# ERROR ESTIMATE OF A QUASI-MONTE CARLO TIME-SPLITTING PSEUDOSPECTRAL METHOD FOR NONLINEAR SCHRÖDINGER EQUATION WITH RANDOM POTENTIALS 

ZHIZHANG WU, ZHIWEN ZHANG, AND XIAOFEI ZHAO


#### Abstract

In this paper, we consider the numerical solution of a nonlinear Schrödinger equation with spatial random potential. The randomly shifted quasi-Monte Carlo (QMC) lattice rule combined with the time-splitting pseudospectral discretization is applied and analyzed. The nonlinearity in the equation induces difficulties in estimating the regularity of the solution in random space. By the technique of weighted Sobolev space, we identify the possible weights and show the existence of QMC that converges optimally at the almost-linear rate without dependence on dimensions. The full error estimate of the scheme is established. We present numerical results to verify the accuracy and investigate the wave propagation.


Keywords: nonlinear Schrödinger (NLS) equation; random potential; quasi-Monte Carlo (QMC) method; time splitting; pseudospectral method; error estimate.

AMS Subject Classification: 65C30; 65D32; 65M15; 65M70; 82B44.

## 1. Introduction

In this paper, we consider the following one-space-dimensional nonlinear Schrödinger (NLS) equation with a spatial random potential on the torus:

$$
\left\{\begin{array}{l}
i \partial_{t} \psi=-\frac{1}{2} \partial_{x}^{2} \psi+V(\omega, x) \psi+\alpha|\psi|^{2} \psi, \quad x \in \mathbb{T}, \omega \in \Omega, t>0  \tag{1.1}\\
\psi(t=0, \omega, x)=\psi_{i n}(x), \quad x \in \mathbb{T}, \omega \in \Omega
\end{array}\right.
$$

Here $t \geq 0$ is the time variable, $x \in \mathbb{T}$ is the space variable with $\mathbb{T}$ the one-dimensional torus (periodic boundary condition in $x$ ), $\omega \in \Omega$ is the random sample with $\Omega$ the sample space, $\psi=\psi(t, \omega, x)$ is the complex-valued unknown with $\psi_{i n}$ the given initial data, $V(\omega, x)$ is a given real-valued random potential, and $\alpha \in \mathbb{R}$ is a given parameter describing the strength of the nonlinearity. We refer to the NLS equation (1.1) as the random NLS.

It is well known that in the one-space-dimensional case in the presence of a random potential and in the absence of nonlinearity (i.e., $\alpha=0$ ) with probability one all the states are exponentially localized [1, 2, 3, 17, 38]. With $\alpha=0$ the equation in (1.1) is a linear Schrödinger equation with random potentials, which belongs to the family of the Anderson models named after P. Anderson. In 1958, P. Anderson considered the discrete linear Schrödinger equations and discovered the localization of waves due to the disorder induced by the random potential [3]. Ever since then, the Anderson localization effect has been widely considered in many applications such as semiconductors and acoustic waves.

When $\alpha \neq 0$, the interplay between disorder and nonlinear effects leads to new interesting physics. In spite of the extensive research, many fundamental problems still remains open. A long-lasting question in mathematics and physics is whether a nonlinearity especially the defocusing nonlinearity, i.e., $\alpha>0$ in (1.1), could break the localization [19. Many analytical and numerical investigations have been done to address this question. The study of the long-time behavior of wave propagation so far still relies on numerical simulations, which are performed on the discrete NLS models [18, 44]. Our model problem (1.1) is the continuous version of the random NLS. It has been considered physically for modeling the Anderson localization of Bose-Einstein Condensates [40, 45] and for the study of nonlinear dispersive wave dynamics in a disordered medium [9, 22, 33]. For theoretical results, we refer readers to [13, 14 ] for the well-posedness results of (1.1) and to [15, 20, 27] for other related mathematical properties. Here we consider the one-space-dimensional case and the torus domain in (1.1) for simplicity.

In this work, we consider the parametrization of the random potential in the manner of the Karhunen-Loève expansion [25, 26, [37, 54]:

$$
\begin{equation*}
V(\omega, x)=v_{0}(x)+\sum_{j=1}^{\infty} \sqrt{\lambda_{j}} \xi_{j}(\omega) v_{j}(x), \quad x \in \mathbb{T}, \omega \in \Omega, \tag{1.2}
\end{equation*}
$$

where $v_{0}(x)$ is a deterministic function, $\left\{v_{j}(x)\right\}_{j \geq 1}$ are the physical components, $\lambda_{1}>\lambda_{2}>$ $\ldots>0$ are the corresponding strengths and $\boldsymbol{\xi}(\omega)=\left(\xi_{1}(\omega), \xi_{2}(\omega), \ldots\right)^{T}$ are independent and identically distributed (i.i.d.) uniform random variables on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. Thus,

$$
\boldsymbol{\xi} \in\left[-\frac{1}{2}, \frac{1}{2}\right]^{\mathbb{N}}=: U
$$

According to the Doob-Dynkin lemma [32], the solution $\psi$ can be represented by parametric functions parameterized by $\boldsymbol{\xi}$. Hence, (1.1) can be rewritten as the following parametric nonlinear Schrödinger equation with a random potential $V(\boldsymbol{\xi}, x)$ given by (1.2):

$$
\left\{\begin{array}{l}
i \partial_{t} \psi=-\frac{1}{2} \partial_{x}^{2} \psi+V(\boldsymbol{\xi}, x) \psi+\alpha|\psi|^{2} \psi, \quad x \in \mathbb{T}, \boldsymbol{\xi} \in U, t>0  \tag{1.3}\\
\psi(t=0, \boldsymbol{\xi}, x)=\psi_{i n}(x), \quad x \in \mathbb{T}, \boldsymbol{\xi} \in U
\end{array}\right.
$$

with $\psi=\psi(t, \boldsymbol{\xi}, x), \psi_{i n}$ being deterministic, and we shall consider that $\alpha \neq 0$.
Along the numerical aspect of (1.1) or (1.3), let us mention several related works. The work of Henning and Peterseim [29] considered the fully deterministic case of (1.1) and addressed the convergence issue of a Crank-Nicolson finite element discretization. Zhao later addressed the corresponding convergence of an exponential integrator spectral method in [55] and numerically investigated the stochastic case. The work of Kachman, Fishman and Soffer [33] proposed an iterative integrator in time for the linear model. However, to our best knowledge, the existing studies on (1.1) so far are yet to address the discretization in the random variable. The sampling method in the aforementioned works as well as in the physical works [18, 40, 44] all consider the classical Monte-Carlo method. It is known that the classical Monte-Carlo offers only a half-order convergence rate and so practical computing would require a large number of samples for accurately capturing the statistical quantities of
interest. To increase the efficiency and accuracy of quantifying the uncertainty, methods like the quasi-Monte Carlo sampling [8, 12, 53], polynomial chaos expansion [10, 21, 30, 54] and stochastic collocation method [6, 42, 52] have later been developed for PDEs with random inputs.

In this work, we shall consider the quasi-Monte Carlo (QMC) sampling method for the NLS (1.3). The origin of QMC dates back to around 1960, which proposes to sample deterministically by using the low-discrepancy sequences [28, 48]. The convergence order of QMC with respect to the number of samples is higher than that of the classical Monte Carlo method; however, it is dependent on the dimension of the random variable [8, 41]. To overcome this problem, some state-of-art randomization techniques of different kinds have been further introduced to the QMC. We refer the interested readers to [12, 36] and the references therein for a detailed review. In this work, we would follow the randomly shifted QMC lattice rule [11, 43, 47]. In such a way, Graham, Kuo, Schwab and Sloan et al. proposed the QMC method with finite element discretizations to solve linear elliptic equations [23, 25, 26, 37]. Via the technique of the weighted Hilbert space with the carefully chosen weights [25, 37], they established the almost-linear convergence rates in $N$ without dependence on the dimension.

By combining the randomly shifted QMC lattice rule with the popular time-splitting Fourier pseudospectral (TSFP) method [4, 7, 31], we propose an efficient numerical method to solve (1.3). Moreover, we aim to analyze for the full convergence result of the scheme. The difficulty/novelty of the analysis mainly come from the following facts: i) the nonlinearity in the equation (1.3) makes it complicated to estimate of the regularity of the solution in the parametric space; ii) the physical observable of interest is a nonlinear functional of the solution; iii) the full scheme involves the discretizations in time, physical space and parametric space. Under suitable assumptions on the decaying rate of the potential (1.2), we point out the possible choice of the weights in the framework of the weighted Sobolev space, and we show the existence of a randomly shifted QMC lattice rule which can achieve an almost-linear convergence rate without dependence on dimensions for the expectation of the physical observable. The optimal root-mean-square error estimate of the QMC-TSFP scheme in time, physical space, and random space will then be established and verified numerically. Finally, we numerically investigate the wave propagation in the random NLS models.

The rest of the paper is organized as follows. In Section 2, we present the detailed QMCTSFP scheme and our main result about its convergence theorem. Section 3 and Section 4 are devoted to rigorously establishing the theorem by analyzing the error on the PDE and on the scheme in a sequel. Numerical results are presented in Section 5. Concluding remarks are made in Section 6.

## 2. Numerical method

In this section, we shall first present the numerical discretization of the NLS (1.3). Then, we shall give the convergence result of the scheme.
2.1. Dimension truncation and discretization. First of all, for practical computation we need to truncate the Karhunen-Loève expansion (1.2) of the random potential $V(\boldsymbol{\xi}, x)$ into a finite sum. That is to say, we choose an integer $m>0$ large enough and truncate the parameterized random potential as:

$$
\begin{equation*}
V_{m}(\boldsymbol{\xi}, x)=v_{0}(x)+\sum_{j=1}^{m} \sqrt{\lambda_{j}} \xi_{j}(\omega) v_{j}(x), \quad x \in \mathbb{T}, \xi=\left(\xi_{1}, \ldots, \xi_{m}\right)^{T} \in U_{m}:=\left[-\frac{1}{2}, \frac{1}{2}\right]^{m} \tag{2.1}
\end{equation*}
$$

We emphasize that the approximation of the truncated random potential $V_{m}(\boldsymbol{\xi}, x)$ to $V(\boldsymbol{\xi}, x)$ depends on the decaying rates of the strengths. With the above finite dimensional random potential, we consider the following truncated NLS problem:

$$
\left\{\begin{array}{l}
i \partial_{t} \psi_{m}=-\frac{1}{2} \partial_{x}^{2} \psi_{m}+V_{m}(\xi, x) \psi_{m}+\alpha\left|\psi_{m}\right|^{2} \psi_{m}, \quad x \in \mathbb{T}, \xi \in U_{m}, t>0  \tag{2.2}\\
\psi_{m}(t=0, \xi, x)=\psi_{i n}(x), \quad x \in \mathbb{T}
\end{array}\right.
$$

which is an approximation to (1.3). Here we have $\psi_{m}(t, \xi, x)=\psi\left(t,\left(\boldsymbol{\xi}_{\{1: m\}}, \mathbf{0}\right), x\right)$ with $\xi=\boldsymbol{\xi}_{\{1: m\}}$.

With one precise sample for $\xi$, (2.2) reads as a cubic NLS with the given potential $V_{m}$, and so it can be numerically solved by any of the classical algorithms. Concerning the periodic boundary condition, we shall consider in this paper the time-splitting Fourier pseudospectral method, which is simple to begin with but undoubtedly one of the most popular numerical methods for NLS [7, 31, 49]. It begins by splitting (2.2) into two subflows $\Psi_{s}^{\mathrm{k}}$ and $\Psi_{s}^{\mathrm{p}}$ as

$$
\begin{align*}
& \Psi_{s}^{\mathrm{k}}: i \partial_{t} \psi_{m}=-\frac{1}{2} \partial_{x}^{2} \psi_{m}, \quad t \in(0, s] ;  \tag{2.3a}\\
& \Psi_{s}^{\mathrm{p}}: i \partial_{t} \psi_{m}=V_{m} \psi_{m}+\alpha\left|\psi_{m}\right|^{2} \psi_{m}, \quad t \in(0, s] . \tag{2.3b}
\end{align*}
$$

Note that $V_{m}$ and $\alpha$ are real-valued, so both of the above equations can be integrated exactly in time, and then the scheme composes as $\psi_{m}\left(t_{n+1}, \xi, x\right) \approx \Psi_{\frac{\tau}{2}}^{\mathrm{k}} \circ \Psi_{\tau}^{\mathrm{p}} \circ \Psi_{\frac{\tau}{2}}^{\mathrm{k}}\left(\psi_{m}\left(t_{n}, \xi, x\right)\right)$, where $\tau=t_{n+1}-t_{n}$ is the time step. Let us briefly present the detailed scheme below.

Denote $\mathbb{T}=[-L, L]$ with some $L>0$ and choose some even integer $M>0$. The physical space is discretized as $x_{k}=-L+k h$ with $h=2 L / M$ for $k=0,1, \ldots, M$. For the time axis, we take $\tau=\Delta t>0$ and denote $t_{n}=n \tau$ for $n=0,1, \ldots$. Denote the numerical solution as $\psi_{m, k}^{n}=\psi_{m, k}^{n}(\xi) \approx \psi_{m}\left(t_{n}, \xi, x_{k}\right)$ with $\psi_{m, k}^{0}=\psi_{i n}\left(x_{k}\right)$, the scheme of the time-splitting Fourier pseudospectral method (TSFP) for solving (2.2) reads for $n \geq 0$,

$$
\begin{align*}
& \psi_{m, k}^{n+1}=\sum_{l=-M / 2}^{M / 2-1} \widetilde{\left(\psi_{m, k}^{n+1}\right)_{l}} \mathrm{e}^{i u_{l}\left(x_{k}+L\right)}, \quad \widetilde{\left(\psi_{m, k}^{n+1}\right)_{l}}=\mathrm{e}^{-i u_{l}^{2} \tau / 4} \widetilde{\left(\psi_{m, k}^{* *}\right)_{l}}, \quad u_{l}=\frac{\pi l}{L},  \tag{2.4a}\\
& \psi_{m, k}^{* *}=\mathrm{e}^{-i\left[V_{m}\left(\xi, x_{k}\right)+\alpha\left|\psi_{m, k}\right|^{2}\right] \tau} \psi_{m, k}^{*}, \quad \widetilde{\left(\psi_{m, k}^{*}\right)_{l}}=\mathrm{e}^{-i u_{l}^{2} \tau / 4} \widetilde{\left(\psi_{m, k}^{n}\right)_{l}}, \quad 0 \leq k \leq M, \tag{2.4b}
\end{align*}
$$

with $\widetilde{(\phi)_{l}}=\frac{1}{M} \sum_{k=0}^{M-1} \phi_{k} \mathrm{e}^{-i u_{l}\left(x_{k}+L\right)}$ being the discrete Fourier coefficients of a discrete function $\phi_{k}$. With the discrete values, we can define a continuous version of the numerical solution
as the interpolation:

$$
\begin{equation*}
I_{M} \psi_{m}^{n}(x)=\sum_{l=-M / 2}^{M / 2-1} \widetilde{\left(\psi_{m, k}^{n}\right)_{l}} \mathrm{e}^{i u_{l}(x+L)}, \quad x \in \mathbb{T}, n \geq 0 \tag{2.5}
\end{equation*}
$$

