

## A QUASI MONTE CARLO-BASED MODEL REDUCTION METHOD FOR SOLVING HELMHOLTZ EQUATION IN RANDOM MEDIA

Dingjiong Ma<sup>⊠</sup> and Zhiwen Zhang<sup>⊠</sup>\*

Department of Mathematics, The University of Hong Kong Pokfulam Road, Hong Kong SAR, China

(Communicated by Hongyu Liu)

ABSTRACT. Wave propagation in random media has broad applications in materials science and engineering. In this paper, we develop a quasi Monte Carlo (qMC)-based model reduction method for solving random Helmholtz equations. In the physical space, we construct multiscale reduced basis functions by using an optimization method together with the proper orthogonal decomposition method. Then, in the random space we employ the qMC method for discretization. Under mild conditions, we prove that the spatial grid size is only proportional to the wave number, and almost a first-order convergence rate is achieved in the random space with respect to the number of samples. Since the exact solution oscillates in both physical and random spaces, our approach provides an efficient strategy to find its numerical approximation. One significant advantage of our approach over existing methods is its applicability to generic random media which cannot be treated as random perturbations of homogeneous media. These are confirmed by a series of numerical examples.

1. Introduction. Random materials are common in nature and their microstructures can be characterized only statistically in many instances [31]. A notable phenomenon is Anderson localization [1], which states the absence of diffusion of waves in disordered media and later experimentally realized in many materials; see [32, 29, 26, 28, 34] for examples. Such a disorder was originally characterized by white noise without any spatial correlation [1], but was later extended to correlated noises [5]. Experimental realizations include electron waves for electronic systems [5, 16, 34] which are modeled by the Schrödinger equation and its analogs, and electromagnetic waves for photonic systems [32, 29, 28] which are modeled by Maxwell's equation and its analogs.

Our focus in this work is the electromagnetic wave propagation in random media. To be specific, we consider the following interior impedance problem associated with Helmholtz equation with random refractive index:

$$\begin{cases} -\Delta u(\mathbf{x},\omega) - k^2 n(\mathbf{x},\omega) u(\mathbf{x},\omega) = f(\mathbf{x}), & \mathbf{x} \in D, \omega \in \Omega, \\ \partial_{\nu} u - ik \sqrt{n(\mathbf{x},\omega)} u(\mathbf{x},\omega) = 0, & \mathbf{x} \in \Gamma, \end{cases}$$
(1)

<sup>2020</sup> Mathematics Subject Classification. Primary: 35J05, 65M60; Secondary: 11K45, 65K10. Key words and phrases. Helmholtz equation, random media, multiscale reduced basis function, quasi-Monte Carlo (qMC) method, convergence analysis.

<sup>\*</sup>Corresponding author: Zhiwen Zhang.

where  $D = [0, 1]^d$  is the spatial domain,  $\Gamma = \partial D$  is the domain boundary,  $\Omega$  is the random space,  $\nu$  is the unit outward normal vector on  $\Gamma$ , k is the wave number, and  $n(\mathbf{x}, \omega) > 0$  is the random refractive index. A precise definition of the random refractive index will be given in Section 3.1.

From the perspective of scientific computation and numerical analysis, there are very few works on (13); see e.g. [13, 14]. Assuming that random media can be expressed as random perturbations of homogeneous media, the authors proposed a multimode representation of the solution and approximated mode functions by classical numerical schemes. Both the multimode representation and the numerical approximation contribute to error estimates. It is expected that such a strategy works well if random media are "close" to homogeneous media, as demonstrated both theoretically and numerically in [13]. As a matter of fact, in such a setting, it was found that there is no drastic and unexpected effect on the behavior of wave motion in the media [8]. Therefore, it is important to consider disordered media with stronger randomness where phases such as wave localization and intermediate states can be expected. This motivates the current work.

There are two major difficulties to solving (13) numerically when the wave number k is high. One is the well-known pollution effect [2], i.e., the solution of the standard finite element method differs significantly from the best approximation with increasing wave number k. The other one is the oscillatory profile of the solution in the random space with frequency proportional to wave number k, which will be demonstrated later.

Motivated by our recent work for random Schrödinger equations [7], we propose a qMC-based model reduction method to solve the Helmholtz equation in random media here. Our method consists of offline and online stages. In the offline stage, we apply an optimization approach to the Helmholtz equation for each realization of the random media and obtain a set of multiscale basis functions. Then, we apply the proper orthogonal decomposition for the multiscale basis functions and extract a small number of multiscale reduced basis functions, which provide a quasi-optimal approximation to the solution of random Helmholtz equation. In the online stage, we use these basis functions in the physical space in the Galerkin formulation and the qMC method to approximate the random space of the solution. Under mild conditions, we provide a convergence analysis of our method. Finally, we present numerical experiments to demonstrate the efficiency and accuracy of the proposed methods. We find the proposed method is efficient in the sense that the number of basis functions is only proportional to the wave number k and the number of samples in the qMC method is inversely proportional to a power of k. Our method still works well for generic random media that have a large fluctuation.

The rest of the paper is organized as follows. In Section 2, we first provide the construction of multiscale basis functions using an optimization approach for deterministic Helmholtz equations, and then in Section 3 we propose a qMC-based model reduction method to solve Helmholtz equations in random media. In Section 4, we provide analysis results of our method. Numerical results in 1D and 2D are given to demonstrate the convergence and efficiency of the proposed method in Section 5. Finally, conclusions and discussions are drawn in Section 6.

## 2. Multiscale basis functions for Helmholtz equations.

2.1. Construction of multiscale basis functions. First of all, we introduce how to construct multiscale basis functions for Helmholtz equations by using an optimization approach. To demonstrate the idea, we consider the following deterministic problem

$$\begin{cases} -\Delta u(\mathbf{x}) - k^2 n(\mathbf{x}) u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in D, \\ \partial_{\nu} u - ik \sqrt{n(\mathbf{x})} u(\mathbf{x}) = 0, & \mathbf{x} \in \Gamma, \end{cases}$$
(2)

where  $n(\mathbf{x}) > 0$  is a deterministic refractive index and the definitions of other notations are the same as before.

We define the Helmholtz operator  $\mathcal{H}(\cdot) \equiv -\Delta(\cdot) - k^2 n(\mathbf{x})(\cdot)$  and introduce the following energy notation  $||\cdot||_V$  for the Helmholtz operator

$$||u||_{V} = (\mathcal{H}u, u)_{D} = \int_{D} |\nabla u|^{2} - k^{2} n(\mathbf{x})|u|^{2} \mathbf{x}.$$
(3)

The construction of (problem-dependent) multiscale basis functions by using an optimization approach has been used to solve many types of deterministic partial differential equations (PDEs), including multiscale elliptic PDEs, Schrödinger equations, and Helmholtz equations; see e.g. [3, 25, 21, 22, 6]. Some local subscale correction idea was used to solve multiscale elliptic PDEs and Helmholtz equations in [23, 27]. Though the energy notation  $|| \cdot ||_V$  in (3) does not define a norm, we can still construct multiscale basis functions for the Helmholtz equation (2) on a coarse mesh by using an optimization approach.

To construct such localized multiscale basis functions, we first partition the physical domain D into a set of regular coarse elements with grid size H. For example, we divide D into a set of non-overlapping triangles  $\mathcal{T}_H = \bigcup \{K\}$ , such that no vertex of one triangle lies in the interior of the edge of another triangle. On each element K, we define a set of nodal basis  $\{\varphi_{j,K}, j = 1, 2, 3\}$ . From now on, we neglect the subscript K for notational convenience.

Let  $\mathcal{N}$  denote the set of vertices of  $\mathcal{T}_H$  (removing the repeated vertices due to the periodic boundary condition) and  $N_H$  be the number of vertices. For every vertex  $\mathbf{x}_j \in \mathcal{N}$ , let  $\varphi_j^H(\mathbf{x})$  denote the corresponding nodal basis function, i.e.,  $\varphi_j^H(\mathbf{x}_{j'}) = \delta_{jj'}$ . Since all the nodal basis functions  $\varphi_j^H(\mathbf{x})$  are continuous across the boundaries of the elements, we have

$$V^H = \{\varphi_j^H(\mathbf{x}) : j = 1, ..., N_H\} \subset H^1(D).$$

Then, we solve optimization problems to obtain the multiscale basis functions. Specifically, let  $\phi_j(\mathbf{x})$ ,  $1 \leq j \leq N_H$ , be the minimizer of the following constrained optimization problem

$$\phi_j = \underset{\phi \in H^1(D)}{\arg\min} ||\phi||_V \tag{4}$$

s.t. 
$$\int_{D} \phi(\mathbf{x}) \varphi_{j'}^{H}(\mathbf{x}) \dot{\mathbf{x}} = \delta_{jj'}, \ \forall 1 \le j' \le N_{H},$$
(5)

where  $\varphi_{j'}^{H}(\mathbf{x})$  are called measurement functions in our method. Notice that we solve the optimization problem in  $H^1(D)$  since the impedance boundary condition is imposed for the Helmholtz equation (2). If the Dirichlet boundary condition is imposed, we need to modify the searching space by incorporating the boundary condition accordingly.

In general, one cannot solve the optimization problem (4)-(5) analytically. Therefore, we use numerical methods to solve it. Specifically, we partition the physical domain D into a set of non-overlapping fine triangles with size  $h = O(k^{-2})$ . Let  $\varphi_s^h(\mathbf{x}), s = 1, ..., N_h$  denote the finite element basis functions defined on fine triangles with size h, where  $N_h$  is the number of basis functions. Then, we discretize  $\phi_j(\mathbf{x}), \varphi_{j'}^H(\mathbf{x}), 1 \leq j, j' \leq N_H$  by using the fine-scale basis functions  $\varphi_s^h(\mathbf{x})$ . After discretization, the optimization problem (4)-(5) reduces to a constrained quadratic optimization problem; see (56) in A, which can be efficiently solved using Lagrange multiplier methods. Finally, with these multiscale basis functions  $\{\phi_j(\mathbf{x})\}_{j=1}^{N_H}$ , we can approximate the solution by  $u(\mathbf{x}) = \sum_{j=1}^{N_H} u_j \phi_j(\mathbf{x})$  and solve the Helmholtz equation (2) combined with Galerkin projection method.

**Remark 2.1.** In analogy to the multiscale finite element method [20, 11], the multiscale basis functions  $\{\phi_j(\mathbf{x})\}_{j=1}^{N_H}$  are defined on coarse elements with size H. However, they are represented by fine-scale basis with size h, which can be precomputed and done in parallel.

