MONODROMY OF FOUR DIMENSIONAL IRREDUCIBLE COMPATIBLE SYSTEMS OF $\mathbb{Q}$

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In honor of Professor Michael Larsen's 60th birthday

Abstract. Let F be a totally real field and $n \le 4$ a natural number. We study the monodromy groups of any n-dimensional strictly compatible system $\{\rho_\lambda\}_\lambda$ of λ-adic representations of F with distinct Hodge-Tate numbers such that ρ_{λ_0} is irreducible for some λ_0. When $F = \mathbb{Q}$, $n = 4$, and ρ_{λ_0} is fully symplectic, the following assertions are obtained.

(i) The representation ρ_λ is fully symplectic for almost all λ.

(ii) If in addition the similitude character μ_{λ_0} of ρ_{λ_0} is odd, then the system $\{\rho_\lambda\}_\lambda$ is potentially automorphic and the residual image $\bar{\rho}_\lambda(\mathrm{Gal}_{\mathbb{Q}})$ has a subgroup conjugate to $\mathrm{Sp}_4(\mathbb{F}_\ell)$ for almost all λ.

1. Introduction

1.1. **Main results.** Let K, E be number fields, and $\{\rho_\lambda\}_\lambda$ an E-rational compatible system of semisimple n-dimensional λ-adic representations of K *in the sense of Serre* (Definition 2.2). Denote by Γ_λ the *monodromy* (i.e., image) of ρ_λ and by $\mathbf{G}_\lambda$ the *algebraic monodromy group* of ρ_λ[1]. A fundamental problem about compatible systems of λ-adic Galois representations concerns whether the absolute irreducibility of ρ_λ is independent of λ, and furthermore, whether the residual representations $\bar{\rho}_\lambda$ are absolutely irreducible for almost all λ (i.e., all but finitely many λ) if some member of $\{\rho_\lambda\}_\lambda$ is absolutely irreducible, to which establishing big images Γ_λ for all (or almost all) λ is pivotal. Two impetuses of the problem are the *Mumford-Tate conjecture* that concerns the λ-independence of $\mathbf{G}_\lambda^\circ$ for motivic compatible systems and the *irreducibility conjecture* that predicts the absolute irreducibility of ρ_λ for compatible systems attached to algebraic cuspidal automorphic forms of $\mathrm{GL}_n(\mathbb{A}_K)$. There have been many studies on the irreducibility and big images of compatible system since 70s, assuming that $\{\rho_\lambda\}_\lambda$ is

- motivic, see [Se72, Se85],[Ri76] for abelian varieties, [DV04],[DW21] for $n = 3$, and [DV08, DV11] for $n = 4$,
- automorphic, see [Ri77, Ri85],[Ta95],[Dim05] for (Hilbert) modular forms, [BR92],[DV04] for $n = 3$, [Die02b, Die07],[Ra13],[DZ20],[We22] for $n = 4$, [CG13],[Xi19],[Hu22] for $n \le 6$, and [BLGGT14] in general,

and more recently,

- a weakly compatible system (Definition 2.1), see [BLGGT14],[PT15],[LY16],[PSW18], and [DW21].

[1] The Zariski closure of the monodromy Γ_λ in $\mathrm{GL}_{n,E_\lambda}$.

In particular in the recent work [Hu22], when $K = F$ is totally real or CM, $n \leq 6$, and

$$\{\rho_\lambda : \mathrm{Gal}_F \to \mathrm{GL}_n(\overline{E}_\lambda)\}_\lambda \tag{1}$$

is the strictly compatible system (Definition 2.1) attached to a regular algebraic, polarized, cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_F)$, we proved ([Hu22, Theorem 1.4]) that

(I) ρ_λ is irreducible for almost all λ and
(II) ρ_λ is residually irreducible for almost all λ if in addition $F = \mathbb{Q}$

by combining various Galois theoretic and potential automorphy results. The innovation of [Hu22] is the development of some big images results for subrepresentations of the compatible system $\{\rho_\lambda\}_\lambda$ inspired by previous works [Hu15] and [HL16, HL20] (see Theorem 2.10 and Proposition 2.11). The strategy of the irreducibility result (I), roughly speaking, is that if there exists an infinite set $\mathcal{L}$ such that ρ_λ is reducible whenever $\lambda \in \mathcal{L}$, then some low dimensional subrepresentations of ρ_λ (or representation constructed from ρ_λ) for some $\lambda \in \mathcal{L}$ have big images so that the potential automorphy theorems in [BLGGT14] can be applied to draw a contradiction to the irreducibility of a member of $\{\rho_\lambda\}_\lambda$ due to Patrikis-Taylor [PT15]. The residual irreducibility result (II) requires an extra input from Serre's modularity conjecture [Se87] (proven in [KW09a, KW09b]). In this article, we use these ideas to study the monodromy of some low dimensional strictly compatible systems of totally real field F. Given an algebraic extension E_ℓ of $\mathbb{Q}_\ell$, the algebraic monodromy group $\mathbf{G}_\ell$ of an ℓ-adic Galois representation $\sigma_\ell : \mathrm{Gal}_K \to \mathrm{GL}_n(E_\ell)$ is the E_ℓ-subgroup of GL_{n,E_ℓ} defined as the Zariski closure of the monodromy $\sigma_\ell(\mathrm{Gal}_K)$ in GL_{n,E_ℓ}.

Theorem 1.1. *Suppose F is a totally real field, $n \leq 4$, and $\{\rho_\lambda : \mathrm{Gal}_F \to \mathrm{GL}_n(\overline{E}_\lambda)\}_\lambda$ is a strictly compatible system of F with distinct τ-Hodge-Tate numbers for each $\tau : F \to \overline{E}_\lambda$. Let $\mathbf{G}_\lambda$ be the algebraic monodromy of ρ_λ. If ρ_{λ_0} is irreducible for some λ_0, then the following assertions hold.*

(i) When $n \leq 3$, the representation ρ_λ is irreducible for all λ.
(ii) When $n \leq 3$, the identity component $\mathbf{G}_\lambda^\circ \subset \mathrm{GL}_{n,\overline{E}_\lambda}$ is independent of λ.
(iii) When $n \leq 3$, the representation ρ_λ is residually irreducible for almost all λ.
(iv) When $n = 4$, the representation ρ_λ is irreducible for almost all λ.

When $F = \mathbb{Q}$ and $n = 4$, we have a better description of the monodromy by Serre's modularity conjecture. If $\mathbf{G}_\lambda \subset \mathrm{GO}_n$ (resp. GSp_n) with respect to a non-degenerate symmetric (resp. skew-symmetric) pairing $\langle\ ,\ \rangle$ on the representation space of ρ_λ, there is a similitude character μ_λ such that $\rho_\lambda \cong \rho_\lambda^\vee \otimes \mu_\lambda$. The character μ_λ is said to be *odd* (resp. *even*) if $\mu_\lambda(c) = -1$ (resp. 1) where $c \in \mathrm{Gal}_\mathbb{Q}$ is a complex multiplication. Denote by $\mathbf{G}_\lambda^{\mathrm{der}}$ the derived group of the identity component $\mathbf{G}_\lambda^\circ$. The representation ρ_λ is said to be

- *fully orthogonal* if $\mathbf{G}_\lambda^{\mathrm{der}} = \mathrm{SO}_n$ with $\langle\ ,\ \rangle$ symmetric;
- *fully symplectic* if $\mathbf{G}_\lambda^{\mathrm{der}} = \mathrm{Sp}_n$ with $\langle\ ,\ \rangle$ skew-symmetric.

Below is our main result on four dimensional fully symplectic Galois representations of $\mathbb{Q}$; see [LY16] for four dimensional fully orthogonal Galois representations.

Theorem 1.2. *Suppose* $\{\rho_\lambda : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_4(\overline{E}_\lambda)\}_\lambda$ *is a strictly compatible system of* $\mathbb{Q}$ *with distinct Hodge-Tate numbers. If* ρ_{λ_0} *is fully symplectic*[2] *for some* λ_0*, then the following assertions hold.*

(i) The representation ρ_λ *is fully symplectic for almost all* λ*.*
(ii) If in addition the similitude character μ_{λ_0} *is odd, then* $\{\rho_\lambda\}_\lambda$ *is potentially automorphic and the residual image* $\bar{\rho}_\lambda(\mathrm{Gal}_{\mathbb{Q}})$ *has a subgroup conjugate to* $\mathrm{Sp}_4(\mathbb{F}_\ell)$ *for almost all* λ*.*

When the strictly compatible system $\{\rho_\lambda\}_\lambda$ is also a Serre compatible system defined over $E = \mathbb{Q}$, the prime λ is a rational prime ℓ, the algebraic monodromy group $\mathbf{G}_\ell$ is defined over $\mathbb{Q}_\ell$, and the monodromy group Γ_ℓ is a compact open subgroup of $\mathbf{G}_\ell(\mathbb{Q}_\ell)$. In [Lar95], Larsen conjectured that for any motivic compatible system the subgroup $\Gamma_\ell^{\mathrm{sc}} \subset \mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ (Definition 2.6) is *hyperspecial maximal compact* (see [Ti79]) when ℓ is sufficiently large. We obtain the corollary below.

