CONSTRUCTION OF SIMPLE QUOTIENTS OF BERNSTEIN-ZELEVINSKY DERIVATIVES AND HIGHEST DERIVATIVE MULTISEGMENTS I: REDUCTION TO COMBINATORICS

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Abstract. Let F be a non-Archimedean local field. A sequence of derivatives of generalized Steinberg representations can be used to construct simple quotients of Bernstein-Zelevinsky derivatives of irreducible representations of $\mathrm{GL}_n(F)$. In the first of a series of two articles, we introduce a notion of a highest derivative multisegment, which in turn gives a combinatorial approach to study problems about those simple quotients. We also prove a double derivative result along the way.

1. Introduction

Let F be a non-Archimedean local field. The Bernstein-Zelevinsky derivative is a twisted Jacquet functor, originally introduced in classifying the irreducible representations of $\mathrm{GL}_n(F)$ in [BZ76, BZ77, Ze80]. This is the first one of a series of two articles in studying the theory of Bernstein-Zelevinsky derivatives. The main result of this article provides a combinatorial approach to the problems of socles and cosocles of Bernstein-Zelevinsky derivatives of irreducible representations. In the second one [Ch22+d], we shall obtain a canonical sequence from some minimality by using results of this one, and then establish properties for such sequence. Applications to branching laws will be considered in [Ch22+b]. Indeed, we shall show in [Ch22+b] that any simple quotients of Bernstein-Zelevinsky derivatives can be constructed from such sequences.

On the other hand, there is a notion of ρ-derivatives introduced and studied by C. Jantzen [Ja07] and independently by Mínguez [Mi09], which will be important in our study. To be more precise, the notion of derivatives in this article is the one using essentially square-integrable representations to replace cuspidal representations ρ in [Ja07, Mi09], which we shall simply call St-derivatives. Such derivative is also used in recent work of Atobe-Mínguez [AM20], and Lapid-Mínguez [LM22] for other studies. A certain sequence of St-derivatives can be used to construct some simple quotients of Bernstein-Zelevinsky (BZ) derivatives (see Section 3). This is based on the observation that any standard module in GL case has unique submodule and such submodule is generic [JS83] (also see [CaSh98, Ch21]).

We recall some classical known results on BZ derivatives. The highest derivative of an arbitrary irreducible representation is determined by Zelevinsky [Ze80]. A complete description for all the derivatives has been previously established for Steinberg representations and their Zelevinsky duals by Zelevinsky [Ze80], and for ladder representations (including Speh representations) by Lapid-Mínguez [LM14] (also see [Ta87, CS19]). An asymmetry property of simple quotients between left and right derivatives is shown in [Ch21]. It could be hard to give a nice explicit description for the general case, and so it may be desirable to study derivatives in terms of some properties and invariants.

1.1. **Two notions of derivatives.** We introduce the two notions of derivatives and more notations will be given in Sections 2 and 3. Let $G_n = \mathrm{GL}_n(F)$, the general linear group over a non-Archimedean field F. For $a, b \in \mathbb{Z}$ with $b-a \geq 0$ and a cuspidal representation ρ of G_m, we shall call $[a,b]_\rho$ to be a segment and define $l_a([a,b]_\rho) = (b-a+1)m$. Zelevinsky [Ze80] showed that essentially square-integrable representations of G_n can be parametrized by those segments. For each segment Δ, we shall denote by $\mathrm{St}(\Delta)$ the corresponding essentially square-integrable representation (see Section 2.6).

Let $\mathrm{Irr}(G_n)$ be the set of (isomorphism classes of) irreducible smooth complex representations of G_n. Let $\mathrm{Irr} = \sqcup_n \mathrm{Irr}(G_n)$. Let $N_i \subset G_n$ (depending on n) be the unipotent radical containing matrices of the form $\begin{pmatrix} I_{n-i} & u \\ & I_i \end{pmatrix}$, where u is a $(n-i) \times i$ matrix. There exists at most one irreducible module $\tau \in \mathrm{Irr}(G_{n-i})$ such that

$$\tau \boxtimes \mathrm{St}(\Delta) \hookrightarrow \pi_{N_i},$$

where π_{N_i} is defined as the (normalized) Jacquet module of π. If such τ exists, we denote such τ by $D_\Delta(\pi)$. Otherwise, we set $D_\Delta(\pi) = 0$. We shall refer D_Δ to be a *St-derivative*.

Let

$$R_i = \left\{ \begin{pmatrix} g & m \\ & u \end{pmatrix} : g \in G_{n-i}, m \in Mat_{(n-i)\times i}, u \in U_i \right\}.$$

The right i-th *Bernstein-Zelevinsky derivative* $\pi^{(i)}$ of π is defined as

$$\delta_{R_i}^{-1/2} \cdot \frac{\pi}{\langle x.v - \psi(x)v : x \in R_i, v \in \pi \rangle}, \tag{1.1}$$

where δ_{R_i} is the modular character of R_{n-i}, and ψ is a non-degenerate character on U_i extended trivially to R_i. Regarding G_{n-i} as a subgroup G_n via $g \mapsto \begin{pmatrix} g & \\ & I_i \end{pmatrix}$, we obtain a natural G_{n-i}-module structure on $\pi^{(i)}$. Here U_i is viewed as a subgroup of R_i via the embedding

$$u \mapsto \begin{pmatrix} I_{n-i} & 0 \\ & u \end{pmatrix}.$$

The *level* of π is the largest integer i^* such that $\pi^{(i^*)} \neq 0$ and, for any $i > i^*$, $\pi^{(i)} = 0$. For the level i^* of π, let $\pi^- = \pi^{(i^*)}$, which is known to be irreducible [Ze80]. We shall call π^- the *highest derivative* of π.

1.2. **Motivations from branching laws.** Our goal is applications towards branching laws for general linear groups or even other classical groups, in view of the recent derivative approach in studying branching laws e.g. [MW12, Ve13, SV17, Pr18, CS21, Gu22, Ch21, GGP20, Ch22, Ch23]. Those applications will appear in elsewhere, see e.g. [Ch22+b], [Ch22+c].

Let $\nu : G_n \to \mathbb{C}^\times$ be the character $\nu(g) = |\det(g)|_F$, where $|.|_F$ is the norm for F. The close relation of derivatives and branching laws underlies in the Bernstein-Zelevinsky theory (see e.g. [Ch21]):

Lemma 1.1. *Let $\pi \in \mathrm{Irr}(G_{n+1})$. Let τ be a simple quotient of $\nu^{1/2} \cdot \pi^{(i)}$. Then, for some cuspidal representation $\sigma \in \mathrm{Irr}(G_{n-i})$,*

$$\mathrm{Hom}_{G_n}(\pi, \tau \times \sigma) \neq 0.$$

One interesting consequence of the above lemma is that the multiplicity at-most-one phenomenon [AGRS10, Ch23] implies the multiplicity-freeness on socles and cosocles of the Bernstein-Zelevinsky derivatives of an irreducible representation (see Sections 3.6 and 3.8).

Instead of asking simple quotients, one may also ask for simple submodules of the Bernstein-Zelevinsky derivative of an irreducible representation. Such two problems are indeed equivalent by the dual structure of the Bernstein-Zelevinsky derivative (see Lemma 3.13, [CS21, Lemma 2.4]).

1.3. **Main results.** Fix a cuspidal representation $\rho \in \mathrm{Irr}$. Let Irr_ρ be the subset of Irr of all irreducible representations which are an irreducible quotient of $\nu^{a_1}\rho \times \ldots \times \nu^{a_k}\rho$, for some integers $a_1, \ldots, a_k \in \mathbb{Z}$. Representations in Irr_ρ are the most interesting case, and the general case can be deduced from that (see Section 3.9).

Let Seg be the set of all segments including the empty set. A *multisegment* is a multiset of non-empty segments (see (2.2)). Let Seg_ρ be the subset of Seg of all segments of the form $[a,b]_\rho$ for some $a, b \in \mathbb{Z}$. Let Mult be the set of all multisegments and let Mult_ρ be the subset of Mult of all multisegments whose segments are in Seg_ρ. The empty set $\emptyset$ is also considered in Mult.

For $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathrm{Mult}$, we write $\mathfrak{m}_2 \leq_Z \mathfrak{m}_1$ if $\mathfrak{m}_2$ can be obtained by a sequence of elementary intersection-union operations from $\mathfrak{m}_1$ (see Section 2.4) or $\mathfrak{m}_1 = \mathfrak{m}_2$. In particular, if any pair of segments in $\mathfrak{m}$ is unlinked, then $\mathfrak{m}$ is a minimal element under $\leq_Z$. We shall equip Mult with the poset structure by $\leq_Z$.

A sequence of segments $[a_1,b_1]_\rho, \ldots, [a_k,b_k]_\rho$ (all $a_j, b_j \in \mathbb{Z}$) is said to be in an *ascending order* if for any $i \leq j$, either $[a_i,b_i]_\rho$ and $[a_j,b_j]_\rho$ are unlinked; or $a_i < a_j$. For a multisegment $\mathfrak{n} \in \mathrm{Mult}_\rho$, which we write the segments in $\mathfrak{n}$ in an ascending order $\Delta_1, \ldots, \Delta_k$. Define

$$D_{\mathfrak{n}}(\pi) := D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi).$$

We will show in Lemma 4.10 that the derivative $D_{\mathfrak{n}}$ is independent of the ordering of an ascending sequence for $\mathfrak{n}$. In particular, one may choose the ordering such that $a_1 \leq \ldots \leq a_k$.

In general, we have the following connection between two notions of derivatives:

Proposition 1.2. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ such that*

$$D_{\mathfrak{n}}(\pi) \neq 0.$$

Then $D_{\mathfrak{n}}(\pi)$ is a simple quotient of $\pi^{(i)}$, where $i = l_a(\Delta_1) + \ldots + l_a(\Delta_k)$.

We remark that when π is generic, those simple quotients have been described in [CS21, Corollary 2.6] by using a suitable filtration on the derivatives from the geometric lemma.

In general, two different sequences can give isomorphic simple quotients. Hence it is natural to study the combinatorial structure of the following set: for $\pi \in \mathrm{Irr}_\rho$ and for a simple quotient τ of $\pi^{(i)}$ for some i, define

$$\mathcal{S}(\pi,\tau) := \{\mathfrak{n} \in \mathrm{Mult}_\rho : D_{\mathfrak{n}}(\pi) \cong \tau\}.$$

The ordering $\leq_Z$ induces a partial ordering on $\mathcal{S}(\pi,\tau)$. We now introduce two ingredients in studying the set: highest derivative multisegments and removal process.

We now explain the first ingredient. A multisegment $\mathfrak{m}$ is said to be *at the point* $\nu^c\rho$ if any segment Δ in $\mathfrak{m}$ takes the form $[c,b]_\rho$ for some $b \geq c$. For $\pi \in \mathrm{Irr}_\rho$, define $\mathfrak{h}_c$ to be the maximal multisegment at the left point $\nu^c\rho$ such that $D_{\mathfrak{h}_c}(\pi) \neq 0$. (We refer to Section 4.2 for the meaning of maximality.) Define the *highest derivative multisegment* of $\pi \in \mathrm{Irr}_\rho$ to be

$$\mathfrak{hd}(\pi) := \sum_{c \in \mathbb{Z}} \mathfrak{h}_c.$$

The highest derivative multisegments of some special cases are given in Section 8.

