ON THE REAL ZEROS OF DEPTH 1 QUASIMODULAR FORMS

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Abstract. We discuss the critical points of modular forms, or more generally the zeros of quasimodular forms of depth 1 for $\mathrm{PSL}_2(\mathbb{Z})$. In particular, we consider the derivatives of the unique weight k modular forms f_k with the maximal number of consecutive zero Fourier coefficients following the constant 1. Our main results state that (1) every zero of a depth 1 quasimodular form near the derivative of the Eisenstein series in the standard fundamental domain lies on the geodesic segment $\{z \in \mathbb{H} : \Re(z) = 1/2\}$, and (2) more than half of zeros of f_k in the standard fundamental domain lie on the geodesic segment $\{z \in \mathbb{H} : \Re(z) = 1/2\}$ for large enough k with $k \equiv 0 \pmod{12}$.

1. Introduction

In this paper, it is assumed that the readers are familiar with the classical theory of holomorphic modular forms, or refer to [DS05] or [Ser73]. Let $\mathbb{H}$ be the upper half-plane of complex numbers and k and p be non-negative integers. A quasimodular form of weight k and depth p for the full modular group $\Gamma := \mathrm{PSL}_2(\mathbb{Z})$ is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following conditions:

(i) There exist holomorphic functions $Q_i(f)$ on $\mathbb{H}$ for $i = 0, 1, \ldots, p$ that satisfy

$$f|_k\gamma = \sum_{i=0}^{p} Q_i(f)X(\gamma)^i, \qquad \text{with } Q_p(f) \not\equiv 0, \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

where the operator $|_k\gamma$ is defined by

$$f|_k\gamma(z) = (cz + d)^{-k} f(\gamma z) \quad \text{for } z \in \mathbb{H},$$

and the function $X(\gamma)$ is defined by

$$X(\gamma)(z) = \frac{z}{cz + d} \quad \text{for } z \in \mathbb{H}.$$

(ii) f is polynomially bounded, i.e., there exists a constant $\alpha > 0$ such that

$$f(z) = O((1 + |z|^2)/y)^{\alpha}$$

as $y \to \infty$ and $y \to 0$, where $z = x + iy$ with $x, y \in \mathbb{R}$.

Instead of the second condition, one may replace it with *holomorphic at cusps* for the functions $Q_i(f)$, which is further discussed in [Roy12]. Throughout this paper, for the sake of brevity we have omitted explicit mention, but it should be noted that all modular and quasimodular forms discussed herein are defined for Γ. The space of quasimodular forms of weight k and depth $\le p$ is denoted by $\widetilde{M_k}^{(\le p)}$, and M_k simply represents $\widetilde{M_k}^{(\le 0)}$, the space of modular forms of weight k.

The Eisenstein series E_2 of weight 2 and the derivatives of modular forms are standard examples of quasimodular forms. More precisely, if f is a quasimodular form of weight k and depth p, then f' is a quasimodular form of weight $k + 2$ and depth $p + 1$. In particular, the derivative of a holomorphic modular form is a depth 1 quasimodular form.

There have been various and extensive studies on the zeros of modular and quasimodular forms in the standard fundamental domain F of Γ, where

$$F = \{z \in \mathbb{H} : -1/2 < \Re(z) \le 1/2, \quad |z| > 1 \text{ if } \Re(z) < 0, \quad |z| \le 1 \text{ if } \Re(z) \ge 0\}.$$

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Rankin and Swinnerton-Dyer's celebrated result [RS70] states that if $k \geq 4$, every zero of the Eisenstein series E_k lies on the unit circle. Getz [Get04] proved that the same property holds for certain class of functions; modular forms f 'near' the Eisenstein series, those are, $f = E_k + \sum a_i E_{k-12i}\Delta^i$ for $a_i \in \mathbb{R}$ with small enough $|a_i|$.

Let us introduce the so-called 'Gap function' which can be viewed as a standard basis element of the space of weakly holomorphic modular forms. These are of the form

$$f_{k,m}(z) = q^{-m} + O(q^{\ell+1})$$

for $m \geq -\ell$, where $q := e^{2\pi i z}$ and ℓ is the dimension of the space of weight k holomorphic modular forms, explicitly given by

$$k = 12\ell + k', \quad \text{for } k' \in \{0, 4, 6, 8, 10, 14\}.$$

We simply write $f_k := f_{k,0}$. Duke and Jenkins [DJ08] proved that any zero of $f_{k,m}$ lies on the unit circle when $m \geq |\ell| - \ell$. This result expands the understanding of the distribution of zeros of modular forms in the vicinity of the unit circle.

On the other hand, there are some results on the zeros lying on another geodesic segment in F. Ghosh and Sarnak [GS12] established estimates for the number of zeros of cuspidal Hecke eigenforms on the union of segments $\delta_1 \cup \delta_2 \cup \delta_3$, where

$$\delta_1 := \{z \in F : \Re(z) = 0\}, \quad \delta_2 := \{z \in F : \Re(z) = 1/2\}, \quad \delta_3 := \{z \in F : |z| = 1\}.$$

They proved that the number of zeros of weight k cuspidal Hecke eigenform on the segment δ_2 is $\gg \log k$, and on the union of the segments $\delta_1 \cup \delta_2$, it is $\gg_\epsilon k^{1/4-1/80-\epsilon}$ as k goes to infinity. Matomäki [Mat16] later improved by showing that each number is $\gg_\epsilon k^{1/8-\epsilon}$ and $\gg_\epsilon k^{1/4-\epsilon}$, respectively.

In accordance with Ghosh and Sarnak's terminology, we use the term *real zeros* to describe the zeros lying on the aforementioned geodesic segments.

In a recent paper [GO22], Gun and Oesterlé proved that the behavior of the derivative of the Eisenstein series E_k for $k \geq 4$ with respect to the real zeros is interesting. They showed that all the zeros of E_k' in F lie on the line segment δ_2. This finding prompts two natural questions, given the fact that Getz's result in [Get04] and Duke and Jenkins' result in [DJ08] focus on the properties of the zeros of the analogue of E_k:

Q1. Is every zero of depth 1 quasimodular forms in F 'near' E_k' lying on δ_2?
Q2. Is every zero of f_k in F lying on δ_2?

In this paper, we investigate the properties of the real zeros of depth 1 quasimodular forms, and address two natural questions related to the behavior of derivatives of modular forms. Our first main result provides an affirmative answer to Q1 when the field of coefficients of forms is restricted to real numbers.

Theorem 1.1. *Let $E_k', g_1, g_2, \ldots, g_n$ be a basis of the space of weight $k+2$ depth ≤ 1 quasimodular forms with real Fourier coefficients, and let $f = E_k' + \sum_{j=1}^n a_j g_j$ be a depth ≤ 1 quasimodular form with $a_j \in \mathbb{R}$. If $|a_j|$'s are small enough, then every zero of f in F lies on δ_2.*

On the other hand, the answer to Q2 is negative. If we consider a weight 98 depth 1 quasimodular form f_{96}', then numerical computations found that it has 4 real zeros on δ_2 and 2 non-real zeros in F, whose real parts are approximately 0.44 and -0.44, respectively. Nonetheless, we provide partial results on the proportion of real zeros for f_k' under certain conditions regarding the residue of weight.

Let $\theta_j := \frac{j\pi}{k+1}$ and $t_j := \frac{1}{2}\cot\theta_j$.

Theorem 1.2. *Let $k \equiv 0 \pmod{12}$.*

(a) *For sufficiently large k, the function $f_k'\left(\frac{1}{2} + it\right)$ in variable t has $\gg k$ sign changes along the interval $(\sqrt{3}/2, \infty)$. More precisely, if $k \geq 1116$, then the sign of $f_k'\left(\frac{1}{2} + it_j\right)$ is $(-1)^j$ for $19(k+1)/50\pi \leq j \leq [k/6] - 1$.*
(b) *For large k, approximately over 54.9% zeros of f_k' in F lie on δ_2.*

Theorem 1.2(a) does not hold for other $k' = 4, 6, 8, 10, 14$. For example, if $k = 12\ell + 14$ is large, our proof of Theorem 1.2 shows that the function $Df_{k,0}(z)$ never vanishes for $0.38 \le \theta \le \theta_0$ for appropriate $\theta_0 < \pi/6$. Note that θ_j for $j = 19(k+1)/50$ tends to 0.38 as k approaches ∞. However, this does not mean that Theorem 1.2(b) does not hold for other k'. Again for $k' = 14$ for instance, we speculate that if $j \ll 19(k+1)/50\pi$, it plays a similar role as in Theorem 1.2(a), and consequently more than half of zeros are real zeros as in the case of $k' = 0$. For further details, see Remark 5.5.

Let us briefly outline the contents of this paper. In Section 2 and 3, we provide the necessary preliminaries, and Section 4 presents the proof of Theorem 1.1 which affirms the answer to the first question. In Section 5, we prove Theorem 1.2, which gives a partial result towards answering the second question. Finally, Section 6 offers additional discussions on the real zeros of quasimodular forms on the line $\{z \in \mathbb{H} : \Re(z) = 1/2\}$ including the existence of such zeros.

2. Valence formula

Hereafter we assume every Fourier coefficient of the quasimodular form discussed in this paper is real. To prove our results, we first introduce the *valence formula* for depth 1 quasimodular forms which is developed in [IR22] recently. The classical valence formula is the equation on the multiplicities of zeros of a weight k modular form f as follows:

$$\sum_{z \in X} \frac{v_z(f)}{e_z} = \frac{k}{12}.$$

Here, X is the (compactified) modular curve $X := \Gamma\backslash(\mathbb{H} \cup \{\infty\})$, e_z is the ramification index of z (which is 1 except for $z = i$, $\rho = e^{i\pi/3}$ and $e_i = 2$, $e_\rho = 3$) and $v_z(f)$ is the multiplicity of f at z.