Corollary 1.3. *Suppose* $\{\rho_\ell : \mathrm{Gal}_{\mathbb{Q}} \to \mathrm{GL}_4(\mathbb{Q}_\ell)\}_\ell$ *is a strictly compatible system of* $\mathbb{Q}$ *with distinct Hodge-Tate numbers. If* ρ_{ℓ_0} *is fully symplectic with odd similitude character* μ_{ℓ_0} *for some* ℓ_0*, then* $\Gamma_\ell^{\mathrm{sc}}$ *is a hyperspecial maximal compact subgroup of* $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell) = \mathbf{G}_\ell^{\mathrm{der}}(\mathbb{Q}_\ell) \cong \mathrm{Sp}_4(\mathbb{Q}_\ell)$ *(that is,* $\Gamma_\ell^{\mathrm{sc}} \cong \mathrm{Sp}_4(\mathbb{Z}_\ell)$*) for all* $\ell \gg 0$*.*

1.2. **Remarks and comparisons of techniques.** An ℓ-adic representation $\sigma_\ell : \mathrm{Gal}_K \to \mathrm{GL}_m(\overline{\mathbb{Q}}_\ell)$ is said to be *of type A* if every simple factor of the Lie algebra $\mathfrak{g}_\ell$ of the algebraic monodromy group of the semisimplification $\sigma_\ell^{\mathrm{ss}}$ is of type A in the Killing-Cartan classification, i.e.,

$$[\mathfrak{g}_\ell, \mathfrak{g}_\ell] \cong \bigoplus_i \mathfrak{sl}_{k_i,\overline{\mathbb{Q}}_\ell}.$$

Note that σ_ℓ is of type A when $\dim \sigma_\ell = m \leq 3$.

For a semisimple Serre compatible system $\{\rho_\lambda\}_\lambda$ satisfying reasonable local conditions (Theorem 2.10(a),(b)), the big images results in [Hu22] give lower bounds for the residual images of subrepresentations σ_λ in terms of the *formal bi-characters* (Definition 2.8) of the algebraic monodromy of σ_λ for almost all λ (Theorem 2.10(i)-(iii)). This result works particularly well if σ_λ is of type A (Theorem 2.10(iv)-(v)) and can be used to streamline some arguments or improve some results in other works.

1.2.1. *Case* $n = 2$. Theorem 1.1(i) (and F is any number field) is essentially due to Ribet [Ri77] (see also [BR92, Proposition 2.3.1]) by using only the fact that if ρ_{λ_1} is reducible for some λ_1 then it is locally algebraic, which implies $\{\rho_\lambda\}_\lambda$ is a sum of two one-dimensional systems. This contradicts that ρ_{λ_0} is irreducible.

Theorem 1.1(iii) for Serre compatible system $\{\rho_\lambda\}_\lambda$ attached to objects like non-CM elliptic curves [Se72] and (Hilbert) modular forms [Ri85],[Dim05] is obtained by analyzing some explicit data of the object, e.g., sizes of Fourier coefficients. Since $\{\rho_\lambda\}_\lambda$ is of type A and is either motivic or strictly compatible, the results follow directly from our big images results (Theorem 2.10(v)).

[2] With respect to a non-degenerate skew-symmetric pairing on the representation space of ρ_{λ_0}.

1.2.2. *Case* $n = 3$. In [DW21], it is proved that some three dimensional, regular self-dual compatible system $\{\rho_\ell\}_\ell$ arising from a surface (an elliptic fibration over the projective line) over $\mathbb{Q}$ is either (a) absolutely irreducible for a Dirichlet density one set of rational primes ℓ or (b) $\{\rho_\ell\}_\ell$ decomposes as a sum of two irreducible compatible systems. The density one set of primes condition appears in case (a) because some results of [CG13] and [BLGGT14] (that hold under density one[3]) are used. Case (a) can be improved to be absolutely irreducible for all ℓ (Theorem 1.1(i)).

In [DV04], it is proved that some three dimensional non-self-dual motivic compatible system $\{\rho_\lambda\}_\lambda$ of $\mathbb{Q}$ have big images (in particular $\bar{\rho}_\lambda$ is absolutely irreducible) if $\ell := \mathrm{char}(\lambda)$ belongs to a Dirichlet density one set of rational primes. The techniques (also in other works of Dieulefait [Die02b],[DV08, DV11],[DZ20]) include studying the tame inertia characters ([Se72]) for large ℓ by relating them to Hodge-Tate weights ([FL82]) and the classification of maximal subgroups of $\underline{G}(\mathbb{F}_q)$ where $\underline{G}$ is some connected $\mathbb{F}_q$-semisimple group. Since $\{\rho_\lambda\}_\lambda$ is of type A and motivic, it follows from Theorem 2.10(v) that residual (absolute) irreducibility holds for almost all λ.

1.2.3. *Case* $n = 4$. Irreducibility and big images for four dimensional compatible system $\{\rho_\lambda\}_\lambda$ of $\mathbb{Q}$ are studied in the works [Die02b],[DV11],[Ra13],[LY16],[DZ20],[We22], in which (potential) automorphy techniques such as Serre's conjecture and [BLGGT14] are used to rule out two dimensional factors of ρ_λ. This is also the main idea for Theorem 1.1(iv) and Theorem 1.2. In order to apply such automorphy theorems, an oddness condition and a big image condition are usually required (see Theorem 2.4(2),(4)). The first condition for some two dimensional representation of a totally real field F is satisfied by a result of Calegari [Ca11] (see Proposition 2.5). The second one can be obtained by Theorem 2.10 if the formal character of $\mathbf{G}_\lambda^{\mathrm{der}}$ is known. For example, if $\mathbf{G}_{\lambda_0}^{\mathrm{der}} = \mathrm{Sp}_4$ then $\mathbf{G}_\lambda^{\mathrm{der}}$ (in GL_4) has only three possibilities: Sp_4, SO_4, $\mathrm{SL}_2 \times \mathrm{SL}_2$ (by Theorem 2.9). Big images results work well in the last two cases because they are of type A. If the strict compatibility of $\{\rho_\lambda\}_\lambda$ and the fully symplectic (for some λ_0) condition are replaced by weak compatibility and purity, then the potential automorphy assertion in Theorem 1.2(ii) is obtained in [PT15, Theorem A].

1.3. **Structure of the article.** Section 2 presents the preliminaries for studying monodromy groups of compatible systems, including some conjectures on compatible systems of Galois representations and some essential results for later use, for example, a potential automorphy theorem [BLGGT14, Theorem C] and some big images results in [Hu22] (summarized as Theorem 2.10). Section 3 is devoted to the proofs of the main results in §1.1.

Acknowledgments

After the collaboration [HL20], Professor Larsen asked if there are some general four dimensional compatible systems $\{\rho_\ell\}_\ell$ with GSp_4 algebraic monodromy that fulfill the big image criterion $(*)$ in Theorem 2.7, i.e., ρ_ℓ has to be residually (absolutely) irreducible for all $\ell \gg 0$. Larsen's question inspired the work [Hu22], which is used in this article essentially to obtain Theorem 1.2 and Corollary 1.3, that to some extent, get back to his question. My debt to the work of Professor Larsen on Galois representations is obvious and I am grateful

[3]This is because works like [BLGGT14] rely on a (Dirichlet density one) big images result of Larsen [Lar95] for Serre compatible systems.

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2. Preliminaries

2.1. **Compatible systems.** Let K and E be number fields and denote their sets of finite places by Σ_K and Σ_E respectively. For any prime number ℓ, denote by S_ℓ the set of elements of Σ_K dividing ℓ. For $\lambda \in \Sigma_E$, denote by ℓ the residue characteristic of λ. Let

$$\{\rho_\lambda : \mathrm{Gal}_K \to \mathrm{GL}_n(\overline{E}_\lambda)\}_{\lambda\in\Sigma_E} \tag{2}$$

be a family of λ-adic representations of K.