One main property of $\mathfrak{hd}$ is to give a new construction of the highest derivative:

Theorem 1.3. *(Theorem 5.2) Let $\pi \in \mathrm{Irr}_\rho$. Then*

$$D_{\mathfrak{hd}(\pi)}(\pi) = \pi^-.$$

In [Ch22+b], we shall show that $\mathcal{S}(\pi, \tau) \neq \emptyset$, giving a converse of Proposition 1.2. Following our development, the exhaustion part (i.e. $\mathcal{S}(\pi, \tau) \neq \emptyset$) seems to be a deeper fact.

We now explain the second ingredient. In Section 6, we define a combinatorial algorithm, called *removal process*, on a pair $(\Delta, \mathfrak{h})$ for a segment Δ and a multisegment $\mathfrak{h}$. The algorithm results a multisegment, denoted $\mathfrak{r}(\Delta, \mathfrak{h})$. The case that we are interested in is when $\mathfrak{h} = \mathfrak{hd}(\pi)$. We also develop some rules and properties for computing $\mathfrak{r}(\Delta, \mathfrak{h})$ in Section 6, and the relation to $D_\Delta(\pi)$ is given in Theorem 6.20.

For a multisegment $\mathfrak{n} \in \mathrm{Mult}_\rho$, which we write the segments in $\mathfrak{n}$ in an ascending order $\Delta_1, \ldots, \Delta_k$, we define

$$\mathfrak{r}(\mathfrak{n}, \pi) := \mathfrak{r}(\Delta_k, \mathfrak{r}(\Delta_{k-1}, \ldots \mathfrak{r}(\Delta_1, \mathfrak{hd}(\pi)) \ldots)).$$

One remarkable property of the multisegment $\mathfrak{r}(\mathfrak{n}, \pi)$ is to measure the difference between the derivative $D_\mathfrak{n}(\pi)$ and the highest derivative π^-:

Theorem 1.4. *(=Theorem 7.1) Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ such that $D_\mathfrak{n}(\pi) \neq 0$. Then*

$$D_{\mathfrak{r}(\mathfrak{n},\pi)} \circ D_\mathfrak{n}(\pi) \cong \pi^-.$$

Theorem 1.4 has applications in [Ch22+b]. On the other hand, by shifting the 'right' branching law in Lemma 1.1 to the 'left' branching law, it gives an interpretation to Theorem 1.4 (more details given in [Ch22+b]), which is also a starting point of this article and [Ch22+d].

The second main property of $\mathfrak{r}(\mathfrak{n}, \pi)$ is to determine when two derivatives gives isomorphic quotients:

Theorem 1.5. *(=Corollary 6.22+Theorem 7.2) Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{n}_1, \mathfrak{n}_2$ be multisegments.*

(1) *Then $D_{\mathfrak{n}_i}(\pi) \neq 0$ if and only if $\mathfrak{n}_i$ is admissible to $\mathfrak{hd}(\pi)$. (Refer the definition of admissibility to Definition 6.14)*
(2) *Suppose $D_{\mathfrak{n}_1}(\pi) \neq 0$ and $D_{\mathfrak{n}_2}(\pi) \neq 0$. Then*

$$D_{\mathfrak{n}_1}(\pi) \cong D_{\mathfrak{n}_2}(\pi) \quad \Longleftrightarrow \quad \mathfrak{r}(\mathfrak{n}_1, \pi) = \mathfrak{r}(\mathfrak{n}_2, \pi).$$

Combining Theorem 1.5 with [Ch22+b], $\mathfrak{r}(\mathfrak{n}, \pi)$ provides a combinatorial invariant for describing the socle and cosocle of the Bernstein-Zelevinsky derivative of an irreducible representation. Theorem 1.5 reduces problems about derivatives of essentially square-integrable representations into combinatorics problems. Applications will appear in the sequel [Ch22+d].

We finally comment on the proof of Theorems 1.4 and 1.5. The multisegment associated to the derivative $D_\mathfrak{n}(\pi)$ can be in general computed via explicit algorithms (see e.g. [LM16]), but our proof does not directly use that. Our proof is more combinatorially soft in the sense that we use some commutation relations between derivatives studied in Section 4. Our proof also uses certain inductions via taking the ρ-derivatives.

1.4. **Remarks.** Apart from branching laws, there are many other applications for both derivatives. For example, it is important for unitarity by Tadić [Ta86], theta correspondence by Mínguez [Mi08, Mi09], L-functions (e.g. work of Matringe, Cogdell–Pietaski-Shapiro, Jo-Krishnamurthy [Ma13, CPS17, JK22]), other distinction problems (e.g. Offen [Of18]), and Aubert-Zelevinsky duals by Mœglin-Waldspurger and Atobe-Mínguez [MW86, AM20], and Arthur packets by Xu [Xu17] and many others.

We finally give some background of our study. In [CS19] joint with Savin, we formulated the analogous Bernstein-Zelevinsky derivative functor for affine Hecke algebras of type A and so one could also formulate the analogous results in such setting, which will be explained in more detail in Section 9. Some parts in this article are originally inspired by the work of Grojnowski-Vazirani [GV01] in Hecke algebra setting few years ago, in which they used ρ-derivatives to study branching problems.

Using ρ-derivatives to study Bernstein-Zelevinsky derivatives also explicitly appears before, for example Deng [De16] studying with orbital varieties and a more recent work of Gurevich [Gu21] studying with RSK model. However, we emphasis that St-derivatives are important in the study of simple quotients of classical Bernstein-Zelevinsky derivatives, while ρ-derivatives seem to be not enough for such purpose. In particular, using machinery in Sections 6 and 7, one can find some simple quotients of Bernstein-Zelevinsky derivatives which cannot be constructed from a sequence of ρ-derivatives (see Example 7.3).

1.5. **Organization of this article.** Section 3 discusses some relations of two notions of derivatives, and the multiplicity freeness of Lemma 3.11. Section 4 studies some preliminary results for commutativity of derivatives (mainly by the geometric lemma). Section 5 defines the highest derivative multisegment and shows that its corresponding derivative gives the highest derivative (Theorem 5.2). Section 6 introduces the removal process, which is used in Section 7 to prove that the effect of ascending sequences of derivatives is determined by the removal process (Theorem 7.1). Explicit descriptions of the highest derivatives of some representations are given in Section 8. In the Appendix, we discuss how to transfer results to affine Hecke algebras of type A.

2. Preliminaries

2.1. **Notations.** All the representations are smooth and over $\mathbb{C}$. We shall usually drop those descriptions. We sometimes do not distinguish representations in the same isomorphism class. We also use the notations in Section 1.1. Let $\mathrm{Alg}(G_n)$ be the category of smooth representations of G_n. For $\pi \in \mathrm{Alg}(G_n)$, denote by $\pi^\vee$ the smooth dual of π.

For $\pi \in \mathrm{Irr}$, $n(\pi)$ is defined to be the number that $\pi \in \mathrm{Irr}(G_{n(\pi)})$. Let $\mathrm{Irr}^c(G_n)$ be the set of (irreducible) cuspidal representations of G_n. Similarly, let $\mathrm{Irr}^c = \sqcup_n \mathrm{Irr}^c(G_n)$.

For any $\pi_1 \in \mathrm{Alg}(G_{n_1})$ and $\pi_2 \in \mathrm{Alg}(G_{n_2})$, define

$$\pi_1 \times \pi_2 = \mathrm{Ind}_{P_{n_1,n_2}}^{G_{n_1+n_2}} \pi_1 \boxtimes \pi_2,$$

where we inflate $\pi_1 \boxtimes \pi_2$ to a P_{n_1,n_2}-representation. Here Ind is the normalized parabolic induction.

For $a, b \in \mathbb{Z}$ with $b - a \in \mathbb{Z}_{\geq 0}$ and a cuspidal representation ρ, we call

$$[a, b]_\rho := \left\{\nu^a \rho, \dots, \nu^b \rho\right\} \tag{2.2}$$

be a *segment*. We also set $[a, a-1]_\rho = \emptyset$ for $a \in \mathbb{Z}$. For a segment $\Delta = [a, b]_\rho$, we write $a(\Delta) = \nu^a \rho$ and $b(\Delta) = \nu^b \rho$. We also write:

$$[a]_\rho := [a, a]_\rho.$$

We may also write $[\nu^a\rho, \nu^b\rho]$ for $[a, b]_\rho$ and write $[\nu^a\rho]$ for $[a]_\rho$. The *relative length* of a segment $[a, b]_\rho$ is defined as $b - a + 1$, and we shall denote by $l_r([a, b]_\rho)$. The *absolute length* of a segment $[a, b]_\rho$ is defined as $(b - a + 1)n(\rho)$, and we shall denote by $l_a([a, b]_\rho)$ as before. Two segments $[a, b]_\rho$ and $[a', b']_{\rho'}$ are said to be equal if $\nu^{a'}\rho' \cong \nu^a \rho$ and $b - a + 1 = b' - a' + 1$. When ρ is the trivial representation of G_1, we simply write a segment $[a, b]$ for $[a, b]_1$.

Two segments Δ and Δ' are said to be *linked* if $\Delta \cup \Delta'$ is still a segment, and $\Delta \not\subset \Delta'$ and $\Delta' \not\subset \Delta$. Otherwise, it is called to be not linked or unlinked.

For any $\pi \in \mathrm{Irr}$, there exist $\rho_1, \ldots, \rho_r \in \mathrm{Irr}^c$ such that π is a simple composition factor for $\rho_1 \times \ldots \times \rho_r$, and we shall denote such multiset to be $\mathrm{csupp}(\pi)$, which is called the *cuspidal support* of π.

2.2. **More notations for multisegments.** For two multisegments $\mathfrak{m}$ and $\mathfrak{n}$, we write $\mathfrak{m}+\mathfrak{n}$ to be the union of two multisegments, counting multiplicities. For a multisegment $\mathfrak{m}$ and a segment Δ, we write

$$\mathfrak{m}+\Delta = \begin{cases} \mathfrak{m}+\{\Delta\} & \text{if } \Delta \neq \emptyset \\ \mathfrak{m} & \text{if } \Delta = \emptyset \end{cases}$$

The notions $\mathfrak{m}-\mathfrak{n}$ and $\mathfrak{m}-\Delta$ are defined in a similar way.

For $\rho_1, \rho_2 \in \mathrm{Irr}^c$, we write $\rho_2 < \rho_1$ if $\rho_1 \cong \nu^a \rho_2$ for some integer $a > 0$. For two segments Δ_1, Δ_2, we write $\Delta_1 < \Delta_2$ if Δ_1 and Δ_2 are linked and $b(\Delta_1) < b(\Delta_2)$.

For an integer c, let $\mathrm{Mult}^a_{\rho,c}$ be the subset of Mult_ρ containing all multisegments $\mathfrak{m}$ such that any segment Δ in $\mathfrak{m}$ satisfies $a(\Delta) \cong \nu^c\rho$. Similarly, define $\mathrm{Mult}^b_{\rho,c}$ to be the subset of Mult_ρ containing all multisegments $\mathfrak{m}$ such that any segment Δ in $\mathfrak{m}$ satisfies $b(\Delta) \cong \nu^c\rho$. The empty sets are also considered in $\mathrm{Mult}^a_{\rho,c}$ and $\mathrm{Mult}^b_{\rho,c}$.

For a multisegment $\mathfrak{m}$ in Mult_ρ and an integer c, let $\mathfrak{m}[c]$ be the submultisegment of $\mathfrak{m}$ containing all the segments Δ satisfying $a(\Delta) \cong \nu^c\rho$; and let $\mathfrak{m}\langle c\rangle$ be the submultisegment of $\mathfrak{m}$ containing all the segments Δ satisfying $b(\Delta) \cong \nu^c\rho$.