2.2. Some properties of the multiscale basis functions. It can be proved that the multiscale basis functions  $\{\phi_j(\mathbf{x})\}_{j=1}^{N_H}$  decay exponentially fast away from its associated vertex  $\mathbf{x}_j \in \mathcal{N}$  under certain conditions. This allows us to localize the basis functions to a relatively smaller domain and reduce the computational cost. We first define a series of nodal patches  $\{D_\ell\}$  associated with  $\mathbf{x}_i \in \mathcal{N}$  as

$$D_0 := \sup\{\varphi_j\} = \bigcup\{K \in \mathcal{T}_H | \mathbf{x}_j \in K\},\tag{6}$$

$$D_{\ell} := \bigcup \{ K \in \mathcal{T}_H | K \cap \overline{D_{\ell-1}} \neq \emptyset \}, \quad \ell = 1, 2, \cdots.$$

$$\tag{7}$$

**Assumption 2.2.** We assume the refractive index  $n(\mathbf{x})$  is bounded, i.e.,  $n_0 := ||n(\mathbf{x})||_{L^{\infty}(D)} < +\infty$  and the grid size H of  $\mathcal{T}_H$  satisfies

$$\sqrt{n_0}kH \lesssim 1,\tag{8}$$

where  $\leq$  means bounded from above by a constant.

The Assumption 2.2 for coarse elements is called the resolution condition. One can see that the number of basis functions in our method is significantly reduced compared with the standard finite element method. Moreover, the multiscale finite element basis functions have the exponentially decaying property, which further reduces the cost of our method.

**Proposition 2.3.** Under the resolution condition of the coarse mesh, i.e., (8), there exist constants C > 0 and  $0 < \beta < 1$  independent of H, such that

$$||\nabla\phi_j(\boldsymbol{x})||_{L^2(D\setminus D_\ell)} \le C\beta^\ell ||\nabla\phi_j(\boldsymbol{x})||_{L^2(D)},\tag{9}$$

for any  $j = 1, 2, ..., N_H$ .

Proof of (9) for a deterministic refractive index can be found in [27]. The main idea is to combine an iterative Caccioppoli-type argument [23, 22] and some refined estimates with respect to the patches index.

The exponential decay property enables us to localize the support sets of the basis functions  $\{\phi_j(\mathbf{x})\}_{j=1}^{N_H}$  so that the corresponding stiffness matrix is sparse and the computational cost is reduced. In practice, we define a modified constrained optimization problem as follows

$$\phi_j^{\text{loc}} = \underset{\phi \in H^1(D)}{\arg\min} ||\phi||_V \tag{10}$$

QMC-BASED MODEL REDUCTION METHOD FOR HELMHOLTZ EQUATION 3

s.t. 
$$\int_{D_{l^*}} \phi \varphi_{j'}^H \dot{\mathbf{x}} = \delta_{jj'}, \ \forall 1 \le j' \le N_H,$$
(11)

$$\phi(\mathbf{x}) = 0, \ \mathbf{x} \in D \backslash D_{l^*},\tag{12}$$

where  $D_{l^*}$  is the support set of the localized multiscale basis function  $\phi_j^{\text{loc}}(\mathbf{x})$  and the choice of the integer  $l^*$  depends on the decaying speed of  $\phi_j^{\text{loc}}(\mathbf{x})$ . In (11) and (12), we have used the fact that  $\phi_j(\mathbf{x})$  has the exponentially decaying property so that we can localize the support set of  $\phi_j(\mathbf{x})$  to a smaller domain  $D_{l^*}$ . In numerical experiments, we find that a small integer  $l^* \sim \log(L/H)$  will give accurate results, where L is the diameter of the domain D. Moreover, the optimization problem (10)-(12) can be solved in parallel. Therefore, the exponentially decaying property significantly reduces the computational cost in constructing basis functions and computing the solution of the Helmholtz equation (2).

## 3. A qMC-based model reduction method for random Helmholtz equation.

3.1. Model reduction using multiscale reduced basis functions. In this section, we present the numerical method to solve the following Helmholtz equation with a random refractive index

$$\begin{cases} -\Delta u(\mathbf{x},\omega) - k^2 n(\mathbf{x},\omega) u(\mathbf{x},\omega) = f(\mathbf{x}), & \mathbf{x} \in D, \omega \in \Omega, \\ \partial_{\nu} u - ik\sqrt{n(\mathbf{x},\omega)} u(\mathbf{x},\omega) = 0, & \mathbf{x} \in \Gamma, \end{cases}$$
(13)

where  $D = [0, 1]^d$  is the spatial domain,  $\Gamma = \partial D$  is the domain boundary,  $\Omega$  is the random space,  $\nu$  is the unit outward normal vector on  $\Gamma$ , k is the wave number, and  $n(\mathbf{x}, \omega) > 0$  is the random refractive index.

We assume  $n(\mathbf{x}, \omega)$  has an affine form and is parameterized by m random variables, i.e.,

$$n(\mathbf{x},\omega) = \bar{n}(\mathbf{x}) + \sum_{j=1}^{m} \sqrt{\lambda_j} \xi_j(\omega) \psi_j(\mathbf{x}), \qquad (14)$$

where  $\xi_j(\omega)$ 's are independent uniform random variables on [0, 1],  $\psi_j(\mathbf{x})$ 's are physical components, and  $\lambda_j$ 's are the corresponding strengths. In addition, we assume  $n(\mathbf{x}, \omega)$  is almost surely positive and bounded. Namely, there exist  $n_{\max} > 0$  and  $n_{\min} > 0$ , such that

$$P(\omega \in \Omega \mid n(\mathbf{x}, \omega) \in [n_{\min}, n_{\max}], \ \forall \mathbf{x} \in D) = 1.$$
(15)

For the random Helmholtz equation (13), it is prohibitively expensive to construct multiscale basis functions for each realization of the refractive index by solving the optimization problem (10)-(12). To address this issue, we use a model reduction method to build a small number of reduced basis functions that enable us to obtain multiscale basis functions in a cheaper way.

For every  $\mathbf{x}_j \in \mathcal{N}$ , we first compute a set of samples of multiscale basis functions associated with the vertex  $\mathbf{x}_j$ . Specifically, let  $\{n(\mathbf{x}, \omega_q)\}_{q=1}^Q$  be samples of the refractive index that are obtained using the Monte Carlo (MC) method or qMC method, where Q is the number of samples. Denote  $\zeta_0^j(\mathbf{x}) = \frac{1}{Q} \sum_{q=1}^Q \phi_j^{\text{loc}}(\mathbf{x}, \omega_q)$ the sample mean of the basis functions, and  $\tilde{\phi}_j^{\text{loc}}(\mathbf{x}, \omega_q) = \phi_j^{\text{loc}}(\mathbf{x}, \omega_q) - \zeta_0^j(\mathbf{x})$  is the fluctuation of the *j*-th basis function.

We apply the proper orthogonal decomposition (POD) method [4, 30] to the snapshot space  $V = \{\tilde{\phi}_j^{\text{loc}}(\mathbf{x}, \omega_q)\}_{q=1}^Q$  and build a set of basis functions  $\{\zeta_1^j(\mathbf{x}), \zeta_2^j(\mathbf{x}), ..., \zeta_{m_j}^j(\mathbf{x})\}$  with  $m_j \ll Q$  that optimally approximates V. We have the following approximating property.

**Proposition 3.1.** Let  $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_{m_j} \geq \lambda_{m_j+1} \geq ... > 0$  be positive eigenvalues of the covariance kernel associated with the snapshot space V and the corresponding eigenfunctions are  $\zeta_1^j(\mathbf{x}), ..., \zeta_{m_j}^j(\mathbf{x}), ...$  Then, the reduced basis functions  $\{\zeta_l^j(\mathbf{x})\}_{l=1}^{m_j}$  have the following approximation property

$$\frac{\sum_{q=1}^{Q} \left\| \left| \tilde{\phi}_{j}^{loc}(\boldsymbol{x}, \omega_{q}) - \sum_{l=1}^{m_{j}} \left( \tilde{\phi}_{j}^{loc}(\boldsymbol{x}, \omega_{q}), \zeta_{l}^{j}(\boldsymbol{x}) \right)_{X} \zeta_{l}^{j}(\boldsymbol{x}) \right\|_{X}^{2}}{\sum_{q=1}^{Q} \left\| \left| \tilde{\phi}_{j}^{loc}(\boldsymbol{x}, \omega_{q}) \right\|_{X}^{2}} = \frac{\sum_{s=m_{j}+1}^{Q} \lambda_{s}}{\sum_{s=1}^{Q} \lambda_{s}}, \quad (16)$$

where  $X = L^2(D)$  or  $X = H^1(D)$  and the number  $m_j$  is determined according to the ratio  $\rho = \frac{\sum_{s=1}^{m_j} \lambda_s}{\sum_{s=1}^{Q} \lambda_s}$ .

In practice, we choose the first  $m_j$  dominant reduced basis functions such that  $\rho$  is close enough to 1, in order to achieve a desired accuracy, say  $\rho = 99\%$ . More details of the POD method can be found in [4, 30]. Notice that reduced basis functions  $\zeta_0^j(\mathbf{x})$  and  $\zeta_l^j(\mathbf{x})$ ,  $l = 1, ..., m_j$  approximately capture the mean profile and fluctuation of multiscale basis functions associated with  $\mathbf{x}_j$ , respectively. Thus, for each realization of the random potential, the associated multiscale basis functions can be approximated by the reduced basis functions.

**Remark 3.2.** Constructing the multiscale reduced basis functions requires additional computational cost in the offline stage. However, the precomputed reduced basis functions can be used repeatedly to solve (13) for each realization of the random potential and different source function  $f(\mathbf{x})$ , which results in considerable savings.

The choice of snapshots is critical to the quality of the reduced basis functions in the POD method. We study the continuous dependence of multiscale basis functions on the refractive index, which provides guidance on how to determine the number of samples in the construction of multiscale basis functions. We summarize the result into the following theorem, whose proof is given in A. For notational simplification, we carry out the analysis for multiscale basis functions without localization.