Definition 2.1. *The family* (2) *is called a weakly compatible system defined over E (in the sense of* [BLGGT14, §5.1]*) if ρ_λ is semisimple for all λ and there exist a finite subset $S \subset \Sigma_K$ and a polynomial $P_v(t) \in E[t]$ for each $v \in \Sigma_K\backslash S$ such that the following conditions (a),(b), and (c) hold.*

- *(a) For each $\lambda \in \Sigma_E$ (with residue characteristic ℓ) and $v \in \Sigma_K\backslash(S \cup S_\ell)$, the representation ρ_λ is unramified at v and the characteristic polynomial of the image $\rho_\lambda(Frob_v)$ of the Frobenius class at v is $P_v(t)$.*
- *(b) Each ρ_λ is de Rham at all places $v \in S_\ell$ and moreover crystalline if $v \notin S$.*
- *(c) For each embedding $\tau : K \hookrightarrow \overline{E}$, the set of τ-Hodge-Tate numbers of ρ_λ is independent of λ and any $\overline{E} \hookrightarrow \overline{E}_\lambda$ over E.*

A weakly compatible system is called a strictly compatible system if condition (d) below holds.

- *(d) If $v \in \Sigma_K$ does not divide the residue characteristic of λ, the semisimplified Weil-Deligne representation $\iota WD(\rho_\lambda|_{\mathrm{Gal}_{K_v}})^{F-ss}$ is independent of λ and $\iota : \overline{E}_\lambda \xrightarrow{\cong} \mathbb{C}$.*

Definition 2.2. *(see* [Se98, Chapter 1]*) The family* (2) *is called a Serre compatible system of K defined over E if $\rho_\lambda(\mathrm{Gal}_K) \subset \mathrm{GL}_n(E_\lambda)$ for all λ and the family satisfies the compatibility condition 2.1(a) for some finite $S \subset \Sigma_K$ and $P_v(t) \in E[t]$ for each $v \in \Sigma_K\backslash S$.*

Under the distinct τ-Hodge-Tate numbers condition, a strictly compatible system (2) can be identified as a Serre compatible system after enlarging E.

Proposition 2.3. [BLGGT14, Lemma 5.3.1(3)] *Let $\{\rho_\lambda : \mathrm{Gal}_K \to \mathrm{GL}_n(\overline{E}_\lambda)\}_\lambda$ be a strictly compatible system of K defined over E with distinct τ-Hodge-Tate numbers. After enlarging E, we may suppose $\{\rho_\lambda\}_\lambda$ is a semisimple Serre compatible system defined over E.*

2.2. **Automorphy of Galois representations.** When K is totally real or CM, one can attach an n-dimensional strictly compatible system of K defined over some CM field E to a regular algebraic, polarized, cuspidal automorphic representation π of $\mathrm{GL}_n(\mathbb{A}_K)$ (see [BLGGT14, §2.1]) such that the L-functions agree. Conversely, the Fontaine-Mazur-Langlands conjecture

[FM95],[Lan79] (see also [Cl90],[Ta04]) asserts that any irreducible ℓ-adic representation $\rho_\ell:\mathrm{Gal}_{\mathbb{Q}}\to\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which is unramified at all but finitely many primes and with $\rho_\ell|_{\mathrm{Gal}_{\mathbb{Q}_\ell}}$ de Rham comes from a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. For $n=2$, many cases of this conjecture are established by Kisin [Ki09] and Emerton [Em11] when ρ_ℓ is residually irreducible and recently by Pan [Pa21] when ρ_ℓ is residually reducible. For general n, we have the following potential automorphy theorem for totally real field. Denote $\zeta_\ell:=e^{2\pi i/\ell}$.

Theorem 2.4. [BLGGT14, Theorem C] *Suppose F is a totally real field. Let m be a positive integer and let $\ell\geq 2(m+1)$ be a prime. Let*

$$\rho_\ell:\mathrm{Gal}_F\to\mathrm{GL}_m(\overline{\mathbb{Q}}_\ell)$$

be a continuous representation. Suppose the following conditions are satisfied.

(1) (Unramified almost everywhere): ρ_ℓ is unramified at all but finitely many primes.
(2) (Odd essential self-duality): either ρ_ℓ maps to GSp_m with totally odd similitude character or it maps to GO_m with totally even similitude character.
(3) (Potential diagonalizability and regularity): ρ_ℓ is potentially diagonalizable (and hence potentially crystalline) at each prime v of F above ℓ and for each $\tau:F\hookrightarrow\overline{\mathbb{Q}}_\ell$ it has m-distinct τ-Hodge-Tate numbers.
(4) (Irreducibility): $\rho_\ell|_{\mathrm{Gal}_{F(\zeta_\ell)}}$ is residually irreducible.

Then we can find a finite Galois totally real extension F'/F such that $\rho_\ell|_{\mathrm{Gal}_{F'}}$ is automorphic (i.e., attached to a regular algebraic, (polarized) cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_{F'})$). Moreover, ρ_ℓ is part of a strictly (pure) compatible system of F.

Given a strictly (and also Serre) compatible system $\{\rho_\lambda:\mathrm{Gal}_F\to\mathrm{GL}_n(E_\lambda)\}_\lambda$ of a totally real field F, in order to apply Theorem 2.4 to a subrepresentation r_λ of ρ_λ we need r_λ to satisfy the four conditions. The first condition is automatic and the third one holds when ℓ is sufficiently large (by conditions 2.1(b),(c) and [BLGGT14, Lemma 1.4.3(2)]). For the second one (odd essential self-duality), we have the following result when $\dim r_\lambda=2$.

Proposition 2.5. [CG13, Proposition 2.5] *Suppose $r_\ell:\mathrm{Gal}_F\to\mathrm{GL}_2(\overline{\mathbb{Q}}_\ell)$ is a continuous representation of a totally real field F and $\ell>7$. Assume that*

(1) r_ℓ is unramified outside of finitely many primes.
(2) $\mathrm{Sym}^2 r_\ell|_{\mathrm{Gal}_{F(\zeta_\ell)}}$ is residually irreducible.
(3) ℓ is unramified in F.
(4) For each place $v|\ell$ of F and each $\tau:F_v\hookrightarrow\overline{\mathbb{Q}}_\ell$, the τ-Hodge-Tate numbers of $r_\ell|_{\mathrm{Gal}_{F_v}}$ is a set of two distinct integers whose difference is less than $(\ell-1)/2$, and $r_\ell|_{\mathrm{Gal}_{F_v}}$ is crystalline.

Then, the pair $(r_\ell,\det r_\ell)$ is essentially self-dual and odd.

Hence, for $\dim r_\lambda=2$ and $\ell\gg 0$ the conditions (1),(3),(4) of Proposition 2.5 hold (again using 2.1(b),(c)). Both conditions Theorem 2.4(4) and Proposition 2.5(2) require r_λ to have big image, which are fulfilled if the algebraic monodromy group of r_λ contains SL_2 and $\ell\gg 0$ by applying Theorem 2.10(iv) and Proposition 2.11 to the compatible system $\{\rho_\lambda\}_\lambda$.

2.3. **Big Galois images and irreducibility.** Our works on big Galois images [HL16],[HL20], and [Hu22] are motivated by a well-known theorem of Serre on Galois actions of non-CM elliptic curves [Se72] and also a conjecture of Larsen [Lar95] on Galois actions of motivic compatible systems (see also [Se94]). Let X be a smooth projective variety defined over a number field K and $w \geq 0$ an integer. The étale cohomology group $V_\ell := H^w(X_{\overline{K}}, \mathbb{Q}_\ell)$ is acted on by Gal_K for all primes ℓ. By [De74], the family of ℓ-adic representations

$$\{\rho_\ell : \mathrm{Gal}_K \to \mathrm{GL}(V_\ell) \cong \mathrm{GL}_n(\mathbb{Q}_\ell)\}_{\ell \in \Sigma_{\mathbb{Q}}} \tag{3}$$

is a Serre compatible system of K defined over $\mathbb{Q}$ (Definition 2.2). Let Γ_ℓ (resp. $\mathbf{G}_\ell$) be the image (resp. algebraic monodromy group) of ρ_ℓ. Consider the diagram

$$\begin{array}{ccc} & & \mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell) \\ & & \Big\downarrow{\scriptstyle \pi_\ell} \\ \mathbf{G}_\ell^{\circ}(\mathbb{Q}_\ell) & \xrightarrow{q_\ell} & \mathbf{G}_\ell^{\mathrm{ss}}(\mathbb{Q}_\ell) \end{array} \tag{4}$$

where $\mathbf{G}_\ell^{\mathrm{ss}}$ is the quotient of the connected linear algebraic group $\mathbf{G}_\ell^{\circ}$ by radical, $\mathbf{G}_\ell^{\mathrm{sc}}$ is the universal covering of $\mathbf{G}_\ell^{\mathrm{ss}}$, and q_ℓ (resp. π_ℓ) is given by the quotient (resp. covering) morphism.