The TSFP (2.4) is fully explicit in time and efficient with the computational cost $\mathcal{O}(M \log M)$ per time level under the help of fast Fourier transform. Moreover, it is time symmetric and preserves the discrete $l^{2}$-norm of the solution. These properties offer it unconditional stability and the reliable long-time performance [16]. Thus, it has been used in [19, 44] to simulate the nonlinear Anderson models. For simulating wave propagation, the torus domain setup for (1.3) or (2.2) is a valid truncation of the whole space problem provided that the domain size $L>0$ is taken large enough and the initially localized solution in $\mathbb{T}$ is away from the boundary within the time of computation.
2.2. Quasi-Monte Carlo sampling. With the solution $\psi$ of (1.3), the physical observables in applications are some functionals of $\psi$, i.e., $G(\psi(t, \boldsymbol{\xi}, \cdot)), t \geq 0, \boldsymbol{\xi} \in U$, for some functional $G$ in the energy space of the solution. Then, the expected value with respect to the random parameter is given by the following integral in infinite dimensions:

$$
\begin{equation*}
\mathbb{E}[G](t):=\lim _{m \rightarrow \infty} \int_{U_{m}} G\left(\psi_{m}(t, \xi, \cdot)\right) d \xi \tag{2.6}
\end{equation*}
$$

with the truncated solution $\psi_{m}$ of (2.2). Therefore, for some sufficiently large $m>0, \mathbb{E}[G]$ can be approximated by $\mathbb{E}[G](t) \approx \int_{U_{m}} G\left(\psi_{m}(t, \xi, \cdot)\right) d \xi$. Note that $|\psi|^{2}$ represents the probability density function in quantum physics. Many widely concerned physical observables are linear functionals of $|\psi|^{2}$. One typical example is the center of mass, which is frequently considered in the context of Anderson localization [19]. Therefore, in the following, we shall focus on the expected value of $G\left(|\psi|^{2}\right)$, where $G$ is a linear functional. In order to estimate the convergence of QMC for computing the expected value of $G\left(|\psi|^{2}\right)$, we shall analyze the parametric regularity of $G\left(|\psi|^{2}\right)$ or equivalently the parametric regularity of $|\psi|^{2}$ in Section 3.3, which is one of our main contributions in this work. Let us denote

$$
\begin{equation*}
\mathcal{G}(t, \boldsymbol{\xi}):=G\left(|\psi(t, \boldsymbol{\xi}, \cdot)|^{2}\right), \quad \mathcal{G}^{m}(t, \xi):=G\left(\left|\psi_{m}(t, \xi, \cdot)\right|^{2}\right), \tag{2.7}
\end{equation*}
$$

for short in the following, and so we look for $\mathbb{E}[\mathcal{G}](t)$.
To evaluate the expectation of a general function $F(\xi)$ under a fixed $m$, the strategy of a quasi-Monte Carlo (QMC) method is an equal weight quadrature rule, i.e.,

$$
\begin{equation*}
\mathbb{E}_{m}[F]:=\int_{U_{m}} F(\xi) d \xi \approx \frac{1}{N} \sum_{p=1}^{N} F\left(\xi^{(p)}\right)=: Q_{m, N}[F], \tag{2.8}
\end{equation*}
$$

where $\xi^{(p)} \in U_{m}$ is the quadrature point and $N>0$ is the total number of samples. Classical QMC is done by constructing some deterministic quadrature points, such as the Halton sequence [28] or the Sobol sequence [48]. However, a drawback is that the approximation error of (2.8) depends on the dimension $m$. When $m$ is large, the QMC might lose its efficiency over the standard Monte Carlo method.

To provide a practical error estimate and reduce the biased error, some randomization techniques have been developed to construct the quadrature points. Here, we adopt the so-called shifted rank-one lattice rule [12] to get $\xi^{(p)}$. That is

$$
\begin{equation*}
\xi^{(p)}=\operatorname{frac}\left(\frac{p z}{N}+\Delta\right)-\frac{1}{2}, \quad p=1, \ldots, N \tag{2.9}
\end{equation*}
$$

where $\operatorname{frac}(w)$ means to take the fractional part of each component of a vector $w, \Delta \in[0,1]^{m}$ is a uniformly distributed random shift and $z \in \mathbb{Z}^{m}$ is known as the generating vector. A specific generating vector $z$ can be constructed by the component-by-component construction ( CBC ) approach [12] to minimize the shifted-averaged worst-case error function, which will be detailed in our analysis later in Section 3.4. For every $\xi^{(p)}$, we solve the NLS (2.2) using the TSFP scheme (2.4). Then, we obtain $\left\{\psi_{m}^{n}\left(\xi^{(p)}\right)\right\}_{p=1}^{N}$ to compute the numerical expectation of some quantity of interest. The whole numerical scheme shall be referred to as QMC-TSFP in the rest of the paper. The practical computing of $\mathbb{E}[\mathcal{G}]$ via the randomly shifted QMC-TSFP method is then implemented as:
(1) Given the truncated dimension $m$ in (2.1), choose $M, \tau$ for discretizations in TSFP, and take $N$ as the number of samples for one random shift and $R$ as the number of random shifts;
(2) Construct the generating vector $z$ using the CBC approach;
(3) Generate i.i.d. random shifts $\Delta_{1}, \ldots, \Delta_{R}$ from the uniform distribution on $[0,1]^{m}$. For each $\Delta_{r}$ with $r=1, \ldots, R$, obtain the sample set $\left\{\xi^{(r, p)}=\operatorname{frac}\left(\frac{p z}{N}+\Delta_{r}\right)-\frac{1}{2}\right.$ : $p=1, \ldots, N\}$;
(4) For each $\xi^{(r, p)}$ with $r=1, \ldots, R$ and $p=1, \ldots, N$, solve (2.2) via (2.4) and obtain $I_{M} \psi_{m}^{n}\left(\xi^{r, p}, x\right)$ in (2.5);
(5) Compute $\left\{Q_{m, N}^{(r)}\left(G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right): r=1, \ldots, R\right\}$, where

$$
\begin{equation*}
Q_{m, N}^{(r)}\left(G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right)=\frac{1}{N} \sum_{p=1}^{N} G\left(\left|I_{M} \psi_{m}^{n}\left(\xi^{(r, p)}, \cdot\right)\right|^{2}\right) \tag{2.10}
\end{equation*}
$$

is the approximation of $\mathbb{E}_{m}\left[G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right]$ with one single shift $\Delta_{r}$ in the QMC rule;
(6) Take the average over all the random shifts to get

$$
\begin{equation*}
\bar{Q}_{m, N, R}\left(G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right)=\frac{1}{R} \sum_{r=1}^{R} Q_{m, N}^{(r)}\left(G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right) \tag{2.11}
\end{equation*}
$$

as the final approximation of $\mathbb{E}\left[G\left(\left|\psi\left(t_{n}, \cdot, \cdot\right)\right|^{2}\right)\right]$.
With a precise shift $\Delta_{r}$, the total computational cost of QMC-TSFP is $\mathcal{O}(N M \log M)$. The number of shifts $R$ is usually kept small in practice, e.g., 10-50 [12].
2.3. Main result. We shall present here the main convergence result. We first introduce the Bochner spaces we will consider for the potential $V$ and the wave functions $\psi$. For the potential $V$, we will consider the Bochner space $L^{\infty}\left(U ; H^{s}(\mathbb{T})\right)$ with the norm

$$
\|V\|_{L^{\infty}\left(U ; H^{s}(\mathbb{T})\right)}=\sup _{\boldsymbol{\xi} \in U}\|V(\boldsymbol{\xi})\|_{H^{s}(\mathbb{T})}
$$

For the wave function $\psi$, we will consider the Bochner space $L^{\infty}\left((0, T) \times U ; H^{s}(\mathbb{T})\right)$ with the norm

$$
\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}(\mathbb{T})\right)}=\sup _{t \in(0, T), \boldsymbol{\xi} \in U}\|\psi(t, \boldsymbol{\xi})\|_{H^{s}(\mathbb{T})} .
$$

Assume that the functional, the potential and the initial data of (1.3) satisfy the following conditions.

Assumption 2.1. Assume the linear functional $G \in\left(H^{1}(\mathbb{T})\right)^{\prime}$, which is the dual space of $H^{1}(\mathbb{T})$. For some fixed $s \geq 1$, assume that $\psi_{\text {in }} \in H^{s}(\mathbb{T}), V \in L^{\infty}\left(U ; H^{s}(\mathbb{T})\right)$ and $\left\|V_{m}\right\|_{L^{\infty}\left(U ; H^{s}\right)} \leq\|V\|_{L^{\infty}\left(U ; H^{s}\right)}$ for any $m>0$. Concerning the relevant physical context, assume further that $\psi_{i n}$ is localized in the torus domain $\mathbb{T}=[-L, L]$ and $L>0$ is large enough such that the wave is yet to reach the boundary within the time of interest.

In addition, to achieve the optimal convergence rate of QMC, we would further ask for the following technical assumption on $V$, which shows the property of the decaying rates of the terms in the series (2.1).

Assumption 2.2. Assume that the potential $V$ in the form of (1.2) satisfies:

$$
\begin{equation*}
\left\|V_{m}-V\right\|_{L^{\infty}\left(U ; H^{1}\right)} \leq C m^{-\chi}, \quad \sum_{j=1}^{\infty}\left[3^{j} \sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}}\right]^{1 / 2}<\infty \tag{2.12}
\end{equation*}
$$

for some constants $C, \chi>0$.

Then, we have the following error estimate result for the QMC-TSFP method.
Theorem 2.3. Assume that Assumption 2.1, Assumption 2.2 hold and $\psi_{i n} \in H^{s}(\mathbb{T})$ with $s \geq 5$. Then, a randomly shifted QMC lattice rule can be constructed to solve (2.2) till any fixed $T>0$, and there exist constants $\tau_{0}, h_{0}>0$ independent of $m$ so that with the numerical solution $I_{M} \psi_{m}^{n}$ from the QMC-TSFP scheme (2.4) for $\tau \leq \tau_{0}$ and $h \leq h_{0}$, the root-mean-square error satisfies: for any $\delta \in(0,1 / 2)$ and $N \leq 10^{30}$,

$$
\begin{equation*}
\sqrt{\mathbb{E}^{\Delta}\left[\left|\mathbb{E}[\mathcal{G}]\left(t_{n}\right)-Q_{m, N}\left[G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right]\right|^{2}\right]} \leq C\left(\tau^{2}+h^{s-1}+m^{-\chi}+N^{\delta-1}\right) \tag{2.13}
\end{equation*}
$$

for all $t_{n} \in[0, T]$ and some constant $C>0$ independent of $m, \tau, h, N$, where $\mathbb{E}^{\Delta}$ denotes the expectation with respect to the random shift.

The rest of the paper is devoted to rigorously establishing the above theorem. It will be done in a sequel by analyzing the QMC error on the PDE in Section 3 and the error of the TSFP scheme in Section 4 .

Remark 1. The condition $N \leq 10^{30}$ is purely a technical condition since we use in the proof of Theorem 2.3 the property that the Euler totient function $\varphi(N)$ satisfies $1 / \varphi(N) \leq 9 / N$ for $N \leq 10^{30}$ [25]; see also the proof of Lemma [3.7] In practice, the number of samples will seldom exceed $10^{30}$.

Remark 2. The Assumption 2.2 basically requires that the eigenpairs in (1.2) are exponentially decaying, which is stronger than the usual algebraic ones in the literature for linear model problems, e.g., [25, 37]. Such exponential decay assumption is a technical condition for the rigorous proof by means of analysis in the weighted Sobolev space. It is essentially raised by the nonlinearity. In practice, it could be satisfied if the random potential $V(\boldsymbol{\xi}, x)$ is a smooth and periodic (or Schwartz for the whole space case) function in terms of $x$ uniformly for all $\boldsymbol{\xi}$. Generally, the decaying speed of the eigenvalues depend on the regularity of the covarivance kernel. We refer interested readers to [46] for more details.

Remark 3. Under weaker assumptions than those in Theorem 2.3, weaker convergence rates can be established similarly. For instance, if we only have $s \geq 3$ then the temporal accuracy of TSFP will drop down to one (see [39] for the deterministic case). On the other hand, if the decaying rate in Assumption 2.2 is slower, then the sampling error from QMC might drop down to $1 / 2$ [37]. Here we omitted these cases for simplicity.

Remark 4. The physical context of Anderson localization in NLS is associated with many physical parameters, including the time interval, spatial domain size, strength of nonlinearity, and randomness of the potential. Our primary focus in Theorem [2.3 is on the convergence rates with respect to numerical parameters. The dependence of the error estimate on physical parameters is encapsulated in the error constant $C>0$ in (2.13). Determining the explicit and particularly optimal dependence of the constant $C$ on these physical parameters requires substantial additional efforts and will be a subject of our future research.

## 3. QMC on the PDE

In this section, we aim to analyze the convergence of the QMC sampling for the NLS model (1.3) or (2.2) on the continuous level. We shall largely follow the framework in [37].

### 3.1. Regularity in physical space.

Lemma 3.1. Under Assumption 2.1, for each $\boldsymbol{\xi} \in U$ the solution $\psi(t, \boldsymbol{\xi}, x)$ of (1.3) is globally well-posed and uniformly bounded in $H^{1}(\mathbb{T})$. Moreover, for any fixed $0<T<\infty$, we have $\psi \in L^{\infty}\left((0, T) \times U ; H^{s}(\mathbb{T})\right)$.

Proof. Following the framework from [24], let us briefly go through the proof to check mainly the effect from the random potential. For simplicity, we shall omit the spatial variable $x$ in the functions. By the Duhamel formula, (1.3) reads

$$
\begin{equation*}
\psi(t, \boldsymbol{\xi})=\mathrm{e}^{i t \partial_{x}^{2} / 2} \psi_{i n}-i \int_{0}^{t} \mathrm{e}^{i(t-\rho) \partial_{x}^{2} / 2}\left[V(\boldsymbol{\xi}) \psi(\rho, \boldsymbol{\xi})+\alpha|\psi(\rho, \boldsymbol{\xi})|^{2} \psi(\rho, \boldsymbol{\xi})\right] d \rho, \quad t \geq 0 \tag{3.1}
\end{equation*}
$$

Note that under condition $s \geq 1$, we have the algebraic property of the Sobolev space $H^{s}$ in the 1D [5, Chapter 4]. Therefore, by taking the $H^{s}$-norm on both sides of (3.1), we get

$$
\|\psi(t, \boldsymbol{\xi})\|_{H^{s}} \leq\left\|\psi_{i n}\right\|_{H^{s}}+C \int_{0}^{t}\left[\|V(\boldsymbol{\xi})\|_{H^{s}}\|\psi(\rho, \boldsymbol{\xi})\|_{H^{s}}+\|\psi(\rho, \boldsymbol{\xi})\|_{H^{s}}^{3}\right] d s
$$

for some constant $C>0$ from the Sobolev embedding which depends only on $s$. Then, a bootstrap-type argument [51] will give the local well-posedness of (1.3) in $H^{s}(\mathbb{T})$ for each
$\boldsymbol{\xi}$ (see Appendix A for details of the bootstrap-type argument). Moreover, there exists a $T_{0}>0$ such that $\psi \in L^{\infty}\left(\left(0, T_{0}\right) \times U ; H^{s}(\mathbb{T})\right)$ owning to the fact that $\psi_{\text {in }}$ is deterministic and the assumption $V \in L^{\infty}\left(U ; H^{s}(\mathbb{T})\right)$.