**Theorem 3.3.** For two realizations  $\omega_1$  and  $\omega_2$  of the random media  $n(\mathbf{x}, \omega)$ , under the assumption that  $n(\mathbf{x}, \omega)$  is almost surely bounded, i.e. (15) is satisfied and grid sizes are small in the sense that: H < 1; hk < 1 and  $k^2h^d || n(\cdot, \omega_1) - n(\cdot, \omega_2) ||_{L^{\infty}(D)} < 1$ , we have

$$\|\phi(\cdot,\omega_1) - \phi(\cdot,\omega_2)\|_{L^{\infty}(D)} \le Ck^2 h^{-d-5} \|n(\cdot,\omega_1) - n(\cdot,\omega_2)\|_{L^{\infty}(D)},$$
(17)

where the constant C is independent of h, k and  $||n(\cdot,\omega_1) - n(\cdot,\omega_2)||_{L^{\infty}(D)}$ .

Equipped with Theorem 3.3, we can estimate the number of samples in the construction of multiscale reduced basis functions. Suppose the random refractive index is of the form (14). For any  $\delta > 0$ , we choose an integer  $Q_{\delta}$  and a set of random samples  $\{n(\mathbf{x}, \omega_q)\}_{q=1}^{Q_{\delta}}$  such that

$$\mathbb{E}\left[\inf_{1 \le q \le Q_{\delta}} \left| \left| n(\mathbf{x}, \omega) - n(\mathbf{x}, \omega_q) \right| \right|_{L^{\infty}(D)} \right] \le \delta,$$
(18)

where the expectation is taken over the random variables in  $n(\mathbf{x}, \omega)$  of the form (14). Since the distribution of the random variables  $\xi_j(\omega)$ , j = 1, ..., m in (14) is known, we can find a way to choose the random samples  $\{n(\mathbf{x}, \omega_q)\}_{q=1}^{Q_{\delta}}$  so that the condition (18) is satisfied.

For every  $\mathbf{x}_j \in \mathcal{N}$ , let  $\{\phi_j(\mathbf{x}, \omega_q)\}_{q=1}^{Q_\delta}$  be the samples of multiscale basis functions associated with  $\mathbf{x}_j$ . Then, we have

$$\mathbb{E}\left[\inf_{1\leq q\leq Q_{\delta}}\left|\left|\phi_{j}(\mathbf{x},\omega)-\phi_{j}(\mathbf{x},\omega_{q})\right|\right|_{L^{\infty}(D)}\right]\leq Ck^{2}h^{-d-5}\delta.$$
(19)

Given parameters k and h, we choose  $\delta$  and  $Q_{\delta}$  so that the right-hand side of (19) is small. Then, the space of multiscale basis functions can be well approximated by the samples of multiscale basis functions  $\{\phi_j(\mathbf{x}, \omega_q)\}_{q=1}^{Q_{\delta}}$  with controllable accuracy and the POD method is further applied to construct multiscale reduced basis functions.

3.2. Derivation of the qMC-based model reduction method. We now present the qMC-based model reduction method for solving the random Helmholtz equation. In our method, we use the multiscale reduced basis functions obtained in Section 3.1 for approximation in the physical space, while in the random space, we use the qMC method for discretization.

The implementation of the qMC method is fairly easy. Given a set of qMC samples, the expectation of the solution is approximated by

$$\mathbb{E}\left[u(\mathbf{x},\omega)\right] \approx \frac{1}{N_{qMC}} \sum_{i=1}^{N_{qMC}} u(\mathbf{x},\omega_i),\tag{20}$$

where  $N_{qMC}$  is the number of qMC samples. Details of the generation of qMC samples and its convergence analysis will be discussed in Section 4.

Now we focus on how to approximate the solution in the physical space for each qMC sample  $\omega_s$ . For each node point  $\mathbf{x}_j \in \mathcal{N}$ , we have constructed a set of multiscale reduced basis functions  $\{\zeta_l^j\}_{l=0}^{m_j}$  and represent the wavefunction by

$$u(\mathbf{x},\omega_s) = \sum_{j=1}^{N_H} \sum_{l=0}^{m_j} c_l^j(\omega_s) \zeta_l^j(\mathbf{x}), \qquad (21)$$

where  $m_j$  is the number of multiscale reduced basis functions associated with node  $\mathbf{x}_j$ .

We apply the Galerkin method to compute the expansion coefficients  $c_l^j(\omega_s)$ . Specifically, we substitute the expansion (21) into the Helmholtz equation (13), multiply both side by the multiscale reduced basis  $\zeta_r^{j'}(\mathbf{x})$  and take integration. This gives us a weak form as follows,

$$\left(\nabla \sum_{j=1}^{N_H} \sum_{l=0}^{m_j} c_l^j(\omega_s) \zeta_l^j(\mathbf{x}) , \nabla \zeta_r^{j'}(\mathbf{x})\right)_D - \left(k^2 n(\mathbf{x},\omega_s) \sum_{j=1}^{N_H} \sum_{l=0}^{m_j} c_l^j(\omega_s) \zeta_l^j(\mathbf{x}) , \zeta_r^{j'}(\mathbf{x})\right)_D - \left(ik\sqrt{n(\mathbf{x},\omega_s)} \sum_{j=1}^{N_H} \sum_{l=0}^{m_j} c_l^j(\omega_s) \zeta_l^j(\mathbf{x}) , \zeta_r^{j'}(\mathbf{x})\right)_\Gamma = \left(f(\mathbf{x}) , \zeta_r^{j'}(\mathbf{x})\right)_D, \\
\mathbf{x} \in D, \quad j' = 1, \cdots, N_H, \quad r = 0, \cdots, m_j,$$
(22)

where the impedance boundary condition has been imposed.

To numerically solve (22), we introduce some notations. Let S,  $M(\omega_s)$  and  $B(\omega_s)$  be matrices with dimension  $\sum_{j=1}^{N_H} (m_j + 1) \times \sum_{j=1}^{N_H} (m_j + 1)$ , and F be a matrix with dimension  $\sum_{j=1}^{N_H} (m_j + 1) \times 1$ . Their entries are given by

$$S_{\sum_{i=1}^{j}(m_{i}+1)+l, \sum_{i=1}^{j'}(m_{i}+1)+r} = \int_{D} \nabla \zeta_{l}^{j} \cdot \nabla \zeta_{r}^{j'} \mathbf{x},$$

$$M_{\sum_{i=1}^{j}(m_{i}+1)+l, \sum_{i=1}^{j'}(m_{i}+1)+r}(\omega_{s}) = \int_{D} \zeta_{l}^{j} n(\mathbf{x}, \omega_{s}) \zeta_{r}^{j'} \mathbf{x},$$

$$B_{\sum_{i=1}^{j}(m_{i}+1)+l, \sum_{i=1}^{j'}(m_{i}+1)+r}(\omega_{s}) = \int_{\Gamma} \zeta_{l}^{j} \sqrt{n(\mathbf{x}, \omega_{s})} \zeta_{r}^{j'} \mathbf{x},$$

$$F_{\sum_{i=1}^{j'}(m_{i}+1)+r, 1} = \int_{D} f(\mathbf{x}) \zeta_{r}^{j'} \mathbf{x}.$$

Then, we can reduce the weak formulation (22) into the following linear algebra system

$$\left(S - k^2 M(\omega_s) - ikB(\omega_s)\right) \mathbf{c}(\omega_s) = F,$$
(23)

where the column vector  $\mathbf{c}(\omega_s) = (c_0^1(\omega_s), ..., c_{m_1}^1(\omega_s), ..., c_0^{N_H}(\omega_s), ..., c_{m_{N_H}}^{N_H}(\omega_s))^T$  consisting of all expansion coefficients of the solution  $u(\mathbf{x}, \omega_s)$  onto the multiscale reduced basis. Since the multiscale basis functions are defined on coarse grids and multiscale reduced basis functions are efficient for model reduction, our method provides considerable savings over the standard finite element methods defined on fine grids, especially for the Helmholtz equation with a high wavenumber.

Before ending this section, we explain why we chose the qMC method to approximate the random space of the solution. Since the parameterization of a random refractive index may have a high dimension, i.e., m is large in (14), non-intrusive methods, such as stochastic collocation methods [24], become expensive to solve PDEs with random coefficients. Polynomial chaos expansion methods [17, 33] are also frequently used in the literature to solve PDEs with random coefficients. This type of method is very efficient if the solution is sufficiently smooth in the random space with small dimensionality. It also suffers from the curse of dimensionality. The performance of the MC method does not depend on the dimension of the random space. However, its convergence rate is merely  $O(\frac{1}{\sqrt{n}})$ . The convergence rate of the qMC method is better both theoretically and numerically; see (48) in Theorem 4.8. Therefore, we choose the qMC method and its implementation is almost the same as the MC method.

4. **Convergence analysis.** In this section, we aim to study the convergence analysis of the proposed method, where the emphasis is put on computing the expectation of functionals of the wavefunction.

4.1. Regularity of the wavefunction with respect to the random variables. The impedance boundary condition ensures the wellposedness of the problem (13). However, the stability constant (denoted by  $C_{stab}$ ) and its possible dependence on the wave number k are not known in general. The following property plays an important role in our analysis, where the analysis can be found in [19].

**Proposition 4.1.** If  $\Gamma$  is convex and  $2n(\mathbf{x}, \omega) + \mathbf{x} \cdot \nabla_{\mathbf{x}} n(\mathbf{x}, \omega) \ge \mu > 0$  for almost all  $x \in D$  with probability 1, then the problem (13) has a unique solution  $u = u(\mathbf{x}, \omega)$ 

and the solution satisfies the following inequality with respect to  $\boldsymbol{x}$  and for each  $\omega$  almost surely.

$$\left|\left|u(\cdot,\omega)\right|\right|_{H^{1}(D)} + k\left|\left|n(\cdot,\omega)u(\cdot,\omega)\right|\right|_{L^{2}(D)} \le C_{stab}||f||_{L^{2}(D)}.$$
(24)

The stability constant  $C_{stab}$  is independent of k,  $f(\mathbf{x})$ ,  $u(\mathbf{x}, \omega)$ , and  $\omega$  but depends on  $\mu$  and the diameter of D.

In practice, we are interested in the expected value of a spatial linear functional  $\mathcal{G}(u(\mathbf{x},\omega))$ . For instance,  $\mathcal{G}(\cdot)$  is an identity operator or an integral operator.

**Assumption 4.2.** The spatial linear functional  $\mathcal{G} \in (H^s(D))'$  for some  $s \ge 0$ , in which  $(H^s(D))'$  means the dual space of  $H^s(D)$ .