To state the valence formula, we let $\lambda(\gamma) := -d/c \in \mathbb{Q} \cup \{\infty\}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$.

Theorem 2.1. [IR22, Theorem 2] *Let $f = f_0 + f_1 E_2$ be a depth 1 quasimodular form, where f_0 and f_1 have no common zeros, and let $\gamma \in \Gamma$. Then, there exist constants $N_1(f)$, $N_2(f)$, and $N_3(f)$ that depend on f such that the following holds:*

$$\sum_{z \in \gamma F} \frac{v_z(f)}{e_z} = \begin{cases} N_1(f) & \text{if } |\lambda(\gamma)| \in (1, \infty], \\ N_2(f) & \text{if } |\lambda(\gamma)| \in (1/2, 1), \\ N_3(f) & \text{if } |\lambda(\gamma)| \in [0, 1/2). \end{cases}$$

The exact formula of three constants $N_1(f), N_2(f), N_3(f)$ are given in [IR22]. It is enough to describe the formula of $N_1(f)$ for our purpose. Let $\theta_1, \ldots, \theta_n$ be real numbers such that $\pi/3 \le \theta_1 \le \ldots \le \theta_n \le \pi/2$ and $e^{i\theta_j}$ are zeros of f for all $1 \le j \le n$, counted with multiplicity. Denote by $r(f_1)$ the sign of the first nonzero coefficient of the Taylor expansion of f_1 around ρ (see [IR22, (11)] for precise definition), and define $w(z)$ by 1 except for $z = i, \rho, i\infty$, and 2 for those exceptional points. Then

$$N_1(f) = \frac{1}{2}\left[\frac{k}{6}\right] - (-1)^{v_\rho(f_1)} r(f_1) \sum_{j=1}^{n} \frac{(-1)^j}{w(e^{i\theta_j})} \operatorname{sgn}\left(e^{\frac{1}{2}ik\theta_j} f(e^{i\theta_j})\right). \tag{1}$$

The following is the special case of Theorem 2.1.

Theorem 2.2. [IR22, Theorem 1] *If f is a modular form of weight k, then*

$$\sum_{z \in \gamma F} \frac{v_z(f')}{e_z} = \begin{cases} C(f) + \frac{1}{3}\delta_{f(\rho)=0} & \text{if } |\lambda(\gamma)| \in (1, \infty], \\ -C(f) & \text{if } |\lambda(\gamma)| \in (1/2, 1), \\ -C(f) + L(f) & \text{if } |\lambda(\gamma)| \in [0, 1/2), \end{cases}$$

where $C(f)$ is the number of distinct zeros z on the unit circle in F with $\Re(z) \ge 0$, counted with weight e_z^{-1}, and $L(f)$ is the number of distinct zeros on $\delta_2 \cup \{\infty\}$.

Theorem 2.2 provides the number of zeros of $f'_{k,0}$ in F as shown in the following proposition:

Proposition 2.3. *The number of zeros z of f_k' in F, counted with multiplicities and with weight e_z^{-1} is $n(k) + \frac{1}{3}\delta_{k\equiv 2 \pmod 6}$, where $n(k) := \left[\frac{k-4}{6}\right] - \ell - 1$.*

Proof. Referring to [GO22, Theorem 2] and Theorem 2.2, we have

$$n(k) + \frac{1}{3}\delta_{k\equiv 2 \pmod 6} + \ell + 1 = \frac{k}{12} + C(E_k) + \frac{1}{3}\delta_{E_k(\rho)=0}.$$

As Getz has pointed out in [Get04], this number is the same as

$$\frac{k}{12} + C(f_k) + \frac{1}{3}\delta_{f_k(\rho)=0},$$

which is the number of zeros of f_k' in $F \cup \{\infty\}$. Since the multiplicity of f_k' at ∞ is $\ell + 1$, this completes the proof. □

3. Serre derivative

Let $D := \frac{1}{2\pi i}\frac{d}{dz} = q\frac{d}{dq}$ be the normalized derivation. We define the Serre derivative for arbitrary depth p quasimodular forms by $\vartheta := D - \frac{k-p}{12}E_2$.

Lemma 3.1. *Let f be a quasimodular form of weight k and depth p. Then ϑf is a quasimodular form of weight $k+2$ and depth p. Furthermore, f is cuspidal if and only if ϑf is cuspidal.*

Proof. For $p = 0$, it is well-known that the lemma holds ([BVHZ08, p.48]). For $p \geq 1$, write $f = f_0E_2^p + f_1$ for a modular form f_0 of weight $k - 2p$ and a quasimodular form f_1 of weight k and depth $\leq p - 1$. Then

$$\begin{aligned} Df &= D(f_0E_2^p) + Df_1 \\ &= (Df_0)E_2^p + \frac{pf_0}{12}E_2^{p-1}(E_2^2 - E_4) + Df_1 \\ &= \frac{k-p}{12}E_2(f_0E_2^p + f_1) + \left((\vartheta f_0)E_2^p + Df_1 - \frac{k-p}{12}E_2f_1 - \frac{p}{12}E_2^{p-1}E_4f_0.\right) \\ &= \frac{k-p}{12}E_2f + g. \end{aligned}$$

One can verify that $\vartheta f = g := (\vartheta f_0)E_2^p + Df_1 - \frac{k-p}{12}E_2f_1 - \frac{p}{12}E_2^{p-1}E_4f_0$ is a quasi-modular form of depth p.

Recall that Df is always cuspidal by the definition of D. Hence

$$\lim_{z\to i\infty}\vartheta f(z) = \lim_{z\to i\infty}\left(Df(z) - \frac{k-p}{12}E_2(z)f(z)\right) = -\frac{k-p}{12}\lim_{z\to i\infty}f(z),$$

so f is cuspidal if and only if ϑf is cuspidal. □

Thus if we write $DE_k = f_0 + f_1E_2$ for modular forms f_0 and f_1, then $f_0 = \vartheta E_k$ and $f_1 = \frac{k}{12}E_k$.

4. proof of Theorem 1.1

Consider the space $\widetilde{M}_{k,\mathbb{R}}^{(\leq p)}$ quasimodular forms of weight k and depth $\leq p$ with real Fourier coefficients, and let C be a subset of $\widetilde{M}_{k,\mathbb{R}}^{(\leq 1)}$ given by

$$C := \{f \in \widetilde{M}_{k,\mathbb{R}}^{(\leq 1)} : \text{if } f(z) = 0 \text{ and } z \in F, \text{ then } \Re(z) = 1/2\}.$$

We equip C with the subspace topology inherited from $\widetilde{M}_{k,\mathbb{R}}^{(\leq 1)}$. Since $DE_k \in C$, it suffices to show the following to prove Theorem 1.1.

Theorem 4.1. *If $k \geq 4$ is an even positive integer, then DE_k belongs to the interior* $\mathrm{int}(C)$ *of C.*

Proof. Let $f_1, \ldots, f_d \in \widetilde{M}_{k,\mathbb{R}}^{(\leq 1)}$ form a basis for $\widetilde{M}_k^{(\leq 1)}$. Define a linear map $f : \mathbb{C}^d \to \widetilde{M}_k^{(\leq 1)}$ by $f(c^1, \ldots, c^d) = c^1f_1 + \cdots + c^df_d$ that is an isomorphism of topological $\mathbb{C}$-vector spaces, and also define $\tilde{f} : \mathbb{H} \times \mathbb{C}^d \to \mathbb{C}$ by $\tilde{f}(z, c^1, \ldots, c^d) = f(c^1, \ldots, c^d)(z)$.

Consider a zero $(z_0, c_0^1, \dots, c_0^d) \in \mathbb{H} \times \mathbb{C}^d$ of $\tilde{f}$, where z_0 is a simple zero of $f(c_0^1, \dots, c_0^d)$. Then,

$$\det\left(\frac{\partial \tilde{f}}{\partial z}(z_0, c_0^1, \dots, c_0^d)\right) = \frac{\partial \tilde{f}}{\partial z}(z_0, c_0^1, \dots, c_0^d) = 0,$$

so by the analytic implicit function theorem, there exist open sets $U \subseteq \mathbb{C}^d$ and $V \subseteq \mathbb{H}$, and an analytic function $\xi : U \to V$, such that $(c_0^1, \dots, c_0^d) \in U$, $z_0 \in V$, and

$$\tilde{f}(z, c^1, \dots, c^d) = \tilde{f}(z_0, c_0^1, \dots, c_0^d) = 0 \quad \text{if and only if} \quad \xi(c^1, \dots, c^d) = z.$$

This means that ξ provides the local zero locus of f near z_0. Identifying $\mathbb{R}^d$ with a subset of $(\mathbb{R} + i \cdot 0)^d$ in $\mathbb{C}^d$, we can restrict ξ to a real analytic function $\xi : U \cap \mathbb{R}^d \to V$.

