{L}} H^w(X, \mathbb{F}_\ell)) \otimes_\mathbb{F} \mathcal{Q}_u$;

$\mathcal{Q}_u$-cohomology $H^w(X, \mathcal{Q}_u) := (\prod_{\ell \in \mathcal{L}} H^w(X, \mathcal{Q}_\ell)) \otimes_\mathcal{Q} \mathcal{Q}_u$.

The following diagram summarizes the relation between the various coefficients:

\[
\begin{array}{c}
\mathcal{Q}_\ell \\
\downarrow \\
\mathcal{Q}_u \\
\downarrow \\
\mathbb{Q} \\
\downarrow \\
\mathbb{F} \\
\downarrow \\
\mathbb{F}_\ell
\end{array}
\]

\[
\begin{array}{c}
\mathbb{Q} \\
\downarrow \\
\hat{\mathbb{Z}} \\
\downarrow \\
\mathbb{Q}_u
\end{array}
\]

From now on, assume the base field $K$ is finitely generated. Let $C$ denote any of $\mathcal{Q}_\ell$, $\mathcal{Q}_u$, $\mathcal{Q}_d$ and write $H_C(X) := H(X, \mathcal{Q}_C)$. The Tannakian target category $T_{H_C}$ is the category of finite-dimensional continuous $C$-representations of $\pi_1(K)$ (as usual, $\mathbb{F}_\ell$ is equipped with the discrete topology, $\mathcal{Q}_\ell$ with the $\ell$-adic topology, $\mathbb{Q}$, $\mathbb{F}$ with the product topology and $\mathcal{Q}_u$, $\mathcal{Q}_d$ with the quotient topology of the product topology on $\mathbb{Q}$, $\mathcal{Q}$). For $X \in \text{SmP}(K)$ the group $G_{H_C}(X)$ is the Zariski-closure of the image of $\pi_1(K)$ acting on $H_C(X)$.

1.2.2. The G-S/T conjecture. For an integer $w \geq 0$ and $X \in \text{SmP}(K)$, consider the following assertions$^1$.

(S, $C$, $w$, $X$) The action of $\pi_1(K)$ on $H^w(X, C)$ is semisimple.

(ws, C, $w$, X) The inclusion $H^{2w}(X, C(w))^\pi_1(K) \hookrightarrow H^{2w}(X, C(w))$ splits $\pi_1(K)$-equivariantly.

(wS', C, $w$, X) The canonical morphism $c_w : H^{2w}(X, C(w))^\pi_1(K) \to H^{2w}(X, C(w))^{\pi_1(K)}$ induced by the identity is an isomorphism.

(F, C, $w$, X) The cycle map $[-] : CH^w(X) \otimes C \to H^{2w}(X, C(w))^\pi_1(K)$ is surjective.

(sF, C, $w$, X) The cycle map $[-] : CH^w(X) \otimes C \to \lim_{K'/K \text{ finite}} H^{2w}(X, C(w))^{\pi_1(K')}$ is surjective.

Apart from sF, the above assertions also make sense with $C$ replaced by $\mathbb{F}_\ell$, $\ell \in \mathcal{L}$; we will use the corresponding notation.

$^1$S stands for ‘semisimplicity’, ws for ‘weak semisimplicity’, F for ‘Fullness’ and sF for ‘stabilized Fullness’. 

With this notation, the classical ( [T66], where it is only formulated for $C = \mathbb{Q}_\ell$) G-S/T conjecture (= Conjecture 1.1.1) for $C$ asserts that $(S, C, \frac{w}{2}, X)$ and $(F, C, w, X)$ hold for every $X \in \text{SmP}(K)$ and integer $w \geq 0$.

1.2.3. Known results. The G-S/T conjecture is widely open. If $p > 0$ and $K$ is finite (resp. $p > 0$, resp. $p = 0$), Tate [T66] (resp. Zarhin [Z75], [Z77], Mori [Mo77], resp. Faltings [FW84]) proved $(S, \mathbb{F}_p, \frac{1}{2}, X)$, $\ell \gg 0$ and $(S, \mathbb{Q}_p, \frac{1}{2}, X)$ for $X$ arbitrary and $(F, \mathbb{F}_p, 1, X)$, $(S, \mathbb{F}_p, \frac{w}{2}, X)$, $\ell \gg 0$ and $(F, \mathbb{Q}_\ell, 1, X)$, $(S, \mathbb{Q}_\ell, \frac{w}{2}, X)$ for $X$ an abelian variety. Their proofs for $F_\ell$, $\ell \gg 0$ mimic their proofs for $\mathbb{Q}_\ell$: they do not deduce one of the statements from the other.

By works of several authors ( [N83], [NO85], [Ma14], [Cha13], [MP15], [KMP16], [MP20], [I18]), $(F, \mathbb{F}_p, w, X)$ and the method of [SkZ15], Ito extended Skorobogatov-Zarhin’s result to the time of [SkZ15]. This missing ingredient was developed by Kim and Madapusi Pera in [KMP16]. Building on [KMP16] and the method of [SkZ15], Ito extended Skorobogatov-Zarhin’s result to the $p = 2$ case.

Eventually, formal arguments allow to deduce a few other cases from the above ones - See e.g. [T94, Thm. 5.2].

1.3. When $p = 0$ and $K$ is embedded into $\mathbb{C}$, the existence of comparison isomorphisms between étale and singular cohomologies (See e.g. [A04, 3.4.2]) implies that $H_{\mathbb{Q}_c}$-homological equivalence is independent of $\dagger \in \mathcal{L} \cup \mathcal{U}$, which ensures that Conjecture 1.1.2 for the $H_{\mathbb{Q}_c}$, $\dagger \in \mathcal{L} \cup \mathcal{U}$ and Conjecture 1.1.1 for one single $H_{\mathbb{Q}_c}$, $\dagger \in \mathcal{L} \cup \mathcal{U}$ imply Conjecture 1.1.1 for every $H_{\mathbb{Q}_c}$, $\dagger \in \mathcal{L} \cup \mathcal{U}$. But, unfortunately, very little is known about Conjecture 1.1.2 when $p = 0$. In contrast, when $p > 0$, and modulo the semisimplicity part of Conjecture 1.1.1, Conjecture 1.1.2 essentially boils down to the Langlands correspondence [L02], [Chi04], [CZ21]. However, in this case, the lack of comparison isomorphisms between the $H_{\mathbb{Q}_c}$, $\dagger \in \mathcal{L} \cup \mathcal{U}$ makes it unclear whether Conjecture 1.1.1 for one single $H_{\mathbb{Q}_c}$, $\dagger \in \mathcal{L} \cup \mathcal{U}$ implies Conjecture 1.1.1 for every $H_{\mathbb{Q}_c}$, $\dagger \in \mathcal{L} \cup \mathcal{U}$.

Let $u \in \mathcal{U}$. The aim of this note is to study a related but easier version of the above problem, namely to relate Conjecture 1.1.1 (in our case, the G-S/T conjecture) for $H_{\mathbb{Q}_c}$, $\ell \in S$ for some $S \in u$, for $H_{\mathbb{Q}_c}$, and for $H_{\mathbb{Q}_c}$. One motivation is to give conceptual and completely general (i.e. working for arbitrary smooth projective varieties) proofs of results like the above mentioned results of Skorobogatov-Zarhin and Ito for K3 surfaces. Another motivation is that we may hope that some new cases of the G-S/T conjecture could be proved more easily for $\mathbb{Q}_u$-coefficients and then transferred to $\mathbb{Q}_\ell$- hence $\mathbb{Q}_c$-coefficients.

Assume $p > 0$. Let $C$ denote any of $\mathbb{Q}_\ell$, $\mathbb{Q}_u$ or $\mathbb{Q}_a$. For any integer $w \geq 0$, $v$ and $X \in \text{SmP}(K)$, let $G_C(X)$ denote the Zariski closure of the image of $\pi_1(K)$ acting on $H^w(X_{\overline{F}}, C(v))$. Before considering Conjecture 1.1.1, we prove the following variant of Conjecture 1.1.2 for the group of connected components.

1.3.1. Theorem. For every $X \in \text{SmP}(K)$ the kernel of the canonical map $\pi_1(K) \to \pi_0(G_C(X))$ is independent of $C = \mathbb{Q}_\ell$, $\mathbb{Q}_u$, $\mathbb{Q}_a$.

For $C = \mathbb{Q}_\ell$, Theorem 1.3.1 is due to Serre [S00, p. 15 sqq] but Serre’s arguments do not transfer as they are to $C = \mathbb{Q}_u$ or $\mathbb{Q}_a$. Instead, we follow the argument of [LaP95, Prop. 2.2] and give a uniform proof of Theorem 1.3.1 (which for $C = \mathbb{Q}_u$, relies on the results of [CHT17]).

When $G_C(X)$ is connected for one of (equivalently every) $C = \mathbb{Q}_\ell$, $\mathbb{Q}_u$, $\mathbb{Q}_a$, one says that $X$ has connected monodromy in degrees $(w, v)$. Under the connected monodromy assumption in degrees $(2w, w)$,
\(H^2(w(X,w,C(w))_{\pi_1(K)}) = H^2(w(X,w,C(w))_{\pi_1(K')})\) for every finite field extension \(K'/K\) and the G-S/T conjecture for \(X\) and \(X' := X \times_K K'\) become equivalent - See Lemma 4.2.

Our second main result is the following statements.

