MacWilliams Extension Property With Respect to Weighted Poset Metric*

Yang Xu[1] Haibin Kan[2] Guangyue Han[3]

June 20, 2023

Abstract—Let $\mathbf{H}$ be the Cartesian product of a family of left modules over a ring S, indexed by a finite set Ω. We study the MacWilliams extension property (MEP) with respect to $(\mathbf{P},\omega)$-weight on $\mathbf{H}$, where $\mathbf{P}=(\Omega,\preccurlyeq_{\mathbf{P}})$ is a poset and $\omega:\Omega\longrightarrow\mathbb{R}^{+}$ is a weight function. We first give a characterization of the group of $(\mathbf{P},\omega)$-weight isometries of $\mathbf{H}$, which is then used to show that MEP implies the unique decomposition property (UDP) of $(\mathbf{P},\omega)$, which, for the case that ω is identically 1, further implies that $\mathbf{P}$ is hierarchical. When $\mathbf{P}$ is hierarchical or ω is identically 1, with some mild additional assumptions, we give necessary and sufficient conditions for $\mathbf{H}$ to satisfy MEP with respect to $(\mathbf{P},\omega)$-weight in terms of MEP with respect to Hamming weight. With the help of these results, when S is a finite field, we compare MEP with various well studied coding-theoretic properties including the property of admitting MacWilliams identity (PAMI), reflexivity of partitions, UDP, transitivity of the group of isometries and whether $(\mathbf{P},\omega)$ induces an association scheme; in particular, we show that MEP is always stronger than all the other properties.

*A preliminary version of this paper has been presented in IEEE International Symposium on Information Theory (ISIT) 2022, DOI: 10.1109/ISIT50566.2022.9834442.

[1]Shanghai Key Laboratory of Intelligent Information Processing, School of Computer Science, Fudan University, Shanghai 200433, China.
Department of Mathematics, Faculty of Science, The University of Hong Kong, Pokfulam Road, Hong Kong, China. E-mail:12110180008@fudan.edu.cn

[2]Shanghai Key Laboratory of Intelligent Information Processing, School of Computer Science, Fudan University, Shanghai 200433, China.
Shanghai Engineering Research Center of Blockchain, Shanghai 200433, China.
Yiwu Research Institute of Fudan University, Yiwu City, Zhejiang 322000, China. E-mail:hbkan@fudan.edu.cn

[3]Department of Mathematics, Faculty of Science, The University of Hong Kong, Pokfulam Road, Hong Kong, China. E-mail:ghan@hku.hk

1 Introduction

In 1962, MacWilliams proved in [38] that for a finite field $\mathbb{F}$ and a positive integer n, any Hamming weight preserving map between two linear codes extends to a Hamming weight isometry of $\mathbb{F}^n$. This celebrated result is henceforth referred to as the MacWilliams extension theorem, and has since been extended, generalized and discussed extensively in the literature with respect to various weights and metrics in coding theory and with respect to codes over various alphabets.

MacWilliams extension property (MEP) with respect to Hamming weight is possibly the most well studied topic in the theory of MEP. In [6], Bogart, Goldberg and Gordon gave a combinatorial proof of MacWilliams extension theorem. In [48], Wood generalized MacWilliams extension theorem to codes over finite Frobenius rings by using group characters (also see [47]). In [49, 50], Wood further improved the result in [48] by proving that for a finite ring R, R^n satisfies MEP with respect to Hamming weight for all $n \in \mathbb{Z}^+$ if and only if R is a Frobenius ring; more generally, for a finite module M, M^n satisfies MEP with respect to Hamming weight for all $n \in \mathbb{Z}^+$ if and only if the sum of all the simple submodules of M is cyclic. The former result was also extended and generalized to infinite rings by Iovanov in [30], and by Schneider and Zumbrägel in [46]. Most of the proofs of the "only if" parts use a strategy proposed by Dinh and López-Permouth in [13] and an explicit construction for codes over matrix module alphabet proposed in [49]. In [15], Dyshko further improved the construction in [49] by deriving the minimal code length for which there exists an unextendable Hamming weight preserving map between two linear codes defined over a matrix module alphabet (also see [14]).

MEP with respect to weights and metrics induced by poset structures has become a topic of interest in recent years. In [3], Barra and Gluesing-Luerssen proved that for a finite Frobenius ring alphabet, a poset weight satisfies MEP if and only if the poset is hierarchical. In [22], Gluesing-Luerssen and Pllaha established MEP with respect to Rosenbloom-Tsfasman weight (i.e., poset weight induced by a chain) and MEP with respect to support (i.e., non-zero bit positions of a codeword) for more general alphabets. In [35], among ten characterizations of hierarchical poset metrics, Machado, Pinheiro and Firer reestablished the corresponding result in [3] for finite field alphabet. In [19], for binary field alphabet, Etzion, Firer and Machado gave a necessary and sufficient condition for a directed graph metric to satisfy MEP, where the reduced canonical form of the graph is a hierarchical poset (also see [34]). In [36], Machado and Firer further generalized the corresponding result in [19] to labeled-poset-block metric. Here we note that in this paper, we will generalize all the above mentioned results in [3, 19, 22, 36] as well as a large part of the above mentioned ten characterizations in [35].

MEP has also been widely studied with respect to other weights and metrics including symmetrized weight composition, homogeneous weight, bi-invariant weight over finite rings, rank metric and combinatorial metric, and moreover, MEP has been proposed and studied with respect to local-global property for subgroups of general linear groups and partitions of finite modules; see, among many others, [3], [14], [18], [22]–[26], [44] and [50].

In this paper, we study MEP with respect to weighted poset metric. Our ambient space $\mathbf{H} = \prod_{i \in \Omega} H_i$ is set to be the Cartesian product of a family of non-zero left modules over a ring S, indexed by a finite set Ω. Moreover, a weighted poset metric is determined by a $(\mathbf{P}, \omega)$-weight defined on $\mathbf{H}$, where $\mathbf{P} = (\Omega, \preccurlyeq_{\mathbf{P}})$ is a poset and $\omega : \Omega \longrightarrow \mathbb{R}^+$ is a weight function. This generalizes the weighted poset metric introduced by Hyun, Kim and Park in [29] for binary field alphabet and the labeled-poset-block metric introduced by Machado and Firer in [36] for finite field alphabet (see Section 2.1 for more details).

The main contributions of the paper are presented as follows.

In Section 3, we characterize the group of isometries for $(\mathbf{P}, \omega)$-weight (Theorem 3.1). Our result generalizes several corresponding results for poset metric, poset-block metric, directed graph metric and labeled-poset-block metric in [41, 1, 19, 36]; moreover, it can be used to derive corresponding results for weighted Hamming metric. Theorem 3.1 also leads to several necessary conditions for $\mathbf{H}$ to satisfy MEP with respect to $(\mathbf{P}, \omega)$-weight (Corollaries 3.1 and 3.2).

In Section 4, we study MEP with respect to $\mathbf{P}$-support, which is in fact a special case of MEP with respect to $(\mathbf{P}, \omega)$-weight. We give a necessary and sufficient condition for $\mathbf{H}$ to satisfy MEP with respect to $\mathbf{P}$-support when $\mathbf{P}$ is hierarchical, and also give a sufficient condition for $\mathbf{H}$ to satisfy MEP with respect to $\mathbf{P}$-support for general $\mathbf{P}$ (Theorems 4.1 and 4.2). Our results generalize several corresponding results in [3, 22] which have been established for the special case that $\mathbf{P}$ is either a chain or an anti-chain.

In Section 5, we study MEP with respect to $(\mathbf{P}, \omega)$-weight when $\mathbf{P}$ is hierarchical or ω is identically 1. For these two cases, with some mild assumptions, we give necessary and sufficient conditions for $\mathbf{H}$ to satisfy MEP with respect to $(\mathbf{P}, \omega)$-weight in terms of MEP with respect to Hamming weight (Theorems 5.1 and 5.2). Our results generalize some corresponding results in [3] which have been established for codes over finite Frobenius rings. Moreover, Theorems 5.1 and 5.2 are of central importance for studying the connections among MEP and other coding-theoretic properties in Section 6.

In Section 6, we assume that S is a finite field and $\mathbf{H}$ is a finite dimensional vector space over S. Using the results established in previous sections, we examine the connections among MEP with respect to $(\mathbf{P},\omega)$-weight and other five well studied coding-theoretic properties including the property of admitting MacWilliams identity (PAMI), reflexivity of partitions, UDP, transitivity of the group of isometries and whether the corresponding weighted poset metric induces an association scheme (see, e.g., [19, 21, 35, 36] and references therein). We establish various relations among these six properties; in particular, we show that MEP is always stronger than all the other five properties (Theorems 6.1 and 6.2). Our results unify and generalize several corresponding results in [19, 35, 36] which have been established for poset metric, directed graph metric and labeled-poset-block metric.

2 Preliminaries

We begin with some notations that are used throughout the remainder of the paper. For any $a,b\in\mathbb{Z}$, we let $[a,b]$ denote the set of all the integers between a and b, i.e.,

$$[a,b]=\{i\in\mathbb{Z}\mid a\leqslant i\leqslant b\}.$$

Let S be an associative ring with the multiplicative identity 1_S, Ω be a nonempty finite set, $(H_i\mid i\in\Omega)$ be a family of left S-modules such that

$$H_i\neq\{0\}\text{ for all }i\in\Omega, \tag{2.1}$$

and let

$$\mathbf{H}=\prod_{i\in\Omega}H_i. \tag{2.2}$$

Any S-submodule of $\mathbf{H}$ is referred to as a *linear code*. For any $\beta\in\mathbf{H}$, we define

$$\mathrm{supp}\,(\beta)\triangleq\{i\in\Omega\mid\beta_i\neq 0\}. \tag{2.3}$$

For $i\in\Omega$, define $\pi_i:\mathbf{H}\longrightarrow H_i$ as

$$\pi_i(\alpha)=\alpha_i, \tag{2.4}$$

and define $\eta_i:H_i\longrightarrow\mathbf{H}$ as

$$\forall\ a\in H_i:\mathrm{supp}\,(\eta_i(a))\subseteq\{i\},\ \ (\eta_i(a))_i=a. \tag{2.5}$$

For any $I \subseteq \Omega$, define

$$\delta(I) \triangleq \{\beta \in \mathbf{H} \mid \mathrm{supp}\,(\beta) \subseteq I\}. \tag{2.6}$$

Finally, we let $\mathbf{1}$ denote the constant one map defined on Ω, i.e.,

$$\mathbf{1}(i) = 1 \text{ for all } i \in \Omega. \tag{2.7}$$

2.1 Weighted poset metric

The notion of weighted poset metric has been introduced by Hyun, Kim and Park in [29] for binary field alphabet, where the authors have classified the weighted posets and directed graphs that admit the extended Hamming code to be a 2-perfect code. It has also been shown in [29] that weighted poset metric can be viewed as an algebraic version of directed graph metric introduced by Etzion, Firer and Machado in [19] (see [29, Sections I, II] and [19, Section III] for more details). More recently in [36], Machado and Firer have proposed and studied labeled-poset-block metric for finite field alphabet, which is a generalization of both weighted poset metric in [29] and directed graph metric in [19]. In [36], the authors have studied the group of linear isometries, the MacWilliams identity and MEP with respect to labeled-poset-block metric. In particular, for binary field alphabet, they have given a necessary and sufficient condition for MEP when the poset is hierarchical.

In this paper, we consider weighted poset metric defined for the left S-module $\mathbf{H}$. This further generalizes the labeled-poset-block metric in [36]. Throughout the remainder of this subsection, let $\mathbf{P} = (\Omega, \preccurlyeq_{\mathbf{P}})$ be a poset. A subset $B \subseteq \Omega$ is said to be an *ideal* of $\mathbf{P}$ if for any $b \in B$ and $a \in \Omega$, $a \preccurlyeq_{\mathbf{P}} b$ implies that $a \in B$. Let $\mathcal{I}(\mathbf{P})$ denote the set of all the ideals of $\mathbf{P}$. For $B \subseteq \Omega$, let $\langle B \rangle_{\mathbf{P}}$ denote the ideal $\{a \in \Omega \mid \exists\ b \in B \ s.t.\ a \preccurlyeq_{\mathbf{P}} b\}$. In addition, B is said to be a *chain* in $\mathbf{P}$ if for any $a, b \in B$, either $a \preccurlyeq_{\mathbf{P}} b$ or $b \preccurlyeq_{\mathbf{P}} a$ holds, and B is said to be an *anti-chain* in $\mathbf{P}$ if for any $a, b \in B$, $a \preccurlyeq_{\mathbf{P}} b$ implies that $a = b$. For any $y \in \Omega$, we let $\mathrm{len}_{\mathbf{P}}(y)$ denote the largest cardinality of a chain in $\mathbf{P}$ containing y as its greatest element. The *dual poset* of $\mathbf{P}$ will be denoted by $\overline{\mathbf{P}}$, where $u \preccurlyeq_{\overline{\mathbf{P}}} v \Longleftrightarrow v \preccurlyeq_{\mathbf{P}} u$ for all $u, v \in \Omega$. The set of all the order automorphisms of $\mathbf{P}$ will be denoted by $\mathrm{Aut}\,(\mathbf{P})$.

Definition 2.1. **(1)** $\mathbf{P}$ *is said to be hierarchical if for any* $u, v \in \Omega$ *with* $\mathrm{len}_{\mathbf{P}}(u) + 1 \leqslant \mathrm{len}_{\mathbf{P}}(v)$, *it holds that* $u \preccurlyeq_{\mathbf{P}} v$.
(2) *For* $\omega : \Omega \longrightarrow \mathbb{R}^{+}$, *we say that* $(\mathbf{P}, \omega)$ *satisfies the unique decomposition property (UDP) if for any* $I, J \in \mathcal{I}(\mathbf{P})$ *with* $\sum_{i \in I} \omega(i) = \sum_{j \in J} \omega(j)$, *there exists* $\lambda \in \mathrm{Aut}\,(\mathbf{P})$ *such that* $J = \lambda[I]$ *and* $\omega(\lambda(i)) = \omega(i)$ *for all* $i \in \Omega$.

We note that hierarchical poset has been extensively studied for poset codes (see, e.g., [3, 19, 35, 36] and references therein), and it also plays a role in most of the results established in the paper. It can be verified that $\mathbf{P}$ is hierarchical if and only if there exists $s \in \mathbb{Z}^+$ and a tuple of non-empty sets $(A_1, \ldots, A_s)$ such that:

(i) $\bigcup_{i=1}^{s} A_i = \Omega$;

(ii) $A_i \cap A_j = \emptyset$ for any $i \neq j \in [1, s]$;

(iii) For any $i \in [1, s-1]$ and $u \in A_i$, $v \in A_{i+1}$, it holds that $u \preccurlyeq_{\mathbf{P}} v$.