Definition 2.6. *Denote by $\Gamma_\ell^{\mathrm{sc}}$ the compact subgroup $\pi_\ell^{-1}(q_\ell(\Gamma_\ell \cap \mathbf{G}_\ell^{\circ}(\mathbb{Q}_\ell)))$ of $\mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ in diagram* (4)*.*

Larsen conjectured [Lar95] that for all $\ell \gg 0$, the subgroup $\Gamma_\ell^{\mathrm{sc}} \subset \mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ is *hyperspecial maximal compact*, i.e., there exists a semisimple group scheme $\mathcal{G}_\ell$ defined over $\mathbb{Z}_\ell$ with generic fiber $\mathbf{G}_\ell^{\mathrm{sc}}$ such that $\mathcal{G}_\ell(\mathbb{Z}_\ell) = \Gamma_\ell^{\mathrm{sc}}$. He proved that for a set of primes ℓ of Dirichlet density one the assertion is true [Lar95, Theorem 3.17]. We proved the conjecture when (the semisimplification of) ρ_ℓ is of type A for $\ell \gg 0$ [HL16] and when X is an abelian variety or hyper-Kähler variety (and $w = 2$) [HL20, Theorem 1.3]. We also established the following criterion.

Theorem 2.7. [HL20, Theorem 1.2] *Let $\{\rho_\ell\}_\ell$ be the Serre compatible system arising from the wth cohomology of a smooth projective variety X/K such that $\mathbf{G}_\ell$ is connected for all ℓ. For all sufficiently large ℓ, the subgroup $\Gamma_\ell^{\mathrm{sc}} \subset \mathbf{G}_\ell^{\mathrm{sc}}(\mathbb{Q}_\ell)$ is hyperspecial maximal compact if and only if*

$$\dim_{\mathbb{Q}_\ell} \mathrm{End}_{\mathrm{Gal}_K}(V_\ell) = \dim_{\mathbb{F}_\ell} \mathrm{End}_{\mathrm{Gal}_K}(\overline{V}_\ell^{\mathrm{ss}}), \tag{$*$}$$

where $\overline{V}_\ell^{\mathrm{ss}}$ is the semisimplified reduction of V_ℓ.

Not only is $\Gamma_\ell \subset \mathbf{G}_\ell(\mathbb{Q}_\ell)$ conjectured to be large for $\ell \gg 0$, but the (general) Mumford-Tate conjecture [Se94, §9] asserts the ℓ-independence of algebraic monodromy group $\mathbf{G}_\ell$ in the sense that if $\mathbf{G}_{MT}$ is the Mumford-Tate group of $H^w(X(\mathbb{C}), \mathbb{Q})$, then for all ℓ we have

$$\mathbf{G}_\ell^{\circ} \cong \mathbf{G}_{MT} \times_{\mathbb{Q}} \mathbb{Q}_\ell.$$

For semisimple Serre compatible systems, we obtained the following λ-independence result (Theorem 2.9) on the algebraic monodromy groups. First we give a definition.

Definition 2.8. *Let F be a field and $\mathbf{H} \subset \mathrm{GL}_{n,F}$ a reductive subgroup.*

(a) The formal character of $\mathbf{H}$ *is a subtorus* $\mathbf{T}$ *in* $\mathrm{GL}_{n,\overline{F}}$ *up to conjugation such that* $\mathbf{T}$ *is a maximal torus of* $\mathbf{H}\times\overline{F}$.

(b) The formal bi-character of $\mathbf{H}$ *is a chain of subtori* $\mathbf{T}'\subset\mathbf{T}$ *in* $\mathrm{GL}_{n,\overline{F}}$ *up to conjugation such that* $\mathbf{T}$ *is a maximal torus of* $\mathbf{H}\times\overline{F}$ *and* $\mathbf{T}'$ *is a maximal torus of* $\mathbf{H}^{\mathrm{der}}\times\overline{F}$, *where* $\mathbf{H}^{\mathrm{der}}$ *denotes the derived group of the identity component* $\mathbf{H}^{\circ}$.

Theorem 2.9. [Hu13, Theorem 3.19, Remark 3.22] *Let* $\{\phi_\lambda\}_\lambda$ *be a semisimple* n*-dimensional Serre compatible system of* K *(defined over* E*). Then the formal bi-character of the algebraic monodromy group* $\mathbf{H}_\lambda\subset\mathrm{GL}_{n,E_\lambda}$ *is independent of* λ[4]. *In particular, the rank (resp. semisimple rank) of* $\mathbf{H}_\lambda$ *is independent of* λ.

Moreover, we extended in [Hu15] this λ-independence result to the mod ℓ motivic compatible system $\{\bar\rho_\ell^{\mathrm{ss}}:\mathrm{Gal}_K\to\mathrm{GL}(\overline{V}_\ell^{\mathrm{ss}})\}_\ell$. More precisely, for $\ell\gg0$ we constructed a connected reductive subgroup $\underline{G}_\ell\subset\mathrm{GL}_{n,\mathbb{F}_\ell}$ called the *algebraic envelope*[5] of ρ_ℓ such that $\underline{G}_\ell(\mathbb{F}_\ell)$ is uniformly commensurate with $\bar\Gamma_\ell$ (the image of $\bar\rho_\ell^{\mathrm{ss}}$) for all $\ell\gg0$ and the formal bi-character of $\mathbf{G}_\ell$ and $\underline{G}_\ell$ coincide for all $\ell\gg0$ [Hu15, Theorem A]. The works [HL16],[HL20], and [Hu22] are all based on [Hu15]. Inspired by some group theoretic techniques developed in [HL20], we constructed *algebraic envelopes of subrepresentations* (see [Hu22, §3]) of Serre compatible systems satisfying certain local conditions (originated from motivic compatible system (3)) to prove the following big images results. Denote by $\bar\epsilon_\ell$ the mod ℓ cyclotomic character.

Theorem 2.10. [Hu22, Theorems 3.12(i),(ii),(iii),(v), and Theorem 1.2] *Let* $\{\rho_\lambda\}_\lambda$ *be a semisimple Serre compatible system of* K *defined over* E. *Suppose there exist some integers* $N_1,N_2\geq0$ *and finite extension* K'/K *such that the following conditions hold.*

(a) (Bounded tame inertia weights): for almost all λ *and each finite place* v *of* K *above* ℓ, *the tame inertia weights of the local representation* $(\bar\rho_\lambda^{\mathrm{ss}}\otimes\bar\epsilon_\ell^{N_1})|_{\mathrm{Gal}_{K_v}}$ *belong to* $[0,N_2]$.

(b) (Potential semistability): for almost all λ *and each finite place* w *of* K' *not above* ℓ, *the semisimplification of the local representation* $\bar\rho_\lambda^{\mathrm{ss}}|_{\mathrm{Gal}_{K'_w}}$ *is unramified.*

Then there exists a finite Galois extension L/K *such that for almost all* λ *and for each subrepresentation* $\sigma_\lambda:\mathrm{Gal}_K\to\mathrm{GL}(W_\lambda)$ *of* $\rho_\lambda\otimes\overline{\mathbb{Q}}_\ell$, *the following assertions hold.*

(i) There is a connected reductive subgroup $\underline{G}_{W_\lambda}\subset\mathrm{GL}_{\overline{W}_\lambda^{\mathrm{ss}}}$ *(called the algebraic envelope of* σ_λ*) that is semisimple on* $\overline{W}_\lambda^{\mathrm{ss}}$ *(semisimplified reduction of* W_λ*) and contains* $\bar\sigma_\lambda^{\mathrm{ss}}(\mathrm{Gal}_L)$.

(ii) The commutants of $\bar\sigma_\lambda^{\mathrm{ss}}(\mathrm{Gal}_L)$ *and* $\underline{G}_{W_\lambda}$ *(resp.* $[\bar\sigma_\lambda^{\mathrm{ss}}(\mathrm{Gal}_L),\bar\sigma_\lambda^{\mathrm{ss}}(\mathrm{Gal}_L)]$ *and* $\underline{G}_{W_\lambda}^{\mathrm{der}}$*) in* $\mathrm{End}(\overline{W}_\lambda^{\mathrm{ss}})$ *are equal. In particular,* $\bar\sigma_\lambda^{\mathrm{ss}}(\mathrm{Gal}_L)$ *(resp.* $[\bar\sigma_\lambda^{\mathrm{ss}}(\mathrm{Gal}_L),\bar\sigma_\lambda^{\mathrm{ss}}(\mathrm{Gal}_L)]$*) is irreducible on* $\overline{W}_\lambda^{\mathrm{ss}}$ *if and only if* $\underline{G}_{W_\lambda}$ *(resp.* $\underline{G}_{W_\lambda}^{\mathrm{der}}$*) is irreducible on* $\overline{W}_\lambda^{\mathrm{ss}}$.