1.3.2. \textbf{Proposition.} For every \(X \in \text{SmP}(K)\), equidimensional of dimension \(d\), and \(u \in \mathcal{U}\), the following hold.

\begin{enumerate}[(1)]
  \item \((F, Q_u)^w, d, X^2) + (S, Q_u, \frac{w}{\tau}, X) \implies (S, Q_u, \frac{w}{\tau}, X);
  \item \((F, Q_u, d, X^2) + (F, Q_u, w, X) + (wS, Q_u, w, X) \implies (wS, Q_u, w, X).
\end{enumerate}

1.3.3. \textbf{Theorem.} Assume \(p > 0\). For every \(X \in \text{SmP}(K)\), equidimensional of dimension \(d\), and \(u \in \mathcal{U}\), the following hold.

\begin{enumerate}[(1)]
  \item \((wS, Q_u, w, X) \implies (wS, Q_u, w, X);
  \item \((S, Q_u, \frac{w}{\tau}, X) \implies (S, Q_u, \frac{w}{\tau}, X);
  \item \((F, Q_u, i, X), i = w, d - w + (wS, Q_u, w, X) \implies (F, Q_u, i, X), i = w, d - w + (wS, Q_u, w, X)).
\end{enumerate}

Proposition 1.3.2 and Theorem 1.3.3 imply formally (See Lemma 4.1) the following.

1.3.4. \textbf{Corollary.} For every \(X \in \text{SmP}(K)\), equidimensional of dimension \(d\),

\begin{enumerate}[(1)]
  \item \((F, x, d, X^2) + (S, \mathbb{F}_\ell, \frac{w}{\tau}, X), \ell \gg 0 \implies (S, \mathbb{F}_\ell, \frac{w}{\tau}, X), \ell \gg 0);
  \item \((F, x, d, X^2) + (F, \mathbb{F}_\ell, w, X) + (\mathbb{F}_\ell, \mathbb{F}_\ell, w, X), \ell \gg 0 \implies (\mathbb{F}_\ell, \mathbb{F}_\ell, w, X), \ell \gg 0.
\end{enumerate}

Assume \(p > 0\). Then

\begin{enumerate}[(3)]
  \item \((wS, \mathbb{F}_\ell, w, X), \ell \gg 0 \implies (wS, \mathbb{F}_\ell, w, X), \ell \gg 0;
  \item \((S, \mathbb{F}_\ell, \frac{w}{\tau}, X), \ell \gg 0 \implies (S, \mathbb{F}_\ell, \frac{w}{\tau}, X), \ell \gg 0;
  \item \((F, \mathbb{F}_\ell, i, X), i = w, d - w + (wS, \mathbb{F}_\ell, w, X), \ell \gg 0 \implies (F, \mathbb{F}_\ell, i, X), i = w, d - w + (wS, \mathbb{F}_\ell, w, X)), \ell \gg 0.
\end{enumerate}

1.3.5. For divisors, Theorem 1.3.3, Corollary 1.3.4 (3)-(5) yield [T94, Prop. 5.1] that for every \(X \in \text{SmP}(K),\)

\begin{enumerate}[(1)]
  \item \((F, Q_u, 1, X) \implies (F, Q_u, 1, X) + (wS, Q_u, 1, X);
  \item \((F, Q_u, 1, X), \ell \gg 0 \implies (F, \mathbb{F}_\ell, 1, X) + (\mathbb{F}_\ell, \mathbb{F}_\ell, 1, X), \ell \gg 0.
\end{enumerate}

In particular, for \(X\) an abelian variety or a K3 surface one can directly deduce \((F, \mathbb{F}_\ell, 1, X) + (\mathbb{F}_\ell, \mathbb{F}_\ell, 1, X), \ell \gg 0\) from \((F, Q_u, 1, X)\) (See Subsection 1.2.3) without resorting to any specific arithmetico-geometric features of \(X\) as in [SkZ15] or [I18].

1.3.6. \textbf{Remark.} The implication \((F, \mathbb{F}_\ell, w, X) \implies (F, Q_u, w, X)\) always holds for \(\ell \gg 0\) (hence the implication \((F, Q_u, w, X) \implies (F, Q_u, w, X))\). This follows from Nakayama’s lemma and the fact that \(H^2(w(X,w,\mathbb{F}_\ell))\) is torsion-free for \(\ell \gg 0\) ([G83] - See Fact 2.2). More precisely, we have the commutative diagram

\[
\begin{array}{ccc}
CH^w(X) & \rightarrow & H^2(w(X,w,\mathbb{F}_\ell))_{\pi_1(K)} \\
\downarrow & & \downarrow \\
H^2(w(X,w,\mathbb{F}_\ell))_{\pi_1(K)} & \leftarrow & H^2(w(X,w,\mathbb{F}_\ell))_{\pi_1(K)} \otimes \mathbb{F}_\ell,
\end{array}
\]

where the bottom arrow is injective for \(\ell \gg 0\). So if the left vertical arrow is surjective, the bottom arrow is an isomorphism hence the diagonal arrow is surjective.

1.4. \textbf{Divisors and finiteness of Brauer groups.} Let \(X \in \text{SmP}(K)\) with connected monodromy in degrees (2,1). Then \((F, Q_u, 1, X)\) is equivalent to the finiteness of the \(\ell\)-primary \(\pi_1(K)\)-invariant part \(Br(X_K)_{\pi_1(K)}[\ell^{\infty}]\) of the Brauer group of \(X_K\) (e.g. [CCh20, Prop. 2.1.1] and the references therein). One has the following strengthening.

\textbf{Corollary.} Assume \(p > 0\). Then for every \(X \in \text{SmP}(K)\) the following assertions are equivalent

\begin{enumerate}[(1)]
  \item \((F, Q_\ell, 1, X), \text{ for some } \ell \neq p;
  \item \((F, Q_\ell, 1, X), \text{ for every } \ell \neq p.
\end{enumerate}
(3) \( \text{Br}(X_{\overline{K}})^{\pi_1(K)}|p' \) is finite,

(where \( \text{Br}(X_{\overline{K}})^{\pi_1(K)}|p' \) denotes the prime-to-\( p \) part of \( \text{Br}(X_{\overline{K}})^{\pi_1(K)} \)).

When \( K \) is finite, Corollary 1.4 was proved by Tate [T94, Prop. 4.3]. In this setting, it is even known that \( (F, \mathbb{Q}_\ell, 1, X) \) is independent of \( \ell \neq p \) and implies that \( \text{Br}(X) \) is finite (See the references in the proof of [T94, Prop. 4.3]). That the equivalence \( (1) \iff (2) \) holds in general was pointed out to us by Yanshuai Qin. This is essentially the same argument as in the finite field case and relies on \[ T94, \text{Prop. 2.9}. \]

Convention.

2.1. closure of the prime field of \( K \)

Let \( K \) be a finitely generated field of characteristic \( p \geq 0 \) and let \( X \in \text{SnP}(K) \). Let \( k \) denote the algebraic closure of the prime field of \( K \) in \( K \).

2.2. Fact.

(1) ([G83]) The \( \mathbb{Z}_\ell \)-local systems \( R^q f_* \mathbb{Z}_\ell(v) \) are torsion-free (of finite constant rank) for \( \ell \neq p \gg 0 \). In particular, for every geometric point \( \overline{s} \) on \( S \), \( R^q f_* \mathbb{Z}_\ell(v)|_{\overline{s}} \otimes \mathbb{F}_\ell \simeq (R^q f_* \mathbb{F}_\ell(v))|_{\overline{s}}, \ell \neq p \gg 0 \);

(2) ([CHT17, Thm. 1.3]) \( H^0(S_{\overline{K}}, R^q f_* \mathbb{Z}_\ell(v)) \otimes \mathbb{F}_\ell \simeq H^0(S_{\overline{K}}, R^q f_* \mathbb{F}_\ell(v)), \ell \neq p \gg 0 \).

2.3. Fact.

(1) ([D80, 3.4.1 (iii)]) \( R^w f_* \mathbb{Q}_\ell(v)|_{S_{\overline{K}}} \) is a semisimple \( \mathbb{Q}_\ell \)-local system on \( S_{\overline{K}}, \ell \neq p \);

(2) ([CHT17, Thm. 1.1]) \( R^w f_* \mathbb{F}_\ell(v)|_{S_{\overline{K}}} \) is a semisimple \( \mathbb{F}_\ell \)-local system on \( S_{\overline{K}} \) for \( \ell \neq p \gg 0 \).
2.4. Fact. 

(1) ([D80, Cor. 3.2.9]) For every closed point \( s \in |S| \) the characteristic polynomial \( P_s := \det(\text{Id}_T - \varphi_s)(R^w f_s \mathbb{Q}_l(v)) \) of the geometric Frobenius \( \varphi_s \in \pi_1(s) \) is in \( \mathbb{Z}[1/p][T] \), independent of \( l(\neq p) \).

(2) (e.g. [LaP95, (proof of) Prop. 2.1]) The characteristic polynomial \( P := \det(\text{Id}_T - \varphi)(H^0(S, R^w f, \mathbb{Q}_l(v))) \) of the geometric Frobenius \( \varphi \in \pi_1(k) \) is in \( \mathbb{Z}[1/p][T] \) and independent of \( l \).

From Fact 2.2 (1), Fact 2.4 (1) implies that, for \( l(\neq p) \geq 0 \), the reduction modulo \( l \) of \( P_\ell \in \mathbb{Z}[1/p][T] \) coincides with the characteristic polynomial \( P_{\ell, u} := \det(\text{Id}_T - \varphi_{\ell, u})(R^w f_{\ell, u} \mathbb{Q}_l(v)) \in \mathbb{F}_\ell[T] \). In turn, this implies that \( P_{\ell, u} \in \mathbb{Z}[1/p][T] \) coincides with the characteristic polynomial \( \det(\text{Id}_T - \varphi_{\ell, u})(H^w(\mathcal{X}_\ell, \mathbb{Q}_u(\mathcal{U}))) \), \( u \in \mathcal{U} \). (It also directly follows from Fact 2.2 (1) that \( P_{\ell, u} \) coincides with the characteristic polynomial \( \det(\text{Id}_T - \varphi_{\ell, u})(H^w(\mathcal{X}_\ell, \mathbb{Q}_u(\mathcal{U}))), u \in \mathcal{U} \).