For a multisegment $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_k\}$, we also set:

$$l_a(\mathfrak{m}) = l_a(\Delta_1) + \ldots + l_a(\Delta_k), \quad l_r(\mathfrak{m}) = l_r(\Delta_1) + \ldots + l_r(\Delta_k).$$

2.3. **Ordering on segments.** For two non-empty segments $[a', b']_\rho$ and $[a'', b'']_\rho$, we write

$$[a', b']_\rho \prec^L [a'', b'']_\rho$$

if either $a' < a''$; or $a' = a''$ and $b' < b''$. We also write $[a', b']_\rho \preceq^L [a'', b'']_\rho$ if $[a', b']_\rho \prec^L [a'', b'']_\rho$ or $[a', b']_\rho = [a'', b'']_\rho$. The ordering $\prec^R$ can be defined in a similar manner by using b-values.

2.4. **Intersection-union operation.** We say that a multisegment $\mathfrak{m}_2$ is obtained from $\mathfrak{m}_1$ by an elementary intersection-union operation if for two segments Δ_1, Δ_2 in $\mathfrak{m}_1$,

$$\mathfrak{m}_2 = \mathfrak{m}_1 - \{\Delta_1, \Delta_2\} + \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2.$$

The ordering $\leq_Z$ is defined in Section 1.3.

2.5. **Ordering on** $\mathrm{Mult}^a_{\rho,c}$ **and** $\mathrm{Mult}^b_{\rho,c}$**.** Fix an integer c. Let $\Delta_1 = [c, b_1]_\rho, \Delta_2 = [c, b_2]_\rho$ be two non-empty segments. We write $\Delta_1 \leq^a_c \Delta_2$ if $b_1 \leq b_2$, and write $\Delta_1 <^a_c \Delta_2$ if $b_1 < b_2$.

For non-empty $\mathfrak{m}_1, \mathfrak{m}_2$ in $\mathrm{Mult}^a_{\rho,c}$, label the segments in $\mathfrak{m}_1$ as: $\Delta_{1,k} \leq^a_c \ldots \leq^a_c \Delta_{1,2} \leq^a_c \Delta_{1,1}$ and label the segments in $\mathfrak{m}_2$ as: $\Delta_{2,r} \leq^a_c \ldots \leq^a_c \Delta_{2,2} \leq^a_c \Delta_{2,1}$. We define the lexicographical ordering: $\mathfrak{m}_1 \leq^a_c \mathfrak{m}_2$ if $k \leq r$ and, for any $i \leq k$, $\Delta_{1,i} \leq^a_c \Delta_{2,i}$. We write $\mathfrak{m}_1 <^a_c \mathfrak{m}_2$ if $\mathfrak{m}_1 \leq^a_c \mathfrak{m}_2$ and $\mathfrak{m}_1 \neq \mathfrak{m}_2$.

We also need a 'right' ordering. One can define $[a_1, c]_\rho \leq^b_c [a_2, c]_\rho$ if $a_1 < a_2$, and similarly define $[a_1, c]_\rho <^b_c [a_2, c]_\rho$. One similarly define $\leq^b_c$ on $\mathrm{Mult}^b_{\rho,c}$.

2.6. **Zelevinsky and Langlands classification.** For a segment $\Delta = [a, b]_\rho \in \mathrm{Mult}$, define $\langle\Delta\rangle$ to be the the unique simple submodule of

$$\nu^a\rho \times \ldots \times \nu^b\rho$$

and define $\mathrm{St}(\Delta)$ to be the unique simple quotient of

$$\nu^a\rho \times \ldots \times \nu^b\rho.$$

For any multisegment $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_k\}$ with the labellings satisfying that, for $i < j$, $\Delta_i \not< \Delta_j$. Define, as in [Ze80, Theorem 6.1], $\langle \mathfrak{m} \rangle$ to be the unique simple submodule of

$$\zeta(\mathfrak{m}) := \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_k \rangle.$$

Define $\mathrm{St}(\mathfrak{m})$ to be the unique simple quotient of

$$\lambda(\mathfrak{m}) := \mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_k).$$

We frequently use the following standard fact (see [Ze80, Theorems 4.2, 6.1]): for two unlinked segments Δ_1 and Δ_2,

$$\langle \Delta_1 \rangle \times \langle \Delta_2 \rangle \cong \langle \Delta_2 \rangle \times \langle \Delta_1 \rangle, \tag{2.3}$$

$$\mathrm{St}(\Delta_1) \times \mathrm{St}(\Delta_2) \cong \mathrm{St}(\Delta_2) \times \mathrm{St}(\Delta_1). \tag{2.4}$$

2.7. **Geometric lemma.** The geometric lemma is a key tool in our study. We shall describe a special case that we frequently use.

For $n_1 + \ldots + n_k = n$, we write $P_{n_1,\ldots,n_k}$ to be the parabolic subgroup generated by the matrices $\mathrm{diag}(g_1, \ldots, g_k)$, where $g_j \in G_{n_j}$, and upper triangular matrices. We shall say that $P_{n_1,\ldots,n_k}$ is the standard parabolic subgroup associated to the partition $(n_1, \ldots, n_k)$. For $i \leq n$, N_i defined in Section 1.1 is the unipotent radical of the parabolic subgroup $P_{n-i,i}$.

Let $\pi_1 \in \mathrm{Alg}(G_{n_1})$ and let $\pi_2 \in \mathrm{Alg}(G_{n_2})$. Let $n = n_1 + n_2$. Then the geometric lemma on $(\pi_1 \times \pi_2)_{N_i}$ asserts that $(\pi_1 \times \pi_2)_{N_i}$ admits a filtration whose successive subquotients take the form:

$$\mathrm{ind}_{P_{n-i,i}}^{G_n}((\pi_1)_{N_{i_1}} \boxtimes (\pi_2)_{N_{i_2}})^{\phi},$$

where $i_1 + i_2 = i$. Here $((\pi_1)_{N_{i_1}} \boxtimes (\pi_2)_{N_{i_2}})^{\phi}$ is a $G_{n_1-i_1} \times G_{n_2-i_2} \times G_{i_1} \times G_{i_2}$-representation with underlying space $(\pi_1)_{N_{i_1}} \boxtimes (\pi_2)_{N_{i_2}}$ determined by the action:

$$(g_1, g_2, g_3, g_4).v_1 \boxtimes v_2 = (g_1, g_3).v_1 \boxtimes (g_2, g_4).v_2,$$

where $v_1 \in (\pi_1)_{N_{i_1}}$ and $v_2 \in (\pi_2)_{N_{i_2}}$.

2.8. **Quotients and submodules of Jacquet functors.** Let $\theta = \theta_n : G_n \to G_n$ be given by $\theta(g) = g^{-t}$, the inverse transpose of g. This induces a self-equivalence exact functor on $\mathrm{Alg}(G_n)$, still denoted by θ. We shall call it the *Gelfand-Kazhdan involution*.

Proposition 2.1. *Let $\pi \in \mathrm{Irr}(G_n)$. Let $n_1 + \ldots + n_r = n$. Let N be the unipotent radical of the parabolic subgroup $P_{n_1,\ldots,n_r}$. Let θ' be the involution on $\mathrm{Alg}(G_{n_1} \times \ldots \times G_{n_r})$ arisen from $\theta'(g_1, \ldots, g_r) = (\theta(g_1), \ldots, \theta(g_r))$. Then $\theta'(\pi_N)^{\vee} \cong \pi_N$. In particular, for an irreducible representation ω of $G_{n_1} \times \ldots \times G_{n_r}$, ω is a simple submodule of π_N if and only if ω is a simple quotient of π_N.*

Proof. Recall that $\theta(\pi)$ and π have the same underlying space, which we refer to V. Note that

$$W := \{\theta(n).v - v : n \in N^-, v \in V\} = \{n.v - v : n \in N, v \in V\} \subset V.$$

Then it induces a natural identification as vector space:

$$\theta_n(\pi)_{N^-} = \theta'(\pi'_N) = V/W.$$

Now one checks the isomorphism lifting to a $G_{n_1} \times \ldots \times G_{n_r}$-morphism. This proves that:

$$(*) \quad \theta(\pi)_{N^-} \cong \theta'(\pi_N).$$

On the other hand, by a result of Bernstein-Casselman [Be92, Page 66] and [Ca95, Corollary 4.2.5],

$$(**) \quad (\theta(\pi)_{N^-})^{\vee} \cong (\theta(\pi)^{\vee})_N \cong \pi_N,$$

where the last isomorphism follows from [BZ76, Theorem 7.3]. The proposition follows by combining (*) and (**). □

2.9. **Jacquet functors on Steinberg representations.** We shall frequently use the following formulas [Ze80]:

$$\langle [a,b]_\rho \rangle_{N_{in(\rho)}} = \langle [a,b-i]_\rho \rangle \boxtimes \langle [b-i+1,b]_\rho \rangle \tag{2.5}$$

and

$$\mathrm{St}([a,b]_\rho)_{N_{in(\rho)}} = \mathrm{St}([a+i,b]_\rho) \boxtimes \mathrm{St}([a,a+i-1]_\rho). \tag{2.6}$$

Later on, we sometimes say to use geometric lemma and compare cuspidal supports, which we mean to use the geometric lemma in Section 2.7 and then use the Jacquet functor computation above. Since those computations are quite routine, we shall not spell out explicitly every time. For instance, some samples of such computations can also be found in [Ch22+].

3. Two notions of derivatives

3.1. **ρ-derivatives.**

Lemma 3.1. [GV01, Ja07, Mi09] *Let $\rho \in \mathrm{Irr}^c(G_r)$ and let $\pi \in \mathrm{Irr}$. For any non-negative integer k,*

$$\pi \times \overbrace{\rho \times \ldots \times \rho}^{k \text{ times}}$$

has unique irreducible submodule and unique irreducible quotient.

In the Jacquet functor version, which follows from Frobenius reciprocity, one has:

Lemma 3.2. *Let $\rho \in \mathrm{Irr}^c(G_r)$ and let $\pi \in \mathrm{Irr}(G_n)$. There is at most one irreducible representation $\tau \in \mathrm{Irr}(G_{n-r})$ such that*

$$\tau \boxtimes \rho \hookrightarrow \pi_{N_r}.$$

We introduce the following notations [Ja07, Mi09]:

Notation 3.3. (1) We shall also write $\pi \times \rho^{\times k}$ for $\pi \times \overbrace{\rho \times \ldots \times \rho}^{k \text{ times}}$ for representations π and ρ.

(2) For $c \in \mathbb{Z}$ and $\pi \in \mathrm{Irr}_\rho$, if such τ in Lemma 3.2, we shall denote $D_c(\pi)$ (depending on ρ) to be such τ. Otherwise, set $D_c(\pi) = 0$. For $k \geq 0$, we shall write $D_c^k(\pi)$ for

$$\overbrace{D_c \circ \ldots \circ D_c}^{k \text{ times}}(\pi).$$

(One may also replace the submodule condition by the corresponding quotient condition, see Proposition 2.1.) When $k = 0$, $D_c^0(\pi) = \pi$. We shall call $D_c(\pi)$ to be a *ρ-derivative* of π (depending on c).

(3) We shall denote the largest non-negative integer k such that $D_c^k(\pi) \neq 0$ by $\varepsilon_c(\pi)$. (We remark that our notation ε_c is motivated from the corresponding notation in the Kashiwara crystal operator theory, see e.g. [GV01] and [Kl10, Section 11].)