On the other hand, since $V$ is real-valued, the model (1.3) enjoys the mass conservation

$$
\begin{equation*}
M(t):=\int_{\mathbb{T}}|\psi(t, \boldsymbol{\xi}, x)|^{2} d x \equiv \int_{\mathbb{T}}\left|\psi_{i n}(x)\right|^{2} d x=M(0), \quad t \geq 0 \tag{3.2}
\end{equation*}
$$

where the mass is independent of $\boldsymbol{\xi}$. Also, we have the energy conservation:

$$
\begin{align*}
E(t, \boldsymbol{\xi}) & :=\int_{\mathbb{T}}\left[\frac{1}{2}\left|\partial_{x} \psi(t, \boldsymbol{\xi}, x)\right|^{2}+V(\boldsymbol{\xi}, x)|\psi(t, \boldsymbol{\xi}, x)|^{2}+\frac{\alpha}{2}|\psi(t, \boldsymbol{\xi}, x)|^{4}\right] d x \\
& \equiv \int_{\mathbb{T}}\left[\frac{1}{2}\left|\partial_{x} \psi_{i n}(x)\right|^{2}+V(\boldsymbol{\xi}, x)\left|\psi_{i n}(x)\right|^{2}+\frac{\alpha}{2}\left|\psi_{i n}(x)\right|^{4}\right] d x=E(0, \boldsymbol{\xi}), \quad t \geq 0, \quad \boldsymbol{\xi} \in U \tag{3.3}
\end{align*}
$$

Note by the Sobolev embedding that $H^{s}(\mathbb{T}) \subseteq L^{\infty}(\mathbb{T})$, then the condition $V \in L^{\infty}\left(U ; H^{s}(\mathbb{T})\right)$ guarantees a uniform upper bound of the initial energy for all the $\boldsymbol{\xi}$, i.e.,

$$
\begin{equation*}
|E(0, \boldsymbol{\xi})| \leq \int_{\mathbb{T}}\left[\frac{1}{2}\left|\partial_{x} \psi_{i n}(x)\right|^{2}+\|V\|_{L^{\infty}(U \times \mathbb{T})}\left|\psi_{i n}(x)\right|^{2}+\frac{|\alpha|}{2}\left|\psi_{i n}(x)\right|^{4}\right] d x=: E_{0}<\infty, \quad \boldsymbol{\xi} \in U . \tag{3.4}
\end{equation*}
$$

In the defocusing case, i.e., $\alpha>0$, it can be seen from the definition of $E(t, \boldsymbol{\xi})$ in (3.3) that

$$
\begin{align*}
E(t, \boldsymbol{\xi}) & \geq \int_{\mathbb{T}}\left[\frac{1}{2}\left|\partial_{x} \psi(t, \boldsymbol{\xi}, x)\right|^{2}+V(\boldsymbol{\xi}, x)|\psi(t, \boldsymbol{\xi}, x)|^{2}\right] d x \\
& \geq \frac{1}{2}\left\|\partial_{x} \psi(t, \boldsymbol{\xi})\right\|_{L^{2}}^{2}-\|V\|_{L^{\infty}(U \times \mathbb{T})}\|\psi(t, \boldsymbol{\xi})\|_{L^{2}}^{2}, \tag{3.5}
\end{align*}
$$

which together with the mass conservation (3.2), the energy conservation (3.3) and the bound of $E(0, \boldsymbol{\xi})$ in (3.4) shows that

$$
\begin{align*}
\left\|\partial_{x} \psi(t, \boldsymbol{\xi})\right\|_{L^{2}}^{2} & \leq 2|E(t, \boldsymbol{\xi})|+2\|V\|_{L^{\infty}(U \times \mathbb{T})}\|\psi(t, \boldsymbol{\xi})\|_{L^{2}}^{2} \\
& \leq 2 E_{0}+2\|V\|_{L^{\infty}(U \times \mathbb{T})}\left\|\psi_{i n}\right\|_{L^{2}}^{2}, \quad \forall t \geq 0, \boldsymbol{\xi} \in U . \tag{3.6}
\end{align*}
$$

In the focusing case, i.e., $\alpha<0$, we derive by the Gagliardo-Nirenberg inequality $\|\psi\|_{L^{4}}^{4} \leq$ $C\left\|\partial_{x} \psi\right\|_{L^{2}}\|\psi\|_{L^{2}}^{3}$ and the Young's inequality that

$$
\begin{equation*}
\|\psi(t, \boldsymbol{\xi})\|_{L^{4}}^{4} \leq C\left\|\partial_{x} \psi(t, \boldsymbol{\xi})\right\|_{L^{2}}\|\psi(t, \boldsymbol{\xi})\|_{L^{2}}^{3} \leq \frac{1}{-2 \alpha}\left\|\partial_{x} \psi(t, \boldsymbol{\xi})\right\|_{L^{2}}^{2}+\frac{-\alpha C^{2}}{2}\|\psi(t, \boldsymbol{\xi})\|_{L^{2}}^{6}, \tag{3.7}
\end{equation*}
$$

which together with the definition of $E(t, \boldsymbol{\xi})$ in (3.3) shows that

$$
\begin{equation*}
E(t, \boldsymbol{\xi}) \geq \frac{1}{4}\left\|\partial_{x} \psi(t, \boldsymbol{\xi})\right\|_{L^{2}}^{2}-\|V\|_{L^{\infty}(U \times \mathbb{T})}\|\psi(t, \boldsymbol{\xi})\|_{L^{2}}^{2}-\frac{\alpha^{2} C^{2}}{4}\|\psi(t, \boldsymbol{\xi})\|_{L^{2}}^{6} . \tag{3.8}
\end{equation*}
$$

Then the inequality (3.8) together with the mass conservation (3.2), the energy conservation (3.3) and the bound of $E(0, \boldsymbol{\xi})$ in (3.4) shows that

$$
\begin{align*}
\left\|\partial_{x} \psi(t, \boldsymbol{\xi})\right\|_{L^{2}}^{2} & \leq 4|E(t, \boldsymbol{\xi})|+4\|V\|_{L^{\infty}(U \times \mathbb{T})}\|\psi(t, \boldsymbol{\xi})\|_{L^{2}}^{2}+\alpha^{2} C^{2}\|\psi(t, \boldsymbol{\xi})\|_{L^{2}}^{6} \\
& \leq 4 E_{0}+4\|V\|_{L^{\infty}(U \times \mathbb{T})}\left\|\psi_{i n}\right\|_{L^{2}}^{2}+\alpha^{2} C^{2}\left\|\psi_{i n}\right\|_{L^{2}}^{6}, \quad \forall t \geq 0, \boldsymbol{\xi} \in U . \tag{3.9}
\end{align*}
$$

Thus, $\|\psi(t, \boldsymbol{\xi})\|_{H^{1}}$ for any $t \geq 0, \boldsymbol{\xi} \in U$ is always bounded from above by a uniform constant depending only on $\left\|\psi_{i n}\right\|_{L^{2}}^{2}, E_{0}$ and $\|V\|_{L^{\infty}(U \times \mathbb{T})}$. Such generic upper bound for $\|\psi(t, \boldsymbol{\xi})\|_{H^{1}}$ together with the local well-posedness leads to the global well-posedness of (1.3) in $H^{1}(\mathbb{T})$ for each $\boldsymbol{\xi}$, and $\psi \in L^{\infty}\left(\mathbb{R}^{+} \times U ; H^{1}(\mathbb{T})\right) \subseteq L^{\infty}\left(\mathbb{R}^{+} \times U \times \mathbb{T}\right)$. Then, (3.1) by the product lemma 51] (see Appendix A for details of the product lemma) gives

$$
\|\psi(t, \boldsymbol{\xi})\|_{H^{s}} \leq\left\|\psi_{i n}\right\|_{H^{s}}+C\left[\|V\|_{L^{\infty}\left(U ; H^{s}\right)}+\|\psi\|_{L^{\infty}\left(\mathbb{R}^{+} \times U \times \mathbb{T}\right)}^{2}\right] \int_{0}^{t}\|\psi(\rho, \boldsymbol{\xi})\|_{H^{s}} d \rho
$$

and the last assertion of the lemma follows from the Gronwall inequality for any fixed $T>$ 0 .

Remark 5. For the truncated NLS problem (2.2), note that the truncation does not change the regularity of the random potential, as stated in Assumption 2.1. Therefore, by the same procedure in Lemma 3.1, we can have $\psi_{m} \in L^{\infty}\left((0, T) \times U_{m} ; H^{s}(\mathbb{T})\right)$ for any finite $T>0$ with $\left\|\psi_{m}\right\|_{L^{\infty}\left((0, T) \times U_{m} ; H^{s}\right)} \leq\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}$.

### 3.2. Error of dimension truncation.

Lemma 3.2. Under Assumption 2.1, the difference between the solution of (2.2) and the solution of the parameterized approximation (1.3) is given by

$$
\left\|\psi_{m}-\psi\right\|_{L^{\infty}\left((0, T) \times U ; H^{1}\right)} \leq C\left\|V_{m}-V\right\|_{L^{\infty}\left(U ; H^{1}\right)},
$$

with a constant $C>0$ depending on $\alpha$, $T$, the norm of $V$ in $L^{\infty}\left(U ; H^{s}(\mathbb{T})\right)$ and the norm of $\psi$ in $L^{\infty}\left((0, T) \times U ; H^{s}(\mathbb{T})\right)$.

Proof. Denoting $\delta \psi=\psi_{m}-\psi$ and taking the difference between (2.2) and (1.3), we get

$$
\left\{\begin{array}{l}
i \partial_{t} \delta \psi=-\frac{1}{2} \partial_{x}^{2} \delta \psi+V \delta \psi+\left(V_{m}-V\right) \psi_{m}+\alpha\left[\left|\psi_{m}\right|^{2} \psi_{m}-|\psi|^{2} \psi\right], \quad x \in \mathbb{T}, t>0 \\
\delta \psi(t=0)=0, \quad x \in \mathbb{T}
\end{array}\right.
$$

The Duhamel formula gives

$$
\begin{aligned}
\delta \psi(t, \boldsymbol{\xi})=-i \int_{0}^{t} \mathrm{e}^{i(t-\rho) \partial_{x}^{2} / 2}[ & V(\boldsymbol{\xi}) \delta \psi(\rho, \boldsymbol{\xi})+\left(V_{m}(\boldsymbol{\xi})-V(\boldsymbol{\xi})\right) \psi_{m}(\rho, \boldsymbol{\xi}) \\
& \left.+\alpha\left(\left|\psi_{m}(\rho, \boldsymbol{\xi})\right|^{2} \psi_{m}(\rho, \boldsymbol{\xi})-|\psi(\rho, \boldsymbol{\xi})|^{2} \psi(\rho, \boldsymbol{\xi})\right)\right] d \rho, \quad t \geq 0
\end{aligned}
$$

which by the product lemma gives

$$
\begin{aligned}
& \|\delta \psi(t, \boldsymbol{\xi})\|_{H^{1}} \\
\leq & C \int_{0}^{t}\left[\|V\|_{L^{\infty}\left(U ; H^{s}\right)}+\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}^{2}+\left\|\psi_{m}\right\|_{L^{\infty}\left((0, T) \times U_{m} ; H^{s}\right)}^{2}\right]\|\delta \psi(\rho, \boldsymbol{\xi})\|_{H^{1}} d \rho \\
& +t C\left\|\psi_{m}\right\|_{L^{\infty}\left((0, T) \times U_{m} ; H^{s}\right)}\left\|V_{m}-V\right\|_{L^{\infty}\left(U ; H^{1}\right)}, \quad 0 \leq t \leq T,
\end{aligned}
$$

with some $C>0$ depending on $\alpha$ only. Note that the function

$$
q(t):=t C\left\|\psi_{m}\right\|_{L^{\infty}\left((0, T) \times U_{m} ; H^{s}\right)}\left\|V_{m}-V\right\|_{L^{\infty}\left(U ; H^{1}\right)}
$$

is non-decreasing in $t$ and the constant

$$
K:=C\left[\|V\|_{L^{\infty}\left(U ; H^{s}\right)}+\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}^{2}+\left\|\psi_{m}\right\|_{L^{\infty}\left((0, T) \times U_{m} ; H^{s}\right)}^{2}\right] \geq 0 .
$$

Then the Gronwall inequality leads to

$$
\begin{aligned}
\|\delta \psi(t, \boldsymbol{\xi})\|_{H^{1}} \leq & q(t)+K \exp (K t) \int_{0}^{t} \exp (-K \rho) q(\rho) d \rho \\
\leq & q(t)\left(1+K \exp (K t) \int_{0}^{t} \exp (-K \rho) d \rho\right)=q(t) \exp (K t) \\
= & t C\left\|\psi_{m}\right\|_{L^{\infty}\left((0, T) \times U_{m} ; H^{s}\right)}\left\|V_{m}-V\right\|_{L^{\infty}\left(U ; H^{1}\right)} \\
& \times \exp \left\{t C\left[\|V\|_{L^{\infty}\left(U ; H^{s}\right)}+\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}^{2}+\left\|\psi_{m}\right\|_{L^{\infty}\left((0, T) \times U_{m} ; H^{s}\right)}^{2}\right]\right\},
\end{aligned}
$$

where we use the fact that $q(t)$ is non-decreasing in $t$ in the second inequality, and the uniform right-hand-side in $\boldsymbol{\xi}$ gives the assertion.