From Fact 2.2 (2), Fact 2.4 (2) implies that, for \( l(\neq p) \geq 0 \), the reduction modulo \( l \) of \( P \in \mathbb{Z}[1/p][T] \) coincides with the characteristic polynomial \( P_{\ell, u} := \det(\text{Id}_T - \varphi)(H^0(\mathcal{S}, R^w f_{\ell, u} \mathbb{Q}_l(v))) \in \mathbb{F}_\ell[T] \). In particular, if \( \delta_{\mathbb{Q}_u}(1) \) (resp. \( \delta_{\mathbb{Z}/l}(1) \)) denotes the multiplicity of 1 as a root of \( P \) (resp. \( P_{\ell, u} \)), \( \delta_{\mathbb{Q}_u}(1) \) is independent of \( l \) and one \( \delta_{\mathbb{Q}_u}(1) = \delta_{\mathbb{Z}/l}(1) \) for \( l(\neq p) \geq 0 \).

2.5. Let \( \Pi \) (resp. \( \Pi \)) denote the image of \( \pi_1(S) \) (resp. \( \pi_1(S) \)) acting on \( \prod_{\ell \in \mathcal{L}}(R^w f_{\ell, u} \mathbb{Q}_l(v))_\mathbb{Q} \).

Then, 

Fact. ([CT19, §3.1]) \( \Pi \) (hence \( \Pi \)) is a topologically finitely generated profinite group.

Let \( \Pi_{\mathbb{Q}_u} \) denote the image of \( \pi_1(S) \) acting on \( H^w(\mathcal{X}_\ell, \mathbb{Q}_u(\mathcal{U}))) \). Fact 2.5 has the following (non-trivial!) consequence

Corollary. For every finite index subgroup \( \Pi_{\mathbb{Q}_u} \subset \Pi_{\mathbb{Q}_u} \) there exists a connected étale cover \( S' \to S \) such that \( \Pi_{\mathbb{Q}_u} \) coincides with the image of \( \pi_1(S') \) acting on \( H^w(\mathcal{X}_\ell, \mathbb{Q}_u(\mathcal{U}))) \).

Proof. From Fact 2.5, \( \Pi \) is a topologically finitely generated profinite group. As the inverse image \( \Pi' \subset \Pi \) of \( \Pi' \) in \( \Pi \) is again of finite index it follows from [NS07a, Thm. 1.1] (which relies on [NS07b]) that \( \Pi' \) is automatically open in \( \Pi \) hence corresponds to a connected étale cover \( S' \to S \).

The fact that \( \Pi \) is topologically finitely generated also ensures (Lemma 8.4.2)

\[
H^w(\mathcal{X}_\ell, \mathbb{Q}_u(\mathcal{U}))) = \left( \prod_{\ell \in \mathcal{L}} H^w(\mathcal{X}_\ell, \mathbb{Q}_l(\mathcal{U})) \right) \otimes \mathbb{Q}_u = \left( \prod_{\ell \in \mathcal{L}} H^w(\mathcal{S}, R^w f_{\ell, u}(\mathbb{Q}_l(v))) \right) \otimes \mathbb{Q}_u
\]

so that, from Fact 2.2 (2) and Fact 2.4 (2), \( P \in \mathbb{Q}[T] \) coincides with the characteristic polynomial \( \det(\text{Id}_T - \varphi)(H^w(\mathcal{X}_\ell, \mathbb{Q}_u(\mathcal{U}))) \), \( u \in \mathcal{U} \).

(From Fact 2.4 (2) and [B96, 6.3.1, 6.3.2], similar results hold for \( \mathbb{Q}_u \)-coefficients).

3. PROOF OF THEOREM 1.3.1

Let \( K \) be a finitely generated field of characteristic \( p > 0 \) and let \( X \in \text{SmP}(K) \). We retain the notation of 2.1. For \( C = \mathbb{Q}_u, \mathbb{Q}_e, \mathbb{Q}_u \), set \( \text{H}_C := H^w(X, C, v) \) and let \( G_C \subset \text{GL}(\text{H}_C) \) denote the Zariski-closure of the image \( \text{H}_C \) of \( \pi_1(K) \) acting on \( \text{H}_C \).

Let \( C_1, C_2 \) be any fields of the form \( \mathbb{Q}_e, \mathbb{Q}_u \) or \( \mathbb{Q}_u \). Since \( \pi_1(S) \)-semisimplification does not change the kernel of \( \pi_1(S) \to \pi_0(G_{C_1}) \), one may assume \( G_{C_1} \) is a semisimple \( \text{H}_C \)-module. Note also that \( \pi_1(S) \)-semisimplification does not affect the action of \( \pi_1(S) \) on \( \text{H}_C \) by Fact 2.3 (and Lemma 8.4.5 if \( C_1 = \mathbb{Q}_u \) or \( \mathbb{Q}_u \)). As \( \pi_1(S) \) acts on \( G_{C_1} \) through a topologically finitely generated quotient, the kernel of \( \pi_1(S) \to \pi_0(G_{C_1}) \) is an (a normal) open subgroup of \( \pi_1(S) \) ([NS07a], [NS07b]) so that, up to replacing \( S \) by \( S \) the corresponding étale (Galois) cover, one may assume \( G_{C_1} \) is connected that is, equivalently ( [D82, Prop. 3.1 (a), (c)]), for every finite index subgroup \( U \subset \pi_1(S) \) and integers \( m, n \geq 0 \),

\[
dim((H^w_{C_1} \otimes H^{\otimes m}_{C_1})^{\pi_1(S)}) = \dim((H^{\otimes m}_{C_1} \otimes H^w_{C_1})^{\pi_1(S)}) = \dim((H^{\otimes m}_{C_1} \otimes H^w_{C_1})^{\pi_1(S)}) = \dim((H^w_{C_1} \otimes H^{\otimes m}_{C_1})^{\pi_1(S)}) \]

[LaP95, Lemma 2.3]. Again, since \( \pi_1(S) \) acts on \( G_{C_1} \) through a topologically finitely generated quotient, one may
restrict to open subgroups \( U \subset \pi_1(S) \). That is, equivalently, one has to show that for every connected étale cover \( S' \to S \) and integers \( m,n \geq 0 \), \( \dim((H^{\otimes m}_{C_2} \otimes H^{\otimes n}_{C_2})_{\pi_1(S)}) = \dim((H^{\otimes m}_{C_2} \otimes H^{\otimes n}_{C_2})_{\pi_1(S)}) \).

But recall that \( H_{C_2} = H^p(X_{\overline{\mathbb{F}}}, C_1(v)) \) so that, by Kunneth formula, \( H^{\otimes m}_{C_2} \otimes H^{\otimes n}_{C_2} \) is a direct factor of \( H^{pm+n+n(2d-w)}(X^m \oplus n_2 \cdot X(n(d-v))) \). In other words, replacing \( X \to S \) with the the \( m+n \)th fibered power \( X^m_n = X \times_S \cdots \times_S X \to S \) (and the Tate twists \(-v\) with \(-n(d-v)\)), it is enough to show that for every connected étale cover \( S' \to S \), \( \dim((H_{C_2})_{\pi_1(S')}^1) = \dim((H_{C_2})_{\pi_1(S')}) \). But as, by assumption, \( \dim(H_{C_2})_{\pi_1(S')} = \dim(H_{C_2})_{\pi_1(S)} \), it is actually enough to show that for every connected étale cover \( S' \to S \), \( \dim(H_{C_2})_{\pi_1(S')} = \dim(H_{C_2})_{\pi_1(S)} \).

Write \( S := S' \) to simplify. As \( H_{C_2} \) is a semisimple \( \Pi_{C_2} \)-module (and using Lemma 8.4.2 for \( C_1 = \mathbb{Q}_p \) or \( \mathbb{Q}_a \)), \( \dim((H_{C_2})_{\pi_1(S)}) \) is the multiplicity of \( 1 \) as an eigenvalue of the Frobenius \( \varphi \in \pi_1(k) \equiv \pi_1(S)/\pi_1(S) \) acting on \( (H_{C_2})_{\pi_1(S')} \). So the assertion follows from the last paragraph of Subsection 2.5.

4. Preliminary observations

Let \( X \in \text{SmP}(K) \). We begin by the following elementary observations, which follow from the formal properties of ultraproducts.

4.1. Lemma. For \( (\mathbb{S}, w_S, \mathbb{S}, w), (F, w) \) we have

1. For every \( u \in U \), \( (\mathbb{S}, w_S, \mathbb{S}, w), (F, w) \) have
2. For every \( u \in U \), \( (\mathbb{S}, w_S, \mathbb{S}, w), (F, w) \) have
3. If \( K'/K \) is Galois, \( (F, w_S, \mathbb{S}, w), (F, w_S, \mathbb{S}, w) \) have
4. Assume furthermore \( X \) has connected monodromy in degrees \( (2w, w) \). Then,
5. The assertions \( (\mathbb{S}, w_S, \mathbb{S}, w), (F, w_S, \mathbb{S}, w), (F, w_S, \mathbb{S}, w) \) are all equivalent.

Proof. By ? =F, see 8.3.3 (with P the property of being surjective) and 8.4.2 (which can be applied by ??

(2)). For ? =S, see 8.4.5 (with P the property of acting semisimply). For ? =wS, see 8.4.6. For ? =wS', see 8.4.2, 8.4.1 and 8.3.3 (with P the property of being an isomorphism).

4.2. Let \( C = \mathbb{Q}_p, \mathbb{Q}_a \) or \( \mathbb{Q}_a \) and let \( K'/K \) be a finite field extension. Write \( X' := X \times_K K' \). Then,

Lemma.

1. \( (\mathbb{S}, w_S, \mathbb{S}, w), (F, w) \) have
2. \( (\mathbb{S}, w_S, \mathbb{S}, w), (F, w) \) have
3. If \( K'/K \) is Galois, \( (F, w_S, \mathbb{S}, w) \) have
4. Assume furthermore \( X \) has connected monodromy in degrees \( (2w, w) \). Then,
5. The assertions \( (\mathbb{S}, w_S, \mathbb{S}, w), (F, w_S, \mathbb{S}, w), (F, w_S, \mathbb{S}, w) \) are all equivalent.