(iii) The algebraic envelope $\underline{G}_{W_\lambda}$ *and algebraic monodromy* $\mathbf{G}_{W_\lambda}$ *of* σ_λ *have the same formal bi-character. Moreover, this formal bi-character has finitely many possibilities depending on* $\{\rho_\lambda\}_\lambda$.

[4]It means that the formal bi-characters have the same $\mathbb{Z}$-form in $\mathrm{GL}_{n,\mathbb{Z}}$.

[5]Serre constructed algebraic envelopes to study Galois actions on ℓ-torsions of abelian varieties for $\ell\gg0$ [Se86].

(iv) If σ_λ is Lie-irreducible[6] *and of type A (see §1.2), then $\underline{G}_{W_\lambda}$ and $\mathrm{Gal}_{K^{ab}}$ are irreducible on $\overline{W}_\lambda^{\mathrm{ss}}$, where K^{ab}/K is the maximal abelian extension.*
(v) If σ_λ is irreducible and of type A, then it is residually irreducible.

2.4. **Potential automorphy of low dimensional subrepresentations.** The following results will be useful in next section.

Proposition 2.11. *Let $\{\rho_\lambda : \mathrm{Gal}_K \to \mathrm{GL}_n(E_\lambda)\}_\lambda$ be a strictly (and also Serre) compatible system of K defined over E. Then it satisfies the conditions (a) and (b) of Theorem 2.10.*

Proof. Verification of conditions (a) and (b) are the same as [Hu22, Theorem 4.1] using the strict compatibility conditions. □

Proposition 2.12. *Let $\{\rho_\lambda : \mathrm{Gal}_F \to \mathrm{GL}_n(E_\lambda)\}_\lambda$ be a strictly (and also Serre) compatible system of a totally real field F defined over E. For almost all λ, if $\sigma_\lambda : \mathrm{Gal}_F \to \mathrm{GL}(W_\lambda)$ is an m-dimensional subrepresentation of $\rho_\lambda \otimes \overline{E}_\lambda$ such that the τ-Hodge-Tate numbers of σ_λ are distinct and the algebraic monodromy group $\mathbf{G}_{W_\lambda}$ is one of the following cases:*
(a) $m = 2$ and $\mathbf{G}_{W_\lambda}^{\mathrm{der}} = \mathrm{SL}_2$;
(b) $m = 3$ and $\mathbf{G}_{W_\lambda}^{\mathrm{der}} = \mathrm{SO}_3$,
then there is a totally real extension F'/F such that $\sigma_\lambda|_{\mathrm{Gal}_{F'}}$ is automorphic (i.e., attached to a regular algebraic, (polarized) cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A}_{F'})$). Moreover, σ_λ is part of a strictly (pure) compatible system of F.

Proof. It suffices to apply Theorem 2.4 on σ_λ for almost all λ. By the facts that $\{\rho_\lambda\}_\lambda$ is strictly compatible and σ_λ has distinct τ-Hodge-Tate numbers, conditions 2.4(1),(3) (resp. conditions 2.5(1),(3),(4)) hold for σ_λ (resp. when $m = 2$) for almost all λ (see §2.2). Since $\mathbf{G}_{W_\lambda}^{\mathrm{der}}$ (of type A) is irreducible on W_λ, Proposition 2.11 and Theorem 2.10(iv) imply that condition 2.4(4) (resp. 2.5(2)) holds for σ_λ (resp. when $m = 2$) for almost all λ.

Therefore, it remains to check condition 2.4(2) (odd essentially self-dual) in each case for almost all λ, i.e., the image of σ_λ is contained in some GSp_m (resp. GO_m) with totally odd (resp. totally even) similitude character. Case (a) follows from Proposition 2.5. Case (b) follows from the facts that the normalizer of SO_3 in GL_3 is GO_3 and the similitude character is totally even [CG13, Lemma 2.1]. □

3. Proofs of the results

By Proposition 2.3, we may assume $\{\rho_\lambda\}_\lambda$ is also a Serre compatible system defined over E, i.e., the image of ρ_λ is contained in $\mathrm{GL}_n(E_\lambda)$.

3.1. **Proof of Theorem 1.1.** It is trivial when $n = 1$. We consider $n = 2, 3, 4$ separately.

(i) and (ii). When $n = 2$, there are two possibilities for the irreducible ρ_{λ_0}. If ρ_{λ_0} is Lie-irreducible, then $\mathbf{G}_{\lambda_0}^{\mathrm{der}} = \mathrm{SL}_2$. If ρ_{λ_0} is not Lie-irreducible, then the distinct τ-Hodge-Tate numbers and Clifford's theorem [Cl37] imply that ρ_{λ_0} is induced from a character, i.e., $\rho_{\lambda_0} = \mathrm{Ind}_K^F \chi_{\lambda_0}$ where $[K : F] = 2$. In the first case, Theorem 2.9 implies that $\mathbf{G}_\lambda^{\mathrm{der}} = \mathrm{SL}_2$

[6] It means $\sigma_\lambda|_{\mathrm{Gal}_M}$ is irreducible for all finite extensions M/K, or equivalently, the identity component $\mathbf{G}_{W_\lambda}^\circ$ is irreducible on W_λ.

and ρ_λ is irreducible for all λ. In the second case, since ρ_{λ_0} and thus χ_{λ_0} are Hodge-Tate at all places above ℓ (condition 2.1(ii)), χ_{λ_0} and thus $\rho_{\lambda_0} = \mathrm{Ind}_K^F \chi_{\lambda_0}$ can be extended to Serre compatible systems $\{\chi_\lambda\}_\lambda$ and $\{\mathrm{Ind}_K^F \chi_\lambda\}_\lambda$ (see e.g., [Se98, Chapter III]) after enlarging E. Since the τ-Hodge-Tate numbers are distinct for all λ (condition 2.1(iii)), $\rho_\lambda = \mathrm{Ind}_K^F \chi_\lambda$ is irreducible for all λ by Mackey's irreducibility criterion [Se77, §7.4].

When $n = 3$, ρ_{λ_0} is induced from a character if it is not Lie-irreducible and the treatment is identical to $n = 2$ by using the regularity condition and Mackey's irreducibility criterion. Hence, it suffices to consider Lie-irreducible ρ_{λ_0}. There are two cases, either $\mathbf{G}_{\lambda_0}^{\mathrm{der}} = \mathrm{SL}_3$ (of rank 2) or $\mathbf{G}_{\lambda_0}^{\mathrm{der}} = \mathrm{SO}_3$ (of rank 1). In the first case, we have $\mathbf{G}_\lambda^{\mathrm{der}} = \mathrm{SL}_3$ (the maximal connected semisimple subgroup of GL_3) for all λ because the rank of $\mathbf{G}_\lambda^{\mathrm{der}}$ is independent of λ by Theorem 2.9. In the second case, suppose ρ_λ is reducible for some λ. Since the formal character of $\mathbf{G}_\lambda^{\mathrm{der}}$ is equal to those of $\mathbf{G}_{\lambda_0}^{\mathrm{der}} = \mathrm{SO}_3$ (Theorem 2.9) which is

$$\begin{pmatrix} x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/x \end{pmatrix}, \tag{5}$$

we obtain an irreducible decomposition

$$\rho_\lambda = W_\lambda \oplus W'_\lambda \tag{6}$$

such that $(\dim W_\lambda, \dim W'_\lambda) = (2, 1)$ and $\mathbf{G}_{W_\lambda}^{\mathrm{der}} = \mathrm{SL}_2$ if $\mathbf{G}_{W_\lambda}$ denotes the algebraic monodromy of W_λ. If there are infinitely many λ such that ρ_λ is reducible, then Proposition 2.12 (case (a)) implies that W_λ is part of a two dimensional strictly compatible system for some λ. This, together with the fact that the character W'_λ is part of a compatible system, contradict that ρ_{λ_0} is irreducible. It follows that ρ_λ is irreducible and $\mathbf{G}_\lambda^{\mathrm{der}} = \mathrm{SO}_3$ for almost all λ. Hence, Proposition 2.12 (case (b)) implies that for some λ and some totally real extension F'/F, $\rho_\lambda|_{\mathrm{Gal}_{F'}}$ is attached to a regular algebraic, polarized, cuspidal automorphic representation of $\mathrm{GL}_3(\mathbb{A}_{F'})$. It follows that $\rho_\lambda(\mathrm{Gal}_{F'}) \subset \mathrm{GO}_3$ for all λ [BLGGT14, §2]. We conclude that for all λ, $\mathbf{G}_\lambda^{\mathrm{der}} = \mathrm{SO}_3$ (Theorem 2.9) and thus ρ_λ is irreducible.