Notice that the random refractive index (14) is parameterized by m random variables. Denote  $\boldsymbol{\xi}(\omega) = (\xi_1(\omega), \cdots, \xi_m(\omega))^T$  for convenience. According to the Doob-Dynkin lemma the solution  $u(\mathbf{x}, \omega)$  can be represented by these random variables, i.e.  $u(\mathbf{x}, \omega) = u(\mathbf{x}, \boldsymbol{\xi}(\omega))$ . Let  $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_m)$  denote a multi-index of non-negative integers, with  $|\boldsymbol{\nu}| = \sum_{j=1}^m \nu_j$  and  $|\boldsymbol{\nu}|_{\infty} = \max_{1 \le j \le m} \nu_j$ . The value of  $\nu_j$  determines the number of derivatives to be taken with respect to  $\xi_j$ .

Let  $F(\cdot, \boldsymbol{\xi}(\omega)) = \mathcal{G}(u(\cdot, \boldsymbol{\xi}(\omega)))$ . Then, we want to compute

$$\mathbb{E}[F(\cdot,\boldsymbol{\xi}(\omega))] = \int_{[0,1]^m} F(\cdot,\boldsymbol{\xi}(\omega)) d\boldsymbol{\xi}.$$
(25)

To carry out the error analysis for the qMC method in computing (25), it is crucial to bound the mixed first derivatives of  $F(\cdot, \boldsymbol{\xi}(\omega))$  with respect to  $\boldsymbol{\xi}(\omega)$ . Let  $\frac{\partial^{\boldsymbol{\nu}}F(\cdot,\boldsymbol{\xi}(\omega))}{\partial \boldsymbol{\xi}^{\boldsymbol{\nu}}}$  denote the mixed derivative of  $F(\cdot,\boldsymbol{\xi}(\omega))$  with respect to all variables specified by the multi-index  $\boldsymbol{\nu}$ , where  $\nu_i = 0$  or 1.

Note that (24) implies (for  $k \ge 1$ ), almost surely we have

$$||u(\cdot,\boldsymbol{\xi}(\omega))||_{L^{2}(D)} \leq (\frac{C_{stab}}{kn_{min}})||f||_{L^{2}(D)}, \quad ||u(\cdot,\boldsymbol{\xi}(\omega))||_{H^{1}(D)} \leq (\frac{C_{stab}}{n_{min}})||f||_{L^{2}(D)}.$$
(26)

Thus, we have the following proposition that

**Proposition 4.3.** For  $k \ge 1$  and s = 0 or s = 1, the solution to the problem (13) satisfies the following inequality with respect to  $\mathbf{x}$  and for each  $\boldsymbol{\xi}(\omega)$  almost surely.

$$||u(\cdot, \boldsymbol{\xi}(\omega))||_{H^{s}(D)} \le k^{s-1} (\frac{C_{stab}}{n_{min}})||f||_{L^{2}(D)}.$$
(27)

The stability constant  $C_{stab}$  is independent of k,  $f(\mathbf{x})$ ,  $u(\mathbf{x}, \omega)$ , and  $\omega$  but depends on  $\mu$  and the diameter of D. The constant  $n_{min}$  is defined in Eq.(15).

Then we can obtain the following theorem about the bound of  $\frac{\partial^{\nu} F(\cdot,\boldsymbol{\xi}(\omega))}{\partial \boldsymbol{\xi}^{\nu}}$ .

**Theorem 4.4.** Under the Proposition 4.1, Assumption 4.2 and the assumption (15) on the refractive index  $n(\mathbf{x}, \omega)$ , the mixed first derivatives of  $F(\cdot, \boldsymbol{\xi}(\omega))$  with respect to  $\boldsymbol{\xi}(\omega)$  can be bounded as follows

$$\frac{\partial^{\boldsymbol{\nu}} F(\cdot, \boldsymbol{\xi}(\omega))}{\partial \boldsymbol{\xi}^{\boldsymbol{\nu}}} \Big| \le ||\mathcal{G}||_{L^{2}(D)'} |\boldsymbol{\nu}|! k^{|\boldsymbol{\nu}|-1} (\frac{C_{stab}}{n_{min}})^{|\boldsymbol{\nu}|+1} (\prod_{j=1}^{m} b_{j}^{\nu_{j}}) ||f||_{L^{2}(D)},$$
(28)

where  $\boldsymbol{\nu} = (\nu_1, \dots, \nu_m)$  is a multi-index with  $\nu_j = 0$  or 1,  $|\boldsymbol{\nu}| = \sum_{j=1}^m \nu_j$ , and  $b_j = ||\psi_j||_{L^{\infty}(D)}$ .

*Proof.* We prove by induction on  $|\nu|$ . First of all, we consider the case  $|\nu| = 0$ . According to the definition  $F(\cdot, \boldsymbol{\xi}(\omega)) = \mathcal{G}(u(\cdot, \boldsymbol{\xi}(\omega)))$ , we have

$$|F(\cdot,\boldsymbol{\xi}(\omega))| = |\mathcal{G}u(\cdot,\boldsymbol{\xi}(\omega))| \le ||\mathcal{G}||_{L^2(D)'} ||u(\cdot,\boldsymbol{\xi}(\omega))||_{L^2(D)}.$$
(29)

Then, by using the estimate (27) for  $||u(\cdot, \boldsymbol{\xi}(\omega))||_{L^2(D)}$  we get

$$|F(\cdot, \boldsymbol{\xi}(\omega))| \le ||\mathcal{G}||_{L^2(D)'} k^{-1} (\frac{C_{stab}}{n_{min}}) ||f||_{L^2(D)}.$$
(30)

Now we consider the case  $|\nu| = 1$ . We compute the first order derivative of  $F(\cdot, \boldsymbol{\xi}(\omega))$  with respect to  $\xi_j$  and obtain

$$\frac{\partial F}{\partial \xi_j}(\cdot, \boldsymbol{\xi}(\omega)) = \frac{\partial}{\partial \xi_j} \mathcal{G}\big(u(\cdot, \boldsymbol{\xi}(\omega))\big) = \mathcal{G}\big(u_j(\cdot, \boldsymbol{\xi}(\omega))\big), \tag{31}$$

where  $u_j(\cdot, \boldsymbol{\xi}(\omega)) = \frac{\partial u(\cdot, \boldsymbol{\xi}(\omega))}{\partial \xi_j}$ . To estimate  $u_j(\cdot, \boldsymbol{\xi}(\omega))$ , we differentiate the Helmholtz Eq.(13) with respect to  $\xi_j$ , where the refractive index is given by (14). Also we assume that  $n \equiv 1$  on  $\Gamma$  so boundary perturbations are forbidden. We get that

$$-\Delta u_j - k^2 n(\mathbf{x}, \omega) u_j = k^2 \psi_j(\mathbf{x}) u, \quad \mathbf{x} \in D, \omega \in \Omega,$$

$$ih_{\mathcal{D}} = (\mathbf{x}, \omega)$$
(32)

$$\partial_{\nu} u_j - ik \sqrt{n(\mathbf{x}, \omega)} u_j = \frac{ik}{2} \frac{n_j(\mathbf{x}, \omega)}{\sqrt{n(\mathbf{x}, \omega)}} = 0, \quad \mathbf{x} \in \Gamma,$$
(33)

where we have used the condition that  $n_j(\mathbf{x}, \omega) = \frac{\partial n(\mathbf{x}, \omega)}{\partial \xi_j} = 0$  on the boundary. Then, together with (32), (33) and (27), we obtain

$$\begin{aligned} ||u_{j}(\cdot,\boldsymbol{\xi}(\omega))||_{L^{2}(D)} &\leq k^{-1}(\frac{C_{stab}}{n_{min}})k^{2}||\psi_{j}||_{L^{\infty}(D)}||u||_{L^{2}(D)}, \\ &\leq k(\frac{C_{stab}}{n_{min}})||\psi_{j}||_{L^{\infty}(D)}(\frac{C_{stab}}{kn_{min}})||f||_{L^{2}(D)}, \\ &= (\frac{C_{stab}}{n_{min}})^{2}||\psi_{j}||_{L^{\infty}(D)}||f||_{L^{2}(D)}. \end{aligned}$$
(34)

Hence combining with (30) and (34), we have

$$\left|\frac{\partial F}{\partial \xi_j}(\cdot,\boldsymbol{\xi}(\omega))\right| \le ||\mathcal{G}||_{L^2(D)'} \left(\frac{C_{stab}}{n_{min}}\right)^2 \left(||\psi_j||_{L^\infty(D)}\right)||f||_{L^2(D)}.$$
(35)

Next, we consider the mixed second-order derivative with respect to  $\xi_j$ ,  $\xi_l$  for  $j \neq l$ ,

$$\left|\frac{\partial^2}{\partial\xi_j\partial\xi_l}F(\cdot,\boldsymbol{\xi}(\omega))\right| = \left|\mathcal{G}(u_{jl}(\cdot,\boldsymbol{\xi}(\omega)))\right| \le \left||\mathcal{G}||_{L^2(D)'}||u_{jl}(\cdot,\boldsymbol{\xi}(\omega))||_{L^2(D)},\tag{36}$$

where  $u_{jl}(\cdot, \boldsymbol{\xi}(\omega)) = \frac{\partial^2 u(\cdot, \boldsymbol{\xi}(\omega))}{\partial \xi_j \partial \xi_l}$ . This enables us to estimate the case  $|\boldsymbol{\nu}| = 2$ . We repeat the same procedure and differentiate (32) and (33) with respect to  $\xi_l$ , which gives us

$$-\Delta u_{jl} - k^2 n(\mathbf{x}, \omega) u_{jl} = k^2 \psi_j(\mathbf{x}) u_l + k^2 \psi_l(\mathbf{x}) u_j, \quad \mathbf{x} \in D, \omega \in \Omega,$$
(37)

$$\partial_{\nu} u_{jl} - ik \sqrt{n(\mathbf{x}, \omega)} u_{jl} = 0, \qquad \mathbf{x} \in \Gamma.$$
(38)

By using the stability condition (27) for the solution of (37) and (38), we have the following estimate

$$||u_{jl}(\cdot,\boldsymbol{\xi}(\omega))||_{L^{2}(D)} \leq k^{-1} (\frac{C_{stab}}{n_{min}}) (k^{2} ||\psi_{j}||_{L^{\infty}(D)} ||u_{l}(\cdot,\boldsymbol{\xi}(\omega))||_{L^{2}(D)} + k^{2} ||\psi_{l}||_{L^{\infty}(D)} ||u_{j}(\cdot,\boldsymbol{\xi}(\omega))||_{L^{2}(D)}).$$
(39)

We substitute the estimates for the first-order derivatives  $||u_l(\cdot, \boldsymbol{\xi}(\omega))||_{L^2(D)}$  and  $||u_j(\cdot, \boldsymbol{\xi}(\omega))||_{L^2(D)}$  (see (34)) into the above inequality and obtain

$$||u_{jl}(\cdot,\boldsymbol{\xi}(\omega))||_{L^{2}(D)} \leq 2k(\frac{C_{stab}}{n_{min}})^{3}||\psi_{j}||_{L^{\infty}(D)}||\psi_{l}||_{L^{\infty}(D)}||f||_{L^{2}(D)}.$$
 (40)

Let us denote  $b_j = ||\psi_j||_{L^{\infty}(D)}$ . We repeat the same argument for the other mixed derivatives and eventually prove the estimate (28).