Proof. We show (3); the other assertions are purely group-theoretic and elementary. Let

\[
\alpha \in H^{2w}(X_K, C(w))_{\pi_1(K)} \subset H^{2w}(X_K, C(w))_{\pi_1(K')}.
\]

Then, from \( (F, w, X') \), one can write \( \alpha = \sum_{1 \leq i \leq r} \lambda_i [Y_i] \) with \( \lambda_i \in C \) and \( Y_i \in Z^w(X') \) an integral cycle. But, then,

\[
\alpha = \frac{1}{[K':K]} \sum_{1 \leq i \leq r} \lambda_i \sum_{\sigma \in \text{Gal}(K'/K)} \sigma[Y_i] = \frac{1}{[K':K]} \sum_{1 \leq i \leq r} \lambda_i \sum_{\sigma \in \text{Gal}(K'/K)} \sigma Y_i.
\]

The conclusion follows from the fact that \( \sum_{\sigma \in \text{Gal}(K'/K)} \sigma Y_i \) is in \( Z^w(X')_{\text{Gal}(K'/K)} = Z^w(X) \).

4.3. Lemma. Assume \( p > 0 \). Then,

1. For \( \ell \neq p \), \( (\mathbb{S}, w_S, \mathbb{S}, w), (F, w_S, \mathbb{S}, w, X) \) have
2. For \( \ell > 0 \), \( (\mathbb{S}, w_S, \mathbb{S}, w, X) \) have
Proof. Let $C = \mathbb{Q}_\ell$ or $\mathbb{F}_\ell$. We retain the notation of 2.1. Write $H := H^{2w}(X, C(w))$ and $\Pi := \pi_1(S_F)$, $\Pi := \pi_1(S)$. The implication ($wS$, $C$, $w$, $X) \Rightarrow (wS, C, w, X)$ is straightforward since the composition of $e^{-1} : H_\Pi \to H_\Pi$ with the canonical projection $H \to H_\Pi$ provides a $\Pi$-equivariant splitting of $H_\Pi \to H$. Conversely, let $\phi \in \Pi$ such that $\phi$ and $\Pi$ generate $\Pi$. As $\Pi$ acts semisimply on $H$ (Fact 2.3) the canonical morphism $H_\Pi \to H_\Pi$ is an isomorphism. Assume ($wS, C, w, X$) and consider a $\Pi$-equivariant decomposition $H = H_\Pi \oplus M$; in particular $M_\Pi = 0$. Then it is enough to show that $0 = M_\Pi = (M_\Pi)_\varphi \in (M_\Pi)_\varphi$ but this follows from the exact sequence

$$0 \to M_\Pi = (M_\Pi)_\varphi \to M_\Pi \varphi_\Pi^{-1} \to M_\Pi \to (M_\Pi)_\varphi \to 0.$$

\[\square\]

5. Proof of Proposition 1.3.2

5.1. Let $X \in \text{SmP}(K)$ of dimension $d$. For $C = \mathbb{Q}_\ell, \mathbb{Z}_\ell, \mathbb{F}_\ell$ write $H_C := H^w(X, C)$ and set $\Pi := \pi_1(K)$. To prove Proposition 1.3.2, one may freely replace $\mathcal{L}$ by a subset in $u$; in particular one may replace $\mathcal{L}$ by a cofinite subset hence assume

$$(5.1.1) \ H_{Z_\ell} \otimes \mathbb{F}_\ell = H_{\mathbb{F}_\ell}, \ \ell \in \mathcal{L}$$

(Fact 2.2 (1)) if $p > 0$; if $p = 0$, this follows from comparison between singular and $\mathbb{Z}_\ell$-cohomology, using the fact that for every embedding $K \subset \mathbb{C}$, $H_{\text{sing}}(X(\mathbb{C}), \mathbb{Z})$ is a finitely generated $\mathbb{Z}$-module). By Künneth formula and Poincaré duality, $(F, \mathbb{Q}_u, d, X^2)$ ensures that up to replacing $\mathcal{L}$ by a subset in $u$ one has

$$(5.1.2) \ \text{End}_\Pi(H_{Z_\ell}) \otimes \mathbb{F}_\ell = (H_{Z_\ell} \otimes H^\vee_{Z_\ell})_\Pi \otimes \mathbb{F}_\ell \to (H_{\mathbb{F}_\ell} \otimes H^\vee_{\mathbb{F}_\ell})_\Pi = \text{End}_\Pi(H_{\mathbb{F}_\ell}), \ \ell \in \mathcal{L}.$$

5.2. Proof of Proposition 1.3.2 (1).

5.2.1. Let $Q$ be a field and $\Gamma$ a group. In this subsection, a $\Gamma$-module means a finite-dimensional $Q$-vector space endowed with an action of $\Gamma$ by $Q$-linear automorphisms. For a $\Gamma$-module $V$, let $V^{ss}$ denote the $\Gamma$-semisimplification of $V$.

Lemma. One has \( \dim(\text{End}_\Gamma(V)) \leq \dim(\text{End}_\Gamma(V^{ss})) \) and \( \dim(\text{End}_\Gamma(V)) = \dim(\text{End}_\Gamma(V^{ss})) \) if and only if $V$ is a semisimple $\Gamma$-module.

Proof. Let $0 \to A \to V \to B \to 0$ be a short exact sequence of $\Gamma$-modules and $W$ a $\Gamma$-module. Then

$$0 \to \text{Hom}_\Gamma(B, W) \to \text{Hom}_\Gamma(V, W) \to \text{Hom}_\Gamma(A, W)$$

and

$$0 \to \text{Hom}_\Gamma(W, A) \to \text{Hom}_\Gamma(W, V) \to \text{Hom}_\Gamma(W, B)$$

are exact and hence we obtain

$$\dim \text{Hom}_\Gamma(V, W) \leq \dim \text{Hom}_\Gamma(A \oplus B, W)$$

and

$$\dim \text{Hom}_\Gamma(W, V) \leq \dim \text{Hom}_\Gamma(W, A \oplus B).$$

By taking $W = V$ in the first inequality and $W = A \oplus B$ in the second, we obtain

$$\dim \text{End}_\Gamma(V) \leq \dim \text{End}_\Gamma(A \oplus B)$$

and induction implies

$$(*) \ \dim \text{End}_\Gamma(V) \leq \dim \text{End}_\Gamma(V^{ss}).$$

When $(*)$ is an equality, $V$ is semisimple. Indeed, all the inequalities become equalities. Hence, the sequence

$$0 \to \text{Hom}_\Gamma(A \oplus B, A) \to \text{Hom}_\Gamma(A \oplus B, V) \to \text{Hom}_\Gamma(A \oplus B, B) \to 0$$

and thus

$$0 \to \text{Hom}_\Gamma(B, A) \to \text{Hom}_\Gamma(B, V) \to \text{Hom}_\Gamma(B, B) \to 0$$

are exact, implying that $0 \to A \to V \to B \to 0$ splits. \[\square\]
Lemma 5.3.1. Given a ring $R$, let $\text{Idem}(R)$ and $\text{CIdem}(R)$ denote respectively the idempotents and central idempotents in $R$.

Let $A$ be a $\mathbb{Z}_\ell$-algebra which, as a $\mathbb{Z}_\ell$-module, is free of finite rank. The following lemma is possibly classical (see e.g. [Do72, Thm. 44.3 (2)] for the surjectivity part of the assertion) but for lack of a suitable complete reference and to keep the exposition self-contained, we include a proof.

Lemma. (Lifting idempotents) The reduction modulo-$\ell$ morphism $A \to A \otimes \mathbb{F}_\ell$ restricts to a surjective map $\text{Idem}(A) \to \text{Idem}(A \otimes \mathbb{F}_\ell)$ and to a bijective map $\text{CIdem}(A) \to \text{CIdem}(A \otimes \mathbb{F}_\ell)$.

Proof. First, observe that for every $a, a' \in A$ such that $[a, a'] = 0$ and $a - a' \in \ell^N A$, we have $a^{m_1} - a'^{m_1} \in \ell^{N+n} A$. Indeed, write $a - a' = \ell^n b_0 \in \ell^N A$. Then, $b_0$ commutes with $a, a'$ and one has

$$a^\ell - a'^\ell = \sum_{1 \leq k \leq \ell} \binom{\ell}{k} \ell^{Nk} a^{\ell - k} b_0^k = \ell^{N+1} \sum_{1 \leq k \leq \ell} \binom{\ell}{k} \ell^{Nk} a^{\ell - k} b_0^k = \ell^{N+1} b_1.$$ 

The conclusion follows by straightforward induction.

- Let $\pi \in \text{Idem}(A \otimes \mathbb{F}_\ell)$ and pick any $a \in A$ such that $[a, \pi] = \pi$. By construction, $a^{m_1} - a \in \ell A$, $m_0 \geq 0$ hence, from the preliminary observation,

$$a^{m_{0+1}} - a^{m_0} = (a^{m_1})^{m_1} - a^{m_1} \in \ell^{m_0+1} A, \quad m_0 \geq 0$$

hence $\{a^{m_0}\}_n$ is a Cauchy sequence. Set $e := \lim_{n \to \infty} a^{m_0}$. By construction, for $n \geq 0$ we have $\pi = \pi^{m_0} = e^{m_0}$. Furthermore, since $a^2 - a \in \ell A$, we get, again, $a^{2m_0} - a^{m_0} \in \ell^{m_0} A$, $n \geq 0$. Since $(-)^2 : A \to A$ is continuous, one gets $e^2 = e$. This shows $\text{Idem}(A) \to \text{Idem}(A \otimes \mathbb{F}_\ell)$.