Therefore up to isomorphism, there is a one-to-one correspondence between all the hierarchical posets over Ω and all the pairs $(s, (a_1, \ldots, a_s))$ where $s \in \mathbb{Z}^+$, and $(a_1, \ldots, a_s)$ is a tuple of positive integers with $a_1 + \cdots + a_s = |\Omega|$. According to [20], up to isomorphism, the number of hierarchical posets over Ω is asymptotically equal to $e^{\pi\sqrt{2|\Omega|/3}}/(4\sqrt{3}|\Omega|)$. So, in this sense, hierarchical posets form a large class of posets. We also note that UDP has been proposed in [19, Definition 2] and [36, Definition 11] in slightly different forms.

The following lemma, whose proof is straightforward and hence omitted, collects some basic properties on hierarchical posets and UDP that will be used frequently in our discussion.

Lemma 2.1. **(1)** *For any* $r \in \mathbb{Z}^+$*, let* $W_r = \{u \in \Omega \mid \mathrm{len}_{\mathbf{P}}(u) = r\}$*. Moreover, fix* $\omega : \Omega \longrightarrow \mathbb{R}^+$*. Suppose that* $(\mathbf{P}, \omega)$ *satisfies UDP. Then, for any* $r \in \mathbb{Z}^+$, $((W_r, =), \omega \mid_{W_r})$ *satisfies UDP.*
(2) $(\mathbf{P}, \mathbf{1})$ *satisfies UDP if and only if* $\mathbf{P}$ *is hierarchical.*
(3) *Suppose that* $\mathbf{P}$ *is hierarchical. Let* B *be a non-empty subset of* Ω*, and let* $r = \max\{\mathrm{len}_{\mathbf{P}}(v) \mid v \in B\}$*. Then, it holds that*

$$\begin{aligned}\langle B \rangle_{\mathbf{P}} &= \langle \{v \in B \mid \mathrm{len}_{\mathbf{P}}(v) = r\} \rangle_{\mathbf{P}} \\ &= \{u \in \Omega \mid \mathrm{len}_{\mathbf{P}}(u) \leqslant r-1\} \cup \{v \in B \mid \mathrm{len}_{\mathbf{P}}(v) = r\}.\end{aligned}$$

Now we propose weighted poset metric defined for $\mathbf{H}$. More precisely, fix a weight function $\omega : \Omega \longrightarrow \mathbb{R}^+$, and consider the *$\omega$-weighted poset* $(\mathbf{P}, \omega)$ (see [29]). For any $\beta \in \mathbf{H}$, the $(\mathbf{P}, \omega)$-weight of β is defined as

$$\mathrm{wt}_{(\mathbf{P},\omega)}(\beta) \triangleq \sum_{i \in \langle \mathrm{supp}(\beta) \rangle_{\mathbf{P}}} \omega(i). \tag{2.8}$$

Similarly as in the proof of [29, Lemma I.2], one can check that $d_{(\mathbf{P},\omega)} : \mathbf{H} \times \mathbf{H} \longrightarrow \mathbb{R}$ defined as

$$d_{(\mathbf{P},\omega)}(\alpha, \beta) = \mathrm{wt}_{(\mathbf{P},\omega)}(\beta - \alpha) \tag{2.9}$$

induces a metric on $\mathbf{H}$, which will henceforth be referred to as *weighted poset metric*. We note that since the weight function ω takes values on each coordinate position, weighted poset metric can be useful to model some specific kind of channels for which the error probability depends on a codeword position (i.e., the distribution of errors is nonuniform), and can also be useful to perform bitwise or messagewise unequal error protection (see, e.g., the abstract of [5] and [19, Section 1, Paragraph 6]).

Weighted poset metric is rather general in the sense that it includes various special cases, as detailed in the following examples.

Example 2.1. *Suppose that S is a finite field, $\mathbf{H}$ is a finite dimensional vector space and ω is integer-valued. Then (2.9) becomes labeled-poset-block metric, first proposed in [36]. Moreover, if $(\mathbf{P},\omega)$ is set to be the reduced canonical form of a directed graph (see [19, Section 3] for more details), then (2.9) recovers the notion of directed graph metric, first proposed in [19].*

Example 2.2. *An application of (2.8) to $(\mathbf{P},\mathbf{1})$ implies that*

$$\forall\ \beta\in\mathbf{H}:\mathrm{wt}_{\mathbf{P}}(\beta)\triangleq\mathrm{wt}_{(\mathbf{P},\mathbf{1})}(\beta)=|\langle\mathrm{supp}\,(\beta)\rangle_{\mathbf{P}}|,\tag{2.10}$$

which recovers the definition of $\mathbf{P}$-weight. Moreover, (2.9) recovers the notion of poset metric and poset-block metric (see [1, 7, 28, 40]). In particular, if $\mathbf{P}$ is a chain, then (2.10) becomes the Rosenbloom-Tsfasman weight (see [3, 21, 45]).

Example 2.3. *Suppose that $\mathbf{P}$ is an anti-chain. Then by (2.8), we have*

$$\forall\ \beta\in\mathbf{H}:\mathrm{wt}_{(\mathbf{P},\omega)}(\beta)=\sum_{i\in\mathrm{supp}\,(\beta)}\omega(i).$$

Moreover, (2.9) recovers the notion of weighted Hamming metric (see [5]).

Example 2.4. *Suppose that $\mathbf{P}$ is hierarchical. Let $\beta\in\mathbf{H}$, $\beta\neq 0$, and let $r=\max\{\mathrm{len}_{\mathbf{P}}(v)\mid v\in\mathrm{supp}\,(\beta)\}$. Then, by (2.8) and (3) of Lemma 2.1, we have*

$$\mathrm{wt}_{(\mathbf{P},\omega)}(\beta)=\left(\sum_{(i\in\Omega,\mathrm{len}_{\mathbf{P}}(i)\leqslant r-1)}\omega(i)\right)+\left(\sum_{(i\in\mathrm{supp}\,(\beta),\mathrm{len}_{\mathbf{P}}(i)=r)}\omega(i)\right).\tag{2.11}$$

Consequently, the $(\mathbf{P},\omega)$-weight of β is determined by r and

$$\{i\in\mathrm{supp}\,(\beta)\mid\mathrm{len}_{\mathbf{P}}(i)=r\}.$$

2.2 MEP with respect to weighted poset metric

Throughout this subsection, we fix a poset $\mathbf{P} = (\Omega, \preccurlyeq_{\mathbf{P}})$ and a weight function $\omega : \Omega \longrightarrow \mathbb{R}^{+}$.

Definition 2.2. **(1)** *For a linear code $C \subseteq \mathbf{H}$ and $f \in \mathrm{Hom}_{S}(C, \mathbf{H})$, we say that f preserves $(\mathbf{P}, \omega)$-weight if*

$$\mathrm{wt}_{(\mathbf{P},\omega)}(f(\alpha)) = \mathrm{wt}_{(\mathbf{P},\omega)}(\alpha) \text{ for all } \alpha \in C.$$

Any S-module automorphism of $\mathbf{H}$ that preserves $(\mathbf{P}, \omega)$-weight is referred to as a $(\mathbf{P}, \omega)$-weight isometry of $\mathbf{H}$. We let $\mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ denote the set of all the $(\mathbf{P}, \omega)$-weight isometries of $\mathbf{H}$.
(2) *We say that $\mathbf{H}$ satisfies MEP with respect to $(\mathbf{P}, \omega)$-weight if for any linear code $C \subseteq \mathbf{H}$ and $f \in \mathrm{Hom}_{S}(C, \mathbf{H})$ such that f preserves $(\mathbf{P}, \omega)$-weight, there exists $\varphi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ with $\varphi \mid_{C}= f$.*

Definition 2.3. *For a linear code $C \subseteq \mathbf{H}$ and $f \in \mathrm{Hom}_{S}(C, \mathbf{H})$, we say that f preserves $\mathbf{P}$-support if*

$$\langle \mathrm{supp}\,(f(\alpha)) \rangle_{\mathbf{P}} = \langle \mathrm{supp}\,(\alpha) \rangle_{\mathbf{P}} \text{ for all } \alpha \in C.$$

We let $\mathrm{GL}_{\mathbf{P}}(\mathbf{H})$ denote the set of all the S-module automorphisms of $\mathbf{H}$ that preserve $\mathbf{P}$-support. Moreover, we say that $\mathbf{H}$ satisfies MEP with respect to $\mathbf{P}$-support if for any linear code $C \subseteq \mathbf{H}$ and $f \in \mathrm{Hom}_{S}(C, \mathbf{H})$ such that f preserves $\mathbf{P}$-support, there exists $\varphi \in \mathrm{GL}_{\mathbf{P}}(\mathbf{H})$ with $\varphi \mid_{C}= f$.

It turns out that with ω appropriately set, Definition 2.3 is a special case of Definition 2.2. Indeed, let $\sigma : \Omega \longrightarrow [0, |\Omega| - 1]$ be any bijection, and set $\omega(i) = 2^{\sigma(i)}$ for all $i \in \Omega$. One can then check from (2.8) that for any $\alpha, \beta \in \mathbf{H}$, it holds that

$$\mathrm{wt}_{(\mathbf{P},\omega)}(\alpha) = \mathrm{wt}_{(\mathbf{P},\omega)}(\beta) \Longleftrightarrow \langle \mathrm{supp}\,(\alpha) \rangle_{\mathbf{P}} = \langle \mathrm{supp}\,(\beta) \rangle_{\mathbf{P}}.$$

Hence a map preserves $(\mathbf{P}, \omega)$-weight if and only if it preserves $\mathbf{P}$-support. Consequently, $\mathbf{H}$ satisfies MEP with respect to $(\mathbf{P}, \omega)$-weight if and only if $\mathbf{H}$ satisfies MEP with respect to $\mathbf{P}$-support.

Now we collect some terminologies for modules which are closely related to MEP, and we refer the reader to [2, 50] for more details. Consider two left S-modules X and Y. We write $X \cong Y$ if X and Y are isomorphic as left S-modules. Y is said to be X*-injective* if for any S-submodule $A \subseteq X$ and $f \in \mathrm{Hom}_{S}(A, Y)$, there exists $g \in \mathrm{Hom}_{S}(X, Y)$ with $g \mid_{A}= f$. We will say that Y is *strong pseudo-injective* if for any S-submodule $B \subseteq Y$ and any injective $h \in \mathrm{Hom}_S(B, Y)$, there exists $\tau \in \mathrm{Aut}_S(Y)$ with $\tau\mid_B = h$.

Below we collect five conditions for $\mathbf{H}$, $\mathbf{P}$ and ω, most of which are in fact necessary conditions for $\mathbf{H}$ to satisfy MEP with respect to $(\mathbf{P}, \omega)$-weight.

Definition 2.4. **(1)** *We say that* $\mathbf{H}$ *satisfies Condition (A) if* H_i *is strong pseudo-injective for all* $i \in \Omega$*.*
(2) *We say that* $(\mathbf{H}, \mathbf{P})$ *satisfies Condition (B) if for any* $k, l \in \Omega$ *such that* $k \preccurlyeq_{\mathbf{P}} l$, $k \neq l$*, it holds that* H_k *is* H_l*-injective.*
(3) *We say that* $\mathbf{H}$ *satisfies Condition (C) if there exists* $\xi \in \mathbf{H}$ *such that* $\xi \neq 0$ *and for any* $k, l \in \Omega$*, it holds that*

$$\{a \in S \mid a \cdot \xi_k = 0\} = \{a \in S \mid a \cdot \xi_l = 0\}.$$

(4) *We say that* $(\mathbf{H}, (\mathbf{P}, \omega))$ *satisfies Condition (D) if* $(\mathbf{P}, \omega)$ *satisfies UDP, and for any* $u, v \in \Omega$ *such that* $\mathrm{len}_{\mathbf{P}}(u) = \mathrm{len}_{\mathbf{P}}(v)$, $\omega(u) = \omega(v)$*, it holds that* $H_u \cong H_v$*.*
(5) *We say that* $(\mathbf{H}, \mathbf{P})$ *satisfies Condition (E) if* $\mathbf{P}$ *is hierarchical, and for any* $u, v \in \Omega$ *such that* $\mathrm{len}_{\mathbf{P}}(u) = \mathrm{len}_{\mathbf{P}}(v)$*, it holds that* $H_u \cong H_v$*.*

One can check that $\mathbf{H}$ satisfies Condition (C) if and only if there exists a left S-module B such that $B \neq \{0\}$ and for any $i \in \Omega$, B is isomorphic to some S-submodule of H_i. Condition (C) is a technical assumption for our discussion which seems to be relatively mild. We will show in Section 3 that Conditions (A) and (B) are necessary conditions for MEP with respect to $(\mathbf{P}, \omega)$-weight; moreover, if $\mathbf{H}$ satisfies Condition (C), then Condition (D) is a necessary condition for MEP with respect to $(\mathbf{P}, \omega)$-weight, and Condition (E) is a necessary condition for MEP with respect to $\mathbf{P}$-weight.

We end this subsection with the following lemma, which immediately follows from (2) of Lemma 2.1.

Lemma 2.2. $(\mathbf{H}, (\mathbf{P}, \mathbf{1}))$ *satisfies Condition (D) if and only if* $(\mathbf{H}, \mathbf{P})$ *satisfies Condition (E).*

3 Group of isometries for $(\mathbf{P}, \omega)$-weight

Groups of isometries for various weights and metrics have been extensively studied in the literature, and have been characterized for Rosenbloom-Tsfasman weight by Lee in [32], for crown weight by Cho and Kim in [9], for poset metric by Panek, Firer, Kim and Hyun in [41], for poset-block metric by Alves, Panek and Firer in [1], for directed graph metric by Etzion, Firer and Machado in [19], for combinatorial metric by Pinheiro, Machado and Firer in [44], and for labeled-poset-block metric by Machado and Firer in [36]. We also refer the reader to [36, 42] for isometries for two general classes of metrics in a poset space.

In this section, we characterize groups of isometries for weighted poset metric. Results established in this section will be used in later sections when we study MEP.

Throughout the rest of this section, we fix a poset $\mathbf{P} = (\Omega, \preccurlyeq_{\mathbf{P}})$ and a weight function $\omega : \Omega \longrightarrow \mathbb{R}^{+}$. Moreover, we define $Q \leqslant \mathrm{Aut}\,(\mathbf{P})$ as

$$Q = \{\mu \in \mathrm{Aut}\,(\mathbf{P}) \mid \omega(i) = \omega(\mu(i)),\, H_i \cong H_{\mu(i)} \text{ for all } i \in \Omega\}.$$

Our goal is to characterize $\mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$. We begin with the following lemma, whose proof is straightforward and hence omitted.