3.2. **Some more results on socle and cosocle.** For a representation π of finite length, we denote by $\mathrm{soc}(\pi)$ and $\mathrm{cosoc}(\pi)$ the socle and cosocle of π respectively. We need the following result later (see e.g. [LM16], c.f. Lemma 3.1) and we refer to [LM16] for a definition of a ladder representation. The particular example of ladder representations, which we shall use, is that $\mathrm{St}(\{\Delta_1, \Delta_2\})$ for two linked segments Δ_1, Δ_2.

Lemma 3.4. [LM16] *Let $\pi \in \mathrm{Irr}_\rho(G_n)$ be a ladder representation or a generic representation. Let $\tau_1 \in \mathrm{Irr}_\rho(G_k)$ and let $\tau_2 \in \mathrm{Irr}_\rho(G_k)$. Then*

- $\mathrm{soc}(\pi \times \tau_i)$ *and* $\mathrm{cosoc}(\pi \times \tau_i)$ *are irreducible* $(i = 1, 2)$*;*
- $\mathrm{soc}(\pi \times \tau_1) \cong \mathrm{soc}(\pi \times \tau_2)$ *if and only if* $\tau_1 \cong \tau_2$*;*
- $\mathrm{cosoc}(\pi \times \tau_1) \cong \mathrm{cosoc}(\pi \times \tau_2)$ *if and only if* $\tau_1 \cong \tau_2$*.*

Remark 3.5. We remark that, by Frobenius reciprocity, the second (resp. third) bullet implies that for any ladder or generic representation τ of G_r and any $\pi \in \mathrm{Irr}(G_n)$, there exists at most one $\omega \in \mathrm{Irr}(G_{n-r})$ such that there exists a surjection $\pi_{N_r} \twoheadrightarrow \omega \boxtimes \tau$ (resp. $\pi_{N_{n-r}} \to \tau \boxtimes \omega$). By Proposition 2.1, if such ω exists, one also has an injection $\omega \boxtimes \tau \hookrightarrow \pi_{N_r}$.

3.3. **Properties of ρ-derivatives.** For the following lemma, see [GV01, Lemma 3.5], [Ja07, Corollary 2.3.2], [Mi09, Corollaire 6.5.]:

Lemma 3.6. [GV01, Ja07, Mi09] *Let $\pi \in \mathrm{Irr}_\rho$. Let c be an integer. Let $\widetilde{\pi}$ be the unique submodule of $\pi \times (\nu^c\rho)^{\times k}$. Then*

(1) $\varepsilon_c(\widetilde{\pi}) = \varepsilon_c(\pi) + k$*;*
(2) $\widetilde{\pi}$ *appears with multiplicity one in* $\pi \times (\nu^c\rho)^{\times k}$*;*
(3) *for any irreducible composition factor τ of $(\pi \times (\nu^c\rho)^{\times k})$ which is not isomorphic to $\widetilde{\pi}$, $\varepsilon_c(\tau) < \varepsilon_c(\pi) + k$.*

The following result follows from an application on geometric lemma:

Lemma 3.7. [GV01, Ja07, Mi09] *Let $\pi \in \mathrm{Irr}_\rho$. Let c be an integer such that $D_c(\pi) \neq 0$. Let $k = \varepsilon_c(\pi)$. Then there is only one simple composition factor in $\pi_{N_{kn(\rho)}}$ of the form*

$$\tau \boxtimes (\nu^c\rho)^{\times k}$$

for some $\tau \in \mathrm{Irr}$. Moreover, such $\tau \cong D_c^k(\pi)$.

As a consequence, we have the following:

Corollary 3.8. *Let $\pi \in \mathrm{Irr}_\rho(G_n)$. Let $k = \varepsilon_c(\pi)$. Suppose $k > 0$. Let ω be an admissible $G_{n-kn(\rho)}$-representation such that*

$$\pi \hookrightarrow \omega \times (\nu^c\rho)^{\times k}$$

Then $D_c^k(\pi) \hookrightarrow \omega$.

Proof. By Frobenius reciprocity, we have a non-zero map:

$$\pi_{N_{kn(\rho)}} \to \omega \boxtimes (\nu^c\rho)^{\times k}.$$

By Lemma 3.7, the only composition factor of the form $\tau \boxtimes (\nu^c\rho)^{\times k}$ is $D_c^k(\pi) \boxtimes (\nu^c\rho)^{\times k}$ and hence is mapped to the submodule of $\omega \boxtimes (\nu^c\rho)^{\times k}$. It follows from Künneth formula (see [Ra07]) that $D_c^k(\pi)$ is a submodule of ω. □

3.4. **Highest derivatives using ρ-derivatives.** Recall that the highest derivative is defined in Section 1.1.

Proposition 3.9. *Let $\mathfrak{m} \in \mathrm{Mult}_\rho$. Let c (resp. d) be the smallest (resp. largest) integer such that $\nu^c\rho \cong b(\Delta)$ (resp. $\nu^c\rho \cong b(\Delta)$ for some $\Delta \in \mathfrak{m}$. For each $e = c, \ldots, d$, let*

$$k_e = |\,\{\Delta \in \mathfrak{m} : b(\Delta) \cong \nu^e\rho\}\,|.$$

Then $D_d^{k_d} \circ \ldots \circ D_{c+1}^{k_{c+1}} \circ D_c^{k_c}(\pi) \cong \pi^-$.

The above proposition can be computed directly by using e.g. [Mi09, Théorème 7.5]. An earlier form of ρ-derivatives is used by Mœglin-Waldspurger [MW86, Lemme II.11] in computing Zelevinsky duals. We refer the reader to [Mi09] for those explicit rules in computing ρ-derivatives and we shall not reproduce here. Indeed it is based on the following lemma:

Lemma 3.10. *Let $\mathfrak{m} \in \mathrm{Mult}_\rho$. Let $\pi = \langle \mathfrak{m} \rangle$. Let $\mathrm{mult}_b(\pi, c)$ be the number of segments Δ in $\mathfrak{m}$ such that $b(\Delta) \cong \nu^c\rho$. Suppose, for some $e \in \mathbb{Z}$ such that $\mathrm{mult}_b(\pi, e-1) = 0$. Then $\varepsilon_e(\pi) = \mathrm{mult}_b(\pi, e)$.*

Proof. This, for example, follows from a simple computation using [Mi09, Théorème 7.5]. $\square$

Proof of Proposition 3.9. Inductively, using Lemmas 3.10 and 3.6, we have that $\mathrm{mult}_b(D_e^{k_e} \circ \ldots \circ D_c^{k_c}(\pi), e) = 0$ and $D_e^{k_e} \circ \ldots \circ D_c^{k_c}(\pi) \neq 0$. Then, $D_d^{k_d} \circ \ldots \circ D_{c+1}^{k_{c+1}} \circ D_c^{k_c}(\pi)$ has the multisegment obtained by removing all the endpoints of the segments in $\mathfrak{m}$. Comparing with the description in [Ze80], we have the desired isomorphism. $\square$

3.5. **Left-right Bernstein-Zelevinsky derivatives.** Recall that the Bernstein-Zelevinsky derivative is defined in Section 1.1. We also define a left version (c.f. [CS21, Ch21, Ch23]). Note that one can use the transpose R_i^t of R_i to define the left derivative as in (1.1). One may then apply a conjugation on an antidiagonal element to obtain the following formulation:

$${}^{(i)}\pi = \delta_{\bar{R}_i}^{-1/2} \cdot \frac{\pi}{\langle x.v - \psi(x)v : x \in \bar{R}_i, v \in \pi \rangle},$$

where $\bar{R}_i = aR_i^t a^{-1}$. Here a is the matrix with 1 in the antidiagonal entries and 0 elsewhere.

Most results will only be stated and proved for the 'right' version, and the 'left' version can be formulated and proved similarly.

3.6. **Properties of Bernstein-Zelevinsky derivatives.** From the multiplicity-one theorem [AGRS10] (see [Ch21, Proposition 2.5], [CS21, Lemma 2.3]) and a self-dual property (see [CS21, Lemma 2.4]), we deduce that:

Lemma 3.11. [Ch21, Proposition 2.5] *Let $\pi \in \mathrm{Irr}(G_n)$. Then $\mathrm{soc}(\pi^{(i)})$ is multiplicity-free. The same holds for $\mathrm{soc}({}^{(i)}\pi)$, $\mathrm{cosoc}(\pi^{(i)})$ and $\mathrm{cosoc}({}^{(i)}\pi)$.*

Using the stronger multiplicity-one theorem in [Ch23], we have the following statement:

Lemma 3.12. *Let π be a standard representation of G_n. Then $\mathrm{cosoc}(\pi^{(i)})$ is multiplicity-free. The same holds for $\mathrm{cosoc}({}^{(i)}\pi)$.*

The proof is similar to [Ch21, Proposition 2.5] and so we only omit the details.

Lemma 3.13. [CS21, Lemma 2.4] *Let $\pi \in \mathrm{Irr}(G_n)$. Then, for any i with $\pi^{(i)} \neq 0$, $\mathrm{soc}(\pi^{(i)}) \cong \mathrm{cosoc}(\pi^{(i)})$.*

For a fixed $\rho \in \mathrm{Irr}^c$, write $k = n(\rho)$. For a segment $\Delta = [a, b]_\rho$, define

$$\Delta^{(ik)} = [a, b-i]_\rho, \quad {}^{(ik)}\Delta = [a+i, b]_\rho$$

for $0 \leq i \leq l_r(\Delta)$. We also define $\Delta^{(j)} = {}^{(j)}\Delta = \emptyset$ if k does not divide j.

Let $\mathfrak{m} \in \mathrm{Mult}_\rho$. For any i,

$$\mathfrak{m}^{(i)} = \left\{ \Delta_1^{(i_1)} + \ldots + \Delta_r^{(i_r)} : i_k = 0 \text{ or } n(\rho), \quad i_1 + \ldots + i_r = i \right\}.$$

The notion ${}^{(i)}\mathfrak{m}$ is defined similarly by using ${}^{(ik)}\Delta$.

Lemma 3.14. *Let $\pi \in \mathrm{Irr}(G_n)$. For any simple quotient or submodule τ of $\pi^{(i)}$ (resp. ${}^{(i)}\pi$), $\tau \cong \langle \mathfrak{n} \rangle$ for some $\mathfrak{n} \in \mathfrak{m}^{(i)}$ (resp. $\mathfrak{n} \in {}^{(i)}\mathfrak{m}$).*

Lemma 3.14 can be proved by embedding π to $\zeta(\mathfrak{m})$ and then applying geometric lemma [BZ77]. See, for example, [Ch21, Lemma 7.3] and [Ch22, Proposition 2.3].

3.7. **Notations for derivatives.** For a segment $\Delta = [a, b]_\rho$, write

$${}^-\Delta = [a+1, b]_\rho, \quad \Delta^- = [a, b-1]_\rho.$$

For a multisegment $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_k\}$ in Mult, write

$$\mathfrak{m}^- = \Delta_1^- + \ldots + \Delta_k^-, \quad {}^-\mathfrak{m} = {}^-\Delta_1 + \ldots + {}^-\Delta_k.$$

3.8. **Submodule of derivatives from Jacquet functor.** The author would like to thank G. Savin for a discussion on the following proposition.

Proposition 3.15. *Let π be a representation of G_n of finite length. Let τ be an irreducible submodule of $\pi^{(i)}$. Then there exists ρ_k of $\mathrm{Irr}^c(G_{n_k})$ ($k = 1, \ldots, r$) such that $\rho_i \not> \rho_j$ for any $i < j$ and*

$$\tau \boxtimes \rho_r \boxtimes \rho_{r-1} \boxtimes \ldots \boxtimes \rho_1 \hookrightarrow \pi_{N'},$$

where N' is the unipotent radical of the standard parabolic subgroup associated to $(n - n_1 - \ldots - n_r, n_1, \ldots, n_r)$.