Under the condition $\left\|V_{m}-V\right\|_{L^{\infty}\left(U ; H^{1}\right)} \leq C m^{-\chi}$ in Assumption 2.2, Lemma 3.2 tells that $\psi_{m} \rightarrow \psi$ in $L^{\infty}\left((0, T) \times U ; H^{1}(\mathbb{T})\right)$ as $m \rightarrow \infty$. Recall that with the given linear functional $G$, we are interested in computing $\mathbb{E}[\mathcal{G}]$ which after the truncation is approximated by

$$
\mathbb{E}_{m}\left[\mathcal{G}^{m}\right]=\int_{U_{m}} G\left(\left|\psi_{m}(t, \xi, \cdot)\right|^{2}\right) d \xi
$$

Its error is given as follows.
Lemma 3.3. Under Assumption 2.1 and Assumption 2.2. with $\mathcal{G}$ and $\mathcal{G}^{m}$ defined in (2.7), we have the estimate

$$
\left|\mathbb{E}[\mathcal{G}]-\mathbb{E}_{m}\left[\mathcal{G}^{m}\right]\right| \leq C m^{-\chi}
$$

where $C$ is independent of $m$.
Proof. By the definition (2.6), $\mathbb{E}[\mathcal{G}]-\mathbb{E}_{m}\left[\mathcal{G}^{m}\right]=\lim _{p \rightarrow \infty} \mathbb{E}_{p}\left[\mathcal{G}^{p}\right]-\mathbb{E}_{m}\left[\mathcal{G}^{m}\right]$. Note that when $p \geq m, \mathbb{E}_{p}\left[\mathcal{G}^{m}\right]=\mathbb{E}_{m}\left[\mathcal{G}^{m}\right]$, so we have

$$
\begin{equation*}
\left|\mathbb{E}[\mathcal{G}]-\mathbb{E}_{m}\left[\mathcal{G}^{m}\right]\right|=\lim _{p \rightarrow \infty}\left|\mathbb{E}_{p}\left[\mathcal{G}^{p}-\mathcal{G}^{m}\right]\right| \tag{3.10}
\end{equation*}
$$

By the assumptions and Lemma 3.2, we have for all $t \in[0, T]$,

$$
\begin{equation*}
\left|\mathcal{G}^{p}\left(t, \boldsymbol{\xi}_{\{1: p\}}\right)-\mathcal{G}^{m}(t, \xi)\right| \leq\|G\|_{H^{1}(\mathbb{T})^{\prime}}\left\|\left|\psi_{p}\left(t, \boldsymbol{\xi}_{\{1: p\}}, \cdot\right)\right|^{2}-\left|\psi_{m}(t, \xi, \cdot)\right|^{2}\right\|_{H^{1}} \tag{3.11}
\end{equation*}
$$

With the estimates in Lemma 3.1 and Remark 5, we get by the triangle inequality and the algebraic property of $H^{1}$,

$$
\left\|\left|\psi_{p}\left(t, \boldsymbol{\xi}_{\{1: p\}}, \cdot\right)\right|^{2}-\left|\psi_{m}(t, \xi, \cdot)\right|^{2}\right\|_{H^{1}} \leq C\left\|\psi_{p}\left(t, \boldsymbol{\xi}_{\{1: p\}}, \cdot\right)-\psi_{m}(t, \xi, \cdot)\right\|_{H^{1}}
$$

Plugging it into (3.11) and then using triangle inequality and Lemma 3.2, we find

$$
\begin{aligned}
\left|\mathcal{G}^{p}\left(t, \boldsymbol{\xi}_{\{1: p\}}\right)-\mathcal{G}^{m}(t, \xi)\right| & \leq C\left\|\psi_{p}\left(t, \boldsymbol{\xi}_{\{1: p\}}, \cdot\right)-\psi(t, \boldsymbol{\xi}, \cdot)\right\|_{H^{1}}+C\left\|\psi(t, \boldsymbol{\xi}, \cdot)-\psi_{m}(t, \xi, \cdot)\right\|_{H^{1}} \\
& \leq C\left(\left\|V_{p}-V\right\|_{L^{\infty}\left(U ; H^{1}\right)}+\left\|V_{m}-V\right\|_{L^{\infty}\left(U ; H^{1}\right)}\right) \leq C\left(m^{-\chi}+p^{-\chi}\right) .
\end{aligned}
$$

Note here $C>0$ is uniform constant in $\boldsymbol{\xi}, p, m$. Then, we find

$$
\left|\mathbb{E}_{p}\left[\mathcal{G}^{p}-\mathcal{G}^{m}\right]\right| \leq \mathbb{E}_{p}\left[\left|\mathcal{G}^{p}-\mathcal{G}^{m}\right|\right] \leq C\left(m^{-\chi}+p^{-\chi}\right)
$$

and so by (3.10) we have the assertion of the lemma.
3.3. Regularity in parametric space. As one key to determine the convergence of the QMC method, we need the regularity of the solution in the parametric space. We now start to estimate the mixed first derivatives of $\psi_{m}$ with respect to $\xi_{j}$. In the following, we shall denote

$$
\Gamma:=\left\{\nu=\left(\nu_{1}, \ldots, \nu_{m}\right) \mid \nu_{j}=0 \text { or } 1, j=1, \ldots, m\right\}
$$

with $|\nu|=\sum_{j=1}^{m} \nu_{j}$, and the mixed first derivative reads

$$
\partial^{\nu} \psi_{m}=\partial_{\xi_{1}}^{\nu_{1}} \ldots \partial_{\xi_{m}}^{\nu_{m}} \psi_{m}, \quad \nu \in \Gamma .
$$

Lemma 3.4. Under Assumption 2.1, for any fixed $T>0, m>0$ and multi-index $0 \neq \nu \in \Gamma$, we have

$$
\begin{equation*}
\left\|\partial^{\nu} \psi_{m}(t, \xi, \cdot)\right\|_{H^{1}} \leq 3^{(|\nu|+2)(|\nu|-1) / 2} C_{T}^{2|\nu|-1} \prod_{j=1}^{m}\left(\sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}}\right)^{\nu_{j}}, \forall \xi \in U_{m}, t \in[0, T] \tag{3.12}
\end{equation*}
$$

where $C_{T}=\max \left\{C_{T}^{0}, 1\right\}$ with

$$
\begin{aligned}
C_{T}^{0}:= & 2 \max \{1,|\alpha|\} C_{0} T \max \left\{1,\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}\right\} \\
& \times \exp \left(T \max \left\{1, C_{0}\left[\|V\|_{L^{\infty}\left(U ; H^{s}\right)}+3|\alpha|\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}^{2}\right]\right\}\right)
\end{aligned}
$$

for some generic constant $C_{0}>0$ independent of $\nu$ and $m$.
Proof. We prove (3.12) by an induction on $|\nu|$. When $|\nu|=1$, we differentiate (1.3) with respect to $\xi_{j}$. By denoting $\partial_{j} \psi_{m}=\partial_{\xi_{j}} \psi_{m}$ and $\partial_{j} V_{m}=\partial_{\xi_{j}} V_{m}$ for short, we get for $j=$ $1, \ldots, m$,

$$
i \partial_{t} \partial_{j} \psi_{m}=-\frac{1}{2} \partial_{x}^{2} \partial_{j} \psi_{m}+\partial_{j} V_{m} \psi_{m}+V_{m} \partial_{j} \psi_{m}+\alpha\left(2\left|\psi_{m}\right|^{2} \partial_{j} \psi_{m}+\psi_{m}^{2} \overline{\partial_{j} \psi_{m}}\right)
$$

Omitting the spatial variable for brevity, the Duhamel formula of the above shows

$$
\begin{aligned}
\partial_{j} \psi_{m}(t, \xi)= & \mathrm{e}^{i t \partial_{x}^{2} / 2} \partial_{j} \psi_{i n}-i \int_{0}^{t} \mathrm{e}^{i(t-\rho) \partial_{x}^{2} / 2}\left[\partial_{j} V_{m}(\xi) \psi_{m}(\rho, \xi)+V_{m}(\xi) \partial_{j} \psi_{m}(\rho, \xi)\right. \\
& \left.+\alpha\left(2\left|\psi_{m}(\rho, \xi)\right|^{2} \partial_{j} \psi_{m}(\rho, \xi)+\psi_{m}(\rho, \xi)^{2} \overline{\partial_{j} \psi_{m}}(\rho, \xi)\right)\right] d \rho
\end{aligned}
$$

Since the initial function is deterministic, so $\partial_{j} \psi_{i n}=0$. Then we find

$$
\begin{gathered}
\left\|\partial_{j} \psi_{m}(t, \xi)\right\|_{H^{1}} \leq C_{0} \int_{0}^{t}\left[\left\|\partial_{j} V_{m}(\xi)\right\|_{H^{1}}\left\|\psi_{m}(\rho, \xi)\right\|_{H^{1}}+\left\|V_{m}(\xi)\right\|_{H^{1}}\left\|\partial_{j} \psi_{m}(\rho, \xi)\right\|_{H^{1}}\right. \\
\left.+3|\alpha|\left\|\psi_{m}(\rho, \xi)\right\|_{H^{1}}^{2}\left\|\partial_{j} \psi_{m}(\rho, \xi)\right\|_{H^{1}}\right] d \rho
\end{gathered}
$$

with a generic constant $C_{0}>0$ coming from the Sobolev embedding. By the fact $\partial_{j} V_{m}=$ $\sqrt{\lambda_{j}} v_{j}$, and the boundedness

$$
\left\|\psi_{m}(t, \xi)\right\|_{H^{1}} \leq\left\|\psi_{m}\right\|_{L^{\infty}\left((0, T) \times U_{m} ; H^{1}\right)} \leq\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}, \quad t \in[0, T],
$$

as stated in Remark [5, we further find for $t \in[0, T]$,

$$
\begin{aligned}
\left\|\partial_{j} \psi_{m}(t, \xi)\right\|_{H^{1}} \leq & C_{0} t \sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}}\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)} \\
& +C_{0}\left[\|V\|_{L^{\infty}\left(U ; H^{s}\right)}+3|\alpha|\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}^{2}\right] \int_{0}^{t}\left\|\partial_{j} \psi_{m}(\rho, \xi)\right\|_{H^{1}} d \rho .
\end{aligned}
$$

By Gronwall's inequality, we get for $t \in[0, T], \xi \in U_{m}$,

$$
\begin{aligned}
& \left\|\partial_{j} \psi_{m}(t, \xi)\right\|_{H^{1}} \\
\leq & \exp \left(t C_{0}\left[\|V\|_{L^{\infty}\left(U ; H^{s}\right)}+3|\alpha|\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}^{2}\right]\right) C_{0} t\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)} \sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}},
\end{aligned}
$$

which verifies (3.12) for $|\nu|=1$.
When $|\nu| \geq 2$, by the Leibniz rule we have

$$
\begin{aligned}
i \partial_{t} \partial^{\nu} \psi_{m}= & -\frac{\partial_{x}^{2}}{2} \partial^{\nu} \psi_{m}+\sum_{\mu \preceq \nu}\binom{\nu}{\mu} \partial^{\nu-\mu} V_{m} \partial^{\mu} \psi_{m}+\alpha \sum_{\mu \preceq \nu}\binom{\nu}{\mu} \partial^{\nu-\mu}\left|\psi_{m}\right|^{2} \partial^{\mu} \psi_{m} \\
= & -\frac{\partial_{x}^{2}}{2} \partial^{\nu} \psi_{m}+V_{m} \partial^{\nu} \psi_{m}+\alpha\left(2\left|\psi_{m}\right|^{2} \partial^{\nu} \psi_{m}+\psi_{m}^{2} \partial^{\nu} \overline{\psi_{m}}\right)+\sum_{|\nu-\mu|=1, \mu \prec \nu} \partial^{\nu-\mu} V_{m} \partial^{\mu} \psi_{m} \\
& +\alpha \sum_{0 \neq \mu \prec \nu} \sum_{\eta \preceq \nu-\mu} \partial^{\nu-\mu-\eta} \psi_{m} \partial^{\mu} \psi_{m} \partial^{\eta} \overline{\psi_{m}} .
\end{aligned}
$$

Here $\binom{\nu}{\mu}=\prod_{j=1}^{m}\binom{\nu_{j}}{\mu_{j}}$. Similarly as before, we can deduce

$$
\begin{aligned}
& \left\|\partial^{\nu} \psi_{m}(t, \xi)\right\|_{H^{1}} \\
\leq & C_{0} \int_{0}^{t}\left[\|V\|_{L^{\infty}\left(U ; H^{s}\right)}+3|\alpha|\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}^{2}\right]\left\|\partial^{\nu} \psi_{m}(\rho, \xi)\right\|_{H^{1}} d \rho \\
& +C_{0} \int_{0}^{t} \sum_{|\nu-\mu|=1, \mu \prec \nu}\left\|\partial^{\nu-\mu} V_{m}\right\|_{H^{1}}\left\|\partial^{\mu} \psi_{m}(\rho, \xi)\right\|_{H^{1}} d \rho \\
& +C_{0}|\alpha| \int_{0}^{t} \sum_{0 \neq \mu \prec \nu} \sum_{\eta \leq \nu-\mu}\left\|\partial^{\nu-\mu-\eta} \psi_{m}(\rho, \xi)\right\|_{H^{1}}\left\|\partial^{\mu} \psi_{m}(\rho, \xi)\right\|_{H^{1}}\left\|\partial^{\eta} \psi_{m}(\rho, \xi)\right\|_{H^{1}} d \rho .
\end{aligned}
$$

By the Gronwall inequality, we have

$$
\begin{align*}
&\left\|\partial^{\nu} \psi_{m}(t, \xi)\right\|_{H^{1}} \leq \tilde{C}\left[\sum_{|\nu-\mu|=1, \mu \prec \nu}\left\|\partial^{\nu-\mu} V_{m}\right\|_{H^{1}} \int_{0}^{T}\left\|\partial^{\mu} \psi_{m}(\rho, \xi)\right\|_{H^{1}} d \rho\right.  \tag{3.13}\\
&\left.+\sum_{0 \neq \mu \prec \nu} \sum_{\eta \leq \nu-\mu} \int_{0}^{T}\left\|\partial^{\nu-\mu-\eta} \psi_{m}(\rho, \xi)\right\|_{H^{1}}\left\|\partial^{\mu} \psi_{m}(\rho, \xi)\right\|_{H^{1}}\left\|\partial^{\eta} \psi_{m}(\rho, \xi)\right\|_{H^{1}} d \rho\right]
\end{align*}
$$

with

$$
\tilde{C}=\max \{1,|\alpha|\} C_{0} \exp \left(T \max \left\{1, C_{0}\left[\|V\|_{L^{\infty}\left(U ; H^{s}\right)}+3|\alpha|\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}^{2}\right]\right\}\right) .
$$

For the first summation term above, noting that $|\mu|=|\nu|-1$ and using an induction argument with the assumption (3.12), we have

$$
\begin{equation*}
\sum_{|\nu-\mu|=1, \mu \prec \nu}\left\|\partial^{\nu-\mu} V_{m}\right\|_{H^{1}}\left\|\partial^{\mu} \psi_{m}(\rho, \xi)\right\|_{H^{1}} \leq|\nu| 3^{(|\nu|+1)(|\nu|-2) / 2} C_{T}^{2|\nu|-3} \prod_{j=1}^{m}\left(\sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}}\right)^{\nu_{j}} \tag{3.14}
\end{equation*}
$$

On the other hand, for the double summation part in (3.13), by the induction assumption (3.12) we find

$$
\begin{aligned}
& \sum_{0 \neq \mu \prec \nu} \sum_{\eta \preceq \nu-\mu}\left\|\partial^{\nu-\mu-\eta} \psi_{m}(\rho, \xi)\right\|_{H^{1}}\left\|\partial^{\mu} \psi_{m}(\rho, \xi)\right\|_{H^{1}}\left\|\partial^{\eta} \psi_{m}(\rho, \xi)\right\|_{H^{1}} \\
\leq & \sum_{0 \neq \mu \prec \nu} \sum_{\eta \preceq \nu-\mu} C_{T}^{2|\nu|-2} \prod_{j=1}^{m}\left(\sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}}\right)^{\nu_{j}} 3^{\left.\frac{|\nu|^{2}+|\nu|-6}{2}+|\mu|^{2}+|\eta|^{2}-|\nu||\mu|-|\nu||\eta|-|\mu| \eta \right\rvert\,},
\end{aligned}
$$

and it is direct to check here that

$$
\frac{|\nu|^{2}+|\nu|-6}{2}+|\mu|^{2}+|\eta|^{2}-|\nu||\mu|-|\nu||\eta|-|\mu||\eta| \leq \frac{|\nu|^{2}-|\nu|-2}{2} .
$$

So by noting that the double summation contains $3^{|\nu|}-3$ terms in total, we further get

$$
\begin{align*}
& \sum_{0 \neq \mu \prec \nu} \sum_{\eta<\nu-\mu}\left\|\partial^{\nu-\mu-\eta} \psi_{m}(\rho, \xi)\right\|_{H^{1}}\left\|\partial^{\mu} \psi_{m}(\rho, \xi)\right\|_{H^{1}}\left\|\partial^{\eta} \psi_{m}(\rho, \xi)\right\|_{H^{1}} \\
\leq & 3^{|\nu|} C_{T}^{2|\nu|-2} \prod_{j=1}^{m}\left(\sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}}\right)^{\nu_{j}} 3^{(|\nu|+1)(|\nu|-2) / 2} . \tag{3.15}
\end{align*}
$$

By plugging (3.15) and (3.14) into (3.13), we obtain

$$
\left\|\partial^{\nu} \psi_{m}(t, \xi)\right\|_{H^{1}} \leq \tilde{C} T C_{T}^{2|\nu|-2} 3^{(|\nu|+1)(|\nu|-2) / 2}\left(|\nu|+3^{|\nu|}\right) \prod_{j=1}^{m}\left(\sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}}\right)^{\nu_{j}}
$$

Further noting that $\tilde{C} T \leq C_{T}^{0} / 2$ and $|\nu|<3^{|\nu|}$, we find

$$
\begin{aligned}
\left\|\partial^{\nu} \psi_{m}(t, \xi)\right\|_{H^{1}} & \leq \frac{1}{2} C_{T}^{2|\nu|-1} 3^{(|\nu|+1)(|\nu|-2) / 2}\left(|\nu|+3^{|\nu|}\right) \prod_{j=1}^{m}\left(\sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}}\right)^{\nu_{j}} \\
& \leq C_{T}^{2|\nu|-1} 3^{(|\nu|+2)(|\nu|-1) / 2} \prod_{j=1}^{m}\left(\sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}}\right)^{\nu_{j}}
\end{aligned}
$$

With the established regularity of the solution in the parametric space, we can now estimate the mixed derivative of the physical observable $\mathcal{G}^{m}(t, \xi)=G\left(\left|\psi_{m}(t, \xi, \cdot)\right|^{2}\right)$ with respect to $\xi$.