Finally, the λ-independence of $\mathbf{G}_\lambda^{\mathrm{der}}$ established above together with Theorem 2.9 imply that the identity component $\mathbf{G}_\lambda^\circ$ is independent of λ. □

(iii). Since $n \leq 3$, the representation ρ_λ is of type A for every λ. The assertion follows directly from Proposition 2.11, Theorem 1.1(i), and Theorem 2.10(v) since $\{\rho_\lambda\}_\lambda$ is a strictly (and Serre) compatible system. □

(iv). When $n = 4$, if ρ_{λ_0} is not Lie-irreducible then again it must be induced by regularity (see [Pat19, Proposition 3.4.1]). Then if it is induced from a character, the treatment is identical to the above. Otherwise, $\rho_{\lambda_0} = \mathrm{Ind}_K^F \phi_{\lambda_0}$ where $[K : F] = 2$ and ϕ_{λ_0} is two dimensional (not induced from a character) and moreover if $\mathbf{G}_{\phi_{\lambda_0}}$ is the algebraic monodromy of ϕ_{λ_0}, the distinct τ-Hodge-Tate numbers condition forces the formal character of $\mathbf{G}_{\phi_{\lambda_0}}^{\mathrm{der}}$ to be

$$\begin{pmatrix} x & 0 \\ 0 & 1/x \end{pmatrix}. \tag{7}$$

If ρ_λ is reducible for some λ, Theorem 2.9 and (7) imply that the irreducible decomposition of such ρ_λ is

$$\rho_\lambda = W_\lambda \oplus W'_\lambda \tag{8}$$

such that $(\dim W_\lambda, \dim W'_\lambda) = (2,2)$ and $\mathbf{G}^{\mathrm{der}}_{W_\lambda} = \mathrm{SL}_2 = \mathbf{G}^{\mathrm{der}}_{W'_\lambda}$ if $\mathbf{G}_{W_\lambda}$ (resp. $\mathbf{G}_{W'_\lambda}$) denotes the algebraic monodromy of W_λ (resp. W'_λ). By twisting $\{\rho_\lambda\}_\lambda$ with a big power of the system $\{\epsilon_\ell\}_\ell$ of ℓ-adic cyclotomic characters, we may assume $\mathbf{G}_{W_\lambda} = \mathrm{GL}_2$. As $\rho_{\lambda_0} = \mathrm{Ind}^F_K \phi_{\lambda_0}$, we obtain $\rho_{\lambda_0} \cong \rho_{\lambda_0} \otimes \eta$ where $\eta : \mathrm{Gal}_F \to \mathrm{Gal}(K/F) \to \{\pm 1\}$ is the non-trivial character. By compatibility, we also obtain

$$\rho_\lambda \cong \rho_\lambda \otimes \eta$$

for all λ. Since $\mathbf{G}_{W_\lambda} = \mathrm{GL}_2$, the representations W_λ and $W_\lambda \otimes \eta$ are not isomorphic and we deduce that $W_\lambda \cong W'_\lambda \otimes \eta$. It follows that $W_\lambda|_{\mathrm{Gal}_K} \cong W'_\lambda|_{\mathrm{Gal}_K}$ but this contradicts the distinct τ-Hodge-Tate numbers condition. We conclude that ρ_λ is irreducible for all λ.

It remains to consider $n = 4$ and that ρ_{λ_0} is Lie-irreducible. We list the four cases:

Case	$(\mathbf{G}^{\mathrm{der}}_{\lambda_0}, \rho_{\lambda_0})$	Rank	Formal character
(a)	$(\mathrm{SL}_2, \mathrm{Sym}^3(\mathrm{std}))$	1	$\begin{pmatrix} x^2 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 1/x & 0 \\ 0 & 0 & 0 & 1/x^2 \end{pmatrix}$
(b)	$(\mathrm{SO}_4, \mathrm{std})$	2	$\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1/x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 1/y \end{pmatrix}$
(c)	$(\mathrm{Sp}_4, \mathrm{std})$	2	$\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & 1/x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & 1/y \end{pmatrix}$
(d)	$(\mathrm{SL}_4, \mathrm{std})$	3	$\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & z & 0 \\ 0 & 0 & 0 & 1/(xyz) \end{pmatrix}$

where "std" stands for standard representation. For cases (a) and (d), the irreducibility of ρ_λ for all λ follows easily from the formal character of $\mathbf{G}^{\mathrm{der}}_{\lambda_0}$ and Theorem 2.9. For cases (b) and (c), the formal characters of $\mathbf{G}^{\mathrm{der}}_{\lambda_0}$ are identical (given on the list). If ρ_λ is reducible for some λ, Theorem 2.9 and the formal character imply that the only possible irreducible decomposition of ρ_λ is

$$\rho_\lambda = W_\lambda \oplus W'_\lambda \tag{9}$$

such that $(\dim W_\lambda, \dim W'_\lambda) = (2,2)$ and $\mathbf{G}^{\mathrm{der}}_{W_\lambda} = \mathrm{SL}_2 = \mathbf{G}^{\mathrm{der}}_{W'_\lambda}$ if $\mathbf{G}_{W_\lambda}$ (resp. $\mathbf{G}_{W'_\lambda}$) denotes the algebraic monodromy of W_λ (resp. W'_λ). If there are infinitely many λ such that ρ_λ is reducible, then Proposition 2.12 (case (a)) implies that W_λ (resp. W'_λ) is part of a two dimensional strictly compatible system for some λ. This contradicts that ρ_{λ_0} is irreducible. It follows that ρ_λ is irreducible for almost all λ. ☐

3.2. **Proof of Theorem 1.2.** By twisting $\{\rho_\lambda\}_\lambda$ with a big power of the system of ℓ-adic cyclotomic characters, we assume $\mathbf{G}_\lambda$ is connected for all λ [Se81] and $\mathbf{G}_{\lambda_0} = \mathrm{GSp}_4$.

(i). Suppose ρ_{λ_0} is fully symplectic for some λ_0. Theorem 1.1(iv) and the information on formal character (Theorem 2.9) imply that for almost all λ, the algebraic monodromy $\mathbf{G}_\lambda$ is either GSp_4 or GO_4°. In the GO_4°-case, the representation $\rho_\lambda \otimes \rho_\lambda^\vee$ has a three dimensional irreducible factor W_λ with algebraic monodromy SO_3 because of the short exact sequence

$$1 \to \mathrm{GL}_1 \to \mathrm{GL}_2 \times \mathrm{GL}_2 \to \mathrm{GO}_4^\circ \to 1 \tag{10}$$

given by the exterior tensor of the standard representation of GL_2 with itself. The factor W_λ has distinct Hodge-Tate numbers. Indeed, if ℓ is the residue characteristic of λ, [Pat19, Corollary 3.2.12] asserts that there is a Hodge-Tate lift of $\rho_\lambda : \mathrm{Gal}_{\mathbb{Q}_\ell} \to \mathrm{GO}_4^\circ(\overline{E}_\lambda)$ to

$$f_\lambda \oplus f'_\lambda : \mathrm{Gal}_{\mathbb{Q}_\ell} \to \mathrm{GL}_2(\overline{E}_\lambda) \times \mathrm{GL}_2(\overline{E}_\lambda).$$

Since $\rho_\lambda|_{\mathrm{Gal}_{\mathbb{Q}_\ell}} = f_\lambda \otimes f'_\lambda$ has distinct Hodge-Tate numbers, both the two dimensional f_λ and f'_λ have distinct Hodge-Tate weights. Hence, $W_\lambda|_{\mathrm{Gal}_{\mathbb{Q}_\ell}} = \mathrm{ad}^0 f_\lambda$ (or $\mathrm{ad}^0 f'_\lambda$) also has distinct Hodge-Tate weights.