Proof. Again we only consider π to be in Irr_ρ. The module τ determines a set of cuspidal representations $\rho_1, \ldots, \rho_r$ such that

$$\mathrm{csupp}(\tau) + \rho_1 + \ldots + \rho_r = \mathrm{csupp}(\pi).$$

We shall relabel $\rho_1, \ldots, \rho_r$ such that $\rho_i \not> \rho_j$ for $i < j$.

Using the Hecke algebra realization [CS19] of Bernstein-Zelevinsky derivatives (see Section 9), there is a submodule of $\pi_{N'}$ of the form

$$\tau \boxtimes \omega,$$

where ω contains a generic representation. Hence we have an embedding

$$\tau \boxtimes \omega \hookrightarrow \pi_{N_i}$$

as $G_{n-i} \times G_i$-modules. Since ω contains a generic representation, it is standard (see [Ch21, Proposition 2.3]) to obtain a non-zero map

$$\rho_1 \times \ldots \times \rho_r \to \omega.$$

Hence, $\omega_{N''}$ has as submodule $\rho_r \boxtimes \ldots \boxtimes \rho_1$, where N'' is the unipotent radical of the parabolic subgroup associated to the partition $(n_1, \ldots, n_r)$. Thus, by Jacquet functors in stages,

$$\tau \boxtimes \rho_r \boxtimes \ldots \boxtimes \rho_1 \hookrightarrow \pi_{N'}$$

as desired. $\square$

We also prove a kind of converse of the above statement.

Definition 3.16. (c.f. [Ze80, Theorem 6.1]) A sequence of segments $\Delta_1, \ldots, \Delta_k$ is said to be *ascending* or in an *ascending order* if for any $i < j$, $\Delta_j \not< \Delta_i$. This is opposite to the ordering which usually defines a standard representation, that means $\mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_k)$ is isomorphic to $\lambda(\{\Delta_1^\vee, \ldots, \Delta_k^\vee\})^\vee$. This also coincides with the one defined in Section 1.3 when all Δ_i are in Seg_ρ for a fixed ρ. Here, for writing $\Delta_i = [a_i, b_i]_\rho$, define $\Delta_i^\vee := [-b_i, -a_i]_{\rho^\vee}$.

Proposition 3.17. *Let $\pi \in \mathrm{Irr}$. Let $\Delta_1, \ldots, \Delta_k$ be an ascending sequence of segments. Let $n_1, \ldots, n_k$ be the absolute lengths of $\Delta_1, \ldots, \Delta_k$. Let N be the unipotent radical associated to the partition $(n - n_1 - \ldots - n_k, n_1, \ldots, n_k)$. Let $n' = n_1 + \ldots + n_k$ and let $N' = N_{n'}$. Then,*

(1) *For any $\tau \in \mathrm{Irr}(G_{n-n'})$,*
$$\dim \mathrm{Hom}_G(\tau \boxtimes \mathrm{St}(\Delta_k) \boxtimes \ldots \boxtimes \mathrm{St}(\Delta_1), \pi_N) \leq 1,$$
where $G = G_{n-n'} \times G_{n_1} \times \ldots \times G_{n_k}$.

(2) *For any $\tau \in \mathrm{Irr}(G_{n-n'})$,*
$$\dim \mathrm{Hom}_{G'}(\tau \boxtimes (\mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_k)), \pi_{N'}) \leq 1,$$
where $G' = G_{n-n'} \times G_{n'}$.

(3) *If the dimensions above are non-zero, then τ is a submodule of $\pi^{(n')}$.*

Proof. Note that (1) and (2) are equivalent by Frobenius reciprocity. (We remark that the ordering of segments in (1) and (2) is switched since we switch from the opposite unipotent subgroup to the usual one defined for $\times$.) We consider (2). Suppose
$$\dim \mathrm{Hom}_{G'}(\tau \boxtimes (\mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_k)), \pi_{N'}) \geq 2.$$

Claim: Let $f, f' \in \mathrm{Hom}_{G'}(\tau \boxtimes (\mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_k)), \pi_{N'})$ with $f \neq cf'$ for some scalar c. Then $\mathrm{im}\, f \neq \mathrm{im}\, f'$.

Proof of claim: For a non-zero $f \in \mathrm{Hom}_{G'}(\tau \boxtimes (\mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_k)), \pi_{N'})$, it suffices to show $\mathrm{End}_{G'}(\mathrm{im}\, f) \cong \mathbb{C}$.

To this end, we observe that $\mathrm{im}\, f \cong \tau \boxtimes \kappa$ for some quotient of $\mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_k)$. Then
$$\mathrm{End}_{G'}(\mathrm{im}\, f) = \mathrm{End}_{G'}(\tau \boxtimes \kappa) = \mathrm{End}_{G_{n-n'}}(\tau) \otimes \mathrm{End}_{G_{n'}}(\kappa) \cong \mathbb{C},$$
where the last equality follows from $\mathrm{End}_{G_{n-n'}}(\tau) \cong \mathbb{C}$ by Schur's lemma and $\mathrm{End}_{G_{n'}}(\kappa) \cong \mathbb{C}$ by that κ has unique simple quotient and other simple composition factors are not isomorphic to that.

By the claim, there exist quotients κ_1, κ_2 of $\mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_k)$ such that
$$\tau \boxtimes \kappa_1 \oplus \tau \boxtimes \kappa_2 \hookrightarrow \pi_{N'}.$$
Note that κ_1 and κ_2 have a generic representation as the unique quotient by [JS83] (also see [Ch21, Proposition 2.3]) and no other composition factor of κ_1 and κ_2 is generic. Hence, now taking the (exact) $(U_{n'}, \psi_{n'})$-twisted Jacquet functor, we obtain an embedding
$$\tau \oplus \tau \hookrightarrow \pi^{(n')}.$$
This contradicts Lemma 3.11. Hence, we have (2). Similar argument above will also give (3). $\square$

3.9. **A reduction to Irr_ρ case.** Let $\pi \in \mathrm{Irr}$. By [Ze80, Proposition 8.5], there exists $\rho_1, \ldots, \rho_r \in \mathrm{Irr}^c$ such that

- for $j \neq k$, $\rho_j \not\cong \nu^c \rho_k$ for any $c \in \mathbb{Z}$;
- $\pi \cong \pi_1 \times \ldots \times \pi_r$ for some $\pi_i \in \mathrm{Irr}_{\rho_i}$.

For any integer i, the geometric lemma gives that

$$\pi^{(i)} \cong \bigoplus_{i_1+\ldots+i_r=i} \pi_1^{(i_1)} \times \ldots \times \pi_r^{(i_r)}, \tag{3.7}$$

where the direct sum is guaranteed by Ext-vanishing from a comparison of cuspidal supports.

Proposition 3.18. *We use the above notations. Suppose τ is a simple quotient of $\pi^{(i)}$. Then there exist integers $i_1, \ldots, i_r$ with $i_1 + \ldots + i_r = i$ such that $\tau \cong \tau_1 \times \ldots \times \tau_r$ for some simple quotients τ_j of $\pi^{(i_j)}$.*

Proof. By (3.7), τ is a simple quotient of $\pi_1^{(i_1)} \times \ldots \times \pi_r^{(i_r)}$ for some integers $i_1 + \ldots + i_r = i$. Now, applying Frobenius reciprocity, we have a non-zero map from $\pi_1^{(i_1)} \boxtimes \ldots \boxtimes \pi_r^{(i_r)}$ to $\tau_{N^-_{i_1,\ldots,i_r}}$. Thus we have representations $\tau_j \in \mathrm{Irr}_{\rho_j}$ $(j = 1, \ldots, r)$ such that there is a non-zero map from $\tau_1 \times \ldots \times \tau_r$ onto τ. Since $\tau_1 \times \ldots \times \tau_r$ is irreducible (by [Ze80, Proposition 8.5]), we have $\tau \cong \tau_1 \times \ldots \times \tau_r$.

Now, we have:

$$\pi_1^{(i_1)} \times \ldots \times \pi_r^{(i_r)} \twoheadrightarrow \tau_1 \times \ldots \times \tau_r.$$

Then, Frobenius reciprocity with some slight cuspidal support arguments gives that:

$$\pi_1^{(i_1)} \boxtimes \ldots \boxtimes \pi_r^{(i_r)} \twoheadrightarrow \tau_1 \boxtimes \ldots \boxtimes \tau_r.$$

This implies that τ_j is a simple quotient of $\pi_j^{(i_j)}$ as desired. □

3.10. **Counting cuspidal representations from derivatives of an admissible ascending sequence of segments.**

Lemma 3.19. *Let $\mathfrak{m} \in \mathrm{Mult}_\rho$ and let $\pi = \langle \mathfrak{m} \rangle \in \mathrm{Irr}_\rho$. Let $\Delta_1, \ldots, \Delta_r$ in Seg_ρ be an ascending sequence of segments such that $D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi) \neq 0$. For an integer c, let $\mathrm{mult}_b(\pi, c)$ be the number of segments Δ in $\mathfrak{m}$ such that $b(\Delta) \cong \nu^c\rho$; and let x_c be the total number of segments Δ in $\{\Delta_1, \ldots, \Delta_r\}$ such that $\nu^c\rho \in \Delta$. Then, for any c,*

$$x_c \leq \mathrm{mult}_b(\pi, c).$$

Proof. Let $\tau = D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi)$. By Proposition 3.17, τ is a simple submodule of $\pi^{(i)}$, where $i = \sum_{k=1}^r l_a(\Delta_k)$. Now the lemma follows from Lemma 3.14 and a cuspidal support condition. □

4. On commutativity of derivatives

4.1. **ε_Δ-invariant.** Let Δ be a segment of absolute length m. Let $\pi \in \mathrm{Irr}(G_n)$ with $n \geq m$. Let τ be the maximal semisimple representation of G_{n-m} such that

$$\tau \boxtimes \mathrm{St}(\Delta) \hookrightarrow \pi_{N_m},$$

or equivalently, by Proposition 2.1, τ is the maximal semisimple representation of G_{n-m} such that

$$\pi_{N_m} \twoheadrightarrow \tau \boxtimes \mathrm{St}(\Delta).$$

It is known that τ is either irreducible or zero (see e.g. Lemma 3.4 or Lemma 3.5, also see [Ch22+]). We shall denote such τ by $D_\Delta(\pi)$ if it is non-zero. If such τ does not exist, we set $D_\Delta(\pi)=0$. Let

$$\varepsilon_\Delta(\pi)$$

be the maximal integer k such that

$$D^k_\Delta(\pi) := \overbrace{D_\Delta \circ \ldots \circ D_\Delta}^{k \text{ times}}(\pi) \neq 0.$$

When $\Delta=[a]_\rho$, ε_Δ coincides with ε_a defined in Section 3.2.