If the fluctuation in the random refractive index (14) is small, we can naturally obtain the following result.

**Corollary 4.5.** If the random refractive index (14) is a small perturbation to a mean refractive index, i.e.,  $k||\psi_j||_{L^{\infty}(D)} = O(1)$ , j = 1, ..., m, the bound for the mixed first derivatives of  $F(\cdot, \boldsymbol{\xi}(\omega))$  with respect to  $\xi_i(\omega)$  can be improved to

$$\frac{\partial^{\boldsymbol{\nu}} F(\cdot, \boldsymbol{\xi}(\omega))}{\partial \boldsymbol{\xi}^{\boldsymbol{\nu}}} \Big| \le C ||\mathcal{G}||_{L^{2}(D)'} |\boldsymbol{\nu}|! k^{-1} (\frac{C_{stab}}{n_{min}})^{|\boldsymbol{\nu}|+1} ||f||_{L^{2}(D)},$$
(41)

where C is a constant independent of k,  $\boldsymbol{\nu} = (\nu_1, \cdots, \nu_m)$  is a multi-index with  $\nu_j = 0$  or 1, and  $|\boldsymbol{\nu}| = \sum_{j=1}^m \nu_j$ .

4.2. Main result of the error analysis. In the framework of uncertainty quantification, we are interested in computing some statistical quantities of the wavefunction  $u(\mathbf{x}, \omega)$ . As such, we provide the error analysis of our method in computing functionals of  $u(\mathbf{x}, \omega)$ .

Let  $\mathcal{G}(\cdot)$  be a continuous linear functional on  $L^2(D)$ , then there exists a constant  $C_{\mathcal{G}}$  such that

$$|\mathcal{G}(u)| \le C_{\mathcal{G}} \|u\|_{L^2(D)},$$

for all  $u(,\omega) \in L^2(D)$  almost surely for all  $\omega \in \Omega$ . Consider the following integral

$$I_m(F) = \int_{\boldsymbol{\xi} \in [0,1]^m} F(\boldsymbol{\xi}) \mathrm{d}\boldsymbol{\xi}$$
(42)

with  $F(\cdot, \boldsymbol{\xi}) = \mathcal{G}(u(\cdot, \boldsymbol{\xi}))$ . We approximate the integral over the unit cube by randomly shifted lattice rules

$$Q_{m,n}(\boldsymbol{\Delta};F) \triangleq \frac{1}{N_{qMC}} \sum_{i=1}^{N_{qMC}} F\left(\operatorname{frac}(\frac{i\boldsymbol{z}}{N_{qMC}} + \boldsymbol{\Delta})\right),$$

where  $\boldsymbol{z} \in \mathbb{N}^m$  is the (deterministic) generating vector and  $\boldsymbol{\Delta} \in [0, 1]^m$  is the random shift which is uniformly distributed over  $[0, 1]^m$ . Notice that m is the dimension of the random vector  $\boldsymbol{\xi}$  in the random potential and n is the number of the sample points in implementing the qMC method. The interested reader is referred to [10] for more details of the randomly shifted lattice rules in the qMC method.

**Lemma 4.6.** Let F be the integrand in (42). Given  $N_{qMC} \in \mathbb{N}$  with  $N_{qMC} \leq 10^{30}$ , weights  $\gamma = (\gamma_{\mathfrak{u}})_{\mathfrak{u} \subset \mathbb{N}}$ , a randomly shifted lattice rule with n points in m dimensions

can be constructed by a component-by-component algorithm such that, for all  $\lambda \in$ (1/2, 1],

$$\sqrt{\mathbb{E}^{\mathbf{\Delta}}}|I_m(F) - Q_{m,N_{qMC}}(\cdot;F)|^2 \le 9C^*C_{\boldsymbol{\gamma},m}(\lambda)N_{qMC}^{-1/(2\lambda)},\tag{43}$$

with

$$C_{\boldsymbol{\gamma},m}(\lambda) = \left(\sum_{\emptyset \neq \boldsymbol{\mathfrak{u}} \subseteq \{1:m\}} \gamma_{\boldsymbol{\mathfrak{u}}}^{\lambda} \prod_{j \in \boldsymbol{\mathfrak{u}}} \varrho(\lambda)\right)^{1/(2\lambda)} \left(\sum_{\boldsymbol{\mathfrak{u}} \subseteq \{1:m\}} \frac{(|\boldsymbol{\mathfrak{u}}|!)^2 T^{2|\boldsymbol{\mathfrak{u}}|}}{\gamma_{\boldsymbol{\mathfrak{u}}} \varepsilon^{2|\boldsymbol{\mathfrak{u}}|}} \prod_{j \in \boldsymbol{\mathfrak{u}}} \lambda_j \|\psi_j\|_{C^0(\bar{D})}^2\right)^{1/2}.$$
(44)

*Proof.* The proof of this result is essentially an application of the Koksma-Hlawka inequality, which is the same as the proofs of Theorem 15, Theorem 16, and Theorem 17 in [18], or Theorem 5.10 in [10] with the following modification of estimates:

$$\varrho(\lambda) = 2\left(\frac{\sqrt{2\pi}}{\pi^{2-2\eta_*}(1-\eta_*)\eta_*}\right)^{\lambda}\zeta(\lambda+\frac{1}{2}), \quad \eta_* = \frac{2\lambda-1}{4\lambda}$$
(45)

with  $\zeta(x) = \sum_{i=1}^{\infty} j^{-x}$  the Riemann zeta function, and  $C^* = \|\mathcal{G}\|_{L^2(D)}$ . 

To analyze the error of our method, we need to make some assumptions on the regularity of the physical components  $\psi_i(\mathbf{x})$  in the random refractive index (14) and the decay rate of  $b_j = \sqrt{\lambda_j} ||\psi_j||_{L^{\infty}(D)}$ .

**Assumption 4.7.** (a) There exist C > 0 and  $\Theta > 1$  such that  $b_j \leq C j^{-\Theta}$  for  $j \geq 1;$ 

- (b) The physical components  $\psi_j(\mathbf{x})$  are continuous and there exist C > 0 and  $\eta \in [0, \frac{\Theta-1}{2\Theta})$  such that  $\|\psi_j\|_{L^{\infty}(D)} \leq C\lambda_j^{-\eta}$  for  $j \geq 1$ ; (c) The sequence defined by  $\sqrt{\lambda_j} \|\psi_j\|_{L^{\infty}(D)}$ ,  $j \geq 1$  satisfies  $\sum_{j\geq 1} \left(\sqrt{\lambda_j} \|\psi_j\|_{L^{\infty}(D)}\right)^p < \infty$  for some  $p \in (0, 1]$ , and  $\sum_{j\geq 1} \sqrt{\lambda_j} \|\psi_j\|_{L^{\infty}(D)}$   $< \frac{1}{k}\sqrt{\varrho(\lambda)}$  for  $\lambda \in (1/2, 1]$ .

Denote  $u_H$  the solution obtained by our method using the multiscale reduced basis functions in the physical space and the qMC method in the random space. Under the Assumption 4.7 for the random potential, we have the following error estimate.

**Theorem 4.8.** Consider the approximation of  $\mathbb{E}[\mathcal{G}(u)]$  via the qMC multiscale finite element method, denoted by  $Q_{N_{qMC}}(\cdot; \mathcal{G}(u_H))$ , where we assume  $u \in L^2(\Omega; H^2(D))$ . A randomly shifted lattice rule  $Q_{N_{qMC}}$  is applied to  $\mathcal{G}(u)$ . Then, we can bound the root-mean-square error with respect to the uniformly distributed shift  $\Delta \in [0,1]^m$  by

$$\sqrt{\mathbb{E}^{\mathbf{\Delta}}\left[\left(\mathbb{E}[\mathcal{G}(u)] - Q_n(\cdot; \mathcal{G}(u_H))\right)^2\right]} \le C\left(H^2 + N_{qMC}^{-r}\right),\tag{46}$$

for  $0 < \chi \leq (1/2 - \eta)\Theta - 1/2$ , and with r = 1/p - 1/2 for  $p \in (2/3, 1]$  and  $r = 1 - \delta$ for  $p \leq 2/3$ , with  $\delta$  arbitrarily small. Here the constant C is independent of k and  $N_{qMC}$ .

*Proof.* Under the assumption  $u \in L^2(\Omega; H^2(D))$ , we have, see for example [27] for Helmholtz equation,

$$|\mathbb{E}[\mathcal{G}(u) - \mathcal{G}(u_H)]| \le CH^2.$$
(47)

When applying the qMC method, we adopt the standard framework, i.e., the Koksma-Hlawka inequality. Under Assumption 4.7, we have, based on Theorem 4.4 and Lemma 4.6,

$$\sqrt{\mathbb{E}^{\boldsymbol{\Delta}}}|I_m(F) - Q_{m,N_{qMC}}(\cdot;F)|^2 \le CN_{qMC}^{-r},\tag{48}$$

where r = 1/p - 1/2 for  $p \in (2/3, 1]$  and  $r = 1 - \delta$  for  $p \leq 2/3$ , with  $\delta$  arbitrarily small. The detailed derivation is essentially the same as the proof of Theorem 20 in [18]. A combination of the above estimates completes the proof.

**Remark 4.9.** In Section 5, we will show that the proposed method works well for a large class of random potentials, even when the eigenvalues in the KL expansion have a relatively slow decay rate. Therefore, Assumption 4.7 is a rather technical assumption for the convergence analysis of the proposed method.

5. Numerical examples. In this section, we conduct numerical experiments to test the accuracy and efficiency of our method. Specifically, we will present convergence tests with respect to the physical grid size, the number of multiscale reduced basis functions on each coarse mesh node, and the number of qMC samples.