Lemma 3.5. Under Assumption 2.1, for any multi-index $\nu \in \Gamma$ with the constant $C_{\nu}:=$ $2^{|\nu|} C_{T}^{2|\nu|-1} 3^{(|\nu|+2)(|\nu|-1) / 2}$, we have

$$
\begin{equation*}
\left|\partial^{\nu} \mathcal{G}^{m}(t, \xi)\right| \leq C_{0} C_{\nu} \prod_{j=1}^{m}\left(\sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}}\right)^{\nu_{j}}\|G\|_{H^{1}(\mathbb{T})^{\prime}}, \quad \forall \xi \in U_{m}, t \in[0, T] . \tag{3.16}
\end{equation*}
$$

Proof. For any $\xi \in U_{m}$ and $t \in[0, T]$, we have

$$
\left|\partial^{\nu} \mathcal{G}^{m}(t, \xi)\right|=\left|G\left(\partial^{\nu}\left|\psi_{m}\right|^{2}\right)\right| \leq\|G\|_{H^{1}(\mathbb{T})^{\prime}}\left\|\partial^{\nu}\left|\psi_{m}\right|^{2}\right\|_{H^{1}(\mathbb{T})} .
$$

By the algebraic property of $H^{1}$ and Lemma 3.4, we find

$$
\begin{aligned}
\left|\partial^{\nu} \mathcal{G}^{m}(t, \xi)\right| & \leq\|G\|_{H^{1}(\mathbb{T})^{\prime}} \sum_{\mu \preceq \nu} C_{0}\left\|\partial^{\nu-\mu} \overline{\psi_{m}}\right\|_{H^{1}}\left\|\partial^{\mu} \psi_{m}\right\|_{H^{1}} \\
& \leq C_{0} C_{\nu}\|G\|_{H^{1}(\mathbb{T})^{\prime}} \prod_{j=1}^{m}\left(\sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}}\right)^{\nu_{j}} .
\end{aligned}
$$

3.4. Analysis of the QMC integration error. With the preparation before, we now specify the QMC method, i.e., the quadrature points in (2.9), and then we shall derive its error bound on the PDE level.

We adopt the weighted Sobolev space technique [37, 47] here. Consider the weighted and unanchored Sobolev space $\mathcal{W}_{m, \gamma}:=\left\{F(\xi): U_{m} \rightarrow \mathbb{R} \mid\|F\|_{W_{m, \gamma}}<\infty\right\}$ with the norm

$$
\begin{equation*}
\|F\|_{W_{m, \gamma}}:=\left[\sum_{\nu \in \Gamma} \gamma_{\nu}^{-1} \int_{U_{|\nu|}}\left(\int_{U_{m-|\nu|}} \partial^{\nu} F(\xi) d \xi_{\nu}^{\dagger}\right)^{2} d \xi_{\nu}\right]^{1 / 2} \tag{3.17}
\end{equation*}
$$

where by the notation, $\xi$ is split into the active part $\xi_{\nu}$ for differentiations and the inactive part $\xi_{\nu}^{\dagger}$, i.e., $\xi_{\nu}$ consists of $\xi_{j}$ with $j$ such that $\nu_{j}=1$ and $\xi_{\nu}^{\dagger}$ consists of $\xi_{k}$ with $k$ such that $\nu_{k}=0$. The notation $\gamma_{\nu}$ denotes the product-type weight, i.e., $\gamma_{\nu}=\Pi_{\nu_{j}=1} \gamma_{j}$ with $\gamma_{0_{\{1: m\}}}=1$, which will be chosen later, and $\gamma:=\left\{\gamma_{\nu} \mid \nu \in \Gamma\right\}$. With the chosen $\gamma_{j}(j=1, \ldots, m)$ and the following worst-case error function

$$
e_{s}^{w t}(z):=-1+\frac{1}{N} \sum_{p=1}^{N} \prod_{j=1}^{s}\left[1+\gamma_{j} B\left(\operatorname{frac}\left(p z_{j} / N\right)\right)\right], \quad z=\left(z_{1}, \ldots, z_{s}\right), 1 \leq s \leq m
$$

where $B(x)=-1 /\left(2 \pi^{2}\right) \sum_{l \in \mathbb{Z} \backslash\{0\}} \mathrm{e}^{2 \pi i l x} / l^{2}$ is the Bernoulli polynomial, the generating vector $z$ in (2.9) is computed by the CBC algorithm [12]. That is to set $z_{1}=1$, and then determine each $z_{s}$ for $s=2, \ldots, m$ in a sequel by minimizing $e_{s}^{w t}(z)$ for $z_{s} \in \mathbb{U}_{N}=\{x \in \mathbb{Z} \mid 1 \leq$ $x \leq N-1, \operatorname{gcd}(x, N)=1\}$. An explicit error bound of the QMC with such constructed quadrature points can be established (see e.g., [36]). Here let us quote it as the following lemma.

Lemma 3.6. ([36, Theorem 4.1]) With fixed $m, N, \gamma$ and some $F(\xi) \in W_{m, \gamma}$, the $Q M C$ method (2.9) given by the CBC algorithm satisfies

$$
\begin{equation*}
\sqrt{\mathbb{E}^{\Delta}\left[\left|E_{m}(F)-Q_{m, N}[F]\right|^{2}\right]} \leq\left[\sum_{\nu \in \Gamma \backslash \mathbf{0}_{\{1: m\}}} \gamma_{\nu}^{\lambda}\left(\frac{2 \zeta(2 \lambda)}{\left(2 \pi^{2}\right)^{\lambda}}\right)^{|\nu|}\right]^{1 /(2 \lambda)} \varphi(N)^{-1 /(2 \lambda)}\|F\|_{W_{m, \gamma},}, \tag{3.18}
\end{equation*}
$$

for any $\lambda \in\left(\frac{1}{2}, 1\right]$. Here $\varphi(N)=\left|\mathbb{U}_{N}\right|$ is the Euler totient function and $\zeta(\cdot)$ is the Riemann zeta function.

The possible choice of the weight $\gamma$ to provide the optimal convergence rate for the QMC is specified in the following lemma.

Lemma 3.7. With fixed $m \in \mathbb{N}_{+}$and $N \leq 10^{30}$, choose $\lambda=1 /(2-2 \delta)$ for any $\delta \in(0,1 / 2)$. Then under Assumption 2.1 and Assumption 2.2, a QMC method (2.9) can be constructed by the CBC algorithm which satisfies

$$
\begin{equation*}
\sqrt{\mathbb{E}^{\Delta}\left[\left|E_{m}\left[\mathcal{G}^{m}\right](t)-Q_{m, N}\left[\mathcal{G}^{m}\right](t)\right|^{2}\right]} \leq C N^{\delta-1}, \quad t \in[0, T] \tag{3.19}
\end{equation*}
$$

where $C>0$ is some constant independent of $m$.

Proof. It is direct to compute from the definition of the weighted norm (3.17) and Lemma 3.5 that

$$
\begin{aligned}
\left\|\mathcal{G}^{m}(t, \cdot)\right\|_{W_{m, \gamma}} & =\left[\sum_{\nu \in \Gamma} \gamma_{\nu}^{-1} \int_{U_{|\nu|}}\left(\int_{U_{m-|\nu|}} \partial^{\nu} \mathcal{G}^{m}(t, \xi) d \xi_{\nu}^{\dagger}\right)^{2} d \xi_{\nu}\right]^{1 / 2} \\
& \leq C_{0}\|G\|_{H^{1}(D)^{\prime}}\left[\sum_{\nu \in \Gamma} \gamma_{\nu}^{-1} C_{\nu}^{2} \prod_{j=1}^{m} \lambda_{j}^{\nu_{j}}\left\|v_{j}\right\|_{H^{1}}^{2 \nu_{j}}\right]^{1 / 2}
\end{aligned}
$$

Meanwhile, it is known that the Euler totient function satisfies $1 / \varphi(N) \leq 9 / N$ for $N \leq 10^{30}$ [25]. Thus, by (3.18) and choosing $\lambda=1 /(2-2 \delta)$ we find

$$
\sqrt{\mathbb{E}^{\Delta}\left[\left|E_{m}\left[\mathcal{G}^{m}\right](t)-Q_{m, N}\left[\mathcal{G}^{m}\right](t)\right|^{2}\right]} \leq 9 C_{\gamma} C_{0}\|G\|_{H^{1}(\mathbb{T})^{\prime}} N^{\delta-1},
$$

with

$$
C_{\gamma}=\left[\sum_{\nu \in \Gamma} \gamma_{\nu}^{-1} C_{\nu}^{2} \prod_{j=1}^{m} \lambda_{j}^{\nu_{j}}\left\|v_{j}\right\|_{H^{1}}^{2 \nu_{j}}\right]^{1 / 2}\left[\sum_{\nu \in \Gamma} \gamma_{\nu}^{\lambda}\left(\frac{2 \zeta(2 \lambda)}{\left(2 \pi^{2}\right)^{\lambda}}\right)^{|\nu|}\right]^{1 /(2 \lambda)}
$$

Denote $b_{j}=\sqrt{\lambda_{j}}\left\|v_{j}\right\|_{H^{1}}$ and $\rho(\lambda)=2 \zeta(2 \lambda) /\left(2 \pi^{2}\right)^{\lambda}$ for simplicity. Now we take

$$
\gamma_{\nu}=\left[C_{\nu} \prod_{j=1}^{m} \frac{b_{j}^{\nu_{j}}}{\rho(\lambda)^{\nu_{j} / 2}}\right]^{2 /(1+\lambda)}
$$

and it is direct to deduce that $C_{\gamma}=\left(A_{\lambda}\right)^{\frac{\lambda+1}{2 \lambda}}$ with

$$
A_{\lambda}=\sum_{\nu \in \Gamma} C_{\nu}^{2 \lambda /(1+\lambda)} \prod_{j=1}^{m}\left[b_{j}^{2 \lambda \nu_{j}} \rho(\lambda)^{\nu_{j}}\right]^{1 /(1+\lambda)} .
$$

It remains to show that $A_{\lambda}$ is bounded $\forall \nu \in \Gamma, m \in \mathbb{N}$.
By the constant $C_{\nu}$ from Lemma 3.5 and using Hölder's inequality, we find

$$
\begin{aligned}
A_{\lambda} & \leq \sum_{\nu \in \Gamma}\left[3^{|\nu|^{2}+|\nu|-1 / 2} \prod_{j=1}^{m} \alpha_{j}^{\nu_{j}}\right]^{2 \lambda /(1+\lambda)} \prod_{j=1}^{m}\left[\frac{\left(2 C_{T}^{2}\right)^{2 \lambda} b_{j}^{2 \lambda} \rho(\lambda)}{\alpha_{j}^{2 \lambda}}\right]^{\nu_{j} /(1+\lambda)} \\
& \leq\left(\sum_{\nu \in \Gamma} \prod_{j=1}^{m} 3^{\nu_{j} s_{j}} \alpha_{j}^{\nu_{j}}\right)^{2 \lambda /(1+\lambda)}\left(\sum_{\nu \in \Gamma} \prod_{j=1}^{m}\left[\frac{\left(2 C_{T}^{2}\right)^{2 \lambda} b_{j}^{2 \lambda} \rho(\lambda)}{\alpha_{j}^{2 \lambda}}\right]^{\nu_{j} /(1-\lambda)}\right)^{(1-\lambda) /(1+\lambda)},
\end{aligned}
$$

where $s_{j}:=\sum_{l=1}^{j} \nu_{l}$ and $\alpha_{j}>0$ is chosen as follows. Note the fact that $\sum_{\nu \in \Gamma} \prod_{j=1}^{m} \beta_{j}^{\nu_{j}} \leq$ $\exp \left(\sum_{j \geq 1} \beta_{j}\right)$ holds for any $m \geq 1$ and any $\beta_{j}>0$ with $\sum_{j \geq 1} \beta_{j}<\infty$ (see [37]), then we have: $\forall \nu \in \Gamma, m \in \mathbb{N}$,

$$
A_{\lambda} \leq \exp \left(\frac{2 \lambda}{1+\lambda} \sum_{j \geq 1} 3^{j} \alpha_{j}\right) \exp \left(\frac{1-\lambda}{1+\lambda} \sum_{j \geq 1}\left[\frac{\left(2 C_{T}^{2}\right)^{2 \lambda} b_{j}^{2 \lambda} \rho(\lambda)}{\alpha_{j}^{2 \lambda}}\right]^{1 /(1-\lambda)}\right)=: A_{\max }
$$

By taking $\alpha_{j}=\sqrt{b_{j} / 3^{j}}$ and under Assumption 2.2, we have $\sum_{j \geq 1} 3^{j} \alpha_{j}<1$. Moreover, with $\lambda \in\left(\frac{1}{2}, 1\right]$ we have $\frac{\lambda}{1-\lambda} \geq 1$ and so

$$
\sum_{j \geq 1}\left(b_{j} / \alpha_{j}\right)^{2 \lambda /(1-\lambda)}=\sum_{j \geq 1}\left(3^{j} b_{j}\right)^{\lambda /(1-\lambda)}<\infty .
$$

Thus, we find the upper bound $A_{\max }$ is finite.
Proposition 3.8. Under the same assumption of Lemma 3.7, there exists a constant $C>0$ independent of $m$ such that

$$
\begin{equation*}
\sqrt{\mathbb{E}^{\Delta}\left[\left|\mathbb{E}[\mathcal{G}](t)-Q_{m, N}\left[\mathcal{G}^{m}\right](t)\right|^{2}\right]} \leq C\left(m^{-\chi}+N^{\delta-1}\right), \quad t \in[0, T] \tag{3.20}
\end{equation*}
$$

for any $\delta \in(0,1 / 2)$.
Proof. Firstly, by triangle inequality we have

$$
\left|\mathbb{E}[\mathcal{G}](t)-Q_{m, N}\left[\mathcal{G}^{m}\right](t)\right| \leq\left|\mathbb{E}[\mathcal{G}](t)-\mathbb{E}_{m}\left[\mathcal{G}^{m}\right]\right|+\left|\mathbb{E}_{m}\left[\mathcal{G}^{m}\right]-Q_{m, N}\left[\mathcal{G}^{m}\right](t)\right| .
$$

The first part of right-hand-side is independent of the random shift $\Delta$, so

$$
\begin{aligned}
\mathbb{E}^{\Delta}\left[\left|\mathbb{E}[\mathcal{G}](t)-Q_{m, N}\left[\mathcal{G}^{m}\right](t)\right|^{2}\right] \leq & 2\left|\mathbb{E}[\mathcal{G}](t)-\mathbb{E}_{m}\left[\mathcal{G}^{m}\right](t)\right|^{2} \\
& +2 \mathbb{E}^{\Delta}\left[\left|\mathbb{E}_{m}\left[\mathcal{G}^{m}\right](t)-Q_{m, N}\left[\mathcal{G}^{m}\right](t)\right|^{2}\right] .
\end{aligned}
$$

Then by Lemma 3.3 and Lemma 3.7, we obtain the result.
Remark 6. We remark the choice of the weight $\gamma$ and the error constant given in our analysis above may be far away from optimal. Our result theoretically serves to show the existence of the randomly shifted QMC lattice rule that can converge at the desired almost-linear and dimension-independent rate for solving the random NLS (1.3).