Lemma 3.1. *Let $\varphi \in \mathrm{Aut}_S(\mathbf{H})$ and $\lambda \in \mathrm{Aut}\,(\mathbf{P})$. Then, the following three statements are equivalent to each other:*
(1) $\langle \mathrm{supp}\,(\varphi(\alpha))\rangle_{\mathbf{P}} = \lambda[\langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}}]$ *for all* $\alpha \in \mathbf{H}$*;*
(2) *For any* $i \in \Omega$ *and* $\alpha \in \mathbf{H}$ *such that* $\mathrm{supp}\,(\alpha) = \{i\}$*, it holds that* $\langle \mathrm{supp}\,(\varphi(\alpha))\rangle_{\mathbf{P}} = \langle \{\lambda(i)\}\rangle_{\mathbf{P}}$*;*
(3) $\pi_j \circ \varphi \circ \eta_i = 0$ *for all* $i, j \in \Omega$ *with* $j \not\preccurlyeq_{\mathbf{P}} \lambda(i)$*, and* $\pi_{\lambda(i)} \circ \varphi \circ \eta_i \in \mathrm{Aut}_S(H_i, H_{\lambda(i)})$ *for all* $i \in \Omega$*.*

The following lemma will be crucial for characterizing $\mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$.

Lemma 3.2. **(1)** *Let* $\theta \in \mathbf{H}$ *such that there exists* $u \in \mathrm{supp}\,(\theta)$ *with* $\langle \mathrm{supp}\,(\theta)\rangle_{\mathbf{P}} = \langle \{u\}\rangle_{\mathbf{P}}$*. Then, for* $\gamma \in \mathbf{H}$*,* $\langle \mathrm{supp}\,(\gamma)\rangle_{\mathbf{P}} \subseteq \langle \mathrm{supp}\,(\theta)\rangle_{\mathbf{P}}$ *if and only if both* $\mathrm{wt}_{(\mathbf{P},\omega)}(\gamma) \leqslant \mathrm{wt}_{(\mathbf{P},\omega)}(\theta)$ *and* $\mathrm{wt}_{(\mathbf{P},\omega)}(\gamma + \theta) \leqslant \mathrm{wt}_{(\mathbf{P},\omega)}(\theta)$ *hold true.*
(2) *Let* $\varphi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$*, and fix* $\theta \in \mathbf{H}$ *such that there exists* $v \in \Omega$ *with* $\langle \mathrm{supp}\,(\varphi(\theta))\rangle_{\mathbf{P}} = \langle \{v\}\rangle_{\mathbf{P}}$*. Then, there exists* $u \in \mathrm{supp}\,(\theta)$ *such that* $\langle \mathrm{supp}\,(\theta)\rangle_{\mathbf{P}} = \langle \{u\}\rangle_{\mathbf{P}}$*. Moreover, for any* $\gamma \in \mathbf{H}$*, we have*

$$\langle \mathrm{supp}\,(\gamma)\rangle_{\mathbf{P}} \subseteq \langle \mathrm{supp}\,(\theta)\rangle_{\mathbf{P}} \Longleftrightarrow \langle \mathrm{supp}\,(\varphi(\gamma))\rangle_{\mathbf{P}} \subseteq \langle \mathrm{supp}\,(\varphi(\theta))\rangle_{\mathbf{P}}.$$

Proof. Throughout the proof, we write $\mathrm{wt}_{(\mathbf{P},\omega)} = \mathrm{wt}$ for notational simplicity. First of all, it follows from (2.8) and the fact $\omega(i) \in \mathbb{R}^{+}$ for all $i \in \Omega$ that the following two statements hold:
(i) Let $\gamma, \theta \in \mathbf{H}$ such that $\mathrm{wt}\,(\gamma) \leqslant \mathrm{wt}\,(\theta)$ and $\langle \mathrm{supp}\,(\theta)\rangle_{\mathbf{P}} \subseteq \langle \mathrm{supp}\,(\gamma)\rangle_{\mathbf{P}}$. Then, it holds that $\langle \mathrm{supp}\,(\theta)\rangle_{\mathbf{P}} = \langle \mathrm{supp}\,(\gamma)\rangle_{\mathbf{P}}$;
(ii) Let $\theta \in \mathbf{H}$ and $u \in \mathrm{supp}\,(\theta)$ such that $\mathrm{wt}\,(\theta) \leqslant \sum_{i \in \langle \{u\}\rangle_{\mathbf{P}}} \omega(i)$. Then, it holds that $\langle \mathrm{supp}\,(\theta)\rangle_{\mathbf{P}} = \langle \{u\}\rangle_{\mathbf{P}}$.

(1) Let $\gamma \in \mathbf{H}$. The "only if" part can be readily derived from (2.8) and the fact that $\mathrm{supp}\,(\gamma+\theta) \subseteq \mathrm{supp}\,(\gamma) \cup \mathrm{supp}\,(\theta)$, and so we only prove the "if" part. First, suppose that $u \in \mathrm{supp}\,(\gamma)$. Then, we have $\langle \mathrm{supp}\,(\theta) \rangle_{\mathbf{P}} \subseteq \langle \mathrm{supp}\,(\gamma) \rangle_{\mathbf{P}}$, which, together with $\mathrm{wt}\,(\gamma) \leqslant \mathrm{wt}\,(\theta)$ and (i), implies that $\langle \mathrm{supp}\,(\gamma) \rangle_{\mathbf{P}} = \langle \mathrm{supp}\,(\theta) \rangle_{\mathbf{P}}$, as desired. Second, suppose that $u \notin \mathrm{supp}\,(\gamma)$. Noticing that $u \in \mathrm{supp}\,(\theta)$, we have $u \in \mathrm{supp}\,(\gamma+\theta)$. Hence we have $\langle \mathrm{supp}\,(\theta) \rangle_{\mathbf{P}} \subseteq \langle \mathrm{supp}\,(\gamma+\theta) \rangle_{\mathbf{P}}$, which, together with $\mathrm{wt}\,(\gamma+\theta) \leqslant \mathrm{wt}\,(\theta)$ and (i), implies that $\langle \mathrm{supp}\,(\gamma+\theta) \rangle_{\mathbf{P}} = \langle \mathrm{supp}\,(\theta) \rangle_{\mathbf{P}}$. It then follows that

$$\mathrm{supp}\,(\gamma) \subseteq \mathrm{supp}\,(\theta) \cup \mathrm{supp}\,(\gamma+\theta) \subseteq \langle \mathrm{supp}\,(\theta) \rangle_{\mathbf{P}},$$

and hence $\langle \mathrm{supp}\,(\gamma) \rangle_{\mathbf{P}} \subseteq \langle \mathrm{supp}\,(\theta) \rangle_{\mathbf{P}}$, as desired.

(2) First of all, by (2.5), we have $\theta = \sum_{i \in \mathrm{supp}\,(\theta)} \eta_i(\theta_i)$, which further implies that

$$\varphi(\theta) = \sum_{i \in \mathrm{supp}\,(\theta)} \varphi(\eta_i(\theta_i)).$$

Noticing that $v \in \mathrm{supp}\,(\varphi(\theta))$, we can choose $u \in \mathrm{supp}\,(\theta)$ such that $v \in \mathrm{supp}\,(\varphi(\eta_u(\theta_u)))$. It then follows from $\mathrm{wt}\,(\theta) = \mathrm{wt}\,(\varphi(\theta))$ and (2.8) that

$$\mathrm{wt}\,(\theta) = \sum_{i \in \langle \{v\} \rangle_{\mathbf{P}}} \omega(i) \leqslant \mathrm{wt}\,(\varphi(\eta_u(\theta_u))) = \mathrm{wt}\,(\eta_u(\theta_u)) = \sum_{i \in \langle \{u\} \rangle_{\mathbf{P}}} \omega(i).$$

This, together with $u \in \mathrm{supp}\,(\theta)$ and (ii), implies that $\langle \mathrm{supp}\,(\theta) \rangle_{\mathbf{P}} = \langle \{u\} \rangle_{\mathbf{P}}$, which further establishes the first assertion. Now we let $\gamma \in \mathbf{H}$. An application of (1) to (γ, θ) and $(\varphi(\gamma), \varphi(\theta))$ respectively yields that

$$\begin{aligned}
&\langle \mathrm{supp}\,(\gamma) \rangle_{\mathbf{P}} \subseteq \langle \mathrm{supp}\,(\theta) \rangle_{\mathbf{P}} \\
&\Longleftrightarrow \mathrm{wt}\,(\gamma) \leqslant \mathrm{wt}\,(\theta) \ \wedge\ \mathrm{wt}\,(\gamma+\theta) \leqslant \mathrm{wt}\,(\theta) \\
&\Longleftrightarrow \mathrm{wt}\,(\varphi(\gamma)) \leqslant \mathrm{wt}\,(\varphi(\theta)) \ \wedge\ \mathrm{wt}\,(\varphi(\gamma+\theta)) \leqslant \mathrm{wt}\,(\varphi(\theta)) \\
&\Longleftrightarrow \mathrm{wt}\,(\varphi(\gamma)) \leqslant \mathrm{wt}\,(\varphi(\theta)) \ \wedge\ \mathrm{wt}\,(\varphi(\gamma)+\varphi(\theta)) \leqslant \mathrm{wt}\,(\varphi(\theta)) \\
&\Longleftrightarrow \langle \mathrm{supp}\,(\varphi(\gamma)) \rangle_{\mathbf{P}} \subseteq \langle \mathrm{supp}\,(\varphi(\theta)) \rangle_{\mathbf{P}},
\end{aligned}$$

which completes the proof of (2). □

Now we are ready to state and prove the main result of this section.

Theorem 3.1. **(1)** *For any $\varphi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$, there uniquely exists $\lambda \in Q$ such that*

$$\langle \mathrm{supp}\,(\varphi(\alpha)) \rangle_{\mathbf{P}} = \lambda[\langle \mathrm{supp}\,(\alpha) \rangle_{\mathbf{P}}] \text{ for all } \alpha \in \mathbf{H};$$

(2) *For any $\psi \in \mathrm{Aut}_S(\mathbf{H})$ such that there exists $\mu \in Q$ with*

$$\langle \mathrm{supp}\,(\psi(\alpha)) \rangle_{\mathbf{P}} = \mu[\langle \mathrm{supp}\,(\alpha) \rangle_{\mathbf{P}}] \text{ for all } \alpha \in \mathbf{H},$$

we have $\psi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$*;*
(3) *There uniquely exists* $\zeta : \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H}) \longrightarrow Q$ *such that for any* $\varphi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$*, it holds that*

$$\langle \mathrm{supp}\,(\varphi(\alpha))\rangle_{\mathbf{P}} = \zeta_{(\varphi)}[\langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}}] \text{ for all } \alpha \in \mathbf{H}.$$

Moreover, ζ *is a group homomorphism,* $\mathrm{ran}\,(\zeta) = Q$ *and* $\ker(\zeta) = \mathrm{GL}_{\mathbf{P}}(\mathbf{H})$*.*

Proof. **(1)** First of all, from (2) of Lemma 3.2, we infer that there exists $\lambda : \Omega \longrightarrow \Omega$ such that for any $i \in \Omega$, it holds that

$$\forall\ \alpha \in \mathbf{H}\ s.t.\ \mathrm{supp}\,(\alpha) = \{i\} : \langle \mathrm{supp}\,(\varphi(\alpha))\rangle_{\mathbf{P}} = \langle \{\lambda(i)\}\rangle_{\mathbf{P}}. \tag{3.1}$$

Next, we show that $\lambda \in \mathrm{Aut}\,(\mathbf{P})$. Indeed, let $i, t \in \Omega$. Since $H_i \neq \{0\}$, $H_t \neq \{0\}$, we can choose $\gamma, \theta \in \mathbf{H}$ such that $\mathrm{supp}\,(\gamma) = \{i\}$, $\mathrm{supp}\,(\theta) = \{t\}$. It then follows from (3.1) and (2) of Lemma 3.2 that

$$i \preccurlyeq_{\mathbf{P}} t \Longleftrightarrow \langle\{i\}\rangle_{\mathbf{P}} \subseteq \langle\{t\}\rangle_{\mathbf{P}} \Longleftrightarrow \langle\{\lambda(i)\}\rangle_{\mathbf{P}} \subseteq \langle\{\lambda(t)\}\rangle_{\mathbf{P}} \Longleftrightarrow \lambda(i) \preccurlyeq_{\mathbf{P}} \lambda(t),$$

as desired.

Now it follows from Lemma 3.1 that $\langle \mathrm{supp}\,(\varphi(\alpha))\rangle_{\mathbf{P}} = \lambda[\langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}}]$ for all $\alpha \in \mathbf{H}$, and $H_i \cong H_{\lambda(i)}$ for all $i \in \Omega$. Moreover, let $u \in \Omega$, and let $I = \langle\{u\}\rangle_{\mathbf{P}}$. Since $H_i \neq \{0\}$ for all $i \in \Omega$, we can choose $\beta, \gamma \in \mathbf{H}$ such that $\mathrm{supp}\,(\beta) = I$, $\mathrm{supp}\,(\gamma) = I - \{u\}$. It immediately follows that $\langle \mathrm{supp}\,(\varphi(\beta))\rangle_{\mathbf{P}} = \lambda[I]$, $\langle \mathrm{supp}\,(\varphi(\gamma))\rangle_{\mathbf{P}} = \lambda[I] - \{\lambda(u)\}$, and hence

$$\omega(u) = \mathrm{wt}_{(\mathbf{P},\omega)}(\beta) - \mathrm{wt}_{(\mathbf{P},\omega)}(\gamma) = \mathrm{wt}_{(\mathbf{P},\omega)}(\varphi(\beta)) - \mathrm{wt}_{(\mathbf{P},\omega)}(\varphi(\gamma)) = \omega(\lambda(u)),$$

which further implies that $\lambda \in Q$, as desired.

Finally, the uniqueness of λ follows from some straightforward verification which we omit.

(2) This can be readily verified and hence we omit the details.

(3) By (1), ζ is well defined and unique. A routine verification yields that ζ is a group homomorphism with $\ker(\zeta) = \mathrm{GL}_{\mathbf{P}}(\mathbf{H})$. Now we show that $\mathrm{ran}\,(\zeta) = Q$. Let $\mu \in Q$. Then, for any $i \in \Omega$, we can choose $\rho_i \in \mathrm{Aut}_S(H_i, H_{\mu(i)})$. Define $\psi : \mathbf{H} \longrightarrow \mathbf{H}$ such that for any $\alpha \in \mathbf{H}$,

$$\psi(\alpha)_{\mu(i)} = \rho_i(\alpha_i) \text{ for all } i \in \Omega.$$

It is straightforward to verify that $\psi \in \mathrm{Aut}_S(\mathbf{H})$ and $\langle \mathrm{supp}\,(\psi(\alpha))\rangle_{\mathbf{P}} = \mu[\langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}}]$ for all $\alpha \in \mathbf{H}$. From (2), we infer that $\psi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$, which further implies that $\mu = \zeta_{(\psi)}$, as desired. □

We present several corollaries of Theorem 3.1 in the following example.