In what follows, we compare the relative error between expectations of the numerical solution  $u_{\text{num}}$  and the reference solution  $u_{\text{ref}}$  in both  $L^2$  norm and  $H^1$  norm

$$\operatorname{Error}_{L^{2}} = \frac{||\mathbb{E}[u_{\operatorname{num}}] - \mathbb{E}[u_{\operatorname{ref}}]||_{L^{2}}}{||\mathbb{E}[u_{\operatorname{ref}}]||_{L^{2}}}, \\ \operatorname{Error}_{H^{1}} = \frac{||\mathbb{E}[u_{\operatorname{num}}] - \mathbb{E}[u_{\operatorname{ref}}]||_{H^{1}}}{||\mathbb{E}[u_{\operatorname{ref}}]||_{H^{1}}}.$$

Here  $\mathbb{E}[u_{\text{num}}] = \int_{\Omega} u_{\text{num}}(\mathbf{x}, \omega) d\rho(\omega)$ ,  $\mathbb{E}[u_{\text{ref}}] = \int_{\Omega} u_{\text{ref}}(\mathbf{x}, \omega) d\rho(\omega)$ ,  $\Omega$  is the random space, and  $\rho(\omega)$  is the probability measure induced by the randomness in  $n(\mathbf{x}, \omega)$ . The reference solution refers to the numerical wavefunction using a very fine mesh and a large amount of qMC samples. In numerical experiments, we use MATLAB's Statistics Toolbox to generate the Sobol sequence to implement the qMC method. When we use the POD method to construct multiscale reduced basis functions, we observe similar decay behaviors of the associated eigenvalues at each coarse grid point. Therefore, we choose the same reduced basis number  $m_j$  for all the coarse grid points.

Consider the 1D Helmholtz equation over D = [0, 1]

$$\begin{cases} -u''(x,\omega) - k^2 n(x,\omega) u(x,\omega) = f(x), & x \in D, \omega \in \Omega, \\ u'(x,\omega) - ik\sqrt{n(x,\omega)} u(x,\omega) = 0, & x = 0, 1, \end{cases}$$
(49)

where the pure robin boundary condition is imposed and the random refractive index  $n(x, \omega)$  is defined as

$$n(x,\omega) = 1 + \sigma \sum_{j=1}^{m} \sin(2\pi j x) \xi_j(\omega).$$
(50)

In the random refractive index (50),  $\sigma$  is used to control the strength of the random media and to make sure  $n(x, \omega) \ge 0$  all the time, and  $\xi_j(\omega)$ 's are independent random variables uniformly distributed in [0, 1]. 5.1. Multi-query results in online stage. In this section, we first demonstrate the property of our multiscale reduced basis. First of all, our basis can be used in the online stage for different source terms f(x), which means that once we have constructed our multiscale reduced basis in the offline stage, we can use the same basis for computing the Helmholtz equation in the multi-query setting. We choose f(x) belongs to  $\mathfrak{F} = \{\sin(a_i\pi x + b_i\pi)\cos(c_i\pi y + d_i\pi)\}_{i=1}^{10}$ , where  $a_i$ ,  $b_i$ ,  $c_i$  and  $d_i$ are uniformly distributed over the interval [0, 1]. The wave number k = 128, the random dimension m = 5 and the random strength  $\sigma = \frac{1}{6}$  in (50). For the reference solution, we choose the POD modes  $m_j = 3$ , the sampling number in the offline training stage to be 200 and the number of qMC samples in the online stage to be 5120. In Figure 1, we can see that our method is stable and accurate for different source terms.



FIGURE 1. Numerical results for 10 different source terms. Left is the relative error in  $L^2$  norm and right is the relative error in  $H^1$ norm.

5.2. Convergence in the physical space. In this experiment, we choose  $f(x) = \sin(\pi x + \pi)\cos(\pi x + \pi)$ . Beside, we choose k = 128, m = 5 and  $\sigma = \frac{1}{6}$  in (50).

Convergence with respect to the coarse grid size H. In our numerical test, for the reference solution, we choose the fine mesh to be  $hk^2 = 1$ , which means that  $h = \frac{1}{16384}$ . And the qMC sample number  $N_{qMC} = 16000$ . In our method, we choose the POD modes  $m_j = 3$ , the sampling number in the offline training stage to be 200 and the number of qMC samples in the online stage to be 5120.

In Table 1, we compute the relative errors of the expectation of the wavefunction in both  $L^2$  norm and  $H^1$  norm for a series of coarse meshes with grid size ranging from  $H = \frac{1}{128}$  to  $H = \frac{1}{1024}$ . We also show the convergence results in Figure 2. Nice convergence in the physical space is observed.

Verification of the exponential decay of multiscale basis functions. For the same problem as above, we choose four different realizations of the multiscale basis functions centered at x = 1/2, i.e.  $\phi(x, \boldsymbol{\xi}(\omega_i)), i = 1, 2, 3, 4$ , which are generated in the offline training stage of our previous experiment when  $H = \frac{1}{256}$ . In Figure 3(a), we plot  $|\nabla \phi(x, \boldsymbol{\xi}(\omega_i))|/||\nabla \phi(x, \boldsymbol{\xi}(\omega_i))||_{L_2(D)}, i = 1, 2, 3, 4$ . In Figure 3(b), we plot the quantity  $E_{\text{relative}} = \frac{||\nabla \phi(x, \boldsymbol{\xi}(\omega_i))||_{L_2(D)} - ||\nabla \phi(x, \boldsymbol{\xi}(\omega_i))||_{L_2(D_\ell)}}{\max(||\nabla \phi(x, \boldsymbol{\xi}(\omega_i))||_{L_2(D)} - ||\nabla \phi(x, \boldsymbol{\xi}(\omega_i))||_{L_2(D_\ell)})}$  with respect to the

H	$\operatorname{Error}_{L^2}$	Order	$\operatorname{Error}_{H^1}$	Order
1/128	0.00191722		0.04086631	
1/256	0.00091108	1.07	0.02404165	0.77
1/512	0.00040158	1.18	0.01436555	0.74
1/1024	0.00006602	2.60	0.00515569	1.48

TABLE 1. Relative  $L^2$  and  $H^1$  errors for the expectation of the wavefunction when k = 128.



FIGURE 2. The numerical test for physical space. The slopes for the  $L^2$  norm and  $H^1$  norm are 1.58 and 0.97, respectively. Left is the relative error in  $L^2$  norm and right is the relative error in  $H^1$ norm.

patch size  $\ell$  (defined in Eq.(7)), which shows the decay rate of  $E_{\text{relative}}$  with respect to  $\ell$ .

One can see that each realization of the multiscale basis functions decays exponentially fast away from the center x = 1/2. Since the multiscale basis functions have exponential decay property, the approximated multiscale basis using the reduced basis functions (see Section 3.1) still has the same property.

Convergence with respect to the number of multiscale reduced basis functions. We study how the approximation error depends on the number of multiscale reduced basis functions used at each coarse mesh node  $\mathbf{x}_k$ , i.e., changing the POD modes  $m_j$ . Again, we solve (49) with the refractive index defined by (50) when k = 128 and  $\sigma = \frac{1}{6}$ . For the reference solution, we choose the grid size to be  $h = \frac{1}{16384}$  and the number of qMC samples to be  $N_{qMC} = 16000$ . In our method, we choose the number of qMC samples in the offline training stage to be 200 and the number of qMC samples in the offline training stage to be 200 and the number of qMC samples in the online stage to be 5120. We fix the coarse grid size  $H = \frac{1}{256}$  and record the relative errors as a function of the number of multiscale reduced basis functions.

In Figure 4, we plot the relative  $L^2$  and  $H^1$  errors with respect to the number of multiscale reduced basis functions. It is observed that results when  $m_j = 3$  have already been good enough in the sense that relative errors are less than 0.1%. These numerical results indicate that multiscale reduced basis functions can efficiently approximate the solution.



FIGURE 3. Exponentially decaying properties of the multiscale basis functions for four different realizations. Left:  $\nabla \phi / ||\phi||_{L_2}$  with respect to the distance to x = 0. Right:  $E_{\text{relative}} = \frac{||\nabla \phi(x,\xi(\omega_i))||_{L_2(D)} - ||\nabla \phi(x,\xi(\omega_i))||_{L_2(D_\ell)}}{\max(||\nabla \phi(x,\xi(\omega_i))||_{L_2(D)} - ||\nabla \phi(x,\xi(\omega_i))||_{L_2(D_\ell)})}$ .



FIGURE 4. Relative errors with respect to the number of the multiscale reduced basis functions. Left is the relative error in  $L^2$  norm and right is the relative error in  $H^1$  norm.

5.3. Convergence in the random space. Again, we solve the 1D Helmholtz equation (49) on D = [0, 1], but we shall focus on the convergence of our method in random space.

Convergence with respect to the number of qMC samples. In this numerical experiment, parameters of the random refractive index are the same as those in Section 5.2, i.e.,  $\sigma = \frac{1}{6}$  and k = 128. For the reference solution, we choose the grid size to be  $h = \frac{1}{16384}$  and the number of qMC samples to be  $N_{qMC} = 16000$ . In our method, we choose the coarse grid size to be  $H = \frac{1}{2048}$  and the number of multiscale reduced basis functions to be  $m_j = 4$ , such that the error in the physical space is small enough. To study the convergence rate of the qMC method, we change the number of the qMC samples successively from  $N_{qMC} = 160$  to  $N_{qMC} = 5120$  and compute the relative  $L^2$  errors. We also compute the relative errors of the MC method with the same setting in the physical space and the same number of samples.

In Figure 5, we show the convergence results of our method. We find that the convergence rate of the qMC method is close to 1, which is consistent with results

in Lemma 4.6 and in Theorem 4.8. Meanwhile, we compare the performance of the qMC method and the MC method. One can see that the convergence rate of the MC method is close to  $\frac{1}{2}$ , which is also consistent with the error estimate of the MC method. This result clearly shows that the qMC method is more accurate and efficient than the MC method.



FIGURE 5. Comparison of the qMC method and the MC method. Convergence rates for qMC and MC are 0.97 and 0.53, respectively.

Estimation of sampling numbers in the construction of multiscale reduced basis functions. In Eqns.(18)-(19), we obtain qualitative estimates on the choice of sampling numbers in the construction of multiscale reduced basis functions. In this experiment, we first generate Q qMC samples of the random refractive index:  $\{n(x, \omega_q)\}_{q=1}^Q$ . Then, for each sample  $n(x, \omega_q)$ , we compute the corresponding multiscale basis functions. Finally, we construct multiscale reduced basis functions using the POD method. In the online stage, we solve (13) using the obtained multiscale reduced basis functions. The numerical setting for the reference solution is the same as before. For our method, we choose  $H = \frac{1}{128}$ ,  $m_j = 3$ , and  $N_{qMC} = 2560$ .