- Let $\pi \in \text{Idem}(A)$ such that $\pi \in \text{CIdem}(A \otimes \mathbb{F}_\ell)$. Then $\pi(A \otimes \mathbb{F}_\ell)(1 - \pi) = 0$ forces $e_\pi(1 - e) = e_\pi A(1 - e)$ hence, by Nakayama’s lemma, $e_\pi(1 - e)A = 0$. Similarly $(1 - e)Ae = 0$. Hence for every $a \in A$,

$$ca = ea(e + (1 - e)) = eae = (e + (1 - e))ae = ae.$$ 

This shows $\text{CIdem}(A) \to \text{CIdem}(A \otimes \mathbb{F}_\ell)$. Let $e, e' \in \text{CIdem}(A)$ such that $\pi = \pi'$ that is, $e - e' \in \ell A$. Since $[e, e'] = 0$, the preliminary observation shows that $e - e' = e^{m_0} - e'^{m_0} \in \ell^{m_0} A$, $n \geq 0$ hence $e = e'$. This shows $\text{CIdem}(A) \to \text{CIdem}(A \otimes \mathbb{F}_\ell)$.

5.3.2. From (5.1.1) one has $\text{End}_\mathbb{F}(\mathbb{H}_{Z_\ell}) \otimes \mathbb{F}_\ell = \text{End}_\mathbb{F}(\mathbb{H}_{F_\ell})$, $\ell \in \mathcal{L}$. On the other hand, $(F, Q_u, w, X)$, $(wS, Q_u, w, X)$ ensure that $\mathbb{F}_\ell$ on replacing $\mathbb{L}$ by a subset in $u$ one also has $(F, \mathbb{F}_\ell, w, X)$, $(wS, \mathbb{F}_\ell, w, X)$, $\ell \in \mathcal{L}$ (Lemma 4.1 (2), Lemma 4.3 (2)). Using (5.1.1) one can apply Lemma 5.3.1 to $A = \text{End}_\mathbb{F}(\mathbb{H}_{Z_\ell})$. Write $M_0 := \mathbb{H}_{Z_\ell}^0$, $M_1 := \ker(H_\mathbb{F} \to \text{End}_\mathbb{F}(\mathbb{H}_{Z_\ell}))$. Then, by $(wS, \mathbb{F}_\ell, w, X)$, one has the canonical decomposition $H_{Z_\ell} = M_1 \oplus M_0$ as $\mathbb{F}_\ell$-modules. By definition of $M_0, M_1$, any element in $\text{End}_\mathbb{F}(H_{F_\ell})$ stabilizes both $M_0$ and $M_1$ hence the elements $e_i : H_{F_\ell} \to M_i \hookrightarrow H_{F_\ell}$ (obtained by composing the canonical projection followed by the canonical injection), $i = 1, 2$ are in $\text{CIdem}(\text{End}_\mathbb{F}(H_{Z_\ell}))$. From Lemma 5.3.1, $e_0, e_1$ lift uniquely to $\tilde{e}_0, \tilde{e}_1 \in \text{CIdem}(\text{End}_\mathbb{F}(H_{Z_\ell}))$ with $Id = \tilde{e}_0 + \tilde{e}_1$. Let $M_{i-1} := \ker(\tilde{e}_i)$, $i = 0, 1$. Then $H_{Z_\ell} = \tilde{M}_1 \oplus \tilde{M}_0$ with $\tilde{M}_i \otimes \mathbb{F}_\ell = M_i$, $i = 0, 1$. It remains to check that $\tilde{M}_0 = \mathbb{H}_{Z_\ell}^0$. 

\[\text{Qr} \text{ versus } \mathbb{F}_r\text{-coefficients in the Grothendieck-Serre/Tate conjectures} 9\]
Since $\tilde{M}_0 \otimes F_\ell = M_0 = (H_{\ell z, z}^\Pi) = H_{\ell z, z}^\Pi \otimes F_\ell$, by Nakayama’s lemma, it is enough to show that $H_{\ell z, z}^{\Pi, \Pi} \subset \tilde{M}_0$. Since $H_{\ell z, z}^{\Pi, \Pi} = (H_{\ell z, z}^{\Pi, \Pi} \cap M_0) \oplus (H_{\ell z, z}^{\Pi, \Pi} \cap M_1)$, this is equivalent to $H_{\ell z, z}^{\Pi, \Pi} \cap M_1 = 0$. Let $h \in H_{\ell z, z}^{\Pi, \Pi} \cap M_1$. Then $h \mod \ell \in H_{\ell z, z}^{\Pi, \Pi} \cap M_1 = 0$ that is $H_{\ell z, z}^{\Pi, \Pi} \cap M_1 \subset \ell H_{\ell z, z}^{\Pi, \Pi}$. But as $H_{z, 0, 1}/(H_{\ell z, z}^{\Pi, \Pi} \cap M_1) \hookrightarrow (H_{z, 0, 1}/H_{\ell z, z}^{\Pi, \Pi}) \times M_0$ is torsion-free (equivalently, $H_{\ell z, z}^{\Pi, \Pi} \cap M_1 \in H_{z, 0, 1}$ is a $\mathbb{Z}_\ell$-direct summand), $(H_{z, 0, 1}/(H_{\ell z, z}^{\Pi, \Pi} \cap M_1) = \ell (H_{\ell z, z}^{\Pi, \Pi} \cap M_1)$. As a result, $H_{\ell z, z}^{\Pi, \Pi} \cap M_1 = \ell (H_{\ell z, z}^{\Pi, \Pi} \cap M_1)$ which, by Nakayama’s lemma, forces $H_{\ell z, z}^{\Pi, \Pi} \cap M_1 = 0$.

### 6. Proof of Theorem 1.3.3

Let $K$ be a finitely generated field of characteristic $p > 0$ and let $X \in \text{SmP}(K)$. We retain the notation of 2.1. Set $\pi_1 := \pi_1(S)$, $\Pi := \pi_1(S)$. Again, to prove Theorem 1.3.3 one may freely replace $\Pi$ by a subset in $\mathfrak{s}$; in particular one may assume $H^0(\mathcal{X}, R^\ast f^\ast F_\ell) \otimes F_\ell \to H^0(\mathcal{X}, R^\ast f^\ast F_\ell), \ell \in \mathcal{L}$ (Fact 2.2 (1)). From Lemma 4.1, it is enough to show

(1') For $\ell \gg 0$, $(wS, \ell, w, X) \iff (wS, F_\ell, w, X)$

(2') For $\ell \gg 0$, $(S, \ell, w, X) \iff (S, F_\ell, w, X)$

(3') For $\ell \gg 0$, $(F, \ell, i, X), i = w, d - w + (wS, \ell, w, X) \iff (F, F_\ell, i, X), i = w, d - w + (wS, F_\ell, w, X)$

#### 6.1. Proof of (1').

For $C = \ell, F_\ell$, write $H_C := H^{|w|\ell}(X_{\overline{K}}, C(w))$ and consider the following seemingly weak variant of $(wS, C, w, X)$.

$(wS^\ast, C, w, X)$ The inclusion $H_{C}^{\Pi} \hookrightarrow H_{C}^{\Pi}$ splits $\pi_1(k)$-equivariantly.

Recall the definition of $\delta_C(1)$ at the end of Paragraph 2.4; by definition this is the dimension of the generalized eigenspace $H_{C}^{\Pi}(1) := \cup_{n \geq 1} \ker((1 - \varphi)^n|H_{C}^{\Pi})$ attached to 1 so that

\begin{equation}
\delta_C(1) = \dim(H_{C}^{\Pi}) \iff \delta_C(1) \leq \dim(H_{C}^{\Pi})
\end{equation}

(where the last equivalence follows from the fact that $\delta_C(1) \geq \dim(H_{C}^{\Pi})$ always holds). One also has

#### 6.1.2 Lemma.

$(wS, \ell, w, X) \iff (wS^\ast, \ell, w, X)$ and $(wS, F_\ell, w, X) \iff (wS^\ast, F_\ell, w, X), \ell \gg 0$.

Proof. The implications $\Rightarrow$ are straightforward. For the converse implications, from Fact 2.3 the canonical $\Pi$-equivariant morphism $H_{C}^{\Pi} \to H_{C}^{\Pi}$ is an isomorphism. So, setting $N := \ker(H_C \to H_C^{\Pi})$, one obtains a direct sum decomposition as $\Pi$-modules $H_C = H_C^{\Pi} \oplus N$.

#### 6.1.3 From 6.1.1

(1”) For $\ell \gg 0$, $0, \delta_{Q_{\ell}}(1) \leq \dim(H_{Q_{\ell}}^{\Pi}) \iff \delta_{Q_{\ell}}(1) \leq \dim(H_{Q_{\ell}}^{\Pi}).$

From the last paragraph of 2.4, $\delta_{Q_{\ell}}(1) = \delta_{Q_{\ell}}(1)$ for $\ell \neq p \gg 0$ so that (1”) follows from $\dim(H_{Q_{\ell}}^{\Pi}) \leq \delta_{Q_{\ell}}(1) \leq \dim(H_{Q_{\ell}}^{\Pi}) \leq \dim(H_{Q_{\ell}}^{\Pi}).$

#### 6.2. Proof of (2’).

This is proved in [CHT17, §11]. We give here a more elementary argument, which avoids Larsen-Pink’s theory of regular semisimple Frobenii. For $C = F_\ell, \ell, F_\ell, \mathbb{Z}_\ell$, write $H_C := H^{\ast\ast}(X_{\overline{K}}, C)$. Also, let $\Pi_\ell$ and $\Pi_\ell$ denote the image of $\Pi$ and $\Pi$ acting on $H_{\mathbb{Z}_\ell}$ respectively.

We begin with the following Lemma. Recall that $(S, \ell, \ast, \mathbb{Z}_\ell, X)$ hence - as this holds for every $u \in \mathcal{U}$ (8.4.5) for $P$ the property of acting semisimply) - $(S, \ell, \ast, \mathbb{Z}_\ell, X)$ for $\ell \gg 0$ are insensitive to finite field extensions of $K$ (Lemma 4.2 (1)).