If there is an infinite set $\mathcal{L}$ of λ such that $\mathbf{G}_\lambda = \mathrm{GO}_4^\circ$, then Proposition 2.12 (case (b) applied to the system $\{\rho_\lambda \otimes \rho_\lambda^\vee\}_\lambda$) asserts that for some $\lambda_1 \in \mathcal{L}$, the three dimensional W_{λ_1} is part of a strictly compatible system $\{\phi_\lambda\}_\lambda$. The algebraic monodromy of ϕ_λ is equal to SO_3 for all λ by Theorem 1.1(ii). Consider the compatible system

$$\{\phi_\lambda \oplus (\rho_\lambda \otimes \rho_\lambda^\vee)\}_\lambda$$

and let $\mathbf{H}_\lambda$ be the algebraic monodromy at λ. By construction, the semisimple ranks of both $\mathbf{H}_{\lambda_1}$ and $\mathbf{G}_{\lambda_1}$ are equal to 2. By $\mathbf{G}_{\lambda_0} = \mathrm{GSp}_4$ and Goursat's lemma, the semisimple rank of $\mathbf{H}_{\lambda_0}$ is equal to 3 which is greater than 2, the semisimple rank of $\mathbf{H}_{\lambda_1}$. This contradicts Theorem 2.9. □

(ii). Theorem 1.2(i) implies that there is a Hodge-Tate character μ_λ such that

$$\rho_\lambda \cong \rho_\lambda^\vee \otimes \mu_\lambda \tag{11}$$

for almost all λ. When $\mathbf{G}_\lambda = \mathrm{GSp}_4$, the trace of $\rho_\lambda(Frob_p) \neq 0$ for a Dirichlet density one set of rational primes p. Hence by (11) and the compatibility (2.1(a)) of $\{\rho_\lambda\}_\lambda$ (resp. $\{\rho_\lambda\}_\lambda^\vee$), for almost all λ the similitude characters μ_λ and μ_{λ_0} are compatible, i.e., for almost all rational primes p both characters are unramified and $\mu_\lambda(Frob_p) = \mu_{\lambda_0}(Frob_p) \in E$. Since μ_{λ_0} is odd, μ_λ is also odd for almost all λ. We break down the proof into three steps:

(ii-1) prove that ρ_λ is residually irreducible for almost all λ;
(ii-2) prove that $\{\rho_\lambda\}_\lambda$ is potentially automorphic;
(ii-3) prove that $\bar{\rho}_\lambda(\mathrm{Gal}_\mathbb{Q})$ has a subgroup conjugate to $\mathrm{Sp}_4(\mathbb{F}_\ell)$ for almost all λ.

(ii-1). By Proposition 2.11, there exist algebraic envelopes $\underline{G}_\lambda$ (Theorem 2.10(i)) of ρ_λ for almost all λ. Then Theorem 2.10(iii) and the construction of $\underline{G}_\lambda$ ([Hu22, Proposition 3.9(iii)]) imply that $\underline{G}_\lambda$ is either GSp_4 or the first group in

$$\mathbb{G}_m(\mathrm{SL}_2 \times \mathrm{SL}_2) \subset \mathrm{GL}_2 \times \mathrm{GL}_2 \subset \mathrm{GL}_4 \quad \text{(reducible action)}, \tag{12}$$

where $\mathbb{G}_m$ denotes the set of scalars in GL_4. If ρ_λ is residually reducible then $\underline{G}_\lambda \neq \mathrm{GSp}_4$ by Theorem 2.10(ii). Thus, $\underline{G}_\lambda$ has to be $\mathbb{G}_m(\mathrm{SL}_2 \times \mathrm{SL}_2)$ in (12) and the semisimplified reduction

$\bar{\rho}_\lambda^{\rm ss}$ decomposes into a sum of two two-dimensional absolutely irreducible representations corresponding to (12),

$$\bar{\rho}_\lambda^{\rm ss} = \overline{W}_\lambda \oplus \overline{W'}_\lambda. \quad (13)$$

By $\bar{\rho}_\lambda^{\rm ss} \cong \bar{\rho}_\lambda^{\vee,\rm ss} \otimes \bar{\mu}_\lambda$ (semisimplified reduction of (11)) and Theorem 2.10(ii), we obtain

$$\overline{W}_\lambda \cong \overline{W}_\lambda^\vee \otimes \bar{\mu}_\lambda \quad \text{and} \quad \overline{W'}_\lambda \cong \overline{W'}_\lambda^\vee \otimes \bar{\mu}_\lambda. \quad (14)$$

Since $\overline{W}_\lambda$ is not induced, it follows that $\det\overline{W}_\lambda = \bar{\mu}_\lambda = \det\overline{W'}_\lambda$ is odd.

Let ϵ_ℓ (resp. $\bar{\epsilon}_\ell$) be the ℓ-adic (resp. mod ℓ) cyclotomic character. Suppose there are infinitely many λ such that $\bar{\rho}_\lambda^{\rm ss}$ is not absolutely irreducible (i.e., (13) holds). Then the two-dimensional Galois representation $\overline{W}_\lambda$ of $\mathbb{Q}$ is odd irreducible. We now follow the arguments in [Hu22, §4.5.1] (also [Die02a, Die02b]) using Serre's modularity conjecture [Se87]. By applying (strong form) Serre's modularity conjecture (proved in [KW09a, KW09b]), there exist an integer m and a cuspidal Hecke eigenform f such that if

$$\{\psi_{\lambda,f} : \mathrm{Gal}_\mathbb{Q} \to \mathrm{GL}_2(E_\lambda)\}_\lambda \quad (15)$$

is the Serre compatible system attached to f (by enlarging E if necessary), then the semisimplified reduction satisfies

$$\bar{\psi}_{\lambda,f}^{\rm ss} \cong \overline{W}_\lambda \otimes \bar{\epsilon}_\ell^m \quad (16)$$

for infinitely many λ[7]. Since $[\mathrm{Gal}_\mathbb{Q}, \mathrm{Gal}_\mathbb{Q}]$ is (absolutely) irreducible on $\overline{W}_\lambda$ by (12) and Theorem 2.10(ii), the algebraic monodromy group of $\psi_{\lambda,f}$ contains SL_2 (for all λ). Consider the 6-dimensional Serre compatible system (satisfying the conditions (a) and (b) of Theorem 2.10) given by direct sum:

$$\{U_\lambda := (\rho_\lambda \otimes \epsilon_\ell^m) \oplus \psi_{\lambda,f}\}_\lambda. \quad (17)$$

Let $\mathbf{M}_\lambda$ be the algebraic monodromy group (resp. $\underline{M}_\lambda$ be the algebraic envelope) of U_λ at λ (resp. for almost all λ). Theorem 2.10(iii) implies that the semisimple groups $\mathbf{M}_\lambda^{\rm der}$ and $\underline{M}_\lambda^{\rm der}$ have the same formal bi-character on respectively U_λ and

$$\overline{U}_\lambda^{\rm ss} := (\bar{\rho}_\lambda^{\rm ss} \otimes \bar{\epsilon}_\ell^m) \oplus \bar{\psi}_{\lambda,f}^{\rm ss} \quad (18)$$

for almost all λ. This is impossible since for infinitely many λ we have $\mathbf{M}_\lambda^{\rm der} = \mathrm{Sp}_4 \times \mathrm{SL}_2$ (by Goursat's lemma) and $\underline{M}_\lambda^{\rm der} \cong \underline{G}_\lambda^{\rm der} = \mathrm{SL}_2 \times \mathrm{SL}_2$ (as the second factor of (18) is a subrepresentation of the first factor) are of different ranks. We conclude that ρ_λ is residually irreducible for almost all λ.

(ii-2). Recall that the algebraic envelope $\underline{G}_\lambda$ is either GSp_4 or $\mathbb{G}_m(\mathrm{SL}_2 \times \mathrm{SL}_2)$ in (12) for almost all λ. If $\underline{G}_\lambda$ is $\mathbb{G}_m(\mathrm{SL}_2 \times \mathrm{SL}_2)$ in (12) for infinitely many λ, then except finitely many such λ the restriction $\bar{\rho}_\lambda|_{\mathrm{Gal}_L}$ is a sum of two non-isomorphic two dimensional irreducible representations for some finite Galois extension $L/\mathbb{Q}$ by Theorem 2.10(i) and (ii). Hence, it follows by [Cl37] that $\bar{\rho}_\lambda$ is induced from a two dimensional representation of a field Q_λ such

[7] Let f_λ be the modular form attached to $\overline{W}_\lambda$ by Serre's recipe (see [Da95, §2]). The point is that, the crystallinity (by 2.1(b)) and λ-independence of τ-Hodge-Tate numbers (by 2.1(c)) give control on the weight of f_λ and the strong local-global compatibility (by 2.1(a,d)) gives control on the level of f_λ. These data allow us to find an integer m and a modular form f attached to $\overline{W}_\lambda \otimes \epsilon_\ell^m$ for infinitely many λ.

that $[Q_\lambda:\mathbb{Q}]=2$ and $Q_\lambda\subset L$[8] for infinitely many λ. Hence, we can fix Q_λ equal to a degree two extension Q of $\mathbb{Q}$ for infinitely many λ. This implies the existence of a half Dirichlet density set of rational primes p such that $\rho_\lambda(Frob_p)$ has zero trace. Since this contradicts $\mathbf{G}_{\lambda_0}=\mathrm{GSp}_4$ by the compatibility condition 2.1(a), this case is impossible.