Example 3.1. **(1)** *Suppose that S is a field and $\mathbf{H}$ is a finite dimensional vector space. Then, we have*

$$Q = \{\mu \in \mathrm{Aut}\,(\mathbf{P}) \mid \omega(i) = \omega(\mu(i)),\ \dim_S(H_i) = \dim_S(H_{\mu(i)}) \text{ for all } i \in \Omega\}.$$

Consequently, Theorem 3.1 recovers [36, Theorem 5] which has been established for labeled-poset-block metric.
(2) *Suppose that $\omega = \mathbf{1}$. Then, we have*

$$Q = \{\mu \in \mathrm{Aut}\,(\mathbf{P}) \mid H_i \cong H_{\mu(i)} \text{ for all } i \in \Omega\}.$$

Consequently, Theorem 3.1 characterizes groups of isometries for $\mathbf{P}$-weight. With S further set to be a field and $\mathbf{H}$ set to be a finite dimensional vector space, we have

$$Q = \{\mu \in \mathrm{Aut}\,(\mathbf{P}) \mid \dim_S(H_i) = \dim_S(H_{\mu(i)}) \text{ for all } i \in \Omega\}.$$

Consequently, Theorem 3.1 recovers [41, Theorems 1.1 and 1.2] and [1, Theorem 4.10], which have been established for poset metric and poset-block metric, respectively.
(3) *Suppose that $\mathbf{P}$ is an anti-chain. Then, we have*

$$Q = \{\mu : \Omega \longrightarrow \Omega \text{ is bijective} \mid \omega(i) = \omega(\mu(i)),\ H_i \cong H_{\mu(i)} \text{ for all } i \in \Omega\}.$$

Consequently, Theorem 3.1 characterizes groups of isometries for weighted Hamming metric.
(4) *Suppose that $(\mathbf{P}, \omega)$ is the reduced canonical form of a directed graph (see Example 2.1). Then, Theorem 3.1 recovers [19, Theorems 8] which has been established for directed graph metric.*

Remark 3.1. *In [36, Section II.A], Machado and Firer have studied the group of isometries for a more general class of metrics including combinatorial metric (see [36, Theorem 1]). Their approach requires $\mathbf{H}$ to be a finite dimensional vector space over a non-binary field, and our Theorem 3.1 applies to more general alphabets.*

As a first application of Theorem 3.1, we now derive several necessary conditions for $\mathbf{H}$ to satisfy MEP with respect to $(\mathbf{P}, \omega)$-weight.

Corollary 3.1. **(1)** *Suppose that $\mathbf{H}$ satisfies MEP with respect to $(\mathbf{P}, \omega)$-weight. Then, $\mathbf{H}$ satisfies Condition (A), $(\mathbf{H}, \mathbf{P})$ satisfies Condition (B).*

Moreover, for any $\gamma, \theta \in \mathbf{H}$ *such that* $\mathrm{wt}_{(\mathbf{P},\omega)}(a \cdot \gamma) = \mathrm{wt}_{(\mathbf{P},\omega)}(a \cdot \theta)$ *for all* $a \in S$*, there exists* $\psi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ *with* $\psi(\gamma) = \theta$*.*
(2) *Suppose that* $\mathbf{H}$ *satisfies Condition (C) and for any* $\gamma, \theta \in \mathbf{H}$ *such that*

$$\mathrm{wt}_{(\mathbf{P},\omega)}(a \cdot \gamma) = \mathrm{wt}_{(\mathbf{P},\omega)}(a \cdot \theta) \textit{ for all } a \in S,$$

there exists $\psi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ *with* $\psi(\gamma) = \theta$*. Then,* $(\mathbf{H}, (\mathbf{P}, \omega))$ *satisfies Condition (D).*

Proof. Throughout the proof, we define $\zeta : \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H}) \longrightarrow Q$ as in (3) of Theorem 3.1.
(1) First, we show that $(\mathbf{H}, \mathbf{P})$ satisfies Condition (B). Let $k, l \in \Omega$ such that $k \preccurlyeq_{\mathbf{P}} l$, $k \neq l$. Consider an S-submodule $B \subseteq H_l$ and $f \in \mathrm{Hom}_S(B, H_k)$. Define $g \in \mathrm{Hom}_S(\eta_l[B], \mathbf{H})$ as

$$g(\eta_l(b)) = \eta_l(b) + \eta_k(f(b)) \text{ for all } b \in B.$$

Noting that g preserves $(\mathbf{P}, \omega)$-weight, we can choose $\varphi \in \mathrm{End}_S(\mathbf{H})$ with $\varphi\,|_{\eta_l[B]} = g$. It follows that $\pi_k \circ \varphi \circ \eta_l \in \mathrm{Hom}_S(H_l, H_k)$ extends f, as desired.

Second, we show that $\mathbf{H}$ satisfies Condition (A). Let $i \in \Omega$. Fix an S-submodule B of H_i with $B \neq \{0\}$, and let $\xi \in \mathrm{Hom}_S(B, H_i)$ be injective. Define $\chi \in \mathrm{Hom}\,(\eta_i[B], \mathbf{H})$ as

$$\chi(\eta_i(b)) = \eta_i(\xi(b)) \text{ for all } b \in B.$$

Since ξ is injective, χ preserves $(\mathbf{P}, \omega)$-weight. Hence we can choose $\varphi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ with $\varphi\,|_{\eta_i[B]} = \chi$. It can be readily verified that $(\pi_i \circ \varphi \circ \eta_i)\,|_B = \xi$, and it remains to show that $\pi_i \circ \varphi \circ \eta_i \in \mathrm{Aut}_S(H_i)$. To this end, let $\lambda = \zeta_{(\varphi)}$. Since $B \neq \{0\}$, we can choose $b \in B$, $b \neq 0$. It then follows that $\langle \mathrm{supp}\,(\varphi(\eta_i(b))) \rangle_{\mathbf{P}} = \langle \{\lambda(i)\} \rangle_{\mathbf{P}}$. Moreover, the injectivity of ξ implies that $\xi(b) \neq 0$, which, together with $\varphi(\eta_i(b)) = \chi(\eta_i(b)) = \eta_i(\xi(b))$, further implies that $\mathrm{supp}\,(\varphi(\eta_i(b))) = \{i\}$. It then follows that $\langle \{\lambda(i)\} \rangle_{\mathbf{P}} = \langle \{i\} \rangle_{\mathbf{P}}$, which further implies that $\lambda(i) = i$. Now it follows from Lemma 3.1 and (3) of Theorem 3.1 that $\pi_i \circ \varphi \circ \eta_i \in \mathrm{Aut}_S(H_i)$, as desired.

Finally, let $\gamma, \theta \in \mathbf{H}$ such that $\mathrm{wt}_{(\mathbf{P},\omega)}(a \cdot \gamma) = \mathrm{wt}_{(\mathbf{P},\omega)}(a \cdot \theta)$ for all $a \in S$. Then, $f \in \mathrm{Hom}_S(S \cdot \gamma, \mathbf{H})$ defined as $f(\gamma) = \theta$ is a $(\mathbf{P}, \omega)$-weight preserving map. Hence we can choose $\psi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ such that $\psi\,|_{S \cdot \gamma} = f$. It follows that $\psi(\gamma) = f(\gamma) = \theta$, as desired.

(2) Since $\mathbf{H}$ satisfies Condition (C), we can choose $\xi \in \mathbf{H}$ such that $\xi \neq 0$ and for any $k, l \in \Omega$, it holds that

$$\{a \in S \mid a \cdot \xi_k = 0\} = \{a \in S \mid a \cdot \xi_l = 0\}.$$

Consider $I, J \in \mathcal{I}(\mathbf{P})$ with $\sum_{i \in I} \omega(i) = \sum_{j \in J} \omega(j)$. Since $\mathrm{supp}\,(\xi) = \Omega$, there uniquely exist $\gamma, \theta \in \mathbf{H}$ such that

$$\mathrm{supp}\,(\gamma) = I,\ \gamma_i = \xi_i \text{ for all } i \in I;\ \mathrm{supp}\,(\theta) = J,\ \theta_j = \xi_j \text{ for all } j \in J.$$

For an arbitrary $a \in S$, considering $I = \emptyset$ and $I \neq \emptyset$ separately, we deduce that either $\mathrm{supp}\,(a \cdot \gamma) = I$, $\mathrm{supp}\,(a \cdot \theta) = J$ or $a \cdot \gamma = a \cdot \theta = 0$ holds true, which further implies that $\mathrm{wt}_{(\mathbf{P},\omega)}(a \cdot \gamma) = \mathrm{wt}_{(\mathbf{P},\omega)}(a \cdot \theta)$. Hence we can choose $\psi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ with $\psi(\gamma) = \theta$. Now let $\lambda = \zeta_{(\psi)}$. It then follows that

$$J = \langle \mathrm{supp}\,(\theta) \rangle_{\mathbf{P}} = \langle \mathrm{supp}\,(\psi(\gamma)) \rangle_{\mathbf{P}} = \lambda[\langle \mathrm{supp}\,(\gamma) \rangle_{\mathbf{P}}] = \lambda[I],$$

which further yields that $(\mathbf{P}, \omega)$ satisfies UDP. Next, we fix $u, v \in \Omega$ such that $\mathrm{len}_{\mathbf{P}}(u) = \mathrm{len}_{\mathbf{P}}(v) \triangleq r$ and $\omega(u) = \omega(v)$. Let

$$I_1 \triangleq \{w \in \Omega \mid \mathrm{len}_{\mathbf{P}}(w) < r\} \cup \{u\},\ J_1 \triangleq \{w \in \Omega \mid \mathrm{len}_{\mathbf{P}}(w) < r\} \cup \{v\}.$$

It follows that $I_1, J_1 \in \mathcal{I}(\mathbf{P})$ and $\sum_{i \in I_1} \omega(i) = \sum_{j \in J_1} \omega(j)$. Hence we can choose $\mu \in Q$ such that $J_1 = \mu[I_1]$. Apparently, we have $v = \mu(u)$, which, together with $\mu \in Q$, further implies that $H_u \cong H_{\mu(u)} = H_v$, as desired. □

As an application of Corollary 3.1, we now show that MEP with respect to $\mathbf{P}$-weight has strong limitations on the structure of $\mathbf{P}$, as detailed in the following corollary.

Corollary 3.2. *If $\mathbf{H}$ satisfies Condition (C) and MEP with respect to $\mathbf{P}$-weight, then $\mathbf{P}$ is hierarchical.*

Proof. It follows from Corollary 3.1 that $(\mathbf{H}, (\mathbf{P}, \mathbf{1}))$ satisfies Condition (D), which, together with Lemma 2.2, further implies that $(\mathbf{H}, \mathbf{P})$ satisfies Condition (E), and therefore, $\mathbf{P}$ is hierarchical, as desired. □

We illustrate Corollary 3.2 with the following example.

Example 3.2. *Apparently, the left S-module S^{Ω} satisfies Condition (C). Hence if S^{Ω} satisfies MEP with respect to $\mathbf{P}$-weight, then $\mathbf{P}$ is hierarchical. This fact generalizes [3, Theorem 7.6] which has been established for codes over finite Frobenius rings.*

4 MEP with respect to P-support

Throughout this section, we fix a poset $\mathbf{P} = (\Omega, \preccurlyeq_{\mathbf{P}})$. We will derive conditions for $\mathbf{H}$ to satisfy MEP with respect to $\mathbf{P}$-support. As remarked in Section 2.2, MEP with respect to $\mathbf{P}$-support is a special case of MEP with respect to general $(\mathbf{P}, \omega)$-weight. On the other hand, results established in this section will be used in Sections 5 and 6 when we study MEP with respect to $(\mathbf{P}, \omega)$-weight.

We begin with a necessary and sufficient condition for the special case that $\mathbf{P}$ is hierarchical.

Theorem 4.1. *Suppose that* $\mathbf{P}$ *is hierarchical. Then,* $\mathbf{H}$ *satisfies MEP with respect to* $\mathbf{P}$*-support if and only if* $\mathbf{H}$ *satisfies Condition (A) and* $(\mathbf{H}, \mathbf{P})$ *satisfies Condition (B).*

Before giving the proof, we first use Theorem 4.1 to generalize several existing results in the literature, as detailed in the following example.

Example 4.1. (1) *First, suppose that* $\mathbf{P}$ *is an anti-chain. Then,* $\mathbf{P}$ *is hierarchical and* $(\mathbf{H}, \mathbf{P})$ *satisfies Condition (B). Hence by Theorem 4.1,* $\mathbf{H}$ *satisfies MEP with respect to* $\mathbf{P}$*-support if and only if* $\mathbf{H}$ *satisfies Condition (A). This result generalizes [3, Theorem 6.3] and [22, Remark 4.21 (a)] which have been established for codes over finite Frobenius rings and finite Frobenius bimodules.*
(2) *Second, suppose that* $\mathbf{P}$ *is a chain. Then,* $\mathbf{P}$ *is hierarchical, and moreover, for any* $\alpha, \beta \in \mathbf{H}$*, it holds that*

$$\mathrm{wt}_{\mathbf{P}}(\alpha) = \mathrm{wt}_{\mathbf{P}}(\beta) \Longleftrightarrow \langle \mathrm{supp}\,(\alpha) \rangle_{\mathbf{P}} = \langle \mathrm{supp}\,(\beta) \rangle_{\mathbf{P}}.$$

Hence by Theorem 4.1 and Example 2.2, $\mathbf{H}$ *satisfies MEP with respect to Rosenbloom-Tsfasman weight if and only if* $\mathbf{H}$ *satisfies Condition (A) and* $(\mathbf{H}, \mathbf{P})$ *satisfies Condition (B). In particular, if one of the following two conditions holds:*
(i) $\mathbf{H} = M^{\Omega}$*, where* M *is a finite left* S*-module such that* M *is* M*-injective;*
(ii) S *is a finite field and* $\mathbf{H}$ *is a finite dimensional vector space,*
then $\mathbf{H}$ *satisfies MEP with respect to Rosenbloom-Tsfasman weight, which further recovers [3, Theorem 6.1], [22, Theorem 4.13] and [22, Theorem 5.1].*

Now we prove Theorem 4.1. Note that the "only if" part is a special case of (1) of Corollary 3.1, it suffices to establish the "if" part, which immediately follows from the following proposition.