Take $c_0^1, \dots, c_0^d \in \mathbb{R}$ so that $f(c_0^1, \dots, c_0^d) = DE_k$. First, suppose $k \not\equiv 2 \pmod 6$. There are $m := \left[\frac{k-4}{6}\right]$ distinct zeros $z_1, \dots, z_m$ in F which all lie on $\{z \in F : \Re(z) = 1/2\}$. These zeros are all simple (see [GO22, Theorem 4]). As we have seen above, one can take open sets $U_i \subseteq \mathbb{C}^d$, $V_i \subseteq \mathbb{H}$, and analytic functions $\xi_i : U_i \to V_i$ for each z_i. By shrinking U_i and V_i, we may assume the following conditions;

(1) V_i's are pairwise disjoint,
(2) all the zeros of $f(c^1, \dots, c^d)$ have moduli $\neq 1$ for any $(c^1, \dots, c^d) \in \bigcap_{i=1}^m U_i$, and
(3) if we write

$$f(c^1, \dots, c^d) = f_0(c^1, \dots, c^d) + f_1(c^1, \dots, c^d)E_2 \in M_k \oplus M_{k-2}E_2,$$

then $f_1(c^1, \dots, c^d)(z)$ has the same number of zeros (counted with multiplicity) on $\{z \in \mathbb{H} : |z| = 1 \text{ and } 0 \le \Re(z) \le 1/2\}$, say n, as of $f_1(c_0^1, \dots, c_0^d) = \frac{k}{12}E_k$, and

$$r(f_1(c^1, \dots, c^d)) = r(E_k),$$
$$\operatorname{sgn}\left(e^{\frac{1}{2}ik\theta_j(c^1,\dots,c^d)} f(c^1, \dots, c^d)(e^{i\theta_j(c^1,\dots,c^d)})\right) = \operatorname{sgn}\left(e^{\frac{1}{2}ik\theta_j(c_0^1,\dots,c_0^d)} E_k(e^{i\theta_j(c_0^1,\dots,c_0^d)})\right)$$

for any $(c^1, \dots, c^d) \in \bigcap_{i=1}^m U_i \cap \mathbb{R}$, where $\theta_j(c^1, \dots, c^d)$ are real numbers for which $e^{i\theta_j(c^1,\dots,c^d)}$ are the zeros of $f(c^1, \dots, c^d)$ on $\{z \in \mathbb{H} : |z| = 1 \text{ and } 0 \le \Re(z) \le 1/2\}$.

The condition (3) is satisfied due to [Get04, Theorem 1]. Indeed, by the classical valence formula, the possible minimum values of the order of weight k modular form at i and ρ are equal to the order of E_k at i and ρ, respectively, so we can take a small enough open set around $(c_0^1, \dots, c_0^d)$ in $\mathbb{R}^d$ such that the vanishing of $f_1(c^1, \dots, c^d)$ at ρ, $r(f_1(c^1, \dots, c^d))$, and the sign of $e^{\frac{1}{2}ik\theta_j(c^1,\dots,c^d)} f(c^1, \dots, c^d)(e^{i\theta_j(c^1,\dots,c^d)})$ do not change as $(c^1, \dots, c^d)$ varies in such an open set. Since the set

$$\left\{(c^1, \dots, c^d) \in \mathbb{C}^d : \frac{\partial \tilde{f}}{\partial z}\left(\xi_i(c^1, \dots, c^d), c^1, \dots, c^d\right) = 0\right\}$$

is closed, we may further assume that $\xi_i(c^1, \dots, c^d)$ is a simple zero of $f(c^1, \dots, c^d)$ for all $(c^1, \dots, c^d) \in U_i$. Let $U := \bigcap_{i=1}^m U_i$ and U_0 be the connected component of $U \cap \mathbb{R}^d$ containing DE_k, which is also open in $\mathbb{R}^d$ due to the local connectedness of $\mathbb{R}^d$.

We claim that $U_0 \subseteq C$ so that $DE_k \in U_0 \subseteq \operatorname{int}(C)$, which completes the proof.

Suppose for contradiction that there exists a point $(c_1^1, \dots, c_1^d) \in U_0$ such that there is some $z_0 \in F \smallsetminus \{z \in F : \Re(z) = 1/2\}$ satisfies $\tilde{f}(z_0, c_1^1, \dots, c_1^d) = 0$. Note that by the condition (3) and Theorem 2.1, $f(c^1, \dots, c^d)$ has m zeros in F for every $(c^1, \dots, c^d) \in U_0$. In other words, $f(c^1, \dots, c^d)$ has the same number of zeros as of $f(c_0^1, \dots, c_0^d) = DE_k$ in F, so we have $z_0 = \xi_a(c_1^1, \dots, c_1^d)$ for some $a \in \{1, \dots, m\}$. Note that $1 - \overline{z_0}$ is also a zero of $f(c_1^1, \dots, c_1^d)$ so that $1 - \overline{z_0} = \xi_b(c_1^1, \dots, c_1^d)$ for some $b \in \{1, \dots, m\} \smallsetminus \{a\}$.

Let $p : [0,1] \to U_0 \cap I_{\mathbb{R}}$ be a path with $p(0) = (c_0^1, \dots, c_0^d)$ and $p(1) = (c_1^1, \dots, c_1^d)$. Define $t_0 := \inf\{t \in [0,1] : \xi_a(p(t)) \in F \smallsetminus \delta_2\}$. Since V_i's are pairwise disjoint and $p((t_0, 1])$ is connected, it follows that

$$1 - \overline{\xi_a}(p(t)) = \xi_b(p(t))$$

for all $t \in (t_0, 1]$. Thus, the continuity of ξ_i's implies that $\xi_a(p(t_0)) = 1 - \overline{\xi_a}(p(t_0)) = \xi_b(p(t_0))$, which contradicts the assumption that the sets V_i are pairwise disjoint.

Now suppose that $k \equiv 2 \pmod 6$. In this case, there is an additional simple zero z_{m+1} of DE_k on δ_2, namely $z_{m+1} = \rho$. We can use the argument as before to show that there is an analytic function $\xi_{m+1} : U_{m+1} \to V_{m+1}$ with $\xi_{m+1}(c_0^1, \dots, c_0^d) = \rho$. Note that a quasimodular form of weight $k+2$ and depth 1 always vanishes at ρ by the classical valence formula. Since the only zero of $f(c^1, \dots, c^d)$ in V_{m+1} is $\xi_{m+1}(c^1, \dots, c^d)$, it means that ξ_{m+1} is a constant function. Therefore, we can use the same argument for the other zeros as before to show that $U_0 \subseteq C$, which completes the proof of the claim. □

We remark that the same argument as above proves the following.

Proposition 4.2. *Let $f = f_0 + f_1 E_2$ be a quasimodular form of weight k, where f_0 and f_1 are modular forms such that $f_1(z) \neq 0$ for all $z \in F$ with $|z| = 1$. If f is on the topological boundary of C, then there is a multiple zero of f in F.*

Proof. Note that either $f \in C$ or $f \in \widetilde{M}_{k,\mathbb{R}}^{(\leq 1)} \smallsetminus C$. Assume that all the zeros of f in F are simple. By the analytic implicit function theorem and the same argument as in the proof of Theorem 4.1, if $f \in C$ (resp. $f \in \widetilde{M}_{k,\mathbb{R}}^{(\leq 1)} \smallsetminus C$) then $f \in \operatorname{int}(C)$ (resp. $f \in \operatorname{cl}\left(\widetilde{M}_{k,\mathbb{R}}^{(\leq 1)} \smallsetminus C\right)$) which implies that $f \notin \operatorname{cl}\left(\widetilde{M}_{k,\mathbb{R}}^{(\leq 1)} \smallsetminus C\right)$ (resp. $f \notin \operatorname{cl}(C)$). □

5. Proof of Theorem 1.2

Our main goal of this section is to determine the value of $Df_{k,m}(z)$ at $z = \frac{1}{2} + \frac{i}{2}\cot\theta = ie^{-i\theta}|z|$ for $\theta \in \left[\frac{(\ell+1)\pi}{k+1}, \frac{([k/6]-1)\pi}{k+1}\right]$. The proof of Theorem 1.2 is based on an integral representation of $f_{k,m}$ which is equivalent to the generating function formula for $f_{k,m}$ established in [DJ08]. Let us define the function $H(\tau, z)$ as follows:

$$\begin{aligned} H(\tau,z) :=& \ell E_2(z)\frac{\Delta^\ell(z)}{\Delta^{\ell+1}(\tau)}\frac{E_{k'}(z)E_{14-k'}(\tau)}{j(\tau)-j(z)}e^{-2\pi i m\tau} \\ &+ \frac{\Delta^\ell(z)}{\Delta^{\ell+1}(\tau)}\frac{DE_{k'}(z)E_{14-k'}(\tau)(j(\tau)-j(z)) - E_{k'}(z)E_{14-k'}(\tau)D(j(\tau)-j(z))}{(j(\tau)-j(z))^2}e^{-2\pi i m\tau}. \end{aligned}$$

Lemma 5.1. *For any $m \in \mathbb{Z}$ and sufficiently large $A > 0$, we have*

$$Df_{k,m}(z) = \int_{-\frac{1}{2}+iA}^{\frac{1}{2}+iA} H(\tau,z)\,d\tau.$$

Proof. The proof can be obtained readily by taking the derivative of the equation in [DJ08, Lemma 2]. □

Lemma 5.2. *If $A' \in \left[\frac{\sqrt{3}}{2}, \frac{1}{2}\cot\theta\right)$, then*

$$\int_{-\frac{1}{2}+iA'}^{\frac{1}{2}+iA'} H(\tau,z)\,d\tau = Df_{k,m}(z) + g_{k'}(z),$$

where

$$\begin{aligned} g_{k'}(z) :=& \frac{e^{-2\pi i m z}}{E_{14}(z)E_{k'}(z)}\big[DE_{14}(z)E_{k'}(z)^3(E_{k'}(z)-1) + DE_{k'}(z)E_{14}(z)(E_{k'}(z)^2-1) \\ &- E_2(z)E_{14}(z)E_{k'}(z)(E_{k'}(z)^3 - (\ell+1)E_{k'}(z)^2 + \ell) + mE_{14}(z)E_{k'}(z)^3\big]. \end{aligned}$$

In particular, if $k' = 0$ we have

$$\int_{-\frac{1}{2}+iA'}^{\frac{1}{2}+iA'} H(\tau,z)\,d\tau = Df_{k,m}(z) + me^{-2\pi i m z}.$$