In Table 2, we show relative errors of numerical solutions obtained using different sampling numbers of the random potential. When the sampling number Q is small, say Q = 10, the error is big and the corresponding multiscale reduced basis functions cannot approximate the random space of the wavefunction well. When we increase Q, i.e., add more samples of the random potential in the construction of multiscale reduced basis functions, we obtain much better results. Notice that  $m_j$  is fixed to be 3. This means when Q is of order 100, the sampling number of the random potential is large enough to ensure the excellent approximation accuracy of multiscale reduced basis functions. One interesting topic on this issue is an optimal sampling strategy in the construction of multiscale reduced basis functions, which will be explored in a subsequent work.

Dependence of the number of qMC samples on wave number k and dimension of the random space m. We choose the random refravtive index  $n(x, \omega)$  as follows,

$$n(x,\omega) = 1 + \frac{1}{9} \sum_{j=1}^{m} \sin(2\pi j x) \xi_j(\omega).$$
(51)

DINGJIONG MA AND ZHIWEN ZHANG

aMC number	$\operatorname{Error}_{L^2}$	$\operatorname{Error}_{H^1}$
10	0.13959960	0.50448116
100	0.02439249	0.07893179
200	0.02438763	0.07892388
400	0.02435734	0.07886723

TABLE 2. Relative  $L^2$  and  $H^1$  errors in terms of sampling numbers of the qMC method in the offline stage.

We set the random dimension to be m = 8. Four values of k = 16, 32, 64 and 128 are tested. In Table 3, we list the number of qMC samples with respect to k for the same accuracy requirement. We find that the number of qMC samples increases linearly with respect to the increase of wave number k.

W	qMC number	$\mathrm{Error}_{L^2}$	$\operatorname{Error}_{H^1}$
16	320	0.00022253	0.00071914
32	1280	0.00024222	0.00085771
64	1280	0.00025691	0.00085576
128	2560	0.00026058	0.00078906

TABLE 3. Number of qMC samples for different k under the same accuracy requirement.

Secondly, we fix k = 128 and change the dimension of the random space from m = 1, m = 2, m = 4, to m = 8. The reference solution and numerical solution are obtained in the same way as above. In Table 4, we list the number of qMC samples with respect to m for the same accuracy requirement. One can see that the number of qMC samples grows quadratically when m increases.

Dimension $m$	qMC number	$\mathrm{Error}_{L^2}$	$\operatorname{Error}_{H^1}$
1	160	0.00002016	0.00021496
2	320	0.00002353	0.00032777
4	1280	0.00003841	0.00024367
8	5120	0.00002785	0.00033547

TABLE 4. Number of qMC samples for different dimension m under the same accuracy requirement.

These numerical results show that the qMC method is very efficient in 1D Helmholtz equation with random refractive index. Moreover, the qMC method can be implemented in a parallel fashion to further improve its efficiency.

5.4. Numerical performance for 2D case. To further demonstrate the performance of our method, we present numerical results for a 2D Helmholtz equation over  $D = [0, 1]^2$ 

$$\begin{cases} -\Delta u(x,\omega) - k^2 n(x,\omega) u(x,\omega) = f(x), & x \in D, \omega \in \Omega, \\ \partial_{\nu} u - ik\sqrt{n(x,\omega)} u(x,\omega) = 0, & x \in \Gamma. \end{cases}$$
(52)

We first choose k = 10 and the random refractive index

$$n(x, y, \omega) = 1 + \sigma \sum_{j=1}^{m} \sin(2\pi j x) \cos(\pi j y) \xi_j(\omega).$$
(53)

The source term f(x) is given by  $f(x) = \sin(\pi x) \cos(\pi y)$ . We test the random dimension to be m = 6 and the random strength  $\sigma = \frac{1}{7}$ . For our numerical solution, we set the coarse mesh  $H = \frac{1}{4k} = \frac{1}{40}$ . We choose the POD modes  $m_j = 2$ , the sampling number in the offline training stage to be 200 and the number of qMC samples in the online stage to be 5120. For the reference solution, we set the fine mesh to be  $h = \frac{1}{2k^2} = \frac{1}{200}$  and the qMC sampling number to be 16000. In Figure 6 and Figure 7, we show the profile of expectation for the numerical solution as well as the error between the solution between the numerical solution and reference solution for both the real part and the imaginary part. The relative error is 0.0015 in  $L^2$  norm and is 0.018 in  $H^1$  norm, respectively.



FIGURE 6. Expectation of numerical solution for both the real part and the imaginary part when k = 10. Left is the real part of numerical solution and right is imaginary part of numerical solution.



FIGURE 7. Difference of the solution between the numerical solution and reference solution for both the real part and imaginary part for k = 10. Left is the difference for real part and right is difference for imaginary part.

Then, we choose k = 32 and the random refractive index

$$n(x, y, \omega) = 1 + \frac{1}{5} \sum_{j=1}^{4} \sin(2\pi j x) \cos(\pi j y) \xi_j(\omega).$$
 (54)

For our numerical solution, we take the coarse mesh  $H = \frac{1}{32}$ . We choose the POD modes  $m_j = 3$ , the sampling number in the offline training stage to be 200 and the number of qMC samples in the online stage to be 4000. In Figure 8, we show the profiles of the expectation for the numerical solutions.



FIGURE 8. Expectation of numerical solution for both the real part and the imaginary part when k = 32. Left is the real part of numerical solution and right is imaginary part of numerical solution.

5.5. Test the dependence of random perturbation. In this section, we use the same random refractive index as in [13] with 1D physical space. Using the same notation as above examples, our random refractive index is defined by:

$$n(x,\omega) = (1 + \sigma\Theta(x,\omega))^2.$$
(55)

In (55),  $\sigma$  is the random perturbation, and  $\Theta(x,\omega)$  is a random field defined by white noise without spatial correlation, which is at each physical point the value of  $\Theta$  is chosen from a uniform distribution in [-1, 1]. The wave number we test is k = 32 and the fine and coarse mesh is  $h = \frac{1}{16384}$  and  $H = \frac{1}{128}$ , respectively. In our method, we choose  $m_j = 3$  and the offline sampling number to be 200. For the reference solution, we choose the online sampling number  $N_{qMC} = 16000$  which is also used for the numerical online stage. In Table 5, we list the relative errors in  $L^2$  norm and  $H^1$  norm for 5 different  $\sigma$ 's. We find that the performance of our multiscale reduced basis method is not severely affected by the random perturbation in the random refractive index.

6. Conclusions and discussions. In this paper, we have proposed a multiscale reduced basis method to solve the Helmholtz equation with random refractive index for different wave numbers. The physical space of the solution is approximated by a set of localized multiscale basis functions based on an optimization approach. The POD method is then applied to extract a smaller number of multiscale reduced basis functions to further reduce the computational cost without loss of approximation accuracy. The number of samples to learn the multiscale reduced basis functions

Perturbation $\sigma$	$\operatorname{Error}_{L^2}$	$\operatorname{Error}_{H^1}$
0.1	0.00068960	0.00889012
0.3	0.00109772	0.01433205
0.5	0.00172031	0.02219552
0.7	0.00275215	0.02849249
0.9	0.00112581	0.01552095

TABLE 5. Relative Error for 5 different random perturbation  $\sigma$ 

is also analyzed, which provides guidance in practical computations. The qMC method is employed to approximate the random space of the solution. The approximation accuracy of the proposed method is analyzed. Finally, we present several numerical examples to demonstrate the accuracy and efficiency of the proposed method.

As the similar oscillatory nature between Helmholtz and Schrödinger equation, in the future we can explore some work in random Schrödinger equation. In the physics community, the random Schrödinger equation in higher dimensions (2D and 3D) has been frequently used to study Anderson localization; see [15] for example. Though the random potential is assumed to be white noise without spatial correlation in the original paper [1], correlated random potentials are also found to generate localized states; see [9] for example. For Anderson localization in 2D Schrödinger equations with random potentials, we have done some exploration in [7]. In the mathematics community, it is also known that the existence or nonexistence of Anderson localization for some types of 3D Schrödinger equations with random potentials remains open [12]. It is thus quite interesting to explore this issue from a numerical perspective.

Acknowledgments. Z. Zhang acknowledges the financial support of Hong Kong RGC grants (Project 17307921), National Natural Science Foundation of China (Project 12171406), and Seed Funding for Strategic Interdisciplinary Research Scheme 2021/22 (HKU). The computations were performed using research computing facilities offered by Information Technology Services at the University of Hong Kong. The authors would like to thank Professors I. Graham and J. Chen for several stimulating discussions.

Appendix A. Continuous dependence of multiscale basis functions on the random media. In this appendix, we prove Theorem 3.3, which plays an important role in determining the number of snapshots in constructing the multiscale reduced basis functions.

*Proof.* Let  $\varphi_s^h(\mathbf{x})$ ,  $s = 1, ..., N_h$  denote the finite element basis functions defined on fine mesh with size h and  $N_h$  is the number of fine-scale finite element basis functions. When we numerically solve (10) - (12), we represent the multiscale basis function as  $\phi_i(\mathbf{x}) = \sum_{s=1}^{N_h} c_s \varphi_s^h(\mathbf{x})$  and obtain the following quadratic programming problem with equality constraints

$$\begin{cases} \min_{\boldsymbol{c}} \frac{1}{2} \boldsymbol{c}^T Q \boldsymbol{c}, \\ \text{s.t. } A \boldsymbol{c} = \boldsymbol{b}, \end{cases}$$
(56)

where  $\boldsymbol{c} = [c_1, ..., c_{N_h}]^T$  is the coefficients and Q is a symmetric positive definite matrix on the fine triangulation  $\mathcal{T}_h$  with the (i, j) component

$$Q_{ij} = \left(\nabla \varphi_i^h, \nabla \varphi_j^h\right) - k^2 \left(n(\mathbf{x}, \omega)\varphi_i^h, \varphi_j^h\right).$$
(57)

In (56), A is an  $N_h$ -by- $N_H$  matrix with  $A_{ij} = (\varphi_i^h, \varphi_j^H)$  and **b** is an  $N_h$ -by-1 vector with only the *i*-th entry being 1 and others being 0.