We have the following reformulation by using Frobenius reciprocity:

Lemma 4.1. *Let $\pi \in \mathrm{Irr}$ and let $\Delta \in \mathrm{Seg}$. Suppose $D_\Delta(\pi)\neq 0$. Then*

$$\pi \hookrightarrow D_\Delta(\pi) \times \mathrm{St}(\Delta).$$

Proof. By definition,

$$D_\Delta(\pi) \boxtimes \mathrm{St}(\Delta) \hookrightarrow \pi_{N_i},$$

where $i=l(\Delta)$, and so, by Proposition 2.1,

$$\pi_{N_i} \to D_\Delta(\pi)\boxtimes \mathrm{St}(\Delta).$$

Now the statement follows from Frobenius reciprocity. □

Indeed, it is known that $\varepsilon_\Delta(\pi)$ coincides with the maximal integer k' such that π is a submodule of

$$\tau' \times \overbrace{\mathrm{St}(\Delta)\times \ldots \times \mathrm{St}(\Delta)}^{k' \text{ times}},$$

for some irreducible representation τ' of $G_{n-k'l_a(\Delta)}$. To see this, we need the fact that $\tau' \times \mathrm{St}(\Delta)\times \ldots \times \mathrm{St}(\Delta)$ has a unique simple submodule (Lemma 3.4), where $\mathrm{St}(\Delta)$ appears for arbitrary times. The uniqueness implies that π is a submodule of

$$\omega_r \times \overbrace{\mathrm{St}(\Delta)\times \ldots \times \mathrm{St}(\Delta)}^{r \text{ times}},$$

where ω_r is the unique submodule of $\tau' \times \overbrace{\mathrm{St}(\Delta)\times \ldots \times \mathrm{St}(\Delta)}^{k'-r \text{ times}}$. Then, inductively, we obtain that $\omega_r \cong D^r_\Delta(\pi)$ and so $k'=k$.

4.2. **Maximal multisegments at a (left) point.** Recall that $\mathrm{Mult}^a_{\rho,c}$ for some integer c is defined in Section 2.5, whose elements will be referred as *multisegments at the point $\nu^c\rho$* (see Section 1.3). An ordering $\leq^a_c$ on $\mathrm{Mult}^a_{\rho,c}$ is also defined in Section 2.5.

For a multisegment $\mathfrak{m}=\{\Delta_1,\ldots,\Delta_r\}$ in $\mathrm{Mult}^a_{\rho,c}$, we define: for any $\pi \in \mathrm{Irr}_\rho(G_n)$,

$$D_{\mathfrak{m}}(\pi) := D_{\Delta_r}\circ \ldots \circ D_{\Delta_1}(\pi).$$

When $\mathfrak{m}$ is the empty set, $D_{\mathfrak{m}}$ is just the identity map. Since $\tau\times\mathrm{St}(\mathfrak{m})$ has unique submodule for any irreducible τ, the same argument as in Section 4.1 shows that $D_{\mathfrak{m}}$ is independent of the choice of ordering of segments.

We also adapt the convention that: $D_\emptyset(\pi)=\pi$.

Lemma 4.2. *Let $\pi\in\mathrm{Irr}_\rho$. Fix $c\in\mathbb{Z}$. There exists a unique $\leq^a_c$-maximal multisegment $\mathfrak{m}$ in $\mathrm{Mult}^a_{\rho,c}$ such that $D_{\mathfrak{m}}(\pi)\neq 0$.*

Proof. Let $\mathfrak{m}_1$ and $\mathfrak{m}_2$ be maximal multisegments at a left point $\nu^c\rho$. Let

$$\mathfrak{m}_1 = \{[c, b_1]_\rho, \ldots, [c, b_r]_\rho\}, \quad \mathfrak{m}_2 = \{[c, b_1']_\rho, \ldots, [c, b_s']\}$$

with $b_1 \geq \ldots \geq b_r$ and $b_1' \geq \ldots \geq b_s'$. Write $\Delta_i = [c, b_i]_\rho$ and $\Delta_i' = [c, b_i']$.

If $b_1 = b_1'$, then one can proceed inductively since $\mathfrak{m}_1 - [c, b_1]_\rho$ and $\mathfrak{m}_2 - [c, b_2]_\rho$ are still maximal for $D_{[c,b_1]_\rho}(\pi)$. If $b_1 > b_1'$, then one considers, by Lemma 4.1,

$$\pi \hookrightarrow D_{\mathfrak{m}_2}(\pi) \times \mathrm{St}(\mathfrak{m}_2).$$

Now one applies the functor $N_{(b_1-c+1)n(\rho)}$ and uses geometric lemma and Section 2.9 to see that the only possible layers of the form $\omega \boxtimes \mathrm{St}([c, b_1]_\rho)$ as a submodule gives the following possible embedding:

$$D_{[c,b_1]_\rho}(\pi) \hookrightarrow \omega' \times \mathrm{St}(\mathfrak{m}_2 - \{\Delta\}),$$

for some $\omega_2 \in \mathrm{Irr}$, where Δ is one of the segments in $\mathfrak{m}_2$. However, then $\mathfrak{m}' := \mathfrak{m}_2 - \{\Delta\} + \{[c, b_1]_\rho\}$ also satisfies $D_{\mathfrak{m}'}(\pi) \neq 0$. This gives a contradiction to the maximality of $\mathfrak{m}_2$. The case for $b_1' > b_1$ is similar. □

Definition 4.3. Let $\pi \in \mathrm{Irr}_\rho$. Let $c \in \mathbb{Z}$. Define $\mathfrak{mxpt}^a(\pi, c)$ to be the unique $\leq_c^a$-maximal multisegment at the point $\nu^c\rho$ such that $D_{\mathfrak{mxpt}^a(\pi,c)}(\pi) \neq 0$.

Proposition 4.4. *Let $\pi \in \mathrm{Irr}_\rho$ and let $\mathfrak{n} \in \mathrm{Mult}_{\rho,c}^a$. Let $c \in \mathbb{Z}$. Suppose $\mathfrak{n} \leq_c^a \mathfrak{mxpt}^a(\pi, c)$. Then*

$$D_{\mathfrak{n}}(\pi) \neq 0.$$

Proof. We have the embedding:

$$\pi \hookrightarrow D_{\mathfrak{mxpt}^a(\pi,c)}(\pi) \times \mathrm{St}(\mathfrak{mxpt}^a(\pi, c)).$$

For $\mathfrak{n} \leq_c^a \mathfrak{mx}(\pi, c)$, we have that:

$$\tau \boxtimes \mathrm{St}(\mathfrak{n}) \hookrightarrow \mathrm{St}(\mathfrak{mxpt}^a(\pi, c))_{N_i},$$

for some $\tau \in \mathrm{Irr}$. Here $i = l_a(\mathfrak{n})$. One may prove the last statement by discussions in Section 6 (or see [Ch21, Corollary 2.6]) and we omit the details. Thus we have a non-zero map:

$$D_{\mathfrak{mxpt}^a(\pi,c)}(\pi) \boxtimes \tau \boxtimes \mathrm{St}(\mathfrak{n}) \hookrightarrow \pi_N,$$

where $N = N_{n(\pi)-l_a(\mathfrak{mxpt}^a(\pi,c)), l_a(\mathfrak{mxpt}^a(\pi,c))-i, i}$. By Frobenius reciprocity, we have

$$\tau' \boxtimes \mathrm{St}(\mathfrak{n}) \hookrightarrow \pi_{N_i}$$

for some $\tau' \in \mathrm{Irr}$, as desired. □

4.3. **Maximal multisegment at a point and ε_Δ.** We introduced the notions of ε_Δ and $\mathfrak{mxpt}^a(\pi, c)$. The relation of the two notions is given as follows:

Proposition 4.5. *Let $\pi \in \mathrm{Irr}_\rho$. Fix $c \in \mathbb{Z}$. For $b \geq c$, let $\mathrm{mult}([c, b]_\rho, \pi)$ be the multiplicity of $[c, b]_\rho$ in $\mathfrak{mxpt}^a(\pi, c)$. Then*

1. *$\mathrm{mult}([c, b]_\rho, \pi) = (\varepsilon_{[c,b]_\rho}(\pi) - \varepsilon_{[c,b+1]_\rho}(\pi))$;*
2. *$\varepsilon_{[c,b]_\rho}(\pi) = \sum_{[c,b']_\rho : b' \geq b} \mathrm{mult}([c, b']_\rho)$, where $[c, b']_\rho$ runs for all the segments in $\mathfrak{mxpt}^a(\pi, c)$.*

Proof. By the definition of $\mathfrak{mxpt}^a(\pi, c)$, we have the inequality:

$$\varepsilon_{[c,b]_\rho}(\pi) \leq \sum_{b' \geq b} \mathrm{mult}([c, b']_\rho, \pi).$$

The opposite inequality follows from Proposition 4.4. The remaining assertion of the proposition then follows by solving $\mathrm{mult}([c, b']_\rho, \pi)$. □

4.4. **Maximal multisegment at a right point.**

Definition 4.6. Recall $\mathrm{Mult}^b_{\rho,c}$ is defined in Section 2.2. We shall call the multisegments in $\mathrm{Mult}^b_{\rho,c}$ are the *multisegments at the right point* $\nu^c\rho$.

Similar to the argument in Lemma 4.2, one can deduce that there is a unique maximal multisegment $\mathfrak{n}$ at the right point $\nu^c\rho$ such that $D_{\mathfrak{n}}(\pi) \neq 0$. We shall denote such multisegment by $\mathfrak{mxpt}^b(\pi, c)$.

Remark 4.7.
- We remark here that the analogous version of Proposition 4.4 for a multisegment at a right point and using $\leq^b_c$ instead of $\leq^a_c$ is not true in general. (c.f. Corollary 6.24 below)
- As we shall see later that, the two notions of multisegments at a point in Definitions 4.3 and 4.6 have quite different uses. The maximal multisegment at a left point is mainly used to define the highest derivative multisegment while the maximal multisegment at a right point is mainly used in some inductions for some proofs. (See one property in Proposition 4.9 below)

For convenience, we define the following notion. Let $c \in \mathbb{Z}$. An irreducible representation $\pi \in \mathrm{Irr}_\rho$ is said to be with the *maximal point* $\nu^c\rho$ if $\nu^c\rho \in \mathrm{csupp}(\pi)$ and for any integer $d > c$, $\nu^d\rho \notin \mathrm{csupp}(\pi)$.

Example 4.8. $\langle [1,2]_\rho \rangle$ and $\mathrm{St}([1,2]_\rho)$ have the maximal point $\nu^2\rho$.

Proposition 4.9. *(c.f.* [LM22, Proposition 7.3]*) Let $\pi \in \mathrm{Irr}_\rho(G_n)$. Let $c \in \mathbb{Z}$. Suppose $\mathfrak{mxpt}^b(\pi, c) = \emptyset$. For any $1 \leq i$, π_{N_i} does not have a simple composition factor of the form $\tau \boxtimes \omega$ for some $\omega \in \mathrm{Irr}_\rho$ with maximal point $\nu^c\rho$ and for some $\tau \in \mathrm{Irr}_\rho$.*

Proof. Suppose not. Then π_{N_i} also has a simple composition factor of the form $\tau \boxtimes \omega$ for some $\omega \in \mathrm{Irr}_\rho(G_i)$ having the maximal point $\nu^c\rho$ and for some $\tau \in \mathrm{Irr}_\rho(G_{n-i})$. Since ω has the maximal point $\nu^c\rho$, there exists $j > 0$ such that ω_{N_j} have an irreducible quotient isomorphic to $\tau' \boxtimes \mathrm{St}(\mathfrak{n})$ for some multisegment $\mathfrak{n}$ at the right point $\nu^c\rho$ and $\tau' \in \mathrm{Irr}(G_{i-j})$.