Proof. Note that A and A' are chosen such that the only poles of $H(\tau,z)$ in the rectangle $\{\tau\in\mathbb{H}:|\Re(\tau)|\le 1/2,\ A'\le\Im(\tau)\le A\}$ are at $\tau=z$ and $\tau=z-1$. By vertically shifting the contour of integration in Lemma 5.1 to a lower height A', while ensuring the avoidance of the poles z and $z-1$ through the inclusion of a clockwise circular path encircling both points, we obtain

$$\begin{aligned}\int_{-\frac{1}{2}+iA'}^{\frac{1}{2}+iA'} H(\tau,z)d\tau &= Df_{k,m}(z)+\pi i\mathrm{Res}_{\tau=z}H(\tau,z)+\pi i\mathrm{Res}_{\tau=z-1}H(\tau,z)\\ &= Df_{k,m}(z)+2\pi i\mathrm{Res}_{\tau=z}H(\tau,z).\end{aligned}$$

Note that $H(\tau,z)$ can be written as

$$\begin{aligned}H(\tau,z)=&-\frac{1}{2\pi i}\ell E_2(z)\frac{\Delta^\ell(z)}{\Delta^\ell(\tau)}\frac{E_{k'}(z)}{E_{k'}(\tau)}\frac{\frac{d}{d\tau}(j(\tau)-j(z))}{j(\tau)-j(z)}e^{-2\pi i m\tau}\\ &-\frac{1}{2\pi i}\frac{\Delta^\ell(z)}{\Delta^\ell(\tau)}\frac{DE_{k'}(z)}{E_{k'}(\tau)}\frac{\frac{d}{d\tau}(j(\tau)-j(z))}{j(\tau)-j(z)}e^{-2\pi i m\tau}+\frac{1}{4\pi^2}\frac{\Delta^\ell(z)}{\Delta^\ell(\tau)}\frac{E_{k'}(z)}{E_{k'}(\tau)}\frac{j'(z)j'(\tau)}{(j(\tau)-j(z))^2}e^{-2\pi i m\tau}.\end{aligned}$$

The proof is completed by invoking the relation $Dj(z)=-E_{14}(z)/\Delta(z)$ and evaluating the residues of $H(\tau,z)$ at $\tau=z$. □

Lemma 5.3. *For $A'\in\left[\frac{\sqrt{3}}{2},\frac{1}{2}\cot\theta\right)$ and $A''\in\left(\frac{1}{3},\sin 2\theta\right)$, we have*

$$\begin{aligned}\int_{-\frac{1}{2}+iA'}^{\frac{1}{2}+iA'} H(\tau,z)d\tau = &\int_{-\frac{1}{2}+iA''}^{\frac{1}{2}+iA''} H(\tau,z)d\tau\\ &-2mi^{-k-2}|z|^{-k-2}e^{2\pi m\sin 2\theta}\cos\left((k+2)\theta+4\pi m\sin^2\theta\right)\\ &+\frac{k}{\pi}i^{-k}|z|^{-k-1}e^{2\pi m\sin 2\theta}\cos\left((k+1)\theta+4\pi m\sin^2\theta\right).\end{aligned}$$

In particular, if $m=0$ we have

$$\int_{-\frac{1}{2}+iA'}^{\frac{1}{2}+iA'} H(\tau,z)d\tau=\int_{-\frac{1}{2}+iA''}^{\frac{1}{2}+iA''} H(\tau,z)d\tau+(-1)^{k/2}\frac{k}{\pi}|z|^{-k-1}\cos\left((k+1)\theta\right).$$

Proof. Note that if $\Im(z)<\frac{1}{2}(3+2\sqrt{2})$, it follows that $\Im(Sz)>\frac{1}{3}$. We observe that

$$\sup_{y\ge\sqrt{3}/2}\Im\left(S\left(\frac{3}{2}+iy\right)\right)=\frac{1}{3},\quad\text{and}\quad\Im\left(ST^{-1}z\right)=\Im\left(Sz\right)=\sin 2\theta.$$

Thus in the region $\{\tau\in\mathbb{H}:|\Re(\tau)|\le 1/2,\ A''\le\Im(\tau)\le A'\}$, the only poles of $H(\tau,z)$ are located at $\tau=ST^{-1}z=\frac{-1}{z-1}$ and $\tau=Sz=\frac{-1}{z}$. By reducing the height of the integration to A'', we obtain

$$\int_{-\frac{1}{2}+iA'}^{\frac{1}{2}+iA'} H(\tau,z)d\tau=\int_{-\frac{1}{2}+iA''}^{\frac{1}{2}+iA''} H(\tau,z)d\tau-2\pi i\left(\mathrm{Res}_{\tau=\frac{-1}{z-1}}H(\tau,z)+\mathrm{Res}_{\tau=\frac{-1}{z}}H(\tau,z)\right).$$

Note that for $z=\frac{1}{2}+\frac{i}{2}\cot\theta$ we have

$$\begin{aligned}z&=ie^{-i\theta}|z|, & z-1&=ie^{i\theta}|z|,\\ -(z-1)^{-1}&=2i(\sin\theta)e^{-i\theta}, & -z^{-1}&=2i(\sin\theta)e^{i\theta},\end{aligned}$$

and hence

$$\begin{aligned}\mathrm{Res}_{\tau=\frac{-1}{z-1}}H_1(\tau,z)&=-\frac{1}{2\pi i}\ell E_2(z)i^{-k}|z|^{-k}e^{-ik\theta+4\pi m(\sin\theta)e^{-i\theta}}-\frac{1}{2\pi i}\frac{DE_{k'}(z)}{E_{k'}(z)}i^{-k}|z|^{-k}e^{-ik\theta+4\pi m(\sin\theta)e^{-i\theta}},\\ \mathrm{Res}_{\tau=\frac{-1}{z}}H_1(\tau,z)&=-\frac{1}{2\pi i}\ell E_2(z)i^{-k}|z|^{-k}e^{ik\theta+4\pi m(\sin\theta)e^{i\theta}}-\frac{1}{2\pi i}\frac{DE_{k'}(z)}{E_{k'}(z)}i^{-k}|z|^{-k}e^{ik\theta+4\pi m(\sin\theta)e^{i\theta}}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\mathrm{Res}_{\tau=\frac{-1}{z}}H_2(\tau,z)&=-\frac{i}{4\pi^2}e^{2\pi im/z}z^{-k-2}\left(2\pi m+2\pi\ell z^2E_2(z)-ikz+2\pi z^2\frac{DE_{k'}(z)}{E_{k'}(z)}\right),\\ \mathrm{Res}_{\tau=\frac{-1}{z-1}}H_2(\tau,z)&=-\frac{i}{4\pi^2}e^{2\pi im/(z-1)}(z-1)^{-k-2}\left(2\pi m+2\pi\ell(z-1)^2E_2(z)-ik(z-1)+2\pi(z-1)^2\frac{DE_{k'}(z)}{E_{k'}(z)}\right),\end{aligned}$$

so that we deduce

$$\left(\mathrm{Res}_{\tau=\frac{-1}{z}}+\mathrm{Res}_{\tau=\frac{-1}{z-1}}\right)\left(H_1(\tau,z)+\frac{i}{4}\left(2\pi\ell E_2(z)+2\pi\frac{DE_{k'}(z)}{E_{k'}(z)}\right)\left(e^{2\pi im/z}z^{-k}+e^{2\pi im/(z-1)}(z-1)^{-k}\right)\right)=0.$$

Calculating the remaining terms of residues leads to the desired result. □

We establish the following proposition using Lemmas 5.1, 5.2 and 5.3, which leads to a direct deduction of Theorem 1.2(a), and when combined with Proposition 2.3, also yields Theorem 1.2(b).

Proposition 5.4. *Let $k'=m=0$ and let $A''=0.49$. If $k\geq 1116$ and $0.38\leq\theta<\pi/6$, then we have*

$$\left|\int_{-\frac{1}{2}+iA''}^{\frac{1}{2}+iA''}H(\tau,z)d\tau\right|<\frac{k}{\pi}|z|^{-k-1}.$$

Consequently, if k is large enough, the sign of $Df_{k,0}\left(\frac{1}{2}+it_j\right)$ is $(-1)^j$ for $19(k+1)/50\pi\leq j\leq [k/6]-1$.