Proposition 4.1. *Suppose that* $\mathbf{P}$ *is hierarchical,* $\mathbf{H}$ *satisfies Condition (A) and* $(\mathbf{H}, \mathbf{P})$ *satisfies Condition (B). Let* m *be the largest cardinality of a chain in* $\mathbf{P}$*, and for any* $s \in [1, m]$*, set*

$$W_s \triangleq \{u \in \Omega \mid \mathrm{len}_{\mathbf{P}}(u) = s\}.$$

Let $r \in [1, m]$*,* $C \subseteq \delta(\bigcup_{j=1}^{r} W_j)$ *be a linear code, and fix* $f \in \mathrm{Hom}_S(C, \mathbf{H})$ *such that* f *preserves* $\mathbf{P}$*-support. Then, there exists* $\varphi \in \mathrm{GL}_{\mathbf{P}}(\mathbf{H})$ *such that* $\varphi\mid_C = f$ *and* $\varphi(\alpha) = \alpha$ *for all* $\alpha \in \delta(\bigcup_{j=r+1}^{m} W_j)$*.*

Proof. Let $D \triangleq C \cap \delta(\bigcup_{j=1}^{r-1} W_j)$. First, applying an induction argument to $r-1$, D and $f\mid_D$, we can choose $\tau \in \mathrm{GL}_{\mathbf{P}}(\mathbf{H})$ such that $\tau\mid_D = f\mid_D$ and $\tau(\alpha) = \alpha$ for all $\alpha \in \delta(\bigcup_{j=r}^{m} W_j)$. Define

$$g \triangleq \tau^{-1} \circ f. \tag{4.1}$$

We claim that there exists $(\mu_i \mid i \in W_r)$ such that for any $i \in W_r$,

$$\mu_i \in \mathrm{Aut}_S(H_i),\ \mu_i(g(\alpha)_i) = \alpha_i \text{ for all } \alpha \in C. \tag{4.2}$$

Indeed, let $i \in W_r$. For any $\alpha \in C$, noticing that $\langle \mathrm{supp}\,(g(\alpha))\rangle_{\mathbf{P}} = \langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}} \subseteq \bigcup_{j=1}^{r} W_j$, we have

$$\alpha \in \ker(\pi_i\mid_C) \Longleftrightarrow i \notin \mathrm{supp}\,(\alpha) \Longleftrightarrow i \notin \mathrm{supp}\,(g(\alpha)) \Longleftrightarrow \alpha \in \ker(\pi_i \circ g).$$

It follows that $\ker(\pi_i\mid_C) = \ker(\pi_i \circ g)$. Hence there exists an injective map $\varsigma_1 \in \mathrm{Hom}_S((\pi_i \circ g)[C], H_i)$ with $\varsigma_1 \circ \pi_i \circ g = \pi_i\mid_C$. Since H_i is strong pseudo-injective, we can choose $\varsigma \in \mathrm{Aut}_S(H_i)$ with $\varsigma\mid_{(\pi_i \circ g)[C]} = \varsigma_1$. It is straightforward to verify that $\varsigma(g(\alpha)_i) = \alpha_i$ for all $\alpha \in C$, as desired.

Next, let $\psi \in \mathrm{End}_S(\mathbf{H})$ such that for any $\alpha \in \mathbf{H}$, it holds that

$$(\forall\ i \in W_r : \psi(\alpha)_i = \mu_i(\alpha_i)) \text{ and } (\forall\ t \in \Omega - W_r : \psi(\alpha)_t = \alpha_t). \tag{4.3}$$

Then, we have $\psi \in \mathrm{GL}_{\mathbf{P}}(\mathbf{H})$ and $\psi(\alpha) = \alpha$ for all $\alpha \in \delta(\Omega - W_r)$. Define

$$g_1 \triangleq \psi \circ g = \psi \circ \tau^{-1} \circ f. \tag{4.4}$$

Then, one can check that $g_1\mid_D = \mathrm{id}_D$; moreover, for any $i \in W_r$ and $\alpha \in C$, it follows from (4.2) and (4.3) that

$$g_1(\alpha)_i = \psi(g(\alpha))_i = \mu_i(g(\alpha)_i) = \alpha_i. \tag{4.5}$$

By $g_1\mid_D = \mathrm{id}_D$ and (4.5), we can define $\rho \in \mathrm{Hom}_S(C/D, \delta(\bigcup_{j=1}^{r-1} W_j))$ as

$$\rho(\alpha + D) = g_1(\alpha) - \alpha \text{ for all } \alpha \in C.$$

Since $C \subseteq \delta(\bigcup_{j=1}^{r} W_j)$ and $D = C \cap \delta(\bigcup_{j=1}^{r-1} W_j)$, we can define an injective map $\varepsilon \in \mathrm{Hom}_S(C/D, \delta(W_r))$ such that

$$\varepsilon(\alpha + D) = \sum_{i \in W_r} \eta_i(\alpha_i) \text{ for all } \alpha \in C.$$

From the facts that $(\mathbf{H}, \mathbf{P})$ satisfies Condition (B) and $\mathbf{P}$ is hierarchical, we infer that H_k is H_l-injective for all $k, l \in \Omega$ with $\mathrm{len}_{\mathbf{P}}(k)+1 \leqslant \mathrm{len}_{\mathbf{P}}(l)$, which further implies that $\delta(\bigcup_{j=1}^{r-1} W_j)$ is $\delta(W_r)$-injective. This, together with the fact that ε is injective, enables us to choose $\lambda \in \mathrm{Hom}_S(\delta(W_r), \delta(\bigcup_{j=1}^{r-1} W_j))$ with $\rho = \lambda \circ \varepsilon$. It then follows from the definition of ρ and ε that

$$g_1(\alpha) = \alpha + \lambda \left(\sum_{i \in W_r} \eta_i(\alpha_i) \right) \text{ for all } \alpha \in C. \tag{4.6}$$

Now define $\sigma \in \mathrm{End}_S(\mathbf{H})$ as

$$\sigma(\gamma) = \gamma + \lambda \left(\sum_{i \in W_r} \eta_i(\gamma_i) \right). \tag{4.7}$$

From Lemma 3.1 and the fact that $\mathbf{P}$ is hierarchical, it is straightforward to verify that $\sigma \in \mathrm{GL}_{\mathbf{P}}(\mathbf{H})$. In addition, by (4.6) and (4.7), we have $\sigma \mid_C = g_1$ and $\sigma(\gamma) = \gamma$ for all $\gamma \in \delta(\Omega - W_r)$. Finally, set $\varphi = \tau \circ \psi^{-1} \circ \sigma$. It then follows from (4.1) and (4.4) that $\varphi \in \mathrm{GL}_{\mathbf{P}}(\mathbf{H})$, $\varphi \mid_C = f$ and $\varphi(\alpha) = \alpha$ for all $\alpha \in \delta(\bigcup_{j=r+1}^{m} W_j)$, as desired. □

Next, we give a sufficient condition for general $\mathbf{P}$. To this end, we first recall the notion of *semi-local ring*. Following [2], for an associative ring R with multiplicative identity, the Jacobson radical of R is the intersection of all the maximal left ideals of R; moreover, R is said to be semi-local if the quotient ring of R by its Jacobson radical is semi-simple.

Lemma 4.1. *([4, Lemma 6.4],[48, Proposition 5.1]) Let R be a semi-local ring, M be a left R-module, and let $x, y \in M$. Suppose that there exists $a, b \in R$ with $y = ax$, $x = by$. Then, there exists a multiplicative invertible element $c \in R$ with $y = cx$.*

Now we are ready to prove the following theorem.

Theorem 4.2. *Define the following subring R of $\mathrm{End}_S(\mathbf{H})$:*

$$R \triangleq \{\varphi \in \mathrm{End}_S(\mathbf{H}) \mid \langle \mathrm{supp}\,(\varphi(\alpha))\rangle_{\mathbf{P}} \subseteq \langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}} \text{ for all } \alpha \in \mathbf{H}\}.$$

Suppose that R is semi-local, and that H_k is H_l-injective for all $k, l \in \Omega$ with $k \preccurlyeq_{\mathbf{P}} l$. Then, $\mathbf{H}$ satisfies MEP with respect to $\mathbf{P}$-support.

Proof. First, we show that for any linear code $D \subseteq \mathbf{H}$ and $\chi \in \mathrm{Hom}_S(D, \mathbf{H})$ such that $\langle \mathrm{supp}\,(\chi(\alpha)) \rangle_{\mathbf{P}} \subseteq \langle \mathrm{supp}\,(\alpha) \rangle_{\mathbf{P}}$ for all $\alpha \in D$, there exists $\varphi \in R$ with $\varphi\,|_D = \chi$. Indeed, for an arbitrary $k \in \Omega$, we define a tuple

$$(\rho_{(i,k)} \mid i \in \Omega) \in \prod_{i \in \Omega} \mathrm{Hom}_S(H_i, H_k)$$

as follows: Let $E = \langle \{k\} \rangle_{\overline{\mathbf{P}}}$. Then, for $\pi_k \circ \chi \in \mathrm{Hom}\,(D, H_k)$ and $\zeta \in \mathrm{Hom}\,(D, \delta(E))$ defined as $\zeta(\alpha) = \sum_{i \in E} \eta_i(\alpha_i)$, we have $\ker(\zeta) \subseteq \ker(\pi_k \circ \chi)$. Hence there exists $\lambda \in \mathrm{Hom}_S(\zeta[D], H_k)$ with $\lambda \circ \zeta = \pi_k \circ \chi$. Now for any $l \in E$, by $k \preccurlyeq_{\mathbf{P}} l$, we have H_k is H_l-injective. It follows that H_k is $\delta(E)$-injective. Hence we can choose $\mu \in \mathrm{Hom}\,(\delta(E), H_k)$ with $\mu\,|_{\zeta[D]} = \lambda$. Finally, for any $i \in \Omega$, we set

$$\rho_{(i,k)} = \begin{cases} \mu \circ \eta_i, & i \in E; \\ 0, & i \in \Omega - E. \end{cases}$$

Now define $\varphi \in \mathrm{End}_S(\mathbf{H})$ as $\pi_k \circ \varphi \circ \eta_i = \rho_{(i,k)}$ for all $(i, k) \in \Omega \times \Omega$. It is then straightforward to verify that $\varphi \in R$ and $\varphi\,|_D = \chi$, as desired.

Now let $C \subseteq \mathbf{H}$ be a linear code. Then, $\mathrm{Hom}_S(C, \mathbf{H})$ is naturally a left R-module. Let $f \in \mathrm{Hom}_S(C, \mathbf{H})$ such that f preserves $\mathbf{P}$-support. Then, we can choose $\sigma \in R$ such that $f = \sigma\,|_C = \sigma \circ \mathrm{id}_C$. Noticing that f is injective and $f^{-1} \in \mathrm{Hom}_S(f[C], \mathbf{H})$ also preserves $\mathbf{P}$-support, we can choose $\tau \in R$ such that $\tau\,|_{f[C]} = f^{-1}$. It follows that $\tau \circ f = \mathrm{id}_C$. Since R is semi-local, by Lemma 4.1, there exists a multiplicative invertible element $\varphi \in R$ such that $f = \varphi \circ \mathrm{id}_C = \varphi\,|_C$; moreover, it follows from the definition of R that $\varphi \in \mathrm{GL}_{\mathbf{P}}(\mathbf{H})$, as desired. □

As an application of Theorem 4.2, in the following example, we consider the special case that $\mathbf{H}$ is finite.

Example 4.2. *Suppose that $\mathbf{H}$ is finite. Then, R defined in Theorem 4.2 is finite, and hence semi-local. Therefore if H_k is H_l-injective for all $k \preccurlyeq_{\mathbf{P}} l \in \Omega$, then $\mathbf{H}$ satisfies MEP with respect to $\mathbf{P}$-support. This general fact again recovers [3, Theorems 6.1 and 6.3] and [22, Theorem 4.13, Theorem 5.1 and Remark 4.21 (a)]. Moreover, it will be used in Section 6 when we study the connections among MEP and other coding-theoretic properties.*

5 MEP with respect to weighted poset metric

Throughout this section, we fix a poset $\mathbf{P} = (\Omega, \preccurlyeq_{\mathbf{P}})$ and a weight function $\omega : \Omega \longrightarrow \mathbb{R}^{+}$. Let m be the largest cardinality of a chain in $\mathbf{P}$, and for any $r \in [1,m]$, we define

$$W_r \triangleq \{u \in \Omega \mid \mathrm{len}_{\mathbf{P}}(u) = r\}.$$

We will consider the case that either $\mathbf{P}$ is hierarchical or $\omega = \mathbf{1}$. For these two scenarios, we will give necessary and sufficient conditions for $\mathbf{H}$ to satisfy MEP with respect to $(\mathbf{P},\omega)$-weight in terms of MEP with respect to Hamming weight.

As a first step, we give a necessary and sufficient condition for $\mathbf{H}$ to satisfy MEP with respect to $(\mathbf{P},\omega)$-weight when $\mathbf{P}$ is hierarchical in terms of MEP with respect to weighted Hamming metric, as detailed in the following proposition.

Proposition 5.1. *Suppose that $\mathbf{P}$ is hierarchical. Then, $\mathbf{H}$ satisfies MEP with respect to $(\mathbf{P},\omega)$-weight if and only if the following two conditions hold:*
(1) *$\mathbf{H}$ satisfies Condition (A), $(\mathbf{H},\mathbf{P})$ satisfies Condition (B);*
(2) *For any $r \in [1,m]$, the left S-module $\prod_{i\in W_r} H_i$ satisfies MEP with respect to $((W_r,=),\omega\mid_{W_r})$-weight.*

Proof. Throughout the proof, for notational simplicity, we define $\varpi : 2^{\Omega} \longrightarrow \mathbb{R}$ as

$$\varpi(A) = \sum_{a\in A}\omega(a). \tag{5.1}$$

For any $\alpha \in \mathbf{H}$ and $J \subseteq \Omega$, we define $\alpha\mid_J \in \prod_{i\in J} H_i$ as

$$\forall\ j \in J : (\alpha\mid_J)_j = \alpha_j. \tag{5.2}$$

For any $J \subseteq \Omega$ and $\tau \in \prod_{i\in J} H_i$, we define $\widetilde{\tau} \in \mathbf{H}$ as

$$\forall\ i \in \Omega : \widetilde{\tau}_i = \begin{cases} \tau_i, & i \in J; \\ 0, & i \notin J. \end{cases} \tag{5.3}$$

First, we prove the "only if" part. Note that (1) follows from Corollary 3.1, we only prove (2). Consider an arbitrary $r \in [1,m]$. Let D be an S-submodule of $\prod_{i\in W_r} H_i$, and let $g \in \mathrm{Hom}_S(D, \prod_{i\in W_r} H_i)$ such that g preserves $((W_r,=),\omega\mid_{W_r})$-weight. Define $C \triangleq \{\widetilde{\gamma} \mid \gamma \in D\}$, and define $f \in \mathrm{Hom}_S(C,\mathbf{H})$ as

$$f(\widetilde{\gamma}) = \widetilde{g(\gamma)} \text{ for all } \gamma \in D.$$

From (2.11), we infer that f preserves $(\mathbf{P},\omega)$-weight. Hence we can choose $\varphi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ with $\varphi\mid_C = f$. Define $\tau \in \mathrm{End}_S(\prod_{i\in W_r} H_i)$ as

$$\tau(\gamma) = \varphi(\widetilde{\gamma})\mid_{W_r}.$$

By $\varphi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ and (2.11), one can check that τ is a $((W_r,=),\omega\mid_{W_r})$-weight isometry of $\prod_{i\in W_r} H_i$. Finally, it follows from $\varphi\mid_C = f$ that $\tau\mid_D = g$, as desired.