## 4. ERror of The scheme

In this section, we continue the analysis to complete the full error estimate result given in Section 2.3. To do so, we first carry out the error estimate of the TSFP scheme on the truncated problem (2.2), which is stated as the following.

Proposition 4.1. Let $\psi_{m}^{n}$ be the numerical solution from the TSFP scheme (2.4) for solving the truncated problem (2.2) till any fixed $T>0$ with any $m \in \mathbb{N}$. Under the same assumption in Theorem [2.3, there exist constants $\tau_{0}, h_{0}>0$ independent of $m$ such that when $\tau \leq \tau_{0}$ and $h \leq h_{0}$, the following error bound holds

$$
\left\|I_{M} \psi_{m}^{n}-\psi_{m}\left(t_{n}, \xi, \cdot\right)\right\|_{H^{1}} \leq C\left(\tau^{2}+h^{s-1}\right), \quad \forall 0 \leq t_{n} \leq T, \quad \xi \in U_{m},
$$

for some constant $C>0$ independent of $\xi, m, \tau$ and $h$.

Proof. The proof largely follows the framework of analysis for the Strang splitting in [7, 39]. Here we shall go through it mainly to check the impact from the parametric space. Let us omit the variables $x, \xi$ and denote $\psi_{m}(t)=\psi_{m}(t, \xi, x)$ for simplicity.

We begin with the estimate of the temporal truncation error. Denoting $v^{n}(\rho):=$ $\mathrm{e}^{\frac{1}{2} i \partial_{x}^{2} \rho} \psi_{m}\left(t_{n}\right)$ for short, by Taylor's expansion we find

$$
\Psi_{\tau}^{\mathrm{p}} \circ \Psi_{\frac{\tau}{2}}^{\mathrm{k}}\left(\psi_{m}\left(t_{n}\right)\right)=\left[1+i \tau\left(V_{m}+\alpha\left|v^{n}(\tau / 2)\right|^{2}\right)-\frac{\tau^{2}}{2}\left(V_{m}+\alpha\left|v^{n}(\tau / 2)\right|^{2}\right)^{2}\right] v^{n}(\tau / 2)+r_{1}^{n},
$$

where $r_{1}^{n}$ depends on $V_{m}$ and $v^{n}$. By Remark 5, we have $\left\|r_{1}^{n}\right\|_{H^{1}} \leq C \tau^{3}$ with $C$ depending on the $H^{1}$-norm of $V_{m}, \psi_{m}$. Then,

$$
\begin{align*}
\Psi_{\frac{\tau}{2}}^{\mathrm{k}} \circ \Psi_{\tau}^{\mathrm{p}} \circ \Psi_{\frac{\tau}{2}}^{\mathrm{k}}\left(\psi_{m}\left(t_{n}\right)\right)= & v^{n}(\tau)+i \tau \mathrm{e}^{\frac{1}{2} i_{x}^{2} \tau / 2}\left(V_{m}+\alpha\left|v^{n}(\tau / 2)\right|^{2}\right) v^{n}(\tau / 2) \\
& -\frac{\tau^{2}}{2}\left(V_{m}+\alpha\left|v^{n}(\tau / 2)\right|^{2}\right)^{2} v^{n}(\tau / 2)+r_{2}^{n}, \tag{4.1}
\end{align*}
$$

where

$$
r_{2}^{n}=\mathrm{e}^{i \partial_{x}^{2} \tau / 4} r_{1}^{n}+\int_{0}^{\tau} \frac{i}{4} \partial_{x}^{2} \mathrm{e}^{i \partial_{x}^{2} s / 4}\left(V_{m}+\alpha\left|v^{n}(\tau / 2)\right|^{2}\right)^{2} v^{n}(\tau / 2) d s
$$

Under the assumption $s \geq 5$, we have $\left\|r_{2}^{n}\right\|_{H^{1}} \leq C \tau^{3}$ with $C$ depending on the $H^{3}$-norm of $V_{m}$ and $\psi_{m}$. On the other hand, by the Duhamel formula of (2.2), we have

$$
\begin{equation*}
\psi_{m}\left(t_{n+1}\right)=\mathrm{e}^{\frac{1}{2} i \tau \partial_{x}^{2}} \psi_{m}\left(t_{n}\right)+i \int_{0}^{\tau} g_{\tau}^{n}(\rho) d \rho, \quad n \geq 0 \tag{4.2}
\end{equation*}
$$

with $g_{\tau}^{n}(\rho):=\mathrm{e}^{\frac{1}{2} i(\tau-\rho) \partial_{x}^{2}}\left[V_{m}+\alpha\left|\psi_{m}\left(t_{n}+\rho\right)\right|^{2}\right] \psi_{m}\left(t_{n}+\rho\right)$. By plugging $\psi_{m}\left(t_{n}+\rho\right)=v^{n}(\rho)+$ $i \int_{0}^{\rho} g_{\rho}^{n}(s) d s$ into (4.2) we find

$$
\begin{aligned}
\psi_{m}\left(t_{n+1}\right)= & v^{n}(\tau)+i \int_{0}^{\tau} \mathrm{e}^{\frac{1}{2} i(\tau-\rho) \partial_{x}^{2}}\left[V_{m}+\alpha\left|v^{n}(\rho)\right|^{2}\right] v^{n}(\rho) d \rho \\
& +i \int_{0}^{\tau} \mathrm{e}^{\frac{1}{2} i(\tau-\rho) \partial_{x}^{2}} \int_{0}^{\rho}\left[i\left(V_{m}+2 \alpha\left|v^{n}(\rho)\right|^{2}\right) g_{\rho}^{n}(s)-i \alpha v^{n}(\rho)^{2} \overline{g_{\rho}^{n}(s)}\right] d s d \rho+r_{3}^{n},
\end{aligned}
$$

where clearly $\left\|r_{3}^{n}\right\|_{H^{1}} \leq C \tau^{3}$ with $C$ depending on the $H^{1}$-norm of $V_{m}, \psi_{m}$. Note $g_{\rho}^{n}(s)=$ $\left[V_{m}+\alpha\left|v^{n}(\rho)\right|^{2}\right] v^{n}(\rho)+\mathcal{O}\left(\tau \partial_{x}^{2}\right)$ for $0 \leq s \leq \rho \leq \tau$, we further get

$$
\begin{aligned}
\psi_{m}\left(t_{n+1}\right)= & v^{n}(\tau)+i \int_{0}^{\tau} \mathrm{e}^{\frac{1}{2} i(\tau-\rho) \partial_{x}^{2}}\left[V_{m}+\alpha\left|v^{n}(\rho)\right|^{2}\right] v^{n}(\rho) d \rho \\
& -\int_{0}^{\tau} \rho\left(V_{m}+\alpha\left|v^{n}(\rho)\right|^{2}\right)^{2} v^{n}(\rho) d \rho+r_{3}^{n}+r_{4}^{n}
\end{aligned}
$$

where $\left\|r_{4}^{n}\right\|_{H^{1}} \leq C \tau^{3}$ with $C$ depending on the $H^{3}$-norm of $V_{m}, \psi_{m}$. Subtracting the above from (4.1) and then using the quadrature error formula of the midpoint rule, we find

$$
\begin{aligned}
& \left\|\psi_{m}\left(t_{n+1}\right)-\Psi_{\frac{\tau}{2}}^{\mathrm{k}} \circ \Psi_{\tau}^{\mathrm{p}} \circ \Psi_{\frac{\tau}{2}}^{\mathrm{k}}\left(\psi_{m}\left(t_{n}\right)\right)\right\|_{H^{1}} \\
\leq & \left\|r_{2}^{n}\right\|_{H^{1}}+\left\|r_{3}^{n}\right\|_{H^{1}}+\left\|r_{4}^{n}\right\|_{H^{1}}+\tau^{3} \int_{0}^{1} \kappa(\theta)\left\|f^{\prime \prime}(\theta \tau)\right\|_{H^{1}} d \theta
\end{aligned}
$$

where $\kappa(\theta)$ is the Peano kernel of the midpoint rule and

$$
f(\rho):=i \mathrm{e}^{\frac{1}{2} i(\tau-\rho) \partial_{x}^{2}}\left[V_{m}+\alpha\left|v^{n}(\rho)\right|^{2}\right] v^{n}(\rho)-\rho\left(V_{m}+\alpha\left|v^{n}(\rho)\right|^{2}\right)^{2} v^{n}(\rho) .
$$

Therefore, we find in total

$$
\begin{equation*}
\left\|\psi_{m}\left(t_{n+1}\right)-\Psi_{\frac{\pi}{2}}^{\mathrm{k}} \circ \Psi_{\tau}^{\mathrm{p}} \circ \Psi_{\frac{\tau}{2}}^{\mathrm{k}}\left(\psi_{m}\left(t_{n}\right)\right)\right\|_{H^{1}} \leq C \tau^{3}, \quad 0 \leq n<T / \tau \tag{4.3}
\end{equation*}
$$

with $C$ depending on the $H^{5}$-norm of $V_{m}, \psi_{m}$. By the uniform bounds of $V_{m}$ and $\psi_{m}$ in $H^{s}$ with $s \geq 5$ from the assumption and the result of Lemma 3.1 and Remark 5, we find the error constant $C>0$ in (4.3) is eventually a constant depending only on $\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}$ and $\|V\|_{L^{\infty}\left(U ; H^{s}\right)}$.

Next, by the triangle inequality and the projection error (denote by $P_{M}$ the projection operator to cutoff the Fourier modes of a function larger than $M / 2$ ) [50], we find

$$
\left\|I_{M} \psi_{m}^{n+1}-\psi_{m}\left(t_{n+1}\right)\right\|_{H^{1}} \leq\left\|I_{M} \psi_{m}^{n+1}-P_{M} \psi_{m}\left(t_{n+1}\right)\right\|_{H^{1}}+C h^{s-1}
$$

with $C$ here depends only on $\left\|\psi_{m}\right\|_{L^{\infty}\left((0, T) \times U_{m} ; H^{s}\right)}$ and so only on $\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}$. Noting that $I_{M} \psi_{m}^{n+1}=\Psi_{\frac{\tau}{2}}^{\mathrm{k}} \circ I_{M}\left[\Psi_{\tau}^{\mathrm{p}} \circ \Psi_{\frac{\tau}{2}}^{\mathrm{k}}\left(I_{M} \psi_{m}^{n}\right)\right]$ and $\Psi_{\frac{\tau}{2}}^{\mathrm{k}}$ is isometric in $H^{1}$, we find by the triangle inequality and (4.3),

$$
\begin{equation*}
\left\|I_{M} \psi_{m}^{n+1}-P_{M} \psi_{m}\left(t_{n+1}\right)\right\|_{H^{1}} \leq\left\|I_{M} \Psi_{\tau}^{\mathrm{p}} \circ \Psi_{\frac{\tau}{2}}^{\mathrm{k}}\left(I_{M} \psi_{m}^{n}\right)-P_{M} \Psi_{\tau}^{\mathrm{p}} \circ \Psi_{\frac{\tau}{2}}^{\mathrm{k}}\left(\psi_{m}\left(t_{n}\right)\right)\right\|_{H^{1}}+C \tau^{3} . \tag{4.4}
\end{equation*}
$$

Then by an induction on the boundedness of $I_{M} \psi_{m}^{n}$ in $H^{1}$-space and the difference between the flows, we find with some $C$ depending only on $\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}$ and $\|V\|_{L^{\infty}\left(U ; H^{s}\right)}$,

$$
\begin{aligned}
& \left\|I_{M} \Psi_{\tau}^{\mathrm{p}} \circ \Psi_{\frac{\tau}{2}}^{\mathrm{k}}\left(I_{M} \psi_{m}^{n}\right)-P_{M} \Psi_{\tau}^{\mathrm{p}} \circ \Psi_{\frac{\tau}{2}}^{\mathrm{k}}\left(\psi_{m}\left(t_{n}\right)\right)\right\|_{H^{1}} \\
\leq & \left\|I_{M} \psi_{m}^{n}-P_{M} \psi_{m}\left(t_{n}\right)\right\|_{H^{1}}+C \tau\left\|I_{M} \psi_{m}^{n}-P_{M} \psi_{m}\left(t_{n}\right)\right\|_{H^{1}}+C \tau h^{s-1}
\end{aligned}
$$

Plugging it into (4.4), summing the inequalities up till $n=0$ and noting $I_{M} \psi_{m}^{0}=I_{M} \psi_{m}(0)$, the Gronwall inequality gives

$$
\left\|I_{M} \psi_{m}^{n+1}-\psi_{m}\left(t_{n+1}\right)\right\|_{H^{1}} \leq C\left(\tau^{2}+h^{s-1}\right),
$$

for some $C$ depending only on $T,\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}$ and $\|V\|_{L^{\infty}\left(U ; H^{s}\right)}$. Thus, there exist constants $\tau_{0}, h_{0}>0$ uniformly in $m$ such that for $\tau \leq \tau_{0}$ and $h \leq h_{0},\left\|I_{M} \psi_{m}^{n+1}\right\|_{H^{1}} \leq$ $\|\psi\|_{L^{\infty}\left((0, T) \times U ; H^{s}\right)}+1$ for all $t_{n+1} \leq T$.

With all the obtained results so far, we are now ready to get the full error bound stated in our main theorem in Section 2.3 for the QMC-TSFP scheme (2.4). The proof is given as follows.