Proof. Since $k'=m=0$, we have the inequality

$$\begin{aligned}&(2\sin\theta)^{-k-1}\left|\int_{-\frac{1}{2}+iA''}^{\frac{1}{2}+iA''}H(\tau,z)d\tau\right|\\&\qquad\leq(2\sin\theta)^{-k-1}\max_{|x|\leq 1/2}|H_1(x+iA'',z)|+(2\sin\theta)^{-k-1}\max_{|x|\leq 1/2}|H_2(x+iA'',z)|.\end{aligned}$$

Note that

$$\begin{aligned}&(2\sin\theta)^{-k-1}|H_1(\tau,z)|+(2\sin\theta)^{-k-1}|H_2(\tau,z)|\\&=(2\sin\theta)^{-1}\left|\frac{\Delta(z)}{(2\sin\theta)^{12}\Delta(\tau)}\right|^{\ell}\left(\left|\ell E_2(z)\frac{E_{14}(\tau)}{\Delta(\tau)(j(\tau)-j(z))}\right|+\left|\frac{E_{14}(\tau)E_{14}(z)}{\Delta(\tau)\Delta(z)(j(\tau)-j(z))^2}\right|\right).\end{aligned}$$

For $z=ie^{-i\theta}/(2\sin\theta)$ for $0.38\leq\theta<\frac{[k/6]\pi}{k+1}$ and $\tau=x+0.49i$, the following bounds can be obtained through numerical computations:

$$|E_2(z)|<1.15,\qquad\left|\frac{E_{14}(\tau)E_{14}(z)}{\Delta(\tau)\Delta(z)(j(\tau)-j(z))^2}\right|<0.6,$$

$$\left|\frac{E_{14}(\tau)}{\Delta(\tau)((j(\tau)-j(z)))}\right|<4.2,\quad\text{and}\quad\left|\frac{\Delta(z)}{(2\sin\theta)^{12}\Delta(\tau)}\right|<0.99.$$

By combining these bounds, we complete the proof. □

Remark 5.5. Proposition 5.4 is specific to $k'=0$, as we have noted in the introduction. To see this, consider the case where $k'=14$. Using the same argument as in the proof of Theorem 1.2 , we find that for $0.38\leq\theta<\pi/6$ and for all $k=12\ell+14$,

$$|z|^{k+1}Df_{k,0}(z)=-|z|^{k+1}g_{14}(z)+R(z),\qquad\text{where }|R(z)|<\frac{2k}{\pi}.$$

Recall that

$$g_{14}(z)=E_2(z)(E_{14}(z)^2-1)\ell+g_{14,0}(z).$$

Numerical computations show that for $z=\frac{1}{2}+\frac{i}{2}\cot\theta=ie^{-i\theta}|z|$ with $\theta\in\left[\frac{(\ell+1)\pi}{k+1},\frac{([k/6]-1)\pi}{k+1}\right]$, $E_2(z)(E_{14}(z)^2-1)$ is negative and bounded away from 0, and $g_{14,0}(z)$ is positive and bounded away from 0 for $0.38\leq\theta<\pi/6$. As θ approaches $\pi/6$, both $|E_{14}(z)^2-1|$ and $g_{14,0}(z)$ tend to ∞. Note that for θ_0 close enough to $\pi/6$ (e.g., $\theta_0=0.511$), if we take $0.38\leq\theta\leq\theta_0$, then $|z|>\epsilon_0$ for some uniform constant $\epsilon_0>1$. Therefore, for large k, the function $|z|^{k+1}Df_{k,0}(z)$ never vanishes.

However, the situation changes if we don't restrict the size of θ. It is possible for the real zeros of $Df_{k,0}$ to occur at z with $k'\neq 0$ and $\pi/12<\theta<0.38$. For example, if we take $k=86$, then there are 6 zeros of $Df_{k,0}$ in F. We can verify numerically that there are 4 real zeros and 2 non-real zeros among them. Such real zeros have an imaginary part larger than $\frac{1}{2}+\frac{i}{2}\cot(0.38)\approx 1.2518$, but smaller than $\frac{1}{2}+\frac{i}{2}\cot(\pi/12)\approx 1.8660$.

6. Other remarks on the zeros

In the previous sections, we have observed that certain types of quasimodular forms have zeros solely on the line δ_2, while other types have more than half of their zeros on this line, but not all of them. In this section, we explore the existence of real zeros.

Note that the space of quasimodular forms of weight k can be written as

$$\widetilde{M}_k = \widetilde{M}_k^{(\leq k/2)} = \bigoplus_{j=0}^{\frac{k}{2}-2} D^j M_{k-2j} \oplus \mathbb{C} D^{\frac{k}{2}-1} E_2.$$

In particular, we have

$$\widetilde{M}_{k,\mathbb{R}} = \widetilde{M}_{k,\mathbb{R}}^{(\leq k/2)} = \bigoplus_{j=0}^{\frac{k}{2}-2} D^j M_{k-2j,\mathbb{R}} \oplus \mathbb{R} D^{\frac{k}{2}-1} E_2.$$

Indeed, if we consider an involution $\imath : f(\tau) \mapsto \overline{f(-\overline{\tau})}$ on $\widetilde{M}_k^{(\leq k/2)}$ which is an anti-linear map, then each direct summand $D^j M_{k-2j}$ and $\mathbb{C} D^{\frac{k}{2}-1} E_2$ in the above are invariant under $\imath$. On the other hand, the subspace $\widetilde{M}_k^{\imath}$ of $\widetilde{M}_k$ consisting of the elements fixed by $\imath$ is exactly $\widetilde{M}_{k,\mathbb{R}}$. Hence if we write $f = \sum_{j=0}^{\frac{k}{2}-2} D^j f_j + c D^{\frac{k}{2}-1} E_2 \in \widetilde{M}_{k,\mathbb{R}}$, then

$$f = \imath(f) = \sum_{j=0}^{\frac{k}{2}-2} D^j \imath(f_j) + \overline{c} D^{\frac{k}{2}-1} E_2,$$

so $f_j = \imath(f_j)$ and $\overline{c} = c$. In particular, we have $\widetilde{M}_{k,\mathbb{R}}^{(\leq 1)} = M_{k,\mathbb{R}} \oplus D M_{k-2,\mathbb{R}}$ if $k \geq 4$.

Theorem 6.1. *Let $S_{k-2,\mathbb{R}}$ be the space of cusp forms of weight $k-2$ with real Fourier coefficients. Let $f \in DS_{k-2,\mathbb{R}}$ be a quasimodular form of weight k and depth 1. Then f has at least two real zeros, each of them lying on the central line $\{z \in \mathbb{H} : \Re(z) = 0\}$ and $\{z \in \mathbb{H} : \Re(z) = 1/2\}$, respectively.*

In other words, any cusp form has at least two critical points z with $\Re(z) = 0$ and $\Re(z) = 1/2$.

Let $\mathfrak{f}$ be a cusp form such that $\mathfrak{f}' = f$. Note that by the valence formula, if $k \equiv 0 \pmod 4$ then i must be a zero of $\mathfrak{f}$. Since $\mathfrak{f}(i\infty) = 0$, we have

$$f(iy_0) = -i \frac{\partial \mathfrak{f}}{\partial y}(iy_0) = 0$$

for some $y_0 \in (1, \infty)$. This argument can be applied to the line $\{z \in F : \Re(z) = \frac{1}{2}\}$, since $\mathfrak{f}\left(\frac{1}{2} + \frac{i}{2}\right) = (i-1)^k \mathfrak{f}(i)$. Thus it is enough to consider $k \equiv 2 \pmod 4$.

For a quasimodular form f, let us denote by $a_\infty(f)$ the first non-zero Fourier coefficient of f. Also, if f is non-cuspidal, we define ϵ'_f and ϵ_f as follows: Let ϵ'_f be the sign of the product of the first two non-zero Fourier coefficients of f, say a_0 and a_n, and

$$\epsilon_f := (-1)^n \epsilon'_f.$$

To prove Theorem 6.1 for the remaining case, we'll show slightly more generally stated as follows.

Proposition 6.2. *Let f be a modular form of weight k and assume that at least one of the following conditions holds:*

(i) *f is cuspidal and $k \equiv 0 \pmod 4$,*
(ii) *f is non-cuspidal with $\epsilon_f = -1$ and $k \equiv 0 \pmod 4$,*
(iii) *f is non-cuspidal with $\epsilon_f = 1$ and $k \equiv 2 \pmod 4$.*

Then f has a critical point lying on the line $\{z \in \mathbb{H} : \Re(z) = 1/2\}$. If we replace the conditions (ii) *and* (iii)*, respectively by*

(iv) *f is non-cuspidal with $\epsilon'_f = -1$ and $k \equiv 0 \pmod 4$,*
(v) *f is non-cuspidal with $\epsilon'_f = 1$ and $k \equiv 2 \pmod 4$,*

then f has a critical point lying on the line $\{z \in \mathbb{H} : \Re(z) = 0\}$.

To prove this, we need the following lemmas.

Lemma 6.3. *Let f and g be quasimodular forms and let $a := v_\infty(f)$ and $b := v_\infty(g)$. If $a-b>0$ then*

$$\lim_{y\to\infty}\frac{y^{-n}g(iy)}{y^{-m}f(iy)} = \mathrm{sgn}(a_\infty(f))\mathrm{sgn}(a_\infty(g))\infty,$$
$$\lim_{y\to\infty}\frac{y^{-n}g(\frac{1}{2}+iy)}{y^{-m}f(\frac{1}{2}+iy)} = (-1)^{a-b}\mathrm{sgn}(a_\infty(f))\mathrm{sgn}(a_\infty(g))\infty,$$

and if $a-b<0$ then

$$\lim_{y\to\infty}\frac{y^{-n}g(iy)}{y^{-m}f(iy)} = \lim_{y\to\infty}\frac{y^{-n}g(\frac{1}{2}+iy)}{y^{-m}f(\frac{1}{2}+iy)} = 0.$$

Lastly if $a-b=0$, then

$$\lim_{y\to\infty}\frac{y^{-n}g(iy)}{y^{-m}f(iy)} = \lim_{y\to\infty}\frac{y^{-n}g(\frac{1}{2}+iy)}{y^{-m}f(\frac{1}{2}+iy)} = \begin{cases} 0 & \text{if } n>m,\\ \frac{a_\infty(g)}{a_\infty(f)} & \text{if } n=m,\\ \mathrm{sgn}(a_\infty(f))\mathrm{sgn}(a_\infty(g))\infty & \text{if } n\le m.\end{cases}$$