Second, we prove the "if" part. Let $C \subseteq \mathbf{H}$ be a linear code, and let $f \in \mathrm{Hom}_S(C,\mathbf{H})$ such that f preserves $(\mathbf{P},\omega)$-weight. Fix $r \in [1,m]$ such that $C \subseteq \delta(\bigcup_{j=1}^{r} W_j)$, and let $C_1 \triangleq C \cap \delta(\bigcup_{j=1}^{r-1} W_j)$. From (2.11), we infer that the following two statements hold true:
(i) $f[C] \subseteq \delta(\bigcup_{j=1}^{r} W_j)$, $f^{-1}[\delta(\bigcup_{j=1}^{r-1} W_j)] = C_1$;
(ii) $\varpi(\mathrm{supp}\,(\alpha)\cap W_r) = \varpi(\mathrm{supp}\,(f(\alpha))\cap W_r)$ for all $\alpha \in C$.
By induction, we assume that $f\mid_{C_1}$ extends to a $(\mathbf{P},\omega)$-weight isometry of $\mathbf{H}$. By Theorem 3.1, we can choose $\mu \in \mathrm{Aut}\,(\mathbf{P})$ such that $\omega(i) = \omega(\mu(i))$, $H_i \cong H_{\mu(i)}$ for all $i \in \Omega$ and

$$\langle \mathrm{supp}\,(f(\alpha))\rangle_{\mathbf{P}} = \mu[\langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}}] \text{ for all } \alpha \in C_1. \tag{5.4}$$

Now let $D = \{\alpha\mid_{W_r} \mid \alpha \in C\}$. Define $g \in \mathrm{Hom}_S(D, \prod_{i\in W_r} H_i)$ as

$$g(\alpha\mid_{W_r}) = f(\alpha)\mid_{W_r} \text{ for all } \alpha \in C,$$

which, by (i) and (ii), is well defined. Moreover, it follows from (2.11) that g preserves $((W_r,=),\omega\mid_{W_r})$-weight. Hence by (2), g extends to a $((W_r,=),\omega\mid_{W_r})$-weight isometry of $\prod_{i\in W_r} H_i$. Applying Theorem 3.1 to $\prod_{i\in W_r} H_i$ and $((W_r,=),\omega\mid_{W_r})$, we choose a permutation σ of W_r such that $\omega(i) = \omega(\sigma(i))$, $H_i \cong H_{\sigma(i)}$ for all $i \in W_r$ and

$$\mathrm{supp}\,(g(\gamma)) = \sigma[\mathrm{supp}\,(\gamma)] \text{ for all } \gamma \in D.$$

From the definitions of D and g, we infer that

$$\mathrm{supp}\,(f(\alpha)) \cap W_r = \sigma[\mathrm{supp}\,(\alpha)\cap W_r] \text{ for all } \alpha \in C. \tag{5.5}$$

Now define $\lambda : \Omega \longrightarrow \Omega$ as $\lambda\mid_{W_r} = \sigma$ and $\lambda\mid_{\Omega - W_r} = \mu\mid_{\Omega - W_r}$. Since $\mathbf{P}$ is hierarchical, we have $\lambda \in \mathrm{Aut}\,(\mathbf{P})$. Moreover, it can be readily verified that $\omega(i) = \omega(\lambda(i))$, $H_i \cong H_{\lambda(i)}$ for all $i \in \Omega$. Now for an arbitrary $\alpha \in C$, we will show that $\langle \mathrm{supp}\,(f(\alpha))\rangle_{\mathbf{P}} = \lambda[\langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}}]$. Indeed, if $\alpha \in C_1$, then by $C_1 \subseteq \delta(\bigcup_{j=1}^{r-1} W_j)$, we have $\langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}} \subseteq \bigcup_{j=1}^{r-1} W_j$, which, together with $\lambda\mid_{\Omega - W_r} = \mu\mid_{\Omega - W_r}$ and (5.4), further implies that

$$\langle \mathrm{supp}\,(f(\alpha))\rangle_{\mathbf{P}} = \mu[\langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}}] = \lambda[\langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}}],$$

as desired. Therefore in what follows, we assume that $\alpha \in C - C_1$. By $\lambda\mid_{W_r} = \sigma$ and (5.5), we have

$$\mathrm{supp}\,(f(\alpha)) \cap W_r = \lambda[\mathrm{supp}\,(\alpha)\cap W_r].$$

Moreover, by $C \subseteq \delta(\bigcup_{j=1}^{r} W_j)$ and (i), we have

$$\mathrm{supp}\,(\alpha) \subseteq \bigcup_{j=1}^{r} W_j,\ \mathrm{supp}\,(\alpha) \nsubseteq \bigcup_{j=1}^{r-1} W_j,$$

$$\mathrm{supp}\,(f(\alpha)) \subseteq \bigcup_{j=1}^{r} W_j,\ \mathrm{supp}\,(f(\alpha)) \nsubseteq \bigcup_{j=1}^{r-1} W_j.$$

It then follows from (3) of Lemma 2.1 and $\lambda \in \mathrm{Aut}\,(\mathbf{P})$ that

$$\begin{aligned}\langle \mathrm{supp}\,(f(\alpha))\rangle_{\mathbf{P}} &= \langle \mathrm{supp}\,(f(\alpha)) \cap W_r\rangle_{\mathbf{P}} = \langle \lambda[\mathrm{supp}\,(\alpha) \cap W_r]\rangle_{\mathbf{P}} \\ &= \lambda[\langle \mathrm{supp}\,(\alpha) \cap W_r\rangle_{\mathbf{P}}] = \lambda[\langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}}],\end{aligned}$$

as desired. Now by Theorem 3.1, we can choose $\varphi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ such that $\langle \mathrm{supp}\,(\varphi(\alpha))\rangle_{\mathbf{P}} = \lambda[\langle \mathrm{supp}\,(\alpha)\rangle_{\mathbf{P}}]$ for all $\alpha \in \mathbf{H}$. It is straightforward to verify that $\varphi^{-1} \circ f \in \mathrm{Hom}_S(C, \mathbf{H})$ preserves $\mathbf{P}$-support. By (1) and Theorem 4.1, we can choose $\sigma \in \mathrm{GL}_{\mathbf{P}}(\mathbf{H})$ such that $\sigma\mid_C = \varphi^{-1} \circ f$. It follows that $\varphi \circ \sigma \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ and $(\varphi \circ \sigma)\mid_C = f$, as desired. □

Next, with the help of Proposition 5.1, we further derive some connections between MEP with respect to $(\mathbf{P}, \omega)$-weight and MEP with respect to Hamming weight, as detailed in the following proposition.

Proposition 5.2. **(1)** *Suppose that* $\mathbf{P}$ *is hierarchical and* $\mathbf{H}$ *satisfies MEP with respect to* $(\mathbf{P}, \omega)$*-weight. Then, for any* $r \in [1, m]$ *and* $b \in \omega[W_r]$*, the left* S*-module* $\prod_{(i \in W_r, \omega(i)=b)} H_i$ *satisfies MEP with respect to Hamming weight.*
(2) *Suppose that* $\mathbf{P}$ *is hierarchical,* $(\mathbf{P}, \omega)$ *satisfies UDP,* $\mathbf{H}$ *satisfies Condition (A),* $(\mathbf{H}, \mathbf{P})$ *satisfies Condition (B), and for any* $r \in [1, m]$, $b \in \omega[W_r]$, $\prod_{(i \in W_r, \omega(i)=b)} H_i$ *satisfies MEP with respect to Hamming weight. Then,* $\mathbf{H}$ *satisfies MEP with respect to* $(\mathbf{P}, \omega)$*-weight.*

Proof. Throughout the proof, we will continue to use the notations $\varpi : 2^{\Omega} \longrightarrow \mathbb{R}$, $\alpha\mid_J$ and $\widetilde{\tau}$ defined in (5.1), (5.2) and (5.3), respectively.

(1) First, we show that if $\mathbf{P}$ is an anti-chain and $\mathbf{H}$ satisfies MEP with respect to $(\mathbf{P}, \omega)$-weight, then for any $b \in \omega[\Omega]$, $\prod_{(i \in \Omega, \omega(i)=b)} H_i$ satisfies MEP with respect to Hamming weight. Indeed, let $c \in \omega[\Omega]$, and let $I \triangleq \{i \in \Omega \mid \omega(i) = c\}$, $M \triangleq \prod_{i \in I} H_i$. Moreover, let B be an S-submodule of M, and let $\rho \in \mathrm{Hom}_S(B, M)$ preserve Hamming weight. Define $C \triangleq \{\widetilde{\beta} \mid \beta \in B\}$, and define $f \in \mathrm{Hom}_S(C, \mathbf{H})$ as

$$f(\widetilde{\beta}) = \widetilde{\rho(\beta)} \text{ for all } \beta \in B.$$

We note that f preserves $(\mathbf{P},\omega)$-weight. Hence we can choose $\varphi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ with $\varphi\mid_C = f$. By (1) of Theorem 3.1, we can define $\varepsilon \in \mathrm{End}_S(M)$ as

$$\varepsilon(\alpha\mid_I) = \varphi(\alpha)\mid_I \text{ for all } \alpha \in \mathbf{H}.$$

Moreover, it is straightforward to verify that ε is a Hamming weight isometry of M with $\varepsilon\mid_B = \rho$, as desired.

Now let $r \in [1,m]$. Then, it follows from Proposition 5.1 that $\prod_{i\in W_r} H_i$ satisfies MEP with respect to $((W_r,=),\omega\mid_{W_r})$-weight, and hence the desired result immediately follows from applying the above claim to $\prod_{i\in W_r} H_i$ and $((W_r,=),\omega\mid_{W_r})$.

(2) First, we show that if $\mathbf{P}$ is an anti-chain, $(\mathbf{P},\omega)$ satisfies UDP, and for any $b \in \omega[\Omega]$, $\prod_{(i\in\Omega,\omega(i)=b)} H_i$ satisfies MEP with respect to Hamming weight, then $\mathbf{H}$ satisfies MEP with respect to $(\mathbf{P},\omega)$-weight. Indeed, let $C \subseteq \mathbf{H}$ be a linear code and let $f \in \mathrm{Hom}_S(C,\mathbf{H})$ preserve $(\mathbf{P},\omega)$-weight. Since $(\mathbf{P},\omega)$ satisfies UDP, f satisfies the following condition:

$$\forall\ \alpha \in C, b \in \mathbb{R} : |\{i \in \mathrm{supp}\,(f(\alpha)) \mid \omega(i) = b\}| = |\{i \in \mathrm{supp}\,(\alpha) \mid \omega(i) = b\}|. \quad (5.6)$$

Now consider an arbitrary $c \in \omega[\Omega]$. Set $I_c \triangleq \{i \in \Omega \mid \omega(i) = c\}$, $A_c \triangleq \{\alpha\mid_{I_c} : \alpha \in C\}$. By (5.6), we can define $\rho_c \in \mathrm{Hom}_S(A_c, \prod_{i\in I_c} H_i)$ as

$$\rho_c(\alpha\mid_{I_c}) = f(\alpha)\mid_{I_c} \text{ for all } \alpha \in C.$$

Moreover, we note that ρ_c preserves Hamming weight. Hence we can choose a Hamming weight isometry ε_c of $\prod_{i\in I_c} H_i$ which extends ρ_c. Now define $\varphi \in \mathrm{End}_S(\mathbf{H})$ such that for any $\alpha \in \mathbf{H}$, it holds that

$$\varphi(\alpha)\mid_{I_c} = \varepsilon_c(\alpha\mid_{I_c}) \text{ for all } c \in \omega[\Omega].$$

It is straightforward to verify that $\varphi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ and $\varphi\mid_C = f$, as desired.

Now consider an arbitrary $r \in [1,m]$. Since $(\mathbf{P},\omega)$ satisfies UDP, by (1) of Lemma 2.1, $((W_r,=),\omega\mid_{W_r})$ satisfies UDP. Hence an application of the above claim to $\prod_{i\in W_r} H_i$ and $((W_r,=),\omega\mid_{W_r})$ implies that $\prod_{i\in W_r} H_i$ satisfies MEP with respect to $((W_r,=),\omega\mid_{W_r})$-weight. Finally, it follows from Proposition 5.1 that $\mathbf{H}$ satisfies MEP with respect to $(\mathbf{P},\omega)$-weight, as desired. □

Now we are in a position to derive the main results of this section. In the following two theorems, for the case that either $\mathbf{P}$ is hierarchical or $\omega = \mathbf{1}$, with the additional assumption that $\mathbf{H}$ satisfies Condition (C), we give necessary and sufficient conditions for $\mathbf{H}$ to satisfy MEP with respect to $(\mathbf{P},\omega)$-weight in terms of MEP with respect to Hamming weight.

Theorem 5.1. *Suppose that* $\mathbf{P}$ *is hierarchical and* $\mathbf{H}$ *satisfies Condition (C). Then,* $\mathbf{H}$ *satisfies MEP with respect to* $(\mathbf{P},\omega)$*-weight if and only if the following two conditions hold:*
(1) $\mathbf{H}$ *satisfies Condition (A),* $(\mathbf{H},\mathbf{P})$ *satisfies Condition (B),* $(\mathbf{H},(\mathbf{P},\omega))$ *satisfies Condition (D);*
(2) *For any* $r\in[1,m]$*,* $b\in\omega[W_r]$*,* $\prod_{(i\in W_r,\omega(i)=b)}H_i$ *satisfies MEP with respect to Hamming weight.*

Proof. This immediately follows from Proposition 5.2 and Corollary 3.1. □

Theorem 5.2. *Suppose that* $\mathbf{H}$ *satisfies Condition (C). Then,* $\mathbf{H}$ *satisfies MEP with respect to* $\mathbf{P}$*-weight if and only if the following two conditions hold:*
(1) $\mathbf{H}$ *satisfies Condition (A),* $(\mathbf{H},\mathbf{P})$ *satisfies Conditions (B) and (E);*
(2) *For any* $r\in[1,m]$*,* $\prod_{i\in W_r}H_i$ *satisfies MEP with respect to Hamming weight.*

Proof. With Corollary 3.2 and Lemma 2.2, the desired result immediately follows from applying Theorem 5.1 to $(\mathbf{P},\mathbf{1})$. □

Theorems 5.1 and 5.2 are of central importance in Section 6 where we show that MEP is always stronger than various other important coding-theoretic properties. Now as a first application of Theorems 5.1 and 5.2, we derive a generalization of [3, Theorems 7.4 and 7.6], as detailed in the following example.