**Lemma.** After replacing $K$ by a finite field extension, there exists a monic polynomial $P \in \mathbb{Q}[T]$ and for every $\ell \neq p$ a semisimple element $\phi_\ell \in \Pi_\ell$ such that, for $\ell \gg 0$, $\Pi_\ell$ is generated by $\Pi_\ell$ and $\phi_\ell$, and $\phi_\ell$ has characteristic polynomial $P_\ell$.

**Proof.** Let $\overline{\mathbb{Z}_\ell}$, $\mathbb{G}_{\mathbb{Z}_\ell}$ denote respectively the Zariski closure of $\Pi_\ell, \Pi_\ell$ in GL($H_{\mathbb{Z}_\ell}$). After possibly replacing $\mathcal{S}$ by a connected étale cover, one may assume $\mathbb{G}_{\mathbb{Q}_\ell}$ is connected for every $\ell \in \mathcal{L}$ (Theorem 1.3.1). One may also assume $\mathcal{S}$ carries a $k$-point $s \in \mathcal{S}(k)$. Let $\varphi_\ell$ denote the image of the geometric Frobenius $\varphi_\ell$ acting on $H_{\mathbb{Z}_\ell}$; recall that its characteristic polynomial $P_\ell$ is in $\mathbb{Q}[T]$ and independent of $\ell$ ([D80]). Write
This shows that and independent of \( \ell \) and the fact that for order. Thus \( (S, F) \)

**Proof.** Lemma. \( p > \delta \).

Thus the conclusion follows from the implications: \( \delta_{Q_\ell}(1) = \dim(H_{Q_\ell}) \)

From [CHT17, Thm. (7.3.2)], there exists an integer \( N \geq 1 \) independent of \( \ell \) such that \( (\varphi_{e, \ell}^*)^N \in \Pi_{\ell} \).

But, then, \((\varphi_{e, \ell}^*)^N - \varphi_{e, \ell}^N \) \( \in \Pi_{\ell} \); after replacing \( k \) by its degree-\( N \) field extension, we may assume \( N = 1 \). Then \( \varphi_{e, \ell}^* - \varphi_{e, \ell}^N \) works. \( \square \)

We can now conclude the proof. The fact that \( \phi_{e, \ell} \) acts semisimply on \( H_{Q_\ell} \) is equivalent to the fact that the minimal polynomial \( Q_{\ell} \) of \( \phi_{e, \ell} \) is separable. Since \( P \) is in \( \mathbb{Q}[T] \) and independent of \( \ell \), \( Q := Q_{\ell} \) is in \( \mathbb{Q}[T] \) and independent of \( \ell \) as well. And since one assumes \( H_{Z_\ell} \) is torsion free, the minimal polynomial of \( \phi_{e, \ell} \) acting on \( H_{Z_{\ell}} \) is the reduction modulo-\( \ell \) of \( Q \) for \( \ell > 0 \); in particular, it is again separable for \( \ell > 0 \).

This shows that \( \phi_{e, \ell} \) acts semisimply on \( H_{Z_{\ell}} \) for \( \ell > 0 \) hence that its image in \( GL(H_{e, \ell}) \) is of prime-to-\( \ell \) order. Thus \((S, F, \frac{w}{\ell}, X) \) follows from Fact 2.3 (2) and [S94a, Lem. 5(b)].

**6.3. Proof of (3).** One retains the notation of Subsection 6.1. Since we may assume \( \ell > 0 \), \((F, Q_{\ell}, i, X), i = w, d - w + (wS, Q_{\ell}, w, X) \) imply that the canonical morphism \( Z^w(X) \otimes Z_{\ell} \to H^H_{Z_{\ell}} \) is surjective ( [MiR04, Lem. 3.1]) and, in particular, that the morphism \( Z^w(X) \otimes Z_{\ell} \to H_{Z_{\ell}} \) has torsion-free cokernel. This and the fact that one assumes \( H_{Z_{\ell}} \) is torsion free show that the images of \( Z^w(X) \otimes C \to H^H_{C} \) for \( C = Q_{\ell}, F, Z_{\ell} \) have the same rank - say \( \delta \). As a result

Thus the conclusion follows from the implications:

\[ \delta_{Q_\ell}(1) = \dim(H_{Q_\ell}^H) \overset{6.1}{\Leftrightarrow} (wS, Q_{\ell}, w, X) \overset{(1)}{\Leftrightarrow} (wS, F, w, X) \overset{6.1}{\Leftrightarrow} \delta_{F_\ell}(1) = \dim(H_{F_\ell}^H), \]

and the fact that for \( \ell > 0 \), \( \delta_{Q_\ell}(1) = \delta_{F_\ell}(1) \) (see the last paragraph of 2.4).

7. PROOF OF COROLLARY 1.4

Assume \( p > 0 \) and let \( X \in SmP(K) \) with dimension \( d \). Let \( Br(X^P) := H^2(X^{P}/P_{m}) \) denote the Brauer group of \( X^P \). For a prime \( \ell \neq p \) and integer \( n \geq 1 \), let \( Br(X^P)[\ell^n] \subset Br(X^P) \) denote the kernel of the multiplication-by-\( \ell^n \) map,

\[ T_\ell(Br(X^P)) := \lim_{\leftarrow} Br(X^P)[\ell^n], \quad V_\ell(Br(X^P)) := T_\ell(Br(X^P)[\ell^n]) \otimes Q_\ell. \]

Recall the following elementary observation.

**Lemma.** For every \( \ell \neq p \), \( Br(X^P)^{\pi_1(K)[\ell^\infty]} \) is finite \( \iff \) \( V_\ell(Br(X^P))^{\pi_1(K)} = 0 \).

**Proof.** As \( Br(X^P)^{\pi_1(K)[\ell^\infty]} \) is finite, \( n \geq 0 \) one has the following equivalences

\[ Br(X^P)^{\pi_1(K)[\ell^\infty]} \text{ is infinite} \iff Br(X^P)^{\pi_1(K)} \text{ contains an element of order exactly } \ell^n \text{ for every } n \geq 1 \]

\[ \overset{(1)}{\Leftrightarrow} T_\ell(Br(X^P))^{\pi_1(K)} \neq 0 \]

\[ \overset{(2)}{\Leftrightarrow} V_\ell(Br(X^P))^{\pi_1(K)} \neq 0, \]

\( ^3 \)This is where we use \((S, Q_\ell, \frac{w}{\ell}, X) \).
Then (1) follows from the fact a projective system of non-empty finite sets is non-empty and (2) follows from the fact $T_f(\Br(X_{\mathbb{P}}))$ is torsion-free. □

7.1. Proof of (1) $\Rightarrow$ (2). We retain, again, the notation of 2.1. Let $\rho(X)$ denote the rank of the Néron-Severi group $NS(X)$ of $X$. For divisors, numerical and algebraic equivalences coincide (e.g. [Gr71, XIII, Thm. 4.6]); in particular $(F, Q_{\ell}, 1, X)$ is equivalent to each of the assertions (a) - (d) in [T94, Prop. 2.9]. From [T94, Prop. 2.9 (a)],

$$\rho(X) = \dim(H^2(X_{\mathbb{P}}, Q_{\ell}(w))) / \dim(H^2(X_{\mathbb{P}}, Q_{\ell}(w)))$$

while, from [T94, Prop. 2.9 (c)], $(S, Q_{\ell}, 1, X)$ holds so that $\rho(X) = \delta_{Q_{\ell}}(1)$. As $P$ is in $\mathbb{Z}[1/p][T]$ and independent of $\ell \neq p$, for every other prime $\ell' \neq p$, one has

$$\rho(X') = \delta_{Q_{\ell}}(1) \geq \dim(H^2(X_{\mathbb{P}}, Q_{\ell}(w))) / \rho(X).$$

So that [T94, Prop. 2.9 (a)] holds for $\ell'$ as well and $(F, Q_{\ell'}, 1, X)$ follows from the implication (a) $\Rightarrow$ (b) in [T94, Prop. 2.9].

7.2. Proof of (2) $\Rightarrow$ (3). From [T94, Prop. (5.1)] and Lemma 4.3, for every $\ell \neq p$, $(F, Q_{\ell}, 1, X)$ implies $(wS, Q_{\ell}, 1, X)$. Whence, in particular, split short exact sequences of $\pi_1(K)$-modules

$$0 \to H^2(X_{\mathbb{P}}, Q_{\ell}(1)) \to H^2(X_{\mathbb{P}}, Q_{\ell}(1)) \to V_\ell(\Br(X_{\mathbb{P}})) \to 0, \quad \ell \neq p,$$

which shows $V_\ell(\Br(X_{\mathbb{P}})) = 0, \ell \neq p$. From the above preliminary Lemma, this is equivalent to the finiteness of $\Br(X_{\mathbb{P}})\pi_1(K)[\ell^\infty]$ for $\ell \neq p$. So, to prove (3), it is enough to show

$$(7.2.1) \quad \Br(X_{\mathbb{P}})\pi_1(K)[\ell] = 0, \quad \ell \gg 0.$$  

From [T94, Prop. (5.1)] and 6 (3'), (2) implies $(F, F_{\ell}, 1, X)$ for $\ell \gg 0$ whence the short exact sequences

$$(7.2.2) \quad 0 \to H^2(X_{\mathbb{P}}, F_{\ell}(1)) \to H^2(X_{\mathbb{P}}, F_{\ell}(1)) \to \Br(X_{\mathbb{P}})[\ell] \to 0, \quad \ell \gg 0.$$  

On the other hand, from [T94, Prop. (5.1)], (2) also implies $(wS', Q_{\ell}, 1, X)$ hence, by Lemma 4.3 (1), $(wS, Q_{\ell}, 1, X)$ which, in turn, by 6 (1'), implies $(wS, F_{\ell}, 1, X)$ for $\ell \gg 0$. This shows (7.2.2) splits $\pi_1(K)$-equivariantly for $\ell \gg 0$, whence (7.2.1).