Proof. Let $x=1/2$ and $\hat{f}\circ q=f$, $\hat{g}\circ q=g$. Note that

$$\begin{aligned}\lim_{y\to\infty}\frac{y^{-n}g(x+iy)}{y^{-m}f(x+iy)} &= \lim_{y\to\infty}\frac{y^{-n}(-e^{2\pi y})^{-b}(-e^{-2\pi y})^{-b}g(x+iy)}{y^{-m}(-e^{2\pi y})^{-a}(-e^{-2\pi y})^{-a}f(x+iy)}\\ &= (-1)^{a-b}\lim_{y\to\infty}\frac{\frac{e^{-2\pi by}}{y^n}q(x+iy)^{-b}\hat{g}(q(x+iy))}{\frac{e^{-2\pi ay}}{y^m}q(x+iy)^{-a}\hat{f}(q(x+iy))}\\ &= (-1)^{a-b}\frac{a_\infty(g)}{a_\infty(f)}\lim_{y\to\infty}\frac{e^{2\pi i(a-b)}}{y^{n-m}}.\end{aligned}$$

Similarly if $x=0$, then

$$\lim_{y\to\infty}\frac{y^{-n}g(x+iy)}{y^{-m}f(x+iy)} = \frac{a_\infty(g)}{a_\infty(f)}\lim_{y\to\infty}\frac{e^{2\pi i(a-b)}}{y^{n-m}}.$$

The assertion follows from the above equations immediately. □

Lemma 6.4. *Let f be a modular form of weight k.*

(a) *If f is cuspidal, then for any positive integer j we have*

$$\begin{cases} D^j f(iy) = \left(\frac{i}{y}\right)^{k+2j} D^j f\left(\frac{i}{y}\right)\times(1+o(1)),\\ D^j f\left(\frac{1}{2}+iy\right) = \left(\frac{i}{2y}\right)^{k+2j} D^j f\left(\frac{1}{2}+\frac{i}{4y}\right)\times(1+o(1)),\end{cases}$$

as y approaches 0.

(b) *If f is non-cuspidal, then*

$$\begin{cases} Df(iy) = \left(\frac{i}{y}\right)^{k+2} Df\left(\frac{i}{y}\right)\times\left(1-\epsilon'_f\omega(1)\right),\\ Df(\frac{1}{2}+iy) = \left(\frac{i}{2y}\right)^{k+2} Df\left(\frac{1}{2}+\frac{i}{4y}\right)\times(1-\epsilon_f\omega(1)),\end{cases}$$

as y approaches 0.

(c) *If f is non-cuspidal, then for any positive integer j*

$$\begin{cases} D^j f(iy) = \left(\frac{i}{y}\right)^{k+2j} D^j f\left(\frac{i}{y}\right)\times\left(1+(-1)^{j-1}\epsilon'_f\omega(1)\right),\\ D^j f(\frac{1}{2}+iy) = \left(\frac{i}{2y}\right)^{k+2j} D^j f\left(\frac{1}{2}+\frac{i}{4y}\right)\times\left(1+(-1)^{j-1}\epsilon_f\omega(1)\right),\end{cases}$$

Here, little-o and little-ω are the asymptotic bounds.

Proof. First, we prove (a) for $j = 1$ and then prove (b). Since Df is a quasimodular form of weight $k+2$ and depth 1, we have

$$Df(\gamma z) = (cz+d)^{k+2} Df(z) + \frac{ck}{2\pi i}(cz+d)^{k+1} f(z)$$

for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$. If we put $\gamma = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ and $z = -\frac{1}{2} + \frac{i}{4y}$, then

$$\begin{aligned} Df\left(\frac{1}{2} + iy\right) &= \left(\frac{i}{2y}\right)^{k+2} Df\left(-\frac{1}{2} + \frac{i}{4y}\right) + \left(\frac{i}{2y}\right)^{k+1} \cdot 2 \cdot \frac{k}{2\pi i} f\left(-\frac{1}{2} + \frac{i}{4y}\right) \\ &= \left(\frac{i}{2y}\right)^{k+2} Df\left(\frac{1}{2} + \frac{i}{4y}\right) \times \left(1 - \frac{k}{2\pi} \frac{f(\frac{1}{2} + \frac{i}{4y})}{\frac{1}{4y} Df(\frac{1}{2} + \frac{i}{4y})}\right). \end{aligned}$$

Recall that Df is always cuspidal, so $v_\infty(Df) > 0$. Let f be cuspidal. Since $D = q\frac{d}{dq}$, it is clear that $v_\infty(f) = v_\infty(Df)$. By Lemma 6.3 we have

$$\lim_{y \to 0} \frac{f(\frac{1}{2} + \frac{i}{4y})}{\frac{1}{4y} Df(\frac{1}{2} + \frac{i}{4y})} = 0.$$

This proves the second equation of (a), and the first one of (a) is obtained by applying the same argument for $\gamma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $z = \frac{i}{y}$. Also we have

$$\lim_{y \to 0} \frac{f(\frac{1}{2} + \frac{i}{4y})}{\frac{1}{4y} Df(\frac{1}{2} + \frac{i}{4y})} = (-1)^{v_\infty(Df)} \mathrm{sgn}(a_\infty(f)) \mathrm{sgn}(a_\infty(Df)) \infty.$$

Note that if f is non-cuspidal and then $a_\infty(f) a_\infty(Df) = n a_0 a_n$, where a_0 and a_n are the first two non-zero Fourier coefficients of f. Thus we have

$$\lim_{y \to 0} \frac{f(\frac{1}{2} + \frac{i}{4y})}{\frac{1}{4y} Df(\frac{1}{2} + \frac{i}{4y})} = \epsilon_f \infty,$$

which implies the second equation of (b), and similarly the first equation is derived as well.

Now suppose $j > 1$. For an arbitrary quasimodular form g of weight k and depth ℓ, there are quasimodular forms $g_0, \ldots, g_\ell$ such that

$$g(\gamma z) = \sum_{m=0}^{\ell} c^j (cz+d)^{k-j} g_m(z)$$

for arbitrary $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathbb{H}$. We denote these g_m by $Q_m(g)$. Thus for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$D^j f(\gamma z) = \sum_{m=0}^{j} c^m (cz+d)^{k+2j-m} Q_m(D^j f)(z),$$

and in particular

$$\begin{aligned} D^j f\left(\frac{1}{2} + iy\right) &= \sum_{m=0}^{j} 2^m \left(\frac{i}{2y}\right)^{k+2j-m} Q_m(D^j f)\left(\frac{1}{2} + \frac{i}{4y}\right) \\ &= \left(\frac{i}{2y}\right)^{k+2j} D^j f\left(\frac{1}{2} + \frac{i}{4y}\right) \times \left(1 + \sum_{m=1}^{j} 2^m \left(\frac{i}{2y}\right)^{-m} \frac{Q_m(D^j f)\left(\frac{1}{2} + \frac{i}{4y}\right)}{D^j f\left(\frac{1}{2} + \frac{i}{4y}\right)}\right). \end{aligned} \tag{2}$$

Referring to [Roy12, Theorem 3.5], it can be verified that $Q_0(D^j f) = D^j f$, $Q_j(D^j f) = \frac{j!}{(2\pi i)^j}\binom{k}{j} f$, and for $1 \le m \le j-1$, $Q_m(D^j f)$ is the $\frac{m!}{(2\pi i)^m}\binom{k}{m}$ times $\mathbb{Z}$-linear combinations of $D^{j-m-1} f$ and $D^{j-m} f$. Thus for $m < j-1$, a function $Q_m(D^j f)$ is cuspidal. Furthermore, $Q_{j-1}(D^j f)$ can be shown inductively to be equal to $\frac{(j-1)!}{(2\pi i)^{j-1}}\binom{k}{j-1}(f + (j-1)Df)$.

Therefore, by applying Lemma 6.3, we conclude that the equation (2) can be expressed as

$$\left(\frac{i}{2y}\right)^{k+2j} D^j f\left(\frac{1}{2}+\frac{i}{4y}\right) \times \left(1+\sum_{m=j-1}^{j} 2^m \left(\frac{i}{2y}\right)^{-m} \frac{Q_m(D^j f)\left(\frac{1}{2}+\frac{i}{4y}\right)}{D^j f\left(\frac{1}{2}+\frac{i}{4y}\right)} + o(1)\right)$$

$$= \left(\frac{i}{2y}\right)^{k+2j} D^j f\left(\frac{1}{2}+\frac{i}{4y}\right) \times \left(1+\left(\frac{(j-1)!}{i^{j-1}(2\pi i)^{j-1}}\binom{k}{j-1}+\frac{j!}{i^j(2\pi i)^j}\binom{k}{j}\cdot 4y\right)\frac{f\left(\frac{1}{2}+\frac{i}{4y}\right)}{\left(\frac{1}{4y}\right)^{j-1} D^j f\left(\frac{1}{2}+\frac{i}{4y}\right)} + o(1)\right)$$

$$= \left(\frac{i}{2y}\right)^{k+2j} D^j f\left(\frac{1}{2}+\frac{i}{4y}\right) \times \left(1+(-1)^{j-1}\epsilon_f \omega(1)\right)$$

as $y \gg 0$. Similarly, we can prove the first equation of (c), and also (a) for $j > 1$. □

Proof of Proposition 6.2. As f has real Fourier coefficients, so does its derivative Df. If f is cuspidal (resp. non-cuspidal), by Lemma 6.4, we have $Df(\frac{1}{2}+iy) = -\frac{1}{2y}Df(\frac{1}{2}+\frac{i}{4y}) \times (1+o(1))$ (resp. $\left(\frac{i}{2y}\right)^{k+2} Df(\frac{1}{2}+\frac{i}{4y}) \times (1-\epsilon_f\omega(1))$), so $Df(\frac{1}{2}+iy)$ and $Df(\frac{1}{2}+\frac{i}{4y})$ are real numbers with opposite signs for some $y \gg 0$.

It immediately follows that there exists $y_0 \in (\frac{1}{4y}, y)$ such that $Df(\frac{1}{2}+iy_0) = 0$. Similarly, one can verify the assertion for δ_1. □

Unless f belongs to $DS_{k-2,\mathbb{R}}$, the existence of a real zero holds under certain weight conditions. Before presenting this, we introduce the following lemma.