Hence, by taking Jacquet functors in stage, we have a surjection:

$$\pi_{N'} \to \tau \boxtimes \tau' \boxtimes \mathrm{St}(\mathfrak{n}),$$

where N' is the unipotent radical of the parabolic subgroup $P_{n-i,i-j,j}$. Now applying Frobenius reciprocity, we have a non-zero map:

$$\pi_{N_j} \to (\tau' \times \tau) \boxtimes \mathrm{St}(\mathfrak{n}).$$

Thus $D_{\mathfrak{n}}(\pi) \neq 0$, giving a contradiction. □

4.5. **First commutativity result.**

Lemma 4.10. *Let $\pi \in \mathrm{Irr}$. Let Δ_1 and Δ_2 be unlinked segments. Then*

$$D_{\Delta_1} \circ D_{\Delta_2}(\pi) \cong D_{\Delta_2} \circ D_{\Delta_1}(\pi).$$

Proof. One can apply geometric lemma to get the lemma. Alternatively, $\mathrm{St}(\Delta_1) \times \mathrm{St}(\Delta_2)$ is a generic representation and so $\mathrm{St}(\Delta_1) \times \mathrm{St}(\Delta_2) \cong \mathrm{St}(\Delta_2) \times \mathrm{St}(\Delta_1)$ [Ze80]. Moreover, one has

$$D_{\Delta_1} \circ D_{\Delta_2}(\pi)$$

is the unique submodule of $\mathrm{Hom}_{G_i}(\mathrm{St}(\Delta_1) \times \mathrm{St}(\Delta_2), \pi_{N_l})$, where $l = l_a(\Delta_1) + l_a(\Delta_2)$, which is regarded as a $G_{n(\pi)-l}$-module. (See [Ch22+] for more discussions on big derivatives.) Then the lemma follows from Lemma 3.4 (also see Remark 3.5). □

Lemma 4.11. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\Delta_1 = [a_1, b_1]_\rho$ and $\Delta_2 = [a_2, b_2]_\rho$ be linked with $\Delta_1 < \Delta_2$. Suppose $D_{\Delta_1}(\pi) \neq 0$ and $D_{\Delta_2}(\pi) \neq 0$. Then*

$$D_{\Delta_1} \circ D_{\Delta_2}(\pi) \neq 0, \quad D_{\Delta_2} \circ D_{\Delta_1}(\pi) \neq 0.$$

Proof. Let $\omega_1 = D_{\Delta_1}(\pi)$ and let $\omega_2 = D_{\Delta_2}(\pi)$. Let $l = n(\rho)$ and let $k = b_1 - a_1 + 1$. Then

$$\pi \hookrightarrow \omega_1 \times \mathrm{St}(\Delta_1), \quad \pi \hookrightarrow \omega_2 \times \mathrm{St}(\Delta_2).$$

Checking the first one is easier by geometric lemma and cuspidal support. Some similar computation appears for the second one and so we omit the details for the first one.

Now we prove the second one. One applies geometric lemma on

$$(\omega_2 \times \mathrm{St}(\Delta_2))_{N_{kl}},$$

to reduce the possibility contributing to a factor of the form $\tau \boxtimes \mathrm{St}(\Delta_1)$ to those of the form

$$(*) \quad \tau' \times \mathrm{St}([b_1+1, b_2]_\rho) \boxtimes \mathrm{St}([a_1, a_2-1]_\rho) \times \mathrm{St}([c+1, b_1]_\rho) \times \mathrm{St}([a_2, c]_\rho),$$

for some c, or simply

$$(**) \quad \tau'' \times \mathrm{St}([a_2, b_2]_\rho) \boxtimes \mathrm{St}([a_1, b_1]_\rho).$$

Here one of the irreducible factor in (*) has to take the form $\mathrm{St}([a_1, a_2-1]_\rho) \times \mathrm{St}([c+1, b_1]_\rho)$ by picking the unique generic representation with given cuspidal support, and another irreducible factor $\mathrm{St}([a_2, c]_\rho)$ comes from the Jacquet module of $\mathrm{St}(\Delta_2)_{N_{(c-a_2+1)l}}$ by Section 2.9.

Here τ' is an irreducible representation such that $\tau' \boxtimes \mathrm{St}([a_1, a_2-1]_\rho)$ is a composition factor of $(\omega_2)_{N_{(a_2-a_1+1)l}}$; and τ'' is an irreducible representation such that $\tau'' \boxtimes \mathrm{St}([a_1, b_1]_\rho)$. However, for Case (*), by Frobenius reciprocity, the $\mathrm{St}([a_1, b_1]_\rho)$ does not appear in the submodule of $\mathrm{St}([a_1, a_2-1]_\rho) \times \mathrm{St}([a_2, b_1]_\rho)$. Thus only (**) can contribute to a submodule of the form $\kappa \boxtimes \mathrm{St}(\Delta_1)$ in $\pi_{N_{(b_1-a_1+1)l}}$ (which we know such submodule exists by $D_{\Delta_1}(\pi) \neq 0$). Thus the socle of $(\omega_2)_{N_{(b_1-a_1+l)l}}$ has a factor of the form $\tau'' \boxtimes \mathrm{St}(\Delta_1)$. This shows $D_{\Delta_2} \circ D_{\Delta_1}(\pi) \neq 0$. □

Proposition 4.12. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\Delta_1 = [a_1, b_1]_\rho$ and $\Delta_2 = [a_2, b_2]_\rho$ be linked with $\Delta_1 < \Delta_2$. Suppose $D_{\Delta_1}(\pi) \neq 0$ and $D_{\Delta_2}(\pi) \neq 0$. Suppose*

$$D_{\Delta_2} \circ D_{\Delta_1}(\pi) \not\cong D_{\Delta_1 \cap \Delta_2} \circ D_{\Delta_1 \cup \Delta_2}(\pi),$$

and

$$D_{\Delta_1} \circ D_{\Delta_2}(\pi) \not\cong D_{\Delta_1 \cap \Delta_2} \circ D_{\Delta_1 \cup \Delta_2}(\pi).$$

Then $D_{\Delta_2} \circ D_{\Delta_1}(\pi) \cong D_{\Delta_1} \circ D_{\Delta_2}(\pi)$.

Proof. By Frobenius reciprocity,

$$\pi \hookrightarrow D_{\Delta_2} \circ D_{\Delta_1}(\pi) \times \mathrm{St}(\Delta_2) \times \mathrm{St}(\Delta_1).$$

There are two composition factors in $\mathrm{St}(\Delta_2) \times \mathrm{St}(\Delta_1)$ [Ze80]: $\mathrm{St}(\Delta_1 + \Delta_2)$ and $\mathrm{St}(\Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2)$. Thus we have either:

$$\pi \hookrightarrow D_{\Delta_2} \circ D_{\Delta_1}(\pi) \times \mathrm{St}(\Delta_1 \cap \Delta_2 + \Delta_1 \cup \Delta_2),$$

or

$$\pi \hookrightarrow D_{\Delta_2} \circ D_{\Delta_1}(\pi) \times \mathrm{St}(\Delta_1 + \Delta_2).$$

In the former case, Lemma 3.4 gives that $D_{\Delta_1 \cap \Delta_2} \circ D_{\Delta_1 \cup \Delta_2}(\pi) \cong D_{\Delta_2} \circ D_{\Delta_1}(\pi)$, giving a contradiction. Thus we must be in the latter case.

We similarly have that

$$\pi \hookrightarrow D_{\Delta_1} \circ D_{\Delta_2}(\pi) \times \mathrm{St}(\Delta_1) \times \mathrm{St}(\Delta_2).$$

With a similar argument as above, we have that

$$\pi \hookrightarrow D_{\Delta_1} \circ D_{\Delta_2}(\pi) \times \mathrm{St}(\Delta_1 + \Delta_2).$$

Now the ladder representation case of Lemma 3.4 gives that

$$D_{\Delta_2} \circ D_{\Delta_1}(\pi) \cong D_{\Delta_1} \circ D_{\Delta_2}(\pi).$$

□

Remark 4.13. As shown in the proof of Proposition 4.12, $D_{\Delta_1} \circ D_{\Delta_2}(\pi)$ is the irreducible submodule of $\pi \times \mathrm{St}(\Delta_1 + \Delta_2)$ or irreducible submodule of

$$\pi \times \mathrm{St}(\Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2).$$

We shall need the following in Section 7.2. We also remark that dropping the condition that $D_{\Delta_1}(\pi) \neq 0$ or $D_{\Delta_2}(\pi) \neq 0$ below will make the statement fail in general (e.g. considering some derivatives on a Speh representation).

Lemma 4.14. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\Delta_1 = [a_1, b_1]_\rho$ and $\Delta_2 = [a_2, b_2]_\rho$ be linked with $\Delta_1 < \Delta_2$. Suppose $D_{\Delta_1}(\pi) \neq 0$ and $D_{\Delta_2}(\pi) \neq 0$. Let $\Delta = \Delta_1 \cup \Delta_2$. If $D_\Delta(\pi) = 0$, then*

$$D_{\Delta_1} \circ D_{\Delta_2}(\pi) \cong D_{\Delta_2} \circ D_{\Delta_1}(\pi).$$

Proof. This is a special case of Proposition 4.12. □

4.6. **Commutations in another form.** As mentioned before, Lemma 4.14 requires the assumption that $D_{\Delta_1}(\pi) \neq 0$ and $D_{\Delta_2}(\pi) \neq 0$. It is not convenient for the purpose of some applications. We now prove another version of commutativity, and one may compare with the proof of Lemma 4.14.

Lemma 4.15. *Let $\pi \in \mathrm{Irr}_\rho$. Let $c \in \mathbb{Z}$. Let $\tau = D_{\mathfrak{mxpt}^a(\pi,c)}(\pi)$. Let $d > c$ be an integer. Let $\mathfrak{n} \in \mathrm{Mult}^a_{\rho,d}$. If $D_\mathfrak{n}(\tau) \neq 0$, then $D_\mathfrak{n}(\pi) \neq 0$.*

Proof. Let $\tau' = D_\mathfrak{n}(\tau)$. By Lemma 4.1, we have embeddings:

$$\pi \hookrightarrow \tau \times \mathrm{St}(\mathfrak{mxpt}^a(\pi, c)), \quad \tau \hookrightarrow \tau' \times \mathrm{St}(\mathfrak{n}).$$

Hence,

$$\pi \hookrightarrow \tau' \times \mathrm{St}(\mathfrak{n}) \times \mathrm{St}(\mathfrak{mxpt}^a(\pi, c)).$$

The lemma will follow from the following claim. The main idea in proving the lemma is to switch a pair of segments respectively in $\mathfrak{n}$ and $\mathfrak{mxpt}^a(\pi, c)$ each time by using the maximality of $\mathfrak{mxpt}^a(\pi, c)$.

Claim: $\pi \hookrightarrow \tau' \times \mathrm{St}(\mathfrak{mxpt}^a(\pi, c)) \times \mathrm{St}(\mathfrak{n})$.
Proof of Claim: We shall write the segments in $\mathfrak{mxpt}^a(\pi, c)$ as:

$$\Delta_k \leq^a_c \ldots \leq^a_c \Delta_1, \tag{4.8}$$

and write the segments in $\mathfrak{n}$ as:

$$\overline{\Delta}_1 \leq^a_d \ldots \leq^a_d \overline{\Delta}_l.$$

(Note that the order is opposite to (4.8).)