If $\underline{G}_\lambda$ is GSp_4 for almost all λ, then the commutants of $\mathrm{Gal}_{\mathbb{Q}(\zeta_\ell)}$ and Sp_4 in $\mathrm{End}(\overline{\mathbb{F}}_\ell)$ are equal for almost all λ by Theorem 2.10(ii). This implies that the condition 2.4(4) holds (and thus $\bar\rho_\lambda^{\mathrm{ss}}$ is absolutely irreducible) for almost all λ. Since the conditions 2.4(1)–(3) also hold for almost all ρ_λ by the strict compatibility of the system, Theorem 1.2(i), and the oddness of similitude character μ_λ, we obtain by Theorem 2.4 that the representation ρ_λ (and thus the system $\{\rho_\lambda\}_\lambda$) is potentially automorphic.

(ii-3). It remains to prove that $\bar\rho_\lambda(\mathrm{Gal}_{\mathbb{Q}})$ contains a group isomorphic to $\mathrm{Sp}_4(\mathbb{F}_\ell)$ for almost all λ. Since the algebraic envelope $\underline{G}_\lambda$ is GSp_4 for almost all λ and $\underline{G}_\lambda$ is the image in the λ-component of the algebraic envelope $\underline{G}_\ell\subset\mathrm{GL}_{4[E:\mathbb{Q}],\mathbb{F}_\ell}$ of

$$\rho_\ell:=\bigoplus_{\lambda|\ell}\rho_\lambda:\mathrm{Gal}_{\mathbb{Q}}\to\prod_{\lambda|\ell}\mathrm{GL}_4(E_\lambda)=(\mathrm{Res}_{E/\mathbb{Q}}\mathrm{GL}_4)(\mathbb{Q}_\ell)\subset\mathrm{GL}_{4[E:\mathbb{Q}]}(\mathbb{Q}_\ell)$$

given by the restriction of scalars [Hu22, §3.4], it follows that the universal cover $\underline{G}_\ell^{\mathrm{sc}}$ of the derived group of $\underline{G}_\ell$ satisfies[9]

$$\underline{G}_\ell^{\mathrm{sc}}\cong\prod_i\mathrm{Res}_{\mathbb{F}_{q_i}/\mathbb{F}_\ell}\mathrm{Sp}_{4,\mathbb{F}_{q_i}}\qquad(19)$$

for all sufficiently large ℓ, where q_i is a power of ℓ. Since for $\ell\gg0$ the index

$$[\underline{G}_\ell^{\mathrm{der}}(\mathbb{F}_\ell):\bar\rho_\ell^{\mathrm{ss}}(\mathrm{Gal}_{\mathbb{Q}})\cap\underline{G}_\ell^{\mathrm{der}}(\mathbb{F}_\ell)]\le C$$

for some $C>0$ independent of ℓ by [Hu22, Theorem 2.11(ii)] and $\underline{G}_\ell^{\mathrm{sc}}(\mathbb{F}_\ell)$ is generated by elements of order ℓ [St68, Theorem 12.4], it follows that $\bar\rho_\ell^{\mathrm{ss}}(\mathrm{Gal}_{\mathbb{Q}})$ contains the image of $\underline{G}_\ell^{\mathrm{sc}}(\mathbb{F}_\ell)$ in $\underline{G}_\ell^{\mathrm{der}}(\mathbb{F}_\ell)$ for $\ell\gg0$. Therefore, (19) and the fact that the representation

$$\prod_j\mathrm{Sp}_{4,\overline{\mathbb{F}}_\ell}\cong\underline{G}_\ell^{\mathrm{sc}}\times\overline{\mathbb{F}}_\ell\to\underline{G}_\ell^{\mathrm{der}}\times\overline{\mathbb{F}}_\ell\twoheadrightarrow\underline{G}_\lambda^{\mathrm{der}}=\mathrm{Sp}_{4,\overline{\mathbb{F}}_\ell}\subset\mathrm{GL}_{4,\overline{\mathbb{F}}_\ell}$$

is given by projection (as very automorphism of $\mathrm{Sp}_{4,\overline{\mathbb{F}}_\ell}$ is inner) to one of the Sp_4-factors imply that the image of the absolutely irreducible $\bar\rho_\lambda$ contains a subgroup conjugate to $\mathrm{Sp}_4(\mathbb{F}_\ell)$ for almost all λ. ☐

3.3. **Proof of Corollary 1.3.** The proof we give here is similar to [HL16, Theorem 8]. Suppose ℓ is sufficiently large. By Theorem 1.2(i), we assume the image Γ_ℓ (resp. $[\Gamma_\ell,\Gamma_\ell]$) is a subgroup of $\mathrm{GSp}_4(\mathbb{Q}_\ell)$ (resp. $\mathrm{Sp}_4(\mathbb{Q}_\ell)$). Let $\Delta_\ell\subset\mathrm{Sp}_4(\mathbb{Q}_\ell)$ be a maximal compact subgroup containing $[\Gamma_\ell,\Gamma_\ell]$. There is an affine smooth group scheme $\mathcal{H}_\ell/\mathbb{Z}_\ell$ such that the generic fiber $\mathcal{H}_\ell\times\mathbb{Q}_\ell=\mathrm{Sp}_{4,\mathbb{Q}_\ell}$ and $\mathcal{H}_\ell(\mathbb{Z}_\ell)=\Delta_\ell$ by Bruhat-Tits theory [Ti79, 3.4.3]. Since $\mathrm{Sp}_{4,\mathbb{Q}_\ell}$ is simply-connected, the special fiber $\mathcal{H}_\ell\times\mathbb{F}_\ell$ is connected [Ti79, 3.5.3]. By $[\Gamma_\ell,\Gamma_\ell]\subset\Delta_\ell$, the big image part of Theorem 1.2(ii), and the facts that $\mathrm{Sp}_4(\mathbb{F}_\ell)$ is perfect for $\ell>2$ and

[8] The field Q_λ corresponds to the index two subgroup $\bar\rho_\lambda(\mathrm{Gal}_{\mathbb{Q}})\cap(\mathrm{GL}_2\times\mathrm{GL}_2)$.

[9] The isomorphism (19) follows from [Kn67, §1.3] and the fact that $\mathrm{Sp}_{4,\mathbb{F}_q}$ has only one $\mathbb{F}_q$-form up to isomorphism.

the kernel of the reduction map $r_\ell : \mathcal{H}_\ell(\mathbb{Z}_\ell) \to \mathcal{H}_\ell(\mathbb{F}_\ell)$ is pro-ℓ, the composition series of the image $r_\ell([\Gamma_\ell, \Gamma_\ell])$ contains the finite simple group of Lie type

$$\mathrm{Sp}_4(\mathbb{F}_\ell)/\{\pm 1\}. \tag{20}$$

Since the *ℓ-dimension* of $\mathrm{Sp}_4(\mathbb{F}_\ell)/\{\pm 1\}$ is $\dim \mathrm{Sp}_{4,\mathbb{F}_\ell}$ (see [HL16, §2] for definition) which is equal to $\dim(\mathcal{H}_\ell \times \mathbb{F}_\ell)$ by the smoothness of $\mathcal{H}_\ell/\mathbb{Z}_\ell$, it follows from [HL16, Corollary 6(iv)] that $\mathcal{H}_\ell \times \mathbb{F}_\ell$ is semisimple. Hence, $\mathcal{H}_\ell$ is a semisimple group scheme over $\mathbb{Z}_\ell$. Moreover, the (simply-connected) special fiber is $\mathrm{Sp}_{4,\mathbb{F}_\ell}$ by footnote 9 and $\Delta_\ell = \mathcal{H}_\ell(\mathbb{Z}_\ell)$ is a hyperspecial maximal compact subgroup of $\mathrm{Sp}_4(\mathbb{Q}_\ell)$ (after [Co14, Remark 7.2.13]) isomorphic to $\mathrm{Sp}_4(\mathbb{Z}_\ell)$ [Ti79, 2.5]. Since (20) is a composition factor of $r_\ell([\Gamma_\ell, \Gamma_\ell])$ and $\mathrm{Sp}_4(\mathbb{F}_\ell)$ does not have an index two subgroup, we obtain

$$r_\ell([\Gamma_\ell, \Gamma_\ell]) = \mathrm{Sp}_4(\mathbb{F}_\ell) = \mathcal{H}_\ell(\mathbb{F}_\ell).$$

It follows from [Va03, Theorem 1.3] that $[\Gamma_\ell, \Gamma_\ell] = \mathcal{H}_\ell(\mathbb{Z}_\ell) = \Delta_\ell$ for $\ell \gg 0$. We are done because the compact subgroup $\Gamma_\ell^{\mathrm{sc}} \subset \mathrm{Sp}_4(\mathbb{Q}_\ell)$ contains the maximal compact subgroup $[\Gamma_\ell, \Gamma_\ell] = \Delta_\ell \cong \mathrm{Sp}_4(\mathbb{Z}_\ell)$. □

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