Lemma 6.5. *Let f be a cuspidal quasimodular form. As $y \gg 0$, the sign of $f\left(\frac{1}{2}+iy\right)$ is $(-1)^{v_\infty(f)}\mathrm{sgn}(a_\infty(f))$, and the sign of $f(iy)$ is $\mathrm{sgn}(a_\infty(f))$.*

Proof. It follows from $f(z) = \sum_{n=v_\infty(f)}^{\infty} a_n q^n = q^{v_\infty(f)}\left(a_{v_\infty(f)} + O(q)\right)$ immediately. □

Proposition 6.6. *Let f be a weight k quasimodular form.*

(a) *If $k \equiv 2 \pmod 4$ and f is non-cuspidal, then f has at least two zeros on $\{z \in \mathbb{H} : \Re(z) = 0 \text{ or } 1/2\}$, one of them lying on $\{z \in \mathbb{H} : \Re(z) = 0\}$ and one of them on $\{z \in \mathbb{H} : \Re(z) = 1/2\}$.*
(b) *If $k \equiv 2 \pmod 4$ and $f = f_0 + Df_1 \in S_{k,\mathbb{R}} \oplus DM_{k-2,\mathbb{R}}$ with $v_\infty(f) \le v_\infty(f_1)$, then f has a zero on the line $\{z \in \mathbb{H} : \Re(z) = 1/2\}$.*
(c) *If the depth of f is not larger than 1 and $k \equiv 6, 10 \pmod{12}$, then f has a zero on the line $\{z \in \mathbb{H} : \Re(z) = 1/2\}$.*
(d) *If the depth of f is not larger than 2 and $k \equiv 6 \pmod{12}$, then f has a zero on the line $\{z \in \mathbb{H} : \Re(z) = 1/2\}$.*

Proof. Let p be the depth of f. Recall that

$$f\left(\frac{1}{2}+iy\right) = \left(\frac{i}{2y}\right)^k f\left(\frac{1}{2}+\frac{i}{4y}\right) \times \left(1+\sum_{j=1}^{p}\frac{1}{i^j}\frac{Q_j(f)}{\left(\frac{1}{4y}\right)^j f}\left(\frac{1}{2}+\frac{i}{4y}\right)\right)$$

and similarly

$$f(iy) = \left(\frac{i}{y}\right)^k f\left(\frac{i}{y}\right) \times \left(1+\sum_{j=1}^{p}\frac{1}{i^j}\frac{Q_j(f)}{\left(\frac{1}{y}\right)^j f}\left(\frac{i}{y}\right)\right).$$

Since f is non-cuspidal, we have $0 = v_\infty(f) \le v_\infty(Q_j(f))$ for any $j \in \{1, 2, \ldots, p\}$ so that both of $\frac{Q_j(f)}{\left(\frac{1}{4y}\right)^j f}\left(\frac{1}{2}+\frac{i}{4y}\right)$ and $\frac{Q_j(f)}{\left(\frac{1}{y}\right)^j f}\left(\frac{i}{y}\right)$ are $o(1)$ as y approaches 0 by Lemma 6.3. In particular, the signs of $f\left(\frac{1}{2}+\frac{i}{4y}\right)$ and of $f\left(\frac{i}{y}\right)$ are opposite to the sign of $f(\infty)$. This completes the proof of (a), and the same arguments can be applied to prove (b).

To prove (c), write $f = f_0 + Df_1$ for $f_0 \in M_{k,\mathbb{R}}$, $f_1 \in M_{k-2,\mathbb{R}}$. We have

$$f\left(\frac{1}{2}+iy\right) = \left(\frac{i}{2y}\right)^k f\left(\frac{1}{2}+\frac{i}{4y}\right) \times \left(1 - \frac{k}{2\pi}\frac{f_1}{\frac{1}{4y}f}\left(\frac{1}{2}+\frac{i}{4y}\right)\right).$$

Since $k-2 \equiv 4, 8 \pmod{12}$, we have $f_1\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) = 0$ by the valence formula. If we take $y = \frac{i}{2\sqrt{3}}$, then we get

$$f\left(\frac{1}{2}+\frac{i}{2\sqrt{3}}\right) = -\left(\frac{5}{2}\right)^k f\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right),$$

hence the desired result follows. A similar argument also proves (d). □

It is natural to question whether there is a quasimodular form with no real zeros on lines δ_1 or δ_2, particularly when the depth is greater than 0. In the case of depth 0, the modular discriminant function Δ is one of the standard examples of such (quasi)modular forms, but the answer is unclear for higher depths.

For quasimodular forms of depth 1, it is clear that the derivative of the Eisenstein series does not vanish on the line $\{z \in \mathbb{H} : \Re(z) = 0\}$. Similarly, there are some examples of quasimodular forms of depth 1 that do not vanish on the line $\{z \in \mathbb{H} : \Re(z) = 1/2\}$. Remarkably, these forms are the derivatives of modular forms, and the corresponding antiderivative functions also lack real zeros on the line δ_2.

Proposition 6.7. *For $k \equiv 0 \pmod{12}$, let f be a non-cuspidal modular form of weight k given by*

$$f(z) = b_1\Delta^{\frac{k}{12}}(z) + b_k\Delta(z)E_4^{\frac{k}{4}-3}(z) + E_4^{\frac{k}{4}}(z).$$

For sufficiently large k, there exists a constant $B > 0$ depending on b_k such that if $(-1)^{\frac{k}{12}}b_1 > B$ then f has no zero lying on the line $\{z \in \mathbb{H} : \Re(z) = 1/2\} \cup \{\infty\}$. Furthermore if we assume $b_k < -60k$, then we can take B for which $f\left(\frac{1}{2}+iy\right)$ is a monotone decreasing function for $y \in (0,\infty)$.

To prove Proposition 6.7, we first investigate the sign changes of the Ramanujan tau function $\tau(n)$ and prove several useful lemmas. Define

$$\tau_m(n) := \sum_{\substack{a_1,a_2,\dots,a_m \geq 1 \\ a_1+a_2+\cdots+a_m = n}} \tau(a_1)\tau(a_2)\cdots\tau(a_m),$$

be the nth Fourier coefficient of Δ^m. For a positive integer $k \equiv 0 \pmod{12}$, let $N_k := \frac{k}{12} + \left[1 + \frac{2}{15}k\right]$. We now introduce the D'Arcais polynomial $P_n(x)$, which is defined recursively by

$$\begin{cases} P_0(x) = 1, \\ P_n(x) = \frac{x}{n}\sum_{j=1}^{n}\sigma_1(j)P_{n-j}(x), & \text{for } n \geq 1. \end{cases}$$

It is well-known that all real roots of $P_n(x)$ are negative. Additionally, the special values of $P_n(x)$ correspond to the q-coefficients of the power of the Dedekind eta function, namely, if we write

$$\prod_{n\geq 1}(1-q^n)^r = \sum_{n=0}^{\infty}\eta_n(r)q^n,$$

for $r \in \mathbb{C}$, then we have $P_n(-r) = \eta_n(r)$ (see [D'Ar13], [New55]).

Furthermore, the sign of each $\eta_n(r)$ is determined by the D'arcais polynomial in the following manner.

Theorem 6.8. [HN20a, Theorem 2] *Let r_n be the number of real roots of $P_n(x)$ which are less than or equal to $-r$. Then we have $(-1)^{n+r_n}\eta_n(r) \geq 0$.*

By virtue of Theorem 6.8, we may reach an immediate conclusion that for $r = 2k$, should any real root of $P_n(x)$ exceed $-2k$, then we have $(-1)^n\tau_{\frac{k}{12}}\left(n+\frac{k}{12}\right) < 0$.

Recently, Heim and Neuhauser [HN20b] established the growth condition for $P_n(x)$, which constitutes an improvement of the results presented in [Kos04] and [Han10].

Theorem 6.9. [HN20b] *If $|x| > 15(n-1)$, then $P_n(x) \neq 0$.*

Combining Theorem 6.8 and 6.9, we get the following lemma.

Lemma 6.10. *Let n be a positive integer. If $\frac{k}{12} \le n \le N_k$, then $\tau_{\frac{k}{12}}(n-1)\tau_{\frac{k}{12}}(n) < 0$.*

Proof. For $n = N_k - \frac{k}{12}$, we have $n < 1 + \frac{2k}{15}$, which is equivalent to $-2k < -15(n-1)$. This implies that there is no real root of $P_n(x)$ less than or equal to $-2k$. Therefore, for $0 \le n \le N_k - \frac{k}{12}$, we have $(-1)^n \tau_{\frac{k}{12}}\left(n + \frac{k}{12}\right) < 0$. □

Proof of Proposition 6.7. Fix a sufficiently large k. We'll specify later how large k should be. Take into account the function f/b_1 by splitting into three partial sums denoted by f_1, f_2, and f_3 as follows;

$$\frac{1}{b_1} f(\tau) = \left(\sum_{n=0}^{\frac{k}{12}-1} + \sum_{n=\frac{k}{12}}^{N_k} + \sum_{n=N_k+1}^{\infty} \right) \left(\frac{a_n(f)}{b_1} q^n \right) =: f_1(z) + f_2(z) + f_3(z).$$

Observing that f_1 is a finite sum, we have $f_1 = O(|b_1|^{-1})$ for any complex number q with $0 < |q| < 1$. Furthermore, for any $n \ge \frac{k}{12}$, we can express $\frac{a_n(f)}{b_1}$ as $\tau_{\frac{k}{12}}(n) + \frac{1}{b_1} g(n)$, where $g(n)$ is given by

$$g(n) = \sum_{\substack{c_1, \ldots, c_{\frac{k}{4}} \ge 0 \\ c_1 + \cdots + c_{\frac{k}{4}} = n}} 240^{\frac{k}{4}} \sigma_3(c_1) \cdots \sigma_3(c_{\frac{k}{4}}) + b_k \sum_{\substack{a > 0, a_1, \ldots, a_{\frac{k}{4}-3} \ge 0 \\ a + a_1 + \cdots + a_{\frac{k}{4}-3} = n}} 240^{\frac{k}{4}-3} \tau(a) \sigma_3(a_1) \cdots \sigma_3(a_{\frac{k}{4}-3}).$$

Here, we adopt the convention that $\sigma_3(0) := \frac{1}{240}$.