7.3. Proof of (3) $\Rightarrow$ (2). From the above preliminary observation the finiteness of $\Br(X_{\mathbb{P}})\pi_1(K)[p']$ implies $V_\ell(\Br(X_{\mathbb{P}})) = 0, \ell \neq p$ so that taking $\pi_1(K)$-invariants in the short exact sequences

$$0 \to NS(X_{\mathbb{P}}) \otimes Q_{\ell} \to H^2(X_{\mathbb{P}}, Q_{\ell}(1)) \to V_\ell(\Br(X_{\mathbb{P}})) \to 0, \quad \ell \neq p$$

(where $NS(X_{\mathbb{P}})$ denotes the Néron-Severi group of $X_{\mathbb{P}}$) one gets

$$(NS(X_{\mathbb{P}}) \otimes Q_{\ell})\pi_1(K) \Rightarrow H^2(X_{\mathbb{P}}, Q_{\ell}(1))\pi_1(K).$$

On the other hand, let $K^{perf} := K^{\pi_1(K)}$ denote the perfect closure of $K$ and write $X^{perf} := X \times_K K^{perf}$. Then

$$(CH^1(X_{\mathbb{P}}) \otimes Q_{\ell})\pi_1(K) = CH^1(X^{perf}) \otimes Q_{\ell} \longrightarrow CH^1(X) \otimes Q_{\ell}$$

(note that, in general, the cokernel of $CH^1(X) \to CH^1(X^{perf})$ is of $p$-primary torsion). Since $\pi_1(K)$ acts through a finite quotient - hence semisimply - on every finite-dimensional $Q_{\ell}$-vector subspace of $CH^1(X_{\mathbb{P}}) \otimes Q_{\ell}$, the morphism $(CH^1(X_{\mathbb{P}}) \otimes Q_{\ell})\pi_1(K) \to (NS(X_{\mathbb{P}}) \otimes Q_{\ell})\pi_1(K)$ is surjective, which yields the surjectivity of

$$[-] : CH^1(X) \otimes Q_{\ell} \twoheadrightarrow (CH^1(X_{\mathbb{P}}) \otimes Q_{\ell})\pi_1(K) \to (NS(X_{\mathbb{P}}) \otimes Q_{\ell})\pi_1(K) \to H^2(X_{\mathbb{P}}, Q_{\ell}(1))\pi_1(K), \quad \ell \neq p$$

8. Appendix: Basic properties of ultraproducts of fields

Let $L$ be an infinite set. For a subset $S \subseteq L$, write $1_S : L \to \{0, 1\}$ for the characteristic function of $S$.

8.1. A filter on $L$ is a family $\mathcal{F}$ of subsets of $L$ such that

1. $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$;
2. $A \in \mathcal{F}, A \subseteq B \subseteq L \Rightarrow B \in \mathcal{F}$;
3. $\emptyset \notin \mathcal{F}$

8.1.1. An ultrafilter is a filter $\mathcal{U}$ which is maximal for $\subseteq$ among all filters that is such that for every filter $\mathcal{F}$ on $L$, $u \subseteq \mathcal{F} \Rightarrow u = \mathcal{F}$. A filter $\mathcal{U}$ on $L$ is an ultrafilter if and only if for every $S \subseteq L$ either $S \in \mathcal{U}$ or $L \setminus S \in \mathcal{U}$. 


8.1.2. An ultrafilter \( u \) on \( \mathcal{L} \) is either principal that is of the form \( u_\ell := \{ S \subset \mathcal{L} \mid \ell \in S \} \) for some \( \ell \in \mathcal{L} \) or contains the filter \( f^\# := \{ S \subset \mathcal{L} \mid |\mathcal{L} \setminus S| < +\infty \} \) of cofinite subsets, and \( f^\# \) is the intersection of all non-principal ultrafilters on \( \mathcal{L} \).

8.1.3. For every \( \ell \in \mathcal{L} \) fix a field \( F_\ell \) and write \( F := \prod_{\ell \in \mathcal{L}} F_\ell \): note that \( F \) is 0-dimensional. For every \( S \subset \mathcal{L} \), write \( e_S := (1 - 1_S(\ell))_\ell \in F \) for the corresponding idempotent. Filters on \( \mathcal{L} \) are in bijection with the ideals of \( F \)

\[
\text{Filters on } \mathcal{L} \quad \leftrightarrow \quad \text{Ideals of } F
\]

\[
\{ S \in \mathcal{P}(\mathcal{L}) \mid e_S \in i \} \quad \leftrightarrow \quad i
\]

This bijection restricts to a bijection between ultrafilters on \( \mathcal{L} \) and the prime (equivalently maximal) spectrum of \( F \)

\[
\text{Ultrafilters on } \mathcal{L} \quad \leftrightarrow \quad \text{Spec}(F)
\]

\[
\mathfrak{u} \quad \leftrightarrow \quad \mathfrak{u}_{\mathfrak{m}} := \{ S \in \mathcal{P}(\mathcal{L}) \mid e_S \in \mathfrak{m} \}
\]

and, via this bijection, principal (prime) ideals corresponds to principal ultrafilters. In particular, 8.1.2 shows that the intersection of all non-principal maximal ideals in \( F \) is the ideal \( \oplus_{\ell \in \mathcal{L}} F_\ell \subset F \).

Let \( \mathcal{U} \) denote the set of all \emph{non-principal} ultrafilters on \( \mathcal{L} \).

8.1.4. For \( u \in \mathcal{U} \) and an \( F \)-module \( M \), write

\[
M_u := M/m_u M = \lim_{s \in u} (1 - e_S) M
\]

(direct limit by reverse inclusion). For \( \ell \in \mathcal{L} \), write \( M_\ell := M_{u_\ell} \) for its ‘\( \ell \)th component’.

Since for every \( S \in u \) the projection \( p_S : F = e_S F \times (1 - e_S) F \to F/e_S F = (1 - e_S) F \) is flat and \( F \to F_u \) is the direct limit of the \( p_S : F \to F/e_S F \), one gets the following.

**Lemma.** For every ultrafilter \( u \) on \( \mathcal{L} \), the morphism \( F \to F_u \) is flat.

8.2. A finitely generated \( F \)-module \( M \) is the direct product \( M = \prod_{\ell \in \mathcal{L}} M_\ell \) of its \( \ell \)th components if and only if it is finitely presented. Write \( \text{Mod}_{F/\ell} \) for the full subcategory of the category of \( F \)-modules whose objects are direct products \( M = \prod_{\ell \in \mathcal{L}} M_\ell \) of their components. One easily checks that \( \text{Mod}_{F/\ell} \) is an abelian category. For \( M \in \text{Mod}_{F/\ell} \), one has

\[
M \text{ is finitely generated} \iff M \text{ is finitely presented} \iff \sup_{\ell \in \mathcal{L}} \text{dim}_{F_\ell} (M_\ell) < +\infty \iff \sup_{u \in \mathcal{U}} \text{dim}_{F_u} (M_u) < +\infty
\]

In particular, for \( M \in \text{Mod}_{F/\ell} \) finitely generated and

- (8.2.1) for \( N \subset M \) an \( F \)-submodule, one has

\[
N \in \text{Mod}_{F/\ell} \iff \text{N is finitely generated} \iff N \text{ is finitely presented}
\]

- (8.2.2) for every \( F \)-module \( N \) and \( u \in \mathcal{U} \), the canonical morphism

\[
\text{Hom}_{F/\ell}(M_u, N) \otimes_F F_u \to \text{Hom}_{F_u}(M_u, N_u)
\]

is an isomorphism ( [Bo85, Chap. I, §2.10, Prop. 11], using 8.1.4).

8.2.1. The full subcategory of finitely generated \( F \)-modules in \( \text{Mod}_{F/\ell} \) is an abelian subcategory of \( \text{Mod}_{F/\ell} \), stable under taking internal Hom and tensor products: for finitely generated \( M, N \in \text{Mod}_{F/\ell} \), the canonical morphisms \( \text{Hom}_{F/\ell}(M, N) \to \prod_{\ell \in \mathcal{L}} \text{Hom}_{F_\ell}(M_\ell, N_\ell) \) and \( M \otimes_F N \to \prod_{\ell \in \mathcal{L}} M_\ell \otimes_{F_\ell} N_\ell \) are isomorphisms.

8.3. For every \( u \in \mathcal{U} \),

8.3.1. **Lemma.** Let \( M \in \text{Mod}_{F/\ell} \) be finitely generated and let \( N_u : N_0 = M_u \supset N_1 \supset \cdots \supset N_r \supset N_{r+1} = 0 \) be a finite filtration by \( F_u \)-submodules. Then there exists a filtration \( N_* : N_0 = M \supset N_1 \supset \cdots \supset N_r \supset N_{r+1} = 0 \) in \( \text{Mod}_{F/\ell} \) such that \( N_{*,u} = N_u \).

**Proof.** One may assume \( r = 1 \); write \( N := N_1 \). Fix an \( F_u \)-basis \( n_1, \ldots, n_r \) of \( N \) and lift it to a family \( \bar{n}_1, \ldots, \bar{n}_r \in M \). Then the \( F \)-submodule \( \bar{N} = \sum_{1 \leq i \leq r} F \bar{n}_i \subset M \) is in \( \text{Mod}_{F/\ell} \) by (8.2.1). \( \square \)
8.3.2. Lemma. Let $M \in \text{Mod}_E$ and consider the following properties.
- (8.3.2.1) $M_u = 0$;
- (8.3.2.2) The set of $\ell \in L$ such that $M_{\ell} = 0$ is in $u$.
Then (8.3.2.2) $\Rightarrow$ (8.3.2.1). If $M$ is finitely generated, (8.3.2.1) $\Rightarrow$ (8.3.2.2).