Under the assumption that  $n(\mathbf{x}, \omega)$  is uniformly bounded and h is small, we know that Q is a positive definite matrix. Moreover, we know that A has full rank, i.e., rank $(A) = N_H$ . Therefore, the quadratic optimization problem (56) has a unique minimizer, satisfying the Karush-Kuhn-Tucker condition. Specifically, the unique minimizer of (56) can be explicitly written as

$$\boldsymbol{c} = Q^{-1} \boldsymbol{A}^T (\boldsymbol{A} Q^{-1} \boldsymbol{A}^T)^{-1} \boldsymbol{b}.$$
 (58)

For two samples  $\omega_1$  and  $\omega_2$ , we define  $\delta Q = Q_1 - Q_2$ . Then

$$\left(\delta Q\right)_{ij} = k^2 \left( \left(n(\cdot,\omega_1) - n(\cdot,\omega_2)\right) \varphi_i^h, \varphi_j^h \right), \tag{59}$$

and thus

$$\|\delta Q\|_{\infty} \le k^2 h^d \|n(\cdot,\omega_1) - n(\cdot,\omega_2)\|_{L^{\infty}(D)}.$$
(60)

By assumption  $\|\delta Q\|_{\infty} \leq 1$ , we have

$$Q_2^{-1} = \sum_{n=0}^{\infty} \left( Q_1^{-1} \delta Q \right)^n Q_1^{-1},$$

and thus

$$c_{2} - c_{1} = \left[Q_{2}^{-1} - Q_{1}^{-1}\right] A^{T} (AQ_{1}^{-1}A^{T})^{-1} \boldsymbol{b} + Q_{2}^{-1}A^{T} \left[ (AQ_{2}^{-1}A^{T})^{-1} - (AQ_{1}^{-1}A^{T})^{-1} \right] \boldsymbol{b}$$
  
$$= Q_{1}^{-1} \delta Q Q_{1}^{-1} A^{T} (AQ_{1}^{-1}A^{T})^{-1} \boldsymbol{b}$$
  
$$- Q_{2}^{-1} A^{T} (AQ_{1}^{-1}A^{T})^{-1} (AQ_{1}^{-1}\delta Q Q_{1}^{-1}A^{T}) (AQ_{1}^{-1}A^{T})^{-1} \boldsymbol{b} + o(\|\delta Q\|_{\infty}),$$
  
$$= Q_{1}^{-1} \delta Q Q_{1}^{-1} A^{T} (AQ_{1}^{-1}A^{T})^{-1} \boldsymbol{b}$$
  
$$- Q_{1}^{-1} A^{T} (AQ_{1}^{-1}A^{T})^{-1} (AQ_{1}^{-1}\delta Q Q_{1}^{-1}A^{T}) (AQ_{1}^{-1}A^{T})^{-1} \boldsymbol{b} + o(\|\delta Q\|_{\infty}).$$

Therefore,

 $|\boldsymbol{c}_2 - \boldsymbol{c}_1|_\infty$ 

$$\leq C \|A\|_{\infty} \|Q_1^{-1}\|_{\infty}^2 \|(AQ_1^{-1}A^T)^{-1}\|_{\infty} |\mathbf{b}|_{\infty} \left(1 + \|A\|_{\infty}^2 \|Q_1^{-1}\|_{\infty} \|(AQ_1^{-1}A^T)^{-1}\|_{\infty}\right) \|\delta Q\|_{\infty}.$$

By their definitions, we have

$$||A||_{\infty} \le Ch^{d-1}H, |\mathbf{b}|_{\infty} = 1, ||Q_1^{-1}||_{\infty} \le Ch^{-d}, ||Q_1||_{\infty} \le C \max\{h^{d-2}, k^2h^d\} \le Ch^{d-2}.$$

Substituting these into the above inequality yields

$$|\boldsymbol{c}_2 - \boldsymbol{c}_1|_{\infty} \le CHh^{-2d-5} \|\delta Q\|_{\infty}.$$

Due to the assumption H < 1, we have

$$\|\phi(\cdot,\omega_2) - \phi(\cdot,\omega_1)\|_{L^{\infty}(D)} \leq |\mathbf{c}_2 - \mathbf{c}_1|_{\infty} \leq Ck^2 h^{-d-5} \|n(\cdot,\omega_2) - n(\cdot,\omega_1)\|_{L^{\infty}(D)},$$
  
which completes the proof.

## REFERENCES

- P. Anderson, Absence of diffusion in certain random lattices, *Physical Review*, **109** (1958), 1492-1505.
- [2] I. M. Babuska and S. A. Sauter, Is the pollution effect of the FEM avoidable for the Helmholtz equation considering high wave numbers?, SIAM Journal on Numerical Analysis, 34 (1997), 2392-2423.
- [3] I. Babuska and R. Lipton, Optimal local approximation spaces for generalized finite element methods with application to multiscale problems, *Multiscale Model. Simul.*, 9 (2011), 373-406.
- [4] G. Berkooz, P. Holmes and J. Lumley, The proper orthogonal decomposition in the analysis of turbulent flows, Annual Review of Fluid Mechanics, 25 (1993), 539-575.
- [5] P. Carpena, P. Bernaola-Galvan, P. Ivanov and H. Stanley, Metal-insulator transition in chains with correlated disorder, *Nature*, **418** (2002), 955.
- [6] J. Chen, D. Ma and Z. Zhang, A multiscale finite element method for the Schrödinger equation with multiscale potentials, SIAM Journal on Scientific Computing, 41 (2019), B1115-B1136.
- [7] J. Chen, D. Ma and Z. Zhang, A multiscale reduced basis method for Schrödinger equation with multiscale and random potentials, *Multiscale Model. Simul.*, 18 (2020), 1409-1434.
- [8] Y. M. Chen, Wave propagation in stratified random media, Radio Science Journal of Research NBS/USNC- URSI, 68D (1964), 743-748.
- [9] T. Devakul and D. Huse, Anderson localization transitions with and without random potentials, *Physical Review B*, 96 (2017), 214201.
- [10] J. Dick, F. Y. Kuo and I. Sloan, High-dimensional integration: The Quasi-Monte Carlo way, Acta Numerica, 22 (2013), 133-288.
- [11] Y. Efendiev and T. Y. Hou, Multiscale Finite Element Methods: Theory and Applications, Surv. Tutor. Appl. Math. Sci., 4. Springer, New York, 2009.
- [12] L. Erdos, Lecture notes on quantum Brownian motion, (2010), arXiv:1009.0843.
- [13] X. Feng, J. Lin and C. Lorton, An efficient numerical method for acoustic wave scattering in random media, SIAM/ASA Journal on Uncertainty Quantification, 3 (2015), 790-822.
- [14] X. Feng and C. Lorton, An efficient monte carlo interior penalty discontinuous galerkin method for elastic wave scattering in random media, Computer Methods in Applied Mechanics and Engineering, **315** (2017), 141-168.
- [15] M. Filoche and S. Mayboroda, Universal mechanism for Anderson and weak localization, Proceedings of the National Academy of Sciences, 109 (2012), 14761-14766.
- [16] V. Gantmakher, *Electrons and Disorder in Solids*, Number 130. Oxford University Press, 2005.
- [17] R. G. Ghanem and P. D. Spanos, *Stochastic Finite Elements: A Spectral Approach*, Springer-Verlag, New York, 1991.
- [18] I. G. Graham, F. Y. Kuo, J. A. Nichols, R. Scheichl, C. Schwab and I. Sloan. Quasi-Monte Carlo finite element methods for elliptic PDEs with lognormal random coefficients, *Numerische Mathematik*, **131** (2015), 329-368.
- [19] I. G. Graham, O. R. Pembery and E. A Spence, The Helmholtz equation in heterogeneous media: A priori bounds, well-posedness, and resonances, *Journal of Differential Equations*, 266 (2019), 2869-2923.
- [20] T. Y. Hou, X.-H. Wu and Z. Cai, Convergence of a multiscale finite element method for elliptic problems with rapidly oscillating coefficients, *Mathematics of Computation*, 68 (1999), 913-943.
- [21] T. Y. Hou and P. Zhang, Sparse operator compression of higher-order elliptic operators with rough coefficients, Research in the Mathematical Sciences, 4 (2017), Paper No. 24, 49 pp.
- [22] S. Li and Z. Zhang, Computing eigenvalues and eigenfunctions of Schrödinger equations using a model reduction approach, *Communications in Computational Physics*, 24 (2018), 1073-1100.
- [23] A. Malqvist and D. Peterseim, Localization of elliptic multiscale problems, Mathematics of Computation, 83 (2014), 2583-2603.
- [24] F. Nobile, R. Tempone and C. Webster, A sparse grid stochastic collocation method for partial differential equations with random input data, SIAM Journal on Numerical Analysis, 46 (2008), 2309-2345.
- [25] H. Owhadi, Multigrid with rough coefficients and multiresolution operator decomposition from hierarchical information games, SIAM Review, 59 (2017), 99-149.

- [26] B. Payne, H. Cao and A. Yamilov, Criterion for light localization in random amplifying media, *Physica B: Condensed Matter*, 405 (2010), 3012-3015.
- [27] D. Peterseim, Eliminating the pollution effect in Helmholtz problems by local subscale correction, Mathematics of Computation, 86 (2017), 1005-1036.
- [28] F. Riboli, N. Caselli, S. Vignolini, F. Intonti, K. Vynck, P. Barthelemy, A. Gerardino, L. Balet, L. Li and A. Fiore, Engineering of light confinement in strongly scattering disordered media, *Nature Materials*, **13** (2014), 720.
- [29] T. Schwartz, G. Bartal, S. Fishman and M. Segev, Transport and Anderson localization in disordered two-dimensional photonic lattices, *Nature*, 446 (2007), 52.
- [30] L. Sirovich, Turbulence and the dynamics of coherent structures. I. Coherent structures, Quarterly of Applied Mathematics, 45 (1987), 561-571.
- [31] N. M. Tanushev, J. Qian and J. V. Ralston, Mountain waves and Gaussian beams, Multiscale Model. Simul., 6 (2007), 688-709.
- [32] D. Wiersma, P. Bartolini, A. Lagendijk and R. Righini, Localization of light in a disordered medium, *Nature*, **390** (1997), 671.
- [33] D. Xiu and G. E. Karniadakis, Modeling uncertainty in flow simulations via generalized polynomial chaos, Journal of Computational Physics, 187 (2003), 137-167.
- [34] T. Ying, Y. Gu, X. Chen, X. Wang, S. Jin, L. Zhao, W. Zhang and X. Chen, Anderson localization of electrons in single crystals: LixFe7Se8, *Science Advances*, 187 (2016), e1501283.

Received August 2023; revised September 2023; early access September 2023.