We shall inductively show that:

$$\pi \hookrightarrow \tau' \times A_{ij} \times \mathrm{St}(\overline{\Delta}_j) \times B_{ij},$$

where

$$A_{ij} = (\mathrm{St}(\overline{\Delta}_l) \times \ldots \times \mathrm{St}(\overline{\Delta}_{j+1})) \times (\mathrm{St}(\Delta_k) \times \ldots \times \mathrm{St}(\Delta_i))$$

and

$$B_{ij} = \mathrm{St}(\Delta_{i-1}) \times \ldots \times \mathrm{St}(\Delta_1) \times \mathrm{St}(\overline{\Delta}_{j-1}) \times \ldots \times \mathrm{St}(\overline{\Delta}_1).$$

The basic case has been given before the claim. Suppose the case is proved for $i=i^*$ and $j=j^*$. To prove the case that $i=i^*-1$ and $j=j^*$ (if $i^*=1$, then we proceed $i=k+1$ and $j=j^*+1$ and the argument is similar), we consider two cases:

(1) $\overline{\Delta}_{j^*} \subset \Delta_{i^*-1}$. Then it follows from (i^*,j^*) case and the fact that $\mathrm{St}(\overline{\Delta}_{j^*}) \times \mathrm{St}(\Delta_{i^*-1}) \cong \mathrm{St}(\Delta_{i^*-1}) \times \mathrm{St}(\overline{\Delta}_{j^*})$.

(2) $\overline{\Delta}_{j^*} \not\subset \Delta_{i^*-1}$. Then there are two composition factors in $\mathrm{St}(\overline{\Delta}_{j^*}) \times \mathrm{St}(\Delta_{i^*-1})$, which are

$$R = \mathrm{St}(\overline{\Delta}_{j^*} \cup \Delta_{i^*-1}) \times \mathrm{St}(\overline{\Delta}_{j^*} \cap \Delta_{i^*-1}),$$

and a non-generic factor denoted by S.

Now, by induction hypothesis,

$$\pi \hookrightarrow \tau' \times A_{i^*,j^*} \times \mathrm{St}(\overline{\Delta}_{j^*}) \times \mathrm{St}(\Delta_{i^*-1}) \times B_{i^*-1,j^*}$$

and so the above discussion implies that

$$(\bullet) \quad \pi \hookrightarrow \tau' \times A_{i^*,j^*} \times R \times B_{i^*-1,j^*},$$

or

$$(*) \quad \pi \hookrightarrow \tau' \times A_{i^*,j^*} \times S \times B_{i^*-1,j^*}.$$

We first prove the former case is impossible. Suppose the former case happens. We write $\underline{\Delta} = \overline{\Delta}_{j^*} \cup \Delta_{i^*-1}$. We choose all the segments $\Delta_1, \ldots, \Delta_p$ in $\mathfrak{mxpt}^a(\pi, c)$ such that those $\underline{\Delta} \subset \Delta_x$ $(x=1,\ldots,p)$. Then, by the ordering above, we also have

$$\overline{\Delta}_{j^*-1} \subset \ldots \subset \overline{\Delta}_1 \subset \underline{\Delta} \subset \Delta_p \subset \ldots \subset \Delta_1.$$

We also further have that $\underline{\Delta}$ is unlinked to Δ_y for any y. Thus, using (2.4) several times, we have that:

$$A_{i^*,j^*} \times R \times B_{i^*-1,j^*} \cong A_{i^*,j^*} \times \widetilde{R} \times \mathrm{St}(\underline{\Delta}) \times \mathrm{St}(\Delta_p) \times \ldots \times \mathrm{St}(\Delta_1),$$

where $\widetilde{R}$ is the product of those $\mathrm{St}(\Delta')$s for the remaining segments.

This implies that, by Frobenius reciprocity,

$$\{\underline{\Delta}, \Delta_1, \ldots, \Delta_p\} \leq^a_c \mathfrak{mxpt}^a(\pi, c),$$

but this gives a contradiction to the uniqueness of the maximality in Lemma 4.2.

Thus we must lie in the (*) case. Now combining

$$S \hookrightarrow \mathrm{St}(\Delta_{i^*-1}) \times \mathrm{St}(\overline{\Delta}_{j^*})$$

with (*), we obtain the case that $i=i^*-1$ and $j=j^*$, as desired.

□

5. Highest derivative multisegments

In this section, we construct the highest derivative of an irreducible representation by a sequence of St-derivatives. One may compare with the construction using ρ-derivatives in Proposition 3.9, and the two situations give two extreme cases: the minimal (shown in [Ch22+d]) and maximal one (Proposition 3.9) under $\leq_Z$.

5.1. **A computation on maximal multisegments at a point.**

Lemma 5.1. *Let $\mathfrak{m} \in \mathrm{Mult}_\rho$ and let $\pi = \langle \mathfrak{m} \rangle$. Let $c \in \mathbb{Z}$ with $\varepsilon_c(\pi) \neq 0$. Then, for any $d > c$,*

$$\mathfrak{mxpt}^a(D_{\mathfrak{mxpt}^a(\pi,c)}(\pi), d) = \mathfrak{mxpt}^a(\pi, d).$$

Proof. We abbreviate $\mathfrak{h}_c$ for $\mathfrak{mxpt}^a(\pi, c)$ and $\mathfrak{h}_d$ for $\mathfrak{mxpt}^a(\pi, d)$. We have that:

$$\pi \hookrightarrow D_{\mathfrak{h}_c}(\pi) \times \mathrm{St}(\mathfrak{h}_c).$$

This implies that

$$D_{\mathfrak{h}_d}(\pi) \boxtimes \mathrm{St}(\mathfrak{h}_d) \hookrightarrow (D_{\mathfrak{h}_c}(\pi) \times \mathrm{St}(\mathfrak{h}_c))_{N_{n_2}},$$

where $n_2 = l_a(\mathfrak{h}_d)$.

By the geometric lemma and comparing cuspidal support at $\nu^c\rho$ (and using (2.6)), $D_{\mathfrak{h}_d}(\pi) \boxtimes \mathrm{St}(\mathfrak{h}_d)$ can only come from the layer

$$\mathrm{Ind}_P^{G_n}(D_{\mathfrak{h}_c}(\pi)_{N_{n_2}} \boxtimes \mathrm{St}(\mathfrak{h}_c))^\phi,$$

where $P = P_{n-n_1-n_2,n_1,n_2}$ for $n_1 = l_a(\mathfrak{m}_c)$. This implies that $D_{\mathfrak{h}_d} \circ D_{\mathfrak{h}_c}(\pi) \neq 0$. Hence, $\mathfrak{h}_d \leq_d^a \mathfrak{n}$.

Now the opposite inequality follows from Lemmas 4.15 and 4.2, and hence we are done. □

5.2. **Highest derivatives by St-derivatives.**

Recall that in Proposition 3.17, we have shown that an ascending sequence of segments can be used to construct a simple quotient of BZ derivatives. On the other hand, the highest derivative of an irreducible representation is known to be irreducible. Thus, the strategy of a proof of the following result is to show an ascending sequence of segments has sum of absolute length of segments equal to the level of the representation.

Theorem 5.2. *Let $\mathfrak{m} \in \mathrm{Mult}_\rho$. Let $\pi = \langle \mathfrak{m} \rangle$ in Irr_ρ. We choose the smallest integer c such that $\nu^c\rho \cong b(\Delta)$ for some Δ in $\mathfrak{m}$ and choose the largest integer d such that $\nu^d\rho \cong b(\Delta)$ for some Δ in $\mathfrak{m}$. Then*

$$D_{\mathfrak{mxpt}^a(\pi,d)} \circ \ldots \circ D_{\mathfrak{mxpt}^a(\pi,c)}(\pi) \cong \pi^-.$$

Remark 5.3. In the statement of Theorem 5.2 above, $\mathfrak{mxpt}^a(\pi, d)$ can be an empty set and we only pick a certain range to guarantee the sequence of derivatives gives the highest derivative of π.

Proof. For simplicity, let $\mathfrak{h}_e = \mathfrak{mxpt}^a(\pi, e)$.

Step 1: *Claim:* The following two conditions:

- $D_{\mathfrak{h}_d} \circ \ldots \circ D_{\mathfrak{h}_c}(\pi) \neq 0$; and
- $\mathfrak{h}_d + \ldots + \mathfrak{h}_c$ is a multisegment whose sum of absolute lengths of all segments is equal to the level of π;

imply the theorem.

Proof of the claim: Let

$$\tau = D_{\mathfrak{h}_d} \circ \ldots \circ D_{\mathfrak{h}_c}(\pi).$$

The first bullet and Lemma 4.1 give that

$$\pi \hookrightarrow \tau \times \mathrm{St}(\mathfrak{h}_d) \times \ldots \times \mathrm{St}(\mathfrak{h}_c),$$

and so, by Frobenius reciprocity again,

$$\pi_{N_k} \hookrightarrow \tau \boxtimes \mathrm{St}(\mathfrak{h}_d) \times \ldots \times \mathrm{St}(\mathfrak{h}_c),$$

where k is the sum of the absolute lengths of all multisegments $\mathfrak{h}_d, \ldots, \mathfrak{h}_c$. By Proposition 3.17, τ is a submodule of $\pi^{(k)}$. By the second bullet, $\pi^{(k)}$ is the highest derivative and so it is irreducible. Thus $\tau \cong \pi^{(k)}$.

Note that the first bullet follows from repeatedly using Lemma 5.1. It remains to prove the second bullet in the claim.

Step 2: We now prove the second bullet. Let x_e be the total number of segments in $\mathfrak{h}_c + \ldots + \mathfrak{h}_{e-1}$ containing $\nu^e\rho$. Let $\mathrm{mult}_b(\pi, e)$ be the number of segments in $\mathfrak{m}$ such that $b(\Delta) \cong \nu^e\rho$ i.e. $\mathrm{mult}_b(\pi, e) = |\mathfrak{m}\langle e\rangle|$. We shall show by induction on e that

$$(*) \quad x_e + \text{number of segments in } \mathfrak{mxpt}^a(\pi, e) = \mathrm{mult}_b(\pi, e)\ .$$

The second bullet will then follow from (*).

When $e = c$, one can compute quite directly by Lemma 3.10 (also see [MW86, Ja07, Mi09]). Now again let

$$\tau := D_{\mathfrak{h}_{e-1}} \circ \ldots \circ D_{\mathfrak{h}_c}(\pi).$$

As argued in Step 1, we have that $\tau \hookrightarrow \pi^{(j)}$, where j is the sum of the lengths of all segments in $\mathfrak{h}_c + \ldots + \mathfrak{h}_{e-1}$. We also have, by the induction hypothesis (and a cuspidal support calculation using Lemma 3.14), that $\mathrm{mult}_b(\pi, e-1)$ (i.e. all) segments in $\mathfrak{m}$ with $b(\Delta) \cong \nu^{e-1}\rho$ are truncated to produce the segments in $\mathfrak{m}(\tau) \in \mathfrak{m}^{(j)}$ (see Section 3.6). In other words, $\mathrm{mult}_b(\tau, e-1) = 0$. By Lemma 3.10 (and Lemma 4.2), this implies that $\mathfrak{h}_e$ contains at least $\mathrm{mult}_b(\tau, e)$ number of segments since $\{ \overbrace{\nu^e\rho, \ldots, \nu^e\rho}^{\mathrm{mult}_b(\tau,e) \text{ times}} \} \leq^a_e \mathfrak{h}_e$ by Lemma 5.1, but indeed contains exactly $\mathrm{mult}_b(\tau, e)$-number by Lemma 3.19.

By Lemma 3.14 and a cuspidal support consideration again, the multisegment associated to $D_{\mathfrak{h}_e}(\tau)$ does not have any segment Δ satisfying $b(\Delta) \cong \nu^e\rho$. Again 3.14 and a cuspidal support consideration on

$$D_{\mathfrak{h}_e} \circ \ldots \circ D_{\mathfrak{h}_c}(\pi) = D_{\