Example 5.1. *Suppose that* $\mathbf{H}=M^{\Omega}$*, where* M *is a finite left* S*-module such that* $|M|\geqslant 2$*,* M *is* M*-injective, and moreover,* M *has a cyclic socle, i.e., the sum of all the simple* S*-submodules of* M *is cyclic.*

Since M *is finite and* M*-injective, it follows from [50, Proposition 5.1] that* M *is strong pseudo-injective, which further implies that* $\mathbf{H}$ *satisfies Condition (A); moreover, it follows from* $|M|\geqslant 2$ *that* $\mathbf{H}$ *satisfies Condition (C), and it follows from the fact* M *is* M*-injective that* $(\mathbf{H},\mathbf{P})$ *satisfies Condition (B). In addition, since* M *is finite, strong pseudo-injective and has a cyclic socle, it follows from [50, Theorem 5.2] that for any* $I\subseteq\Omega$*,* M^I *satisfies MEP with respect to Hamming weight. Therefore an application of Theorems 5.1 and 5.2 implies that the following two statements hold:*
(1) *If* $\mathbf{P}$ *is hierarchical, then* $\mathbf{H}$ *satisfies MEP with respect to* $(\mathbf{P},\omega)$*-weight if and only if* $(\mathbf{P},\omega)$ *satisfies UDP;*
(2) $\mathbf{H}$ *satisfies MEP with respect to* $\mathbf{P}$*-weight if and only if* $\mathbf{P}$ *is hierarchical.*

We note that the second statement generalizes [3, Theorems 7.4 and 7.6] which have been established for the case that $H = S^{\Omega}$ where S is supposed to be a finite Frobenius ring.

6 MEP v.s. other coding-theoretic properties

In this section, we consider the special case that $S = \mathbb{F}$ is a finite field with $|\mathbb{F}| = q$, $(k_i \mid i \in \Omega)$ is a family of positive integers, and

$$\mathbf{H} = \prod_{i \in \Omega} \mathbb{F}^{k_i}.$$

As usual, the inner product $\langle \ , \ \rangle : \mathbf{H} \times \mathbf{H} \longrightarrow \mathbb{F}$ is defined as

$$\langle \alpha, \beta \rangle = \sum_{i \in \Omega} \sum_{t=1}^{k_i} \alpha_{i,t} \beta_{i,t},$$

where for $\alpha \in \mathbf{H}$ and $i \in \Omega$, $\alpha_{i,t}$ denotes the t-th entry of $\alpha_i \in \mathbb{F}^{k_i}$. For any linear code $C \subseteq \mathbf{H}$, we let

$$C^{\perp} \triangleq \{\beta \in \mathbf{H} \mid \langle \alpha, \beta \rangle = 0 \text{ for all } \alpha \in C\}$$

denote the dual code of C. With respect to $(\mathbf{P}, \omega)$-weight, we will compare MEP with the following five coding-theoretic properties:

- the property of admitting MacWilliams identity (PAMI);
- reflexivity of partitions;
- UDP;
- whether the group of isometries acts transitively on codewords with the same weight;
- whether the corresponding weighted poset metric induces an association scheme.

We begin with a few terminologies and notations. For a finite set Y, a *partition* of Y is a collection of nonempty disjoint subsets of Y whose union is Y; moreover, for a partition Γ of Y and any $D \subseteq Y$, we refer to the tuple

$$(|D \cap B| \mid B \in \Gamma)$$

as the Γ-*distribution* of D. From now on, we let $\mathcal{Q}(\mathbf{H},\mathbf{P},\omega)$, $\mathcal{Q}(\mathbf{H},\mathbf{P})$, $\mathcal{Q}(\mathbf{H},\overline{\mathbf{P}},\omega)$ and $\mathcal{Q}(\mathbf{H},\overline{\mathbf{P}})$ denote the partitions of $\mathbf{H}$ induced by $(\mathbf{P},\omega)$-weight, $\mathbf{P}$-weight, $(\overline{\mathbf{P}},\omega)$-weight and $\overline{\mathbf{P}}$-weight, respectively.

PAMI was first introduced by Kim and Oh in [31], where the authors proved that being hierarchical is a necessary and sufficient condition for a poset to admit MacWilliams identity, generalizing the classical MacWilliams identity for Hamming weight distributions. The original property has since been extensively studied and has been extended and generalized to poset-block metric, combinatorial metric, directed graph metric, labeled-poset-block metric, MacWilliams-type equivalence relations, as well as general partitions of a finite vector space (see [43, 44, 19, 36, 10, 52] for more details). In this section, we will consider PAMI for $(\mathbf{P},\omega)$-weight, as detailed in the following definition.

Definition 6.1. *We say that $(\mathbf{P},\omega)$ satisfies PAMI if for any two linear codes C_1, C_2 with the same $\mathcal{Q}(\mathbf{H},\overline{\mathbf{P}},\omega)$-distribution, ${C_1}^{\perp}$ and ${C_2}^{\perp}$ have the same $\mathcal{Q}(\mathbf{H},\mathbf{P},\omega)$-distribution. Moreover, we simply say that $\mathbf{P}$ satisfies PAMI if $(\mathbf{P},\mathbf{1})$ satisfies PAMI.*

We remark that Definition 6.1 is in fact equivalent to [36, Definition 12] which has been proposed for labeled-poset-block metric (see Example 2.1).

Reflexive partition was introduced by Gluesing-Luerssen in [21], and it can be characterized in terms of Fourier-invariant pair of partitions and association scheme (see [53]). Reflexive partitions provide a general framework for addressing invertible MacWilliams identities, and they arise naturally from various weights and metrics in coding theory such as poset metric, rank metric and homogeneous weight (see, e.g., [21, 22, 23, 51]). Now we follow [23, Definition 1.2] and recall the definition of reflexive partition.

Definition 6.2. *Let χ be a nontrivial additive character of $\mathbb{F}$. For a partition Γ of $\mathbf{H}$, let $\boldsymbol{l}(\Gamma)$ denote the partition of $\mathbf{H}$ such that for any $\alpha,\gamma\in\mathbf{H}$, α and γ belong to the same member of $\boldsymbol{l}(\Gamma)$ if and only if*

$$\sum_{\beta\in B}\chi(\langle\alpha,\beta\rangle)=\sum_{\beta\in B}\chi(\langle\gamma,\beta\rangle)\ \textit{for all}\ B\in\Gamma.$$

Moreover, a partition Γ of $\mathbf{H}$ is said to be reflexive if $\boldsymbol{l}(\boldsymbol{l}(\Gamma))=\Gamma$.

We note that reflexivity of a partition is independent of the choice of the nontrivial additive character (see [21, Theorem 2.4], [23, Page 4]). Hence from now on, we fix a nontrivial additive character χ of $\mathbb{F}$.

Now we consider transitivity of the group of isometries, i.e., whether the group of isometries acts transitively on codewords with the same weight. Such a property has been studied for poset metric and directed graph metric (see [35, Theorem 3], [19, Proposition 4]), and we now propose it for $(\mathbf{P},\omega)$-weight.

Definition 6.3. *We say that $(\mathbf{P},\omega)$ is transitive if for any $\gamma,\theta\in\mathbf{H}$ with $\mathrm{wt}_{(\mathbf{P},\omega)}(\gamma)=\mathrm{wt}_{(\mathbf{P},\omega)}(\theta)$, there exists $\psi\in\mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ with $\psi(\gamma)=\theta$. Moreover, we simply say that $\mathbf{P}$ is transitive if $(\mathbf{P},\mathbf{1})$ is transitive.*

Association scheme is a classical notion in algebra and combinatorics; moreover, it has numerous applications in coding theory at large (see [11, 12]). Association schemes induced by poset metric have been studied in [35, 39, 51]. Now we follow [11, Section 2.1] and recall the definition of association scheme.

Definition 6.4. *Let Δ be a partition of $\mathbf{H}^2$. Then, $(\mathbf{H},\Delta)$ is said to be an association scheme if the following three conditions hold:*
(1) $\{(\alpha,\alpha)\mid\alpha\in\mathbf{H}\}\in\Delta$;
(2) *For any $R\in\Delta$, $R^{-1}\triangleq\{(\alpha,\beta)\mid(\beta,\alpha)\in R\}\in\Delta$;*
(3) *For any $R,S,T\in\Delta$ and for any $(\alpha,\beta),(\gamma,\theta)\in T$, it holds that*

$$|\{\tau\in\mathbf{H}\mid(\alpha,\tau)\in R,(\tau,\beta)\in S\}|=|\{\tau\in\mathbf{H}\mid(\gamma,\tau)\in R,(\tau,\theta)\in S\}|.$$

Following (2.9), $d_{(\mathbf{P},\omega)}:\mathbf{H}^2\longrightarrow\mathbb{R}$ naturally induces a partition of $\mathbf{H}^2$. More specifically, for any $c\in\mathbb{R}$, set

$$R_c\triangleq\{(\alpha,\beta)\in\mathbf{H}^2\mid d_{(\mathbf{P},\omega)}(\alpha,\beta)=c\},\tag{6.1}$$

and then we have the partition

$$\Delta\triangleq\{R_c\mid c\in\mathbb{R},R_c\neq\emptyset\}.\tag{6.2}$$

It is then natural to consider when $(\mathbf{H},\Delta)$ is an association scheme.

Definition 6.5. *We say that $(\mathbf{P},\omega)$ induces an association scheme if $(\mathbf{H},\Delta)$ is an association scheme. Moreover, we simply say that $\mathbf{P}$ induces an association scheme if $(\mathbf{P},\mathbf{1})$ induces an association scheme.*

We remark that via some straightforward computation, one can check that $(\mathbf{P},\omega)$ induces an association scheme if and only if for any $b,c\in\mathbb{R}$ and any $\gamma,\theta\in\mathbf{H}$ with $\mathrm{wt}_{(\mathbf{P},\omega)}(\gamma)=\mathrm{wt}_{(\mathbf{P},\omega)}(\theta)$, it holds that

$$\begin{aligned}&|\{(\alpha,\beta)\in\mathbf{H}^2\mid\mathrm{wt}_{(\mathbf{P},\omega)}(\alpha)=b,\mathrm{wt}_{(\mathbf{P},\omega)}(\beta)=c,\alpha+\beta=\gamma\}|\\&=|\{(\alpha,\beta)\in\mathbf{H}^2\mid\mathrm{wt}_{(\mathbf{P},\omega)}(\alpha)=b,\mathrm{wt}_{(\mathbf{P},\omega)}(\beta)=c,\alpha+\beta=\theta\}|.\end{aligned}\tag{6.3}$$

For $\mathbf{P}$-weight, it has been proven in [35, Theorem 3] that if $k_i = 1$ for all $i \in \Omega$, then each of the five properties is equivalent to $\mathbf{P}$ being hierarchical. For $(\mathbf{P}, \omega)$-weight with $\mathbf{P}$ hierarchical, Machado and Firer have given a necessary and sufficient condition for $(\mathbf{P}, \omega)$ to satisfy PAMI in [36, Theorem 7], and when $q = 2$, they have given a necessary and sufficient condition for MEP in [36, Theorem 8] (also see [19, Theorems 10 and 12]). Their results show that for binary field alphabet, MEP is always stronger than PAMI.

Now we generalize all the above mentioned results by gathering several properties together and compare them with each other, and showing that MEP is stronger than all the others. The proof of our results will rely on Theorems 5.1 and 5.2, together with the following lemma, which is a special case of [15, Propositions 2, 3 and Theorem 3].

Lemma 6.1. *Let $s, n \in \mathbb{Z}^+$. Then, the $\mathbb{F}$-vector space $(\mathbb{F}^s)^n$ satisfies MEP with respect to Hamming weight if and only if either $s = 1$ or $n \leqslant q$.*

Now we are ready to state and prove the two main results of this section.

Theorem 6.1. *Consider the following eight statements:*
(1) $\mathbf{H}$ *satisfies MEP with respect to* $(\mathbf{P}, \omega)$*-weight;*
(2) $(\mathbf{P}, \omega)$ *is transitive;*
(3) $(\mathbf{H}, (\mathbf{P}, \omega))$ *satisfies Condition (D);*
(4) $\mathcal{Q}(\mathbf{H}, \overline{\mathbf{P}}, \omega) = \boldsymbol{\ell}(\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega))$;
(5) $(\mathbf{P}, \omega)$ *satisfies PAMI;*
(6) $\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)$ *is reflexive;*
(7) $(\mathbf{P}, \omega)$ *induces an association scheme;*
(8) *For any $r \in [1, m]$ and $b \in \omega[W_r]$, either $(\forall\ i \in W_r \text{ s.t. } \omega(i) = b : k_i = 1)$ or $|\{i \in W_r \mid \omega(i) = b\}| \leqslant q$ holds true.*
Then, it holds true in general that

$$(1) \Longrightarrow (2),\ (2) \Longleftrightarrow (3),\ (3) \Longrightarrow (4),\ (4) \Longleftrightarrow (5),\ (5) \Longrightarrow (6),\ (6) \Longleftrightarrow (7).$$

If $\mathbf{P}$ is hierarchical, then we have

$$(1) \Longleftrightarrow ((3) \wedge (8)).$$

If $\mathbf{P}$ is hierarchical and ω is integer-valued, then we have

$$(2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longleftrightarrow (5) \Longleftrightarrow (6) \Longleftrightarrow (7).$$

Proof. (1) $\Longrightarrow$ (2) This follows from (1) of Corollary 3.1.

(2) $\Longrightarrow$ (3) Noticing that $\mathbf{H}$ satisfies Condition (C), the desired result follows from (2) of Corollary 3.1.