Proof of the Theorem 2.3: By the triangle inequality, we have

$$
\begin{align*}
& \mathbb{E}^{\Delta}\left[\left|\mathbb{E}[\mathcal{G}]\left(t_{n}\right)-Q_{m, N}\left[G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right]\right|^{2}\right] \\
\leq & 2 \mathbb{E}^{\Delta}\left[\left|\mathbb{E}[\mathcal{G}]\left(t_{n}\right)-Q_{m, N}\left[\mathcal{G}^{m}\right]\left(t_{n}\right)\right|^{2}\right]+2 \mathbb{E}^{\Delta}\left[\left|Q_{m, N}\left[G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right]-Q_{m, N}\left[\mathcal{G}^{m}\right]\left(t_{n}\right)\right|^{2}\right] \\
\leq & C\left(m^{-\chi}+N^{\delta-1}\right)^{2}+2 \mathbb{E}^{\Delta}\left[\left|Q_{m, N}\left[G\left(\left|I_{M} \psi_{m}^{n}(\xi, \cdot)\right|^{2}\right)-G\left(\left|\psi_{m}\left(t_{n}, \xi, \cdot\right)\right|^{2}\right)\right]\right|^{2}\right], \tag{4.5}
\end{align*}
$$

where we used Proposition 3.8, By (2.8) and Assumption 2.1, we find

$$
\begin{aligned}
& \mathbb{E}^{\Delta}\left[\left|Q_{m, N}\left[G\left(\left|I_{M} \psi_{m}^{n}(\xi, \cdot)\right|^{2}\right)-G\left(\left|\psi_{m}\left(t_{n}, \xi, \cdot\right)\right|^{2}\right)\right]\right|\right] \\
\leq & \sup _{\xi \in U_{m}}\left|G\left(\left|I_{M} \psi_{m}^{n}(\xi, \cdot)\right|^{2}\right)-G\left(\left|\psi_{m}\left(t_{n}, \xi, \cdot\right)\right|^{2}\right)\right| \\
\leq & \|G\|_{H^{1}(D)^{\prime}} \sup _{\xi \in U_{m}}\left\|\left|I_{M} \psi_{m}^{n}(\xi, \cdot)\right|^{2}-\left|\psi_{m}\left(t_{n}, \xi, \cdot\right)\right|^{2}\right\|_{H^{1}} .
\end{aligned}
$$

Then, by the error estimate in Proposition 4.1 which is uniform in $\xi$ and $m$, we plug the above into (4.5) to get

$$
\sqrt{\mathbb{E}^{\Delta}\left[\left|\mathbb{E}[\mathcal{G}]\left(t_{n}\right)-Q_{m, N}\left[G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right]\right|^{2}\right]} \leq C\left(\tau^{2}+h^{s-1}+m^{-\chi}+N^{\delta-1}\right), t_{n} \in[0, T]
$$

for some constant $C>0$ independent of $m$. This completes the proof.

## 5. Numerical examples

In this section, we carry out numerical experiments for the QMC-TSFP scheme. We shall test its accuracy in the random space, physical space and time to verify the theoretical result. The generating vector (2.9) for QMC is constructed by the code in [34] (see also [35]). Some simulations on the wave propagation are provided in the end.

### 5.1. Convergence test.

Example 5.1 (Convergence in random space). Consider (2.2) with $\alpha=1, \mathbb{T}=[-\pi, \pi]$, the initial data $\psi_{\text {in }}(x)=\sqrt{8 / \pi} \exp \left(-8 x^{2}\right)$ and the random potential

$$
\begin{equation*}
V_{m}(\boldsymbol{\xi}, x)=1+\sum_{j=1}^{m} \frac{1}{j^{2}} \xi_{j} \cos (j x), \tag{5.1}
\end{equation*}
$$

where $\left\{\xi_{j}\right\}$ are i.i.d. uniformly distributed random variables on $[-1,1]$. Firstly, we test and compare the performance of the Monte Carlo (MC) method and the QMC method. One set of the numerical solutions are computed by TSFP combined with MC under $\tau=10^{-4}, h=\frac{\pi}{64}$ and $N_{\mathrm{MC}}$ samples $\left\{\xi_{\mathrm{MC}}^{(p)}: p=1, \ldots, N_{\mathrm{MC}}\right\}$. The other set of solutions are computed by QMCTSFP under $\tau=10^{-4}, h=\frac{\pi}{64}$ and $R N_{\mathrm{QMC}}$ samples $\left\{\xi_{\mathrm{QMC}}^{(r, p)}: r=1, \ldots, R, p=1, \ldots, N_{\mathrm{QMC}}\right\}$, where $R$ is the number of shifts and $N_{\mathrm{QMC}}$ is the number of samples for each shift. To have a fair comparison between these two methods, we set $N_{\text {tot }}=N_{\mathrm{MC}}=R N_{\mathrm{QMC}}$. We choose $R=10$ and $N_{\mathrm{QMC}}=2^{10}, 2^{11}, \ldots, 2^{16}$. The final time is fixed as $t_{n}=T=1$.

For $m$, we consider two choices: $m=6$ and $m=16$. For $m=6$, we compute the reference solution using TSFP combined with the multidimensional stochastic collocation method [6] with $\tau=10^{-4}, h=\frac{\pi}{64}$ and 15 Gauss-Legendre points in each of the $m$ dimensions of $\xi$. We consider the $L^{2}$ relative error of the expectation of the density

$$
\begin{equation*}
L^{2} \text { relative error }=\frac{\left\|\mathbb{E}_{\text {num }}\left[\left|\psi_{\text {num }}(x)\right|^{2}\right]-\mathbb{E}_{\text {ref }}\left[\left|\psi_{\text {ref }}(x)\right|^{2}\right]\right\|_{L^{2}}}{\left\|\mathbb{E}_{\text {ref }}\left[\left|\psi_{\text {ref }}(x)\right|^{2}\right]\right\|_{L^{2}}}, \tag{5.2}
\end{equation*}
$$



Figure 1. Convergence of QMC-TSFP in random space.
where for the MC method

$$
\begin{equation*}
\mathbb{E}_{\mathrm{num}}\left[\left|\psi_{\mathrm{num}}(x)\right|^{2}\right]=\frac{1}{N_{\mathrm{MC}}} \sum_{p=1}^{N_{\mathrm{MC}}}\left|I_{M} \psi_{m}^{n}\left(\xi_{\mathrm{MC}}^{(p)}, x\right)\right|^{2} \tag{5.3}
\end{equation*}
$$

and for the QMC method

$$
\begin{equation*}
\mathbb{E}_{\mathrm{num}}\left[\left|\psi_{\mathrm{num}}(x)\right|^{2}\right]=\frac{1}{R N_{\mathrm{QMC}}} \sum_{r=1}^{R} \sum_{p=1}^{N_{\mathrm{QMC}}}\left|I_{M} \psi_{m}^{n}\left(\xi_{\mathrm{QMC}}^{(r, p)}, x\right)\right|^{2}, \tag{5.4}
\end{equation*}
$$

with $n=T / \tau$ and $M=2 \pi / h$. The result for $m=6$ is shown in Figure 1a. For $m=16$, a reference solution is not easy to obtain. Here we consider a physical observable given by the following functional $G$ :

$$
\begin{equation*}
G\left(|\psi(t, \xi, x)|^{2}\right)=\int_{\mathbb{T}}|x|^{2}|\psi(t, \xi, x)|^{2} d x \tag{5.5}
\end{equation*}
$$

Note that $G\left(|\psi|^{2}\right)$ is the second spatial moment of the density function $|\psi|^{2}$, which is important in checking the localization phenomenon (see also Example 5.3). For both MC and QMC, we consider the root-mean-square (RMS) error

$$
R M S:=\sqrt{\mathbb{E}^{\Delta}\left[\left|\mathbb{E}[\mathcal{G}]\left(t_{n}\right)-Q_{m, N}\left[G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right]\right|^{2}\right]}
$$

As given in [12], an unbiased estimator of the RMS for MC reads

$$
\begin{equation*}
R M S \approx \sqrt{\frac{1}{N_{\mathrm{MC}}\left(N_{\mathrm{MC}}-1\right)} \sum_{p=1}^{N_{\mathrm{MC}}}\left(G\left(\left|I_{M} \psi_{m}^{n}\left(\xi_{\mathrm{MC}}^{(p)}\right)\right|^{2}\right)-Q_{m, N_{\mathrm{MC}}}^{\mathrm{MC}}\left[G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right]\right)^{2}} \tag{5.6}
\end{equation*}
$$



Figure 2. Convergence of QMC-TSFP in time and physical space.
where $Q_{m, N_{\mathrm{MC}}}^{\mathrm{MC}}\left[G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right]=\sum_{p=1}^{N_{\mathrm{MC}}} G\left(\left|I_{M} \psi_{m}^{n}\left(\xi_{\mathrm{MC}}^{(p)}\right)\right|^{2}\right)$, and the one for QMC reads

$$
\begin{equation*}
R M S \approx \sqrt{\frac{1}{R(R-1)} \sum_{r=1}^{R}\left(Q_{n, N_{\mathrm{QMC}}}^{(r)}\left[G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right]-\bar{Q}_{m, N_{\mathrm{QMC}, R}}\left[G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right]\right)^{2}} \tag{5.7}
\end{equation*}
$$

where $Q_{n, N_{\mathrm{QMC}}}^{(r)}$ and $\bar{Q}_{m, N_{\mathrm{QMC}, R}}$ are defined in (2.10) and (2.11) respectively. The result for $m=16$ is shown in Figure 1b,

From the results in Figure 1, we observe approximately $\mathcal{O}\left(N_{\text {tot }}^{-1 / 2}\right)$ convergence for MC and approximately $\mathcal{O}\left(N_{\text {tot }}^{-1}\right)$ convergence for QMC. We remark that the decaying rate of the potential (5.1) in this example is in fact weaker than what is imposed in Assumption 2.2. The almost-linear convergence rate of QMC is still observed, and this indicates that the condition on the potential for analysis might be relieved.

Example 5.2 (Convergence in time \& physical space). Next, we test the accuracy of QMCTSFP in time and physical space by considering the same setting in Example 5.1with $m=6$, where the reference solution is obtained as before. To investigate the convergence in time, we fix $R=10, N_{\mathrm{QMC}}=2^{20}, h=\frac{\pi}{64}$ in QMC-TSFP and compute the numerical solution by $\tau=\frac{1}{40}, \frac{1}{80}, \frac{1}{160}, \frac{1}{320}$. For the convergence in space, we fix $R=10, N_{\mathrm{QMC}}=2^{20}, \tau=10^{-4}$ and take $h=\frac{\pi}{4}, \frac{\pi}{8}, \frac{\pi}{16}, \frac{\pi}{32}$. We still consider the $L^{2}$ relative error (5.2) at $t_{n}=1$. Temporal convergence is shown in Figure 2a, and spatial convergence is shown in Figure 2b, Clearly from Figure 2, we can observe the second-order convergence rate in time and the exponential convergence rate in space.

### 5.2. Simulation of wave propagation.

Example 5.3. Consider (2.2) with $\alpha>0, \mathbb{T}=[-6 \pi, 6 \pi], \psi_{i n}(x)=1 / \sqrt{\pi} \exp \left(-x^{2}\right)$ and

$$
\begin{equation*}
V_{m}(\boldsymbol{\xi}, x)=1+\sigma \sum_{j=1}^{32} \frac{1}{j^{\frac{3}{2}}} \xi_{j} \cos (j x), \tag{5.8}
\end{equation*}
$$



Figure 3. Simulation of wave propagation in random NLS with $\alpha=1$.


Figure 4. Sample variances of $G\left(|\psi(t, \xi, x)|^{2}\right)$ with $\alpha=1$.
with $\left\{\xi_{j}\right\}$ the i.i.d. random variables uniformly distributed on $[-1,1]$ and $\sigma \geq 0$ denoting the intensity of the randomness. Here the NLS (2.2) is equipped with a defocusing nonlinearity and the initial density function is exponentially localized in the physical domain. We solve (2.2) to simulate the wave propagations till the time $t_{n}=T=10$ by using the QMCTSFP method with $R=10, N_{Q M C}=2^{16}, \tau=\frac{1}{400}$ and $h=\frac{3 \pi}{256}$. The physical observable $G\left(|\psi(t, \xi, x)|^{2}\right)$ in (5.5) is now considered to measure the spreading of the density function in the physical space at time $t \in[0, T]$, and the expectation of $G\left(|\psi(t, \xi, x)|^{2}\right)$ is approximated by

$$
\mathbb{E}[G](t) \approx A(t):=\bar{Q}_{m, N_{\mathrm{QMC}, R}}\left[G\left(\left|I_{M} \psi_{m}^{n}\right|^{2}\right)\right](t)
$$

with $\bar{Q}_{m, N_{\mathrm{QMC}, R}}$ in (2.11).


Figure 5. Sample variances of $|\psi(T, \xi, x)|^{2}$ with $\alpha=1$.


Figure 6. Simulation of wave propagation in random NLS with $\sigma=4$.


Figure 7. Sample variances of $G\left(|\psi(t, \xi, x)|^{2}\right)$ with $\sigma=4$.
Fixing $\alpha=1$, the temporal behaviour of $A(t)$ is shown in Figure 3a for different $\sigma$, and the quantity

$$
\Psi(x):=\frac{1}{R N} \sum_{r=1}^{R} \sum_{p=1}^{N}\left|I_{M} \psi_{m}^{n}\left(\xi^{(r, p)}, x\right)\right|^{2},
$$



Figure 8. Sample variances of $|\psi(T, \xi, x)|^{2}$ with $\sigma=4$.
which approximates the expectation of the density $|\psi(T, \xi, x)|^{2}$ at the final time $T=10$, is shown in Figure 3b, In addition, the sample variances of the physical observable $G\left(|\psi(t, \xi, x)|^{2}\right)$ and the density $|\psi(T, \xi, x)|^{2}$ at the final time $T=10$ are shown in Figures 4 and 5, respectively. In this case, we observe pure diffusion when randomness is absent in the potential (5.8), and we observe localization of the expected density function within the computational time when randomness presents in the potential (5.8). When the randomness intensity $\sigma$ increases, the profile becomes more localized. By fixing $\sigma=4$ in (2.2), the corresponding results for different $\alpha$ are provided in Figures 6, 7 and 8. In this case, the profile of the expected density expands more as $\alpha$ becomes larger. This shows the defocusing nonlinear effect on the wave propagation.

## 6. Conclusion

We proposed to combine the randomly shifted QMC lattice rule with the time-splitting Fourier pseudospectral method for solving the nonlinear Schrödinger equation with random potential. We analysed the convergence in random space by using the technique of the weighted Sobolev space. The nonlinearity in the equation introduces difficulties in estimating the parametric regularity of the solution, and the physical observable considered here is typically a nonlinear functional of the solution. We propose sufficient conditions to show the existence of a QMC rule that can achieve the almost-linear and dimension-independent convergence rate for the expected value of the physical observable. The full error estimate of the scheme is established which also covers the convergence in time and in the physical space. Numerical examples are presented to verify the theoretical result and simulations are done to investigate the wave propagation in the random NLS.

## Acknowledgement

Z. Zhang is supported by Hong Kong RGC grant (projects 17300318 and 17307921), NSFC 12171406, a Seed Funding from the HKU-TCL Joint Research Centre for Artificial Intelligence, and the outstanding young researcher award of HKU (2020-21). X. Zhao is supported by NSFC 12271413, 11901440 and the Natural Science Foundation of Hubei Province

2019CFA007. The computations were performed using research computing facilities provided by Information Technology Services, the University of Hong Kong.

## Appendix A. Some useful tools for studying NLS

To study the regularity of the solution of (1.3) in the physical space and analyze the error resulting from dimension truncation, we need two useful tools introduced in 51]. We briefly review these two tools in this section.

The bootstrap-type argument. The bootstrap-type argument is an application of the bootstrap principle or the continuity method, which can be viewed as a continuous analogue of the principle of mathematical induction. The abstract bootstrap principle works in the following way.