To estimate the functions $\tau_{\frac{k}{12}}(n)$ and $g(n)$, we use the Ramanujan-Petersson bound $|\tau(n)| \le d(n) n^{\frac{11}{2}}$. Specifically, we have

$$\begin{aligned}
|\tau_{\frac{k}{12}}(n)| &\le \sum_{\substack{c_1, \ldots, c_{\frac{k}{12}} \ge 0 \\ c_1 + \cdots + c_{\frac{k}{12}} = n}} |\tau(c_1) \cdots \tau(c_{\frac{k}{12}})| \\
&\le \sum_{\substack{c_1, \ldots, c_{\frac{k}{12}} \ge 0 \\ c_1 + \cdots + c_{\frac{k}{12}} = n}} d(c_1) \cdots d(c_{\frac{k}{12}}) (c_1 \cdots c_{\frac{k}{12}})^{\frac{11}{2}} \\
&\le \sum_{\substack{c_1, \ldots, c_{\frac{k}{12}} \ge 0 \\ c_1 + \cdots + c_{\frac{k}{12}} = n}} (c_1 \cdots c_{\frac{k}{12}})^{\frac{k+1}{2}} \le p\left(n - \frac{k}{12}\right) \left(\frac{12n}{k}\right)^{\frac{13k}{24}},
\end{aligned}$$

where $p(n)$ is the partition number of n. Similarly for $g(n)$, we have

$$\begin{aligned}
|g(n)| &\le \sum_{\substack{c_1, \ldots, c_{\frac{k}{12}} \ge 0 \\ c_1 + \cdots + c_{\frac{k}{4}} = n}} 240^{\frac{k}{4}} (c_1 \cdots c_{\frac{k}{4}})^4 + |b_k| \sum_{\substack{a > 0, a_1, \ldots, a_{\frac{k}{4}-3} \ge 0 \\ a + a_1 + \cdots + a_{\frac{k}{4}-3} = n}} 240^{\frac{k}{4}-3} d(a) a^{\frac{k-1}{2}} (a_1 \cdots a_{\frac{k}{4}-3})^4 \\
&\le 240^{\frac{k}{4}} p(n) \left(\frac{4n}{k}\right)^{\frac{k}{4}} \left(1 + 240^{-3} |b_k| n^{\frac{k-7}{2}}\right).
\end{aligned}$$

By applying the well-known estimate $p(n) \sim \frac{1}{4\sqrt{3} n^{3/2}} e^{\pi \sqrt{\frac{2}{3} n}}$ as $n \to \infty$ by Hardy and Ramanujan, we choose a sufficiently large positive number B such that if $|b_1| > B$, then

$$\frac{1}{|b_1|} \sum_{n=N_k+1}^{\infty} 240^{\frac{k}{4}} p(n) \left(\frac{4n}{k}\right)^{\frac{k}{4}} \left|1 + 240^{-3} |b_k| n^{\frac{k-7}{2}}\right| < \epsilon.$$

Hence for $z = \frac{1}{2} + iy$ with $y \geq \frac{1}{2}$, we have

$$\begin{aligned} |f_3(z)| &\leq \sum_{n=N_k+1}^{\infty} |\tau_{\frac{k}{12}}(n)||q|^n + \epsilon \\ &\leq \sum_{n=N_k+1}^{\infty} p\left(n - \frac{k}{12}\right)\left(\frac{12n}{k}\right)^{\frac{k(k+1)}{24}} |q|^n + \epsilon \\ &\leq \sum_{n=N_k+1}^{\infty} \frac{C}{4\left(n-\frac{k}{12}\right)\sqrt{3}} e^{\pi\sqrt{\frac{2}{3}\left(n-\frac{k}{12}\right)} - 2\pi y n} \left(\frac{12n}{k}\right)^{\frac{13k}{24}} + \epsilon \end{aligned}$$

for some constant $C > 0$. Note that the last summation in the above inequality converges. We can ensure by choosing a sufficiently large k that

$$\sum_{n=N_k+1}^{\infty} \frac{C}{4\left(n-\frac{k}{12}\right)\sqrt{3}} e^{\pi\sqrt{\frac{2}{3}\left(n-\frac{k}{12}\right)} - \pi n} \left(\frac{12n}{k}\right)^{\frac{13k}{24}} + \epsilon < 2\epsilon.$$

Also, we have $|f_1(z)| < \epsilon$ by taking large B.

We have found that the primary component in evaluating $f(z)$ is $f_2(z)$. As $|b_1|$ becomes large, the sign of $a_n(f)$ for $\frac{k}{12} \leq n \leq N_k$ is determined by the sign of $\tau_{\frac{K}{12}}(n)$, more precisely,

$$\operatorname{sgn}(a_n(f)) = \operatorname{sgn}\left(\tau_{\frac{k}{12}}(n)\right).$$

Thus, according to Lemma 6.10, the sign of $a_n(f)$ is $(-1)^{n-\frac{k}{12}}$ for $\frac{k}{12} \leq n \leq N_k$.

We choose a positive real number y_0 such that if $y > y_0$ then $\left|f\left(\frac{1}{2}+iy\right) - 1\right| < \epsilon$. Then for $z = \frac{1}{2} + iy$ with $y > y_0$, we have

$$f_2(z) = \sum_{n=\frac{k}{12}}^{N_k} \frac{1}{b_1} a_n(f) (e^{-2\pi y})^n > \sum_{n=\frac{k}{12}}^{N_k} \frac{1}{b_1} (-1)^n a_n(f) e^{2\pi y_0 n}.$$

Since we have chosen b_1 such that $(-1)^{\frac{k}{12}} b_1 > B > 0$, each term $\frac{1}{b_1}(-1)^n a_n(f) e^{2\pi y_0 n}$ in the above summation is positive. If we let $M := \sum_{n=\frac{k}{12}}^{N_k} \frac{1}{b_1}(-1)^n a_n(f) e^{2\pi y_0 n}$, then

$$M = \sum_{n=\frac{k}{12}}^{N_k} (-1)^n \left(\tau_{\frac{k}{12}}(n) + \frac{1}{b_1} g(n)\right) e^{2\pi y_0 n} > e^{2\pi y_0}, \quad \text{as } |b_1| > B.$$

Therefore, we conclude that

$$\begin{cases} b_1 f\left(\frac{1}{2}+iy\right) > M - 3\epsilon & \text{if } \frac{1}{2} \leq y \leq y_0, \\ b_1 f\left(\frac{1}{2}+iy\right) > 1 - \epsilon & \text{if } y > y_0, \end{cases}$$

which implies that $f\left(\frac{1}{2}+iy\right)$ is non-zero for any $y \in (0, \infty)$.

It only remains to prove that if $b_k < -60k$, then $f\left(\frac{1}{2}+iy\right)$ is a monotone decreasing function for $y \in (0,\infty)$. Since f is holomorphic, it suffices to show that $Df\left(\frac{1}{2}+iy\right) > 0$ for all $y \in (0,\infty)$. Note that $b_k < -60k$ implies that $v_\infty(Df) = 1$ and $\epsilon_f = 1$. Therefore, there exists $y_0 > 1/2$ such that if $y > y_0$, then $Df\left(\frac{1}{2}+iy\right) > 0$.

Recall Lemma 6.4(b), which implies the existence of y_1 with $0 < y_1 < 1/2$ such that $Df\left(\frac{1}{2}+iy\right) > 0$ for $0 < y < y_1$ in this case. Thus, it suffices to show $Df\left(\frac{1}{2}+iy\right) > 0$ for $y \in [y_1, y_0]$. Since $Df = \sum_{n=1}^{\infty} n a_n(f) q^n$, this follows by the same argument we used to prove the positivity of $f\left(\frac{1}{2}+iy\right)$. □

Remark 6.11. The given weight k in Proposition 6.7 does not need to be very large. In fact, it is enough to take $k \geq 24$. One example of such a modular form f of weight 24 is

$$\begin{aligned} f(z) &:= 2 \cdot 1728 \cdot 2880 \Delta(z)^2 - 2880 \Delta(z) E_4(z)^3 + E_4(z)^6 \\ &= 9953280 \Delta(z)^2 - \frac{3}{225\zeta(4)^3} g_2(z)^3 + \frac{1}{216000\zeta(4)^6} g_2(z)^6. \end{aligned}$$

Remark 6.12. Proposition 6.7 provides that if $k \equiv 0 \pmod{12}$, there are infinitely many non-cuspidal modular forms of weight k and quasimodular forms of weight $k+2$ and depth 1 which do not vanish on the line $\{z \in \mathbb{H} : \Re(z) = 1/2\}$. However, it is not possible to construct such forms for depth ≥ 2 in the same way as in the case of f and Df. In fact, if $j > 1$, $D^j f$ always has a zero on the line $\{z \in \mathbb{H} : \Re(z) = 1/2\}$, since the signs of $D^j f\left(\frac{1}{2} + iy\right)$ and $D^j f\left(\frac{1}{2} + \frac{i}{4y}\right)$ are opposite for small y as shown in Lemma 6.4.

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