(3) $\Longrightarrow$ (2) Let $\gamma, \theta \in \mathbf{H}$ with $\mathrm{wt}_{(\mathbf{P},\omega)}(\gamma) = \mathrm{wt}_{(\mathbf{P},\omega)}(\theta)$. Since $(\mathbf{P}, \omega)$ satisfies UDP, we can choose $\mu \in \mathrm{Aut}(\mathbf{P})$ such that $\langle \mathrm{supp}(\theta) \rangle_{\mathbf{P}} = \mu[\langle \mathrm{supp}(\gamma) \rangle_{\mathbf{P}}]$ and $\omega(i) = \omega(\mu(i))$ for all $i \in \Omega$. For any $i \in \Omega$, it follows from $\omega(i) = \omega(\mu(i))$, $\mathrm{len}_{\mathbf{P}}(i) = \mathrm{len}_{\mathbf{P}}(\mu(i))$ and (3) that $k_i = k_{\mu(i)}$. By Theorem 3.1, we can choose $\varphi \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ such that $\langle \mathrm{supp}(\varphi(\alpha)) \rangle_{\mathbf{P}} = \mu[\langle \mathrm{supp}(\alpha) \rangle_{\mathbf{P}}]$ for all $\alpha \in \mathbf{H}$. Define $f \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F} \cdot \gamma, \mathbf{H})$ as $f(\gamma) = \theta$. Then, it is straightforward to verify that $\varphi^{-1} \circ f \in \mathrm{Hom}_{\mathbb{F}}(\mathbb{F} \cdot \gamma, \mathbf{H})$ preserves $\mathbf{P}$-support. By Example 4.2, we can choose $\sigma \in \mathrm{GL}_{\mathbf{P}}(\mathbf{H})$ such that $\sigma \mid_{\mathbb{F} \cdot \gamma} = \varphi^{-1} \circ f$. It follows that $\varphi \circ \sigma \in \mathrm{GL}_{(\mathbf{P},\omega)}(\mathbf{H})$ and $(\varphi \circ \sigma)(\gamma) = f(\gamma) = \theta$, as desired.

(3) $\Longrightarrow$ (4) This follows from [52, Theorem 10].

(4) $\Longleftrightarrow$ (5) This follows from [52, Proposition 16].

(4) $\Longrightarrow$ (6) Noticing that $|\mathcal{Q}(\mathbf{H}, \overline{\mathbf{P}}, \omega)| = |\mathcal{Q}(\mathbf{H}, \mathbf{P}, \omega)|$, the desired result follows from [21, Theorem 2.4].

(6) $\Longleftrightarrow$ (7) This is a special case of [53, Theorem 1].

Now assume that $\mathbf{P}$ is hierarchical. Then, since $\mathbf{H}$ satisfies Conditions (A) and (C), and $(\mathbf{H}, \mathbf{P})$ satisfies Condition (B), (1) $\Longleftrightarrow$ ((3)$\wedge$(8)) immediately follows from Theorem 5.1 and Lemma 6.1.

Finally, assume that $\mathbf{P}$ is hierarchical and ω is integer-valued. Then, by [52, Proposition 16], we have (6) $\Longleftrightarrow$ (3), which further establishes the equivalence among (2)–(7), as desired. □

Theorem 6.2. *Consider the following eight statements:*

(1) $\mathbf{H}$ *satisfies MEP with respect to* $\mathbf{P}$*-weight;*

(2) $\mathbf{P}$ *is transitive;*

(3) $(\mathbf{H}, \mathbf{P})$ *satisfies Condition (E);*

(4) $\mathcal{Q}(\mathbf{H}, \overline{\mathbf{P}}) = \boldsymbol{l}(\mathcal{Q}(\mathbf{H}, \mathbf{P}))$*;*

(5) $\mathbf{P}$ *satisfies PAMI;*

(6) $\mathcal{Q}(\mathbf{H}, \mathbf{P})$ *is reflexive;*

(7) $\mathbf{P}$ *induces an association scheme;*

(8) *For any* $r \in [1, m]$*, either* $(\forall\ i \in W_r : k_i = 1)$ *or* $|W_r| \leqslant q$ *holds true.*

Then, we have (1) $\Longleftrightarrow$ ((3) $\wedge$ (8)) *and*

$$(2) \Longleftrightarrow (3) \Longleftrightarrow (4) \Longleftrightarrow (5) \Longleftrightarrow (6) \Longleftrightarrow (7).$$

Proof. First, since $\mathbf{H}$ satisfies Conditions (A) and (C), and $(\mathbf{H}, \mathbf{P})$ satisfies Condition (B), (1) $\Longleftrightarrow$ ((3)$\wedge$(8)) immediately follows from Theorem 5.2 and Lemma 6.1. Second, by [51, Theorem II.4], we have (6) $\Longleftrightarrow$ (3). Hence with

Lemma 2.2, an application of Theorem 6.1 to $(\mathbf{P}, \mathbf{1})$ immediately implies the equivalence among (2)–(7), as desired. □

Remark 6.1. *In Theorem 6.1, if* $\mathbf{P}$ *is hierarchical and* ω *is integer-valued, then* $(2) \Longleftrightarrow (3)$ *has been established in [19, Proposition 4] for the case that* $\omega(i) = k_i$ *for all* $i \in \Omega$*,* $(3) \Longleftrightarrow (5)$ *has been established in [36, Theorem 7], and* $(1) \Longleftrightarrow ((3) \wedge (8))$ *has been established in [36, Theorem 8] when* $q = 2$*. In Theorem 6.2,* $(3) \Longleftrightarrow (4)$ *is a special case of [21, Theorems 5.4 and 5.5], and* $(3) \Longleftrightarrow (5)$ *has been established in [43, Theorems 1 and 2]; moreover, if we set* $k_i = 1$ *for all* $i \in \Omega$*, then Theorem 6.2 recovers the equivalence between* $\mathbf{P}$ *to be hierarchical and Parts 1, 2, 3, 4, 6 of [35, Theorem 3]. Based on Theorems 6.1 and 6.2, we conclude that for weighted poset metric, MEP is always stronger than all the other five properties considered in this section.*

References

[1] M. M. S. Alves, L. Panek, M. Firer, Error-block codes and poset metrics, *Advances in Mathematics of Communications*, vol. 2, no. 1 (2008), 95-111.

[2] F. W. Anderson, K. R. Fuller, *Rings and Categories of Modules* (second edition), Springer, 1992.

[3] A. Barra, H. Gluesing-Luerssen, MacWilliams extension theorems and the local-global property for codes over Frobenius rings, *Journal of Pure and Applied Algebra*, vol. 219, no. 4 (2015), 703-728.

[4] H. Bass, K-theory and stable algebra, *Publications Mathématiques de l'I.H.É.S.*, tome 22 (1964), 5-60.

[5] S. Bezzateev, N. Shekhunova, Class of binary generalized Goppa codes perfect in weighted Hamming metric, *Designs, Codes and Cryptography*, vol. 66, no. 1-3 (2013), 391-399.

[6] K. Bogart, D. Goldberg, J. Gordon, An elementary proof of the MacWilliams theorem on equivalence of codes, *Information and Control*, vol. 37 (1978), 19-22.

[7] R. A. Brualdi, J. S. Graves, K. M. Lawrence, Codes with a poset metric, *Discrete Mathematics*, vol. 147, no. 1-3 (1995), 57-72.

[8] E. Byrne, M. Greferath, M. E. O'Sullivan, The linear programming bound for codes over finite Frobenius rings, *Designs, Codes and Cryptography*, vol. 42 (2007), 289-301.

[9] S. H. Cho, D. S. Kim, Automorphism group of the crown-weight space, *European Journal of Combinatorics*, vol. 27, no. 1 (2006), 90-100.

[10] S. Choi, J. Y. Hyun, D. Y. Oh, H. K. Kim, MacWilliams-type equivalence relations, arXiv:1205.1090v2 (2013), 20 pages.

[11] P. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Research Reports Supplements*, No. 10, 1973.

[12] P. Delsarte, V. I. Levenshtein, Association schemes and coding theory, *IEEE Transactions on Information Theory*, vol. 44, no. 6 (1998), 2477-2504.

[13] H. Q. Dinh, S. R. López-Permouth, On the equivalence of codes over rings and modules, *Finite Fields and Their Applications*, vol. 10, no. 4 (2004), 615-625.

[14] S. Dyshko, Generalizations of the MacWilliams extension theorem, General Mathematics [math.GM]. Université de Toulon, 2016. English. NNT: 2016TOUL0018.

[15] S. Dyshko, Geometric approach to the MacWilliams extension theorem for codes over modules, *Applicable Algebra in Engineering, Communication and Computing*, vol. 28, no. 4 (2017), 295-309.

[16] S. Dyshko, Minimal solutions of the isometry equation, *Discrete Mathematics*, vol. 341, no. 11 (2018), 2995-3002.

[17] S. Dyshko, On extendibility of additive code isometries, *Advances in Mathematics of Communications*, vol. 10, no. 1 (2016), 45-52.

[18] S. Dyshko, When the extension property does not hold, *Journal of Algebra and its Applications*, vol. 16, no. 5 (2017).

[19] T. Etzion, M. Firer, R. A. Machado, Metrics based on finite directed graphs and coding invariants, *IEEE Transactions on Information Theory*, vol. 64, no. 4 (2018), 2398-2409.

[20] L. Felix, M. Firer, Canonical-systematic form for codes in hierarchical poset metrics, *Advances in Mathematics of Communications*, vol. 6, no. 3 (2012), 315-328.

[21] H. Gluesing-Luerssen, Fourier-reflexive partitions and MacWilliams identities for additive codes, *Designs, Codes and Cryptography*, vol. 75, no. 3 (2015), 543-563.

[22] H. Gluesing-Luerssen, T. Pllaha, Extension theorems for various weight functions over Frobenius bimodules, *Journal of Algebra and Its Applications*, vol. 17, no. 3 (2018).

[23] H. Gluesing-Luerssen, A. Ravagnani, Partitions of matrix spaces with an application to q-rook polynomials, *European Journal of Combinatorics*, vol. 89, no. 3 (2020), 103-120.

[24] E. Gorla, Rank-metric codes, arXiv:1902.02650 (2019).

[25] M. Greferath, T. Honold, C. Mc Fadden, J. A. Wood, J. Zumbrägel, MacWilliams' extension theorem for bi-invariant weights over finite principal ideal rings, *Journal of Combinatorial Theory, Series A*, vol. 125 (2014), 177-193.

[26] M. Greferath, S. E. Schmidt, Finite-ring combinatorics and MacWilliams' equivalence theorem, *Journal of Combinatorial Theory, Series A*, vol. 92, no. 1 (2000), 17-28.

[27] T. Honold, Characterization of finite Frobenius rings, *Archiv der Mathematik*, vol. 76 (2001), 406-415.

[28] J. Y. Hyun, H. K. Kim, Maximum distance separable poset codes, *Designs, Codes and Cryptography*, vol. 48, no. 3 (2008), 247-261.

[29] J. Y. Hyun, H. K. Kim, J. R. Park, Weighted posets and digraphs admitting the extended Hamming code to be a perfect code, *IEEE Transactions on Information Theory*, vol. 65, no. 8 (2019), 4664-4672.

[30] M. C. Iovanov, Frobenius-Artin algebras and infinite linear codes, *Journal of Pure and Applied Algebra*, vol. 220, no. 2 (2016), 560-576.

[31] H. K. Kim, D. Y. Oh, A classification of posets admmiting MacWilliams identity, *IEEE Transactions on Information Theory*, vol. 51, no. 4 (2005), 1424-1431.

[32] K. Lee, The automorphism group of a linear space with the Rosenbloom-Tsfasman metric, *European Journal of Combinatorics*, vol. 24, no. 6 (2003), 607-612.

[33] R. Lidl, H. Niederreiter, *Finite Fields*, Cambridge University Press, 1997.

[34] R. A. Machado, M. Firer, MacWilliams' identity for metrics determined by directed graphs, *2016 IEEE Information Theory Workshop (ITW)*, 2016, DOI: 10.1109/ITW.2016.7606803.

[35] R. A. Machado, J. A. Pinheiro, M. Firer, Characterization of metrics induced by hierarchical posets, *IEEE Transactions on Information Theory*, vol. 63, no. 6 (2017), 3630-3640.

[36] R. A. Machado, M. Firer, Weights which respect support and NN-decoding, *IEEE Transactions on Information Theory*, vol. 66, no. 6 (2020), 3664-3674.

[37] F. J. MacWilliams, N. J. A. Sloane, *The Theory of Error Correcting Codes*, North-Holland Publishing Company, Amsterdam, the Netherlands, 1977.

[38] F. J. MacWilliams, Combinatorial problems of elementary abelian groups, Ph.D. Dissertation, Harvard University, Cambridge, Mass, 1962.

[39] D. Y. Oh, Posets metrics admmiting association schemes and a new proof of MacWilliams Identity, *Journal of the Korean Mathematical Society*, vol. 50, no. 5 (2013), 9117-931.

[40] A. de Oliveira Moura, M. Firer, Duality for poset codes, *IEEE Transactions on Information Theory*, vol. 56, no. 7 (2010), 3180-3186.

[41] L. Panek, M. Firer, H. K. Kim, J. Y. Hyun, Groups of linear isometries on poset structures, *Discrete Mathematics*, vol. 308, no. 18 (2008), 4116-4123.

[42] L. Panek, J. A. Pinheiro, General approach to poset and additive metrics, *IEEE Transactions on Information Theory*, vol. 66, no. 11 (2020), 6823-6834.

[43] J. A. Pinheiro, M. Firer, Classification of poset-block spaces admitting MacWilliams-type identity, *IEEE Transactions on Information Theory*, vol. 58, no. 12 (2012), 7246-7252.

[44] J. A. Pinheiro, R. A. Machado, M. Firer, Combinatorial metrics: MacWilliams-type identities, isometries and extension property, *Designs, Codes and Cryptography*, vol. 87, no. 2-3 (2019), 327-340.

[45] M. Yu. Rosenbloom, M. A. Tsfasman, Codes for the m-metric, *Problems of Information Transmission*, vol. 33, no. 1 (1997), 45-52.

[46] F. M. Schneider, J. Zumbrägel, MacWilliams' extension theorem for infinite rings, *Proceedings of the American Mathematical Society*, vol. 147, no. 3 (2018), 947-961.

[47] H. N. Ward, J. A. Wood, Characters and the equivalence of codes, *Journal of Combinatorial Theory, Series A*, vol. 73, no. 2 (1996), 348-352.

[48] J. A. Wood, Duality for modules over finite rings and applications to coding theory, *American Journal of Mathematics*, vol. 121, no. 3 (1999), 555-575.

[49] J. A. Wood, Code equivalences characterizes finite Frobenius rings, *Proceedings of the American Mathematical Society*, vol. 136, no. 2 (2008), 699-706.

[50] J. A. Wood, Foundations of linear codes defined over finite modules: the extension theorem and the MacWilliams identities, *Codes Over Rings*, World Scientific Pub. Co. Inc., (2009), 124-190.

[51] Y. Xu, H. B. Kan, G. Y. Han, Fourier-reflexive partitions induced by poset metric, *IEEE Transactions on Information Theory*, vol. 68, no. 4 (2022), 2319-2330.

[52] Y. Xu, H. B. Kan, G. Y. Han, Fourier-reflexive partitions and group of linear isometries with respect to weighted poset metric, 2022 IEEE International Symposium on Information Theory (ISIT), 2022, DOI: 10.1109/ISIT50566.2022.9834567.

[53] V. A. Zinoviev, T. Ericson, Fourier-invariant pairs of partitions of finite abelian groups and association schemes, *Problems of Information Transmission*, vol. 45, no. 3 (2009), 221-231.