Proposition A.1. Let I be a time interval, and for each $t \in I$ suppose we have two statements, a "hypothesis" $\mathbf{H}(t)$ and a "conclusion" $\mathbf{C}(t)$. Suppose we can verify the following four assertions:
(a) (Hypothesis implies conclusion) If $\mathbf{H}(t)$ is true for some time $t \in I$, then $\mathbf{C}(t)$ is also true for for that time $t$.
(b) (Conclusion is stronger than hypothesis) If $\mathbf{C}(t)$ is true for some $t \in I$, then $\mathbf{H}\left(t^{\prime}\right)$ is true for all $t^{\prime} \in I$ in a neighbourhood of $t$.
(c) (Conclusion is closed) If $t_{1}, t_{2}, \ldots$ is a sequence of times in I which converges to another time $t \in I$, and $\mathbf{C}\left(t_{n}\right)$ is true for all $t_{n}$, then $\mathbf{C}(t)$ is true.
(d) (Base case) $\mathbf{H}(t)$ is true for at least one time $t \in I$.

Then $\mathbf{C}(t)$ is true for all $t \in I$.
The proof of this proposition can be found in [51, Chapter 1.3]. Moreover, for an illustration of how the bootstrap-type argument is applied to proving the local well-posedness of NLS, we refer readers to the proof of Proposition 3.8 in [51].

The product lemma. As a useful tool from harmonic analysis, the product lemma reads as follows:

Lemma A.2. For all $f, g \in H^{s}(\mathbb{T})$ with $s \geq 1$, we have

$$
\begin{equation*}
\|f g\|_{H^{s}(\mathbb{T})} \leq C_{s}\left(\|f\|_{H^{s}(\mathbb{T})}\|g\|_{L^{\infty}(\mathbb{T})}+\|f\|_{L^{\infty}(\mathbb{T})}\|g\|_{H^{s}(\mathbb{T})}\right), \tag{A.1}
\end{equation*}
$$

where $C_{s}>0$ is a constant dependent on $s$.
Proof. Let $\mathbb{T}=(0,2 \pi)$ for simplicity, and the Fourier expansion reads $f(x)=\sum_{l \in \mathbb{Z}} \hat{f}_{l} e^{i l x}$ where $\hat{f}_{l}$ denotes the Fourier coefficient. For some $l \in \mathbb{Z}$, we consider

$$
\left(1+l^{2}\right)^{s / 2}\left|\widehat{(f g)_{l}}\right|=\left(1+l^{2}\right)^{s / 2}\left|\sum_{l_{1}+l_{2}=l} \hat{f}_{l_{1}} \hat{g}_{l_{2}}\right| \leq \sum_{l_{1}+l_{2}=l}\left[1+\left(l_{1}+l_{2}\right)^{2}\right]^{s / 2}\left|\hat{f}_{l_{1}} \hat{g}_{l_{2}}\right| .
$$

It is direct to check that

$$
\left[1+\left(l_{1}+l_{2}\right)^{2}\right]^{s / 2} \leq 2^{s / 2}\left[\sqrt{1+l_{1}^{2}}+\sqrt{1+l_{2}^{2}}\right]^{s} \leq C_{s}\left[\left(1+l_{1}^{2}\right)^{s / 2}+\left(1+l_{2}^{2}\right)^{s / 2}\right]
$$

where the last inequality used the convexity of the function $x^{s}$ for $x>0, s \geq 1$. In the following, the constant $C_{s}>0$ may be different at each occurrence. Combining the above, we deduce $\left(1+l^{2}\right)^{s / 2}\left|\widehat{(f g)_{l}}\right| \leq C_{s} \sum_{l_{1}+l_{2}=l}\left(1+l_{1}^{2}\right)^{s / 2}\left|\hat{f}_{l_{1}} \hat{g}_{l_{2}}\right|+C_{s} \sum_{l_{1}+l_{2}=l}\left(1+l_{2}^{2}\right)^{s / 2}\left|\hat{f}_{l_{1}} \hat{g}_{l_{2}}\right|$. By taking the square, we find

$$
\left(1+l^{2}\right)^{s}\left|\widehat{(f g)_{l}}\right|^{2} \leq C_{s} \sum_{l_{1}+l_{2}=l}\left(1+l_{1}^{2}\right)^{s}\left|{\hat{l_{1}}}_{l_{1}} \hat{l}_{l_{2}}\right|^{2}+C_{s} \sum_{l_{1}+l_{2}=l}\left(1+l_{2}^{2}\right)^{s}\left|{\hat{l_{1}}}_{l_{1}} \hat{g}_{2}\right|^{2}
$$

By Parseval's identity and note $\|f g\|_{H^{s}}^{2}=\sum_{l \in \mathbb{Z}}\left(1+l^{2}\right)^{s}\left|\widehat{(f g)_{l}}\right|^{2}$, we get

$$
\|f g\|_{H^{s}}^{2} \leq C_{s}\left(\|f\|_{H^{s}}^{2}\|g\|_{L^{2}}^{2}+\|g\|_{H^{s}}^{2}\|f\|_{L^{2}}^{2}\right.
$$

The assertion of the lemma is obtained by noting $\|f\|_{L^{2}} \leq C\|f\|_{L^{\infty}}$ and Sobolev's embedding $\|f\|_{L^{\infty}} \leq C_{s}\|f\|_{H^{s}}$. For more general statement of product lemma, see e.g., 51.

## References

[1] R. Altmann, P. Henning, D. Peterseim, Quantitative Anderson localization of Schrödinger eigenstates under disorder potentials, Math. Models Methods Appl. Sci., 30 (2020) pp. 917-955.
[2] R. Altmann, P. Henning, D. Peterseim, Localization and delocalization of ground states of BoseEinstein condensates under disorder, SIAM J. Appl. Math., 82 (2022) pp. 330-358.
[3] P.W. Anderson, Absence of diffusion in certain random lattices, Phys. Rev., 109 (1958) 1492.
[4] X. Antoine, W. Bao, C. Besse, Computational methods for the dynamics of the nonlinear Schrödinger/Gross-Pitaevskii equations, Comput. Phys. Commun., 184 (2013) pp. 2621-2633.
[5] R.A. Adams, J.J.F. Fournier, Sobolev Spaces, Academic Press, 2021.
[6] I. Babuška, F. Nobile, R. Tempone, A stochastic collocation method for elliptic partial differential equations with random input data, SIAM J. Numer. Anal., 45 (2007) pp. 1005-1034.
[7] W. Bao, Y. Cai, Mathematical theory and numerical methods for Bose-Einstein condensation, Kinet. Relat. Models, 6 (2013) pp. 1-135.
[8] R.E. Caflisch, Monte Carlo and quasi-Monte Carlo methods, Acta Numer., 7 (1998) pp. 1-49.
[9] C. Conti, Solitonization of the Anderson localization, Phy. Rev. A, 86 (2012) 061801.
[10] A. Cohen, R. De Vore, Ch. Schwab, Convergence rates of best N-term Galerkin approximations for a class of elliptic sPDEs, Found. Comput. Math., 10 (2010) pp. 615-646.
[11] J. Dick, On the convergence rate of the component-by-component construction of good lattice rules, J. Complexity, 20 (2004) pp. 493-522.
[12] J. Dick, F.Y. Kuo, I.H. Sloan, High-dimensional integration: the quasi-Monte Carlo way, Acta Numer., 22 (2013) pp. 133-288.
[13] A. Debussche, J. Martin, Solution to the stochastic Schrödinger equation on the full space, Nonlinearity, 32 (2019) pp. 1147-1174.
[14] A. Debussche, H. Weber, The Schrödinger equation with spatial white noise potential, Electron. J. Probab., 23 (2018) pp. 1-16.
[15] L. Dumaz, C. Labbé, Localization of the continuous Anderson Hamiltonian in 1-D, Probab. Theory Relat. Fields, 176 (2020) pp. 353-419.
[16] E. Faou, Geometric Numerical Integration and Schrödinger Equations, European Math. Soc. Publishing House, Zürich 2012.
[17] M. Filoche, S. Mayboroda, Universal mechanism for Anderson and weak localization, Proc. Natl. Acad. Sci., 109 (2012) pp. 14761-14766.
[18] S. Flach, D.O. Krimer, Ch. Skokos, Universal spreading of wave packets in disordered nonlinear systems, Phys. Rev. Lett., 102 (2009) 024101.
[19] S. Fishman, Y. Krivolapov, A. Soffer, The nonlinear Schrödinger equation with a random potential: results and puzzles, Nonlinearity, 25 (2012) pp. 53-72.
[20] F. Germinet, A. Klein, A comprehensive proof of localization for continuous Anderson models with singular random potentials, J. Eur. Math. Soc., 15 (2013) pp. 53-143.
[21] R.G. Ghanem, P.D. Spanos, Stochastic Finite Elements, Dover, 1991.
[22] N. Ghofraniha, S. Gentilini, V. Folli, E. Delre, C. Conti, Shock waves in disordered media, Phys. Rev. Lett., 109 (2012) 243902.
[23] A.D. Gilber, I.G. Graham, F.Y. Kuo, R. Scheichl, I.H. Sloan, Analysis of quasi-Monte Carlo methods for elliptic eigenvalue problems with stochastic coefficients, Numer. Math., 142 (2019) pp. 863915.
[24] J. Ginibre, G. Velo, On a class of non linear Schrödinger equations. III. Special theories in dimensions 1, 2 and 3, Annales del'IHP Physique théorique, 28 (1978) pp. 287-316.
[25] I.G. Graham, F.Y. Kuo, J.A. Nichols, R. Scheichl, Ch. Schwab, I.H. Sloan, Quasi-Monte Carlo finite element methods for elliptic PDEs with lognormal random coefficients, Numer. Math., 131 (2015) pp. 329-368.
[26] I.G. Graham, F.Y. Kuo, D. Nuyens, R. Scheichl, I.H. Sloan, Quasi-Monte Carlo methods for elliptic PDEs with random coefficients and applications, J. Comput. Phys., 230 (2011) pp. 3668-3694.
[27] Y. Gu, T. Komorowski, L. Ryzhik, The Schrödinger equation with spatial white noise: The average wave function, J. Funct. Anal., 274 (2018) pp. 2113-2138.
[28] J.H. Halton, On the efficiency of certain quasi-random sequences of points in evaluating multidimensional integrals, Numer. Math., 2 (1960) pp. 84-90.
[29] P. Henning, D. Peterseim, Crank-Nicolson Galerkin approximations to nonlinear Schrödinger equations with rough potentials, Math. Models Methods Appl. Sci., 27 (2017) pp. 2147-2184.
[30] J. Hu, S. Jin, D. Xiu, A Stochastic Galerkin Method for Hamilton-Jacobi Equations with Uncertainty, SIAM J. Sci. Comput., 37 (2015) pp. A2246-A2269.
[31] S. Jin, P. Markowich, C. Sparber, Mathematical and computational methods for semiclassical Schrödinger equations, Acta Numer., 20 (2011) pp. 121-209.
[32] O. Kallenberg, Foundations of modern probability, vol. 2, Springer, 1997.
[33] T. Kachman, S. Fishman, A. Soffer, Numerical implementation of the multiscale and averaging methods for quasi periodic systems, Comput. Phys. Comm., 221 (2017) pp. 235-245.
[34] F.Y. Kuo, webpage, https://people.cs.kuleuven.be/ dirk.nuyens/qmc4pde/
[35] F.Y. Kuo, D. Nuyens, Application of quasi-Monte Carlo methods to elliptic PDEs with random diffusion coefficients-a survey of analysis and implementation, Found. Comput. Math., 16 (2016) pp. 1631-1696.
[36] F.Y. Kuo, C. Schwab, I.H. Sloan, Quasi-Monte Carlo methods for high-dimensional integration: the standard (weighted Hilbert space) setting and beyond, ANZIAM J., 53 (2011) pp. 1-37.
[37] F.Y. Kuo, C. Schwab, I.H. Sloan, Quasi-Monte Carlo finite element methods for a class of elliptic partial differential equations with random coefficients, SIAM J. Numer. Anal., 50 (2012) pp. 3351-3374.
[38] P. Lee, T. Ramakrishnan, Disordered electronic systems, Rev. Modern Phys., 57 (1985) pp. 287-337.
[39] Ch. Lubich, On splitting methods for Schrödinger-Poisson and cubic nonlinear Schrödinger equations, Math. Comp., 77 (2008) pp. 2141-2153.
[40] M. Piraud, P. Lugan, P. Bouyer, A. Aspect, L. Sanchez-Palencia, Localization of a matter wave packet in a disordered potential, Phys. Rev. A, 83 (2011) 031603.
[41] H. Niederreiter, Random number generation and quasi-Monte Carlo methods, SIAM, 1992.
[42] F. Nobile, R. Tempone, C. Webster, A sparse grid stochastic collocation method for partial differential equations with random input data, SIAM J. Numer. Anal., 46 (2008) pp. 2309-2345.
[43] D. Nuyens, R. Cools, Fast algorithms for component-by-component construction of rank-1 lattice rules in shift-invariant reproducing kernal Hilbert spaces, Math. Comp., 75 (2006) pp. 903-920.
[44] A.S. Pikovsky, D.L. Shepelyansky, Destruction of Anderson localization by a weak nonlinearity, Phys. Rev. Lett., 100 (2008) 094101.
[45] L. Sanchez-Palencia, D. Clément, P. Lugan, P. Bouyer, G. V. Shlyapnikov, A. Aspect, Anderson localization of expanding Bose-Einstein condensates in random potentials, Phys. Rev. Lett., 98 (2007) 210401.
[46] C. Schwab, R. Todor, Karhunen-Loève approximation of random fields by generalized fast multipole methods, J. Comput. Phys., 217 (2006) pp. 100-122.
[47] I.H. Sloan, F.Y. Kuo, S. Joe, Constructing randomly shifted lattice rules in weighted Sobolev spaces, SIAM J. Numer. Anal., 40 (2002) pp. 1650-1665.
[48] I.M. Sobol, On the distribution of points in a cube and the approximate evaluation of integrals, Zh. Vychisl. Mat. i Mat. Fiz., 7 (1967) pp. 784-802.
[49] Y. Suzuki, G. Suryanarayana, D. Nuyens, Strang splitting in combination with rank-1 and rank-r lattices for the time-dependent Schrödinger equation, SIAM J. Sci. Comput. 41 (2019) pp. B1254-B1283.
[50] J. Shen, T. Tang, L. Wang, Spectral Methods: Algorithms, Analysis and Applications, Springer, 2011.
[51] T. TAO, Nonlinear dispersive equations: local and global analysis, Amer. Math. Soc., 2006.
[52] T. TANG, T. Zhou, Convergence analysis for stochastic collocation methods to scalar hyperbolic equations with a random wave speed, Commun. Comput. Phys., 8 (2010) pp. 226-248.
[53] X. Wang, Strong tractability of multivariate integration using quasi-Monte Carlo algorithms, Math. Comp., 72 (2003) pp. 823-838.
[54] D. Xiu, G.E. Karniadakis, The Wiener-Askey polynomial chaos for stochastic differential equations, SIAM J. Sci. Comput., 24 (2002) pp. 619-644.
[55] X. Zhao, Numerical integrators for continuous disordered nonlinear Schrödinger equation, J. Sci. Comput., 89 (2021) 40.
Z. Wu: Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong, China.

Email address: wuzz@hku.hk
Z. Zhang: Department of Mathematics, The University of Hong Kong, Pokfulam Road, Hong Kong, China.

Email address: zhangzw@hku.hk
X. Zhao: School of Mathematics and Statistics \& Computational Sciences Hubei Key Laboratory, Wuhan University, Wuhan, 430072, China

Email address: matzhxf@whu.edu.cn
URL: http://jszy.whu.edu.cn/zhaoxiaofei/en/index.htm