Proof. (8.3.2.2) $\Rightarrow$ (8.3.2.1) is straightforward. Conversely, if $u M = M$ and $M$ is finitely generated with $E$-generators $m_1, \ldots, m_r$ then for every $i = 1, \ldots, r$, there exists $S_i \in u$ such that $m_i \in e_{S_i} M$ hence $M = e_{S} M$ with $S = S_1 \cap \cdots \cap S_r \in u$. \hfill \Box

8.3.3. Lemma. Let $\phi : M \to N$ be a morphism in $\text{Mod}_E$ and consider the following properties.
- (8.3.3.1) $\phi_u : M_u \to N_u$ has $P$;
- (8.3.3.2) The set $S$ of all $\ell \in L$ such that $\phi_{\ell} : M_\ell \to N_\ell$ has $P$ is in $u$, where $P$ is one of the properties of being injective, surjective, an isomorphism. Then (8.3.3.2) $\Rightarrow$ (8.3.3.1). If the conditions below are satisfied, (8.3.3.1) $\Rightarrow$ (8.3.3.2).

<table>
<thead>
<tr>
<th>$P$</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi$ has finitely generated cokernel</td>
<td></td>
</tr>
<tr>
<td>$\phi$ has finitely generated kernel</td>
<td></td>
</tr>
<tr>
<td>Isomorphism</td>
<td>$\phi$ has finitely generated kernel and cokernel</td>
</tr>
</tbody>
</table>

Proof. By right-exactness (resp. left-exactness - 8.1.4) of $- \otimes_E Q_u$, $\ker(\phi)_u = \ker(\phi_u)$ (resp. $\coker(\phi)_u = \coker(\phi_u)$). So the conclusion follows from 8.3.2. \hfill \Box

8.4. Let $M \in \text{Mod}_E$ and $\Pi$ be a group acting on $M$. For every $u \in U$, $\Pi u \\ 4.1. Lemma. (M_u)_{\Pi} = (M_{\Pi u})$.

Proof. This follows from the exact sequence $M_\Pi \oplus \Pi \sum_{\pi \in \Pi} (\id - \pi) M \to M_{\Pi u} \to 0$, right-exactness of $- \otimes_E F_u$ and the fact that tensor products commute with direct sums. \hfill \Box

From now on, assume furthermore that $M \in \text{Mod}_E$ is finitely generated, that for every $\ell \in L$, $F_{\ell}$ is a Hausdorff topological field, that $\Pi$ is a topological group which acts continuously on $M$ for $M$ equipped with the product topology of the topologies of the $M_{\ell}$ (recall $M_{\ell}$ is a finitely generated $F_{\ell}$-module) and that $\Pi$ is topologically finitely generated with topological generators $\pi_1, \ldots, \pi_s$. Let $\Pi^o \subset \Pi$ denote the abstract group generated by $\pi_1, \ldots, \pi_s$.

8.4.2. Lemma. (M_{\Pi u}) = (M_{\Pi u}).

Proof. The exact sequence $0 \to M_{\Pi u} \to M_{\Pi} (\id - \pi_1, \ldots, \id - \pi_s) \to M_{\Pi u} \to 0$, 8.1.4 and the fact that tensor products commute with finite direct products (=direct sums) yield $(M_{\Pi u}) = (M_{\Pi})_u$. So the assertion follows from the obvious inclusions $(M_{\Pi u}) \subset (M_{\Pi})_u = (M_{\Pi}) \subset (M_{\Pi})_u$. \hfill \Box

In particular, if $N \in \text{Mod}_E$ is also finitely generated and equipped with a continuous action of $\Pi$, (8.2.2) and 8.4.2 yield

(8.4.2.1) $\Hom_{\Pi}(M_{\Pi}, N_u) = \Hom_{\Pi}(M, N_{\Pi})$

8.4.3. Lemma. For every finite filtration $N_\bullet : N_0 = N_1 \supset N_1 \supset \cdots \supset N_r \supset N_{r+1} = 0$ in $\text{Mod}_E$, map $\sigma : \{0, \ldots, r + 1\} \to \{0, \ldots, r + 1\}$ and subset $X \subset F[\Pi]$, consider the following assertions.
- (8.4.3.1) $X N_{i, u} \subset N_{\sigma(i), u}$, $i = 0, \ldots, r + 1$;
- (8.4.3.2) The set of all $\ell \in L$ such that $X N_{\ell, i} \subset N_{\ell, \sigma(i)}$, $i = 0, \ldots, r + 1$ is in $u$.
Then (8.4.3.2) $\Rightarrow$ (8.4.3.1). If $X$ is finite (8.4.3.1) $\Rightarrow$ (8.4.3.2).

Proof. (8.4.3.1) $\Rightarrow$ (8.4.3.2) is straightforward. For (8.4.3.1) $\Rightarrow$ (8.4.3.2), write $X = \{x_1, \ldots, x_t\}$. Then for every $i = 0, \ldots, r + 1$, one has $X N_{\ell, i} \subset N_{\ell, \sigma(i)}$ if and only if $(x_1, \ldots, x_t) \subset N_{\ell, \sigma(i)} \subset M_{\ell, i}$. Let $\nu_1, \ldots, \nu_{t, i, \ell}$ be a set of $E$-generators for $N_{\ell}$ (8.2.1). By (8.4.3.1), for every $1 \leq j \leq t_i$, there exists $S_{i, j} \in u$ such that $x_j \in e_{S_{i, j}} M_{\ell, i}$. Hence $(x_1, \ldots, x_t) \subset e_{S_{i, j}} M_{\ell, i}$. The set of $\ell \in L$ satisfying (8.4.3.2) then contains $S_0 \cap \cdots \cap S_r \in u$.

In particular,
- (8.4.3.3) $(\sigma = Id, X = \{\pi_1, \ldots, \pi_s\}) N_{\bullet u}$ is $\Pi$-stable if and only if the set of all $\ell \in L$ such that $N_{\bullet, \ell}$ is $\Pi$-stable is in $u$.\hfill \Box
which is $\Pi$-stable. As $u := \{1 - \pi_1, \ldots, 1 - \pi_s\}$ - See 8.3.1) $\Pi$ acts unipotently on $M_u$ if and only if the set of all $\ell \in \mathcal{L}$ such that $\Pi$ acts unipotently on $M_{\ell}$ is in $u$.

8.4.4. 8.3.1 and 8.4.3 imply that, every $\Pi$-submodule $N \subset M_u$ (hence resp. every $\Pi$-quotient $M_u \rightarrow N$) lifts to a $\Pi$-submodule $\overline{N} \subset \overline{M}$ (resp. a $\Pi$-quotient $\overline{M} \rightarrow \overline{N}$) in $\text{Mod}_{F}$. From this, one immediately deduces that any $\Pi$-module $N$ in the Tannakian category generated by the $\Pi$-module $M_u$ lifts to some $\overline{N}$ in $\text{Mod}_{F}$ which is a $\Pi$-subquotient of a $\Pi$-module of the form $\bigoplus_{(m,n) \in \mathbb{Z}_{\geq 0}^2} (M^m \otimes \overline{M}^n)_{\oplus \mu(m,n)}$ for some function $\mu : \mathbb{Z}_{\geq 0}^2 \rightarrow \mathbb{Z}_{\geq 0}$ with finite support.

8.4.5. Lemma. The following assertions are equivalent.
- (8.4.5.1) $\Pi$ acting on $M_u$ has $P$;
- (8.4.5.2) The set of all $\ell \in \mathcal{L}$ such that $\Pi$ acting on $M_{\ell}$ has $P$ is in $u$, where $P$ is one of the properties of acting irreducibly or semisimply.

Proof. The assertion for the property of acting irreducibly follows from 8.3.2 and 8.4.4. Let $P$ be the property of acting semisimply and assume (8.4.5.2). Let $N \subset M_u$ be a $\Pi$-submodule. By 8.4.4, $N$ lifts to an $F$-submodule $\overline{N}$ in $\text{Mod}_{F}$ which is $\Pi$-stable. As $S \in u$, one may take $N_{\ell} = 0$ for $\ell \notin \mathcal{L} \setminus S$. By (8.4.5.2), the projection $\overline{M} \rightarrow \overline{M}/N$ splits $\Pi$-equivariantly. The conclusion follows by applying $- \otimes F_u$. Conversely, assume (8.4.5.1). If $S \notin u$ then $\mathcal{L} \setminus S \in u$ and for every $\ell \notin \mathcal{L} \setminus S$ there exists a $\Pi$-submodule $N_{\ell} \subset M_{\ell}$ such that $Q_{\ell} := \text{coker}(\text{Hom}_{\Pi}(M_{\ell}/N_{\ell}, M_{\ell}) \rightarrow \text{Hom}_{\Pi}(M_{\ell}/N_{\ell}, \overline{M}_{\ell}/N_{\ell}))$

is non-zero, where $p_{\ell} : M_{\ell} \rightarrow M_{\ell}/N_{\ell}$ is the canonical quotient morphism. In particular, $Q_{u} \neq 0$, where $Q := \prod_{\ell \in \mathcal{L}} Q_{\ell}$. Write also $\overline{N} := \prod_{\ell \notin \mathcal{L} \setminus S} N_{\ell}$ and let $\overline{p} : \overline{M} \rightarrow \overline{M}/\overline{N}$ denote the canonical quotient morphism. By right-exactness of $- \otimes F_u$, one obtains an exact sequence

$$\text{Hom}_{\Pi}(\overline{M}/\overline{N}, \overline{M}/\overline{N}) \rightarrow Q_u \rightarrow 0,$$

which, by (8.4.2.1), identifies with $\text{Hom}_{\Pi}(M_u/N_u, M_u) \rightarrow \text{Hom}_{\Pi}(M_u/N_u, M_u/N_u) \rightarrow Q_u \rightarrow 0$.

The same arguments show the following.

8.4.6. Lemma. Let $N \subset M$ be an $F$-submodule in $\text{Mod}_{F}$ which is $\Pi$-stable. The following assertions are equivalent.
- (8.4.6.1) The inclusion $N_u \hookrightarrow M_u$ splits $\Pi$-equivariantly;
- (8.4.6.2) The set of all $\ell \in \mathcal{L}$ such that the inclusion $N_{\ell} \hookrightarrow M_{\ell}$ splits $\Pi$-equivariantly is in $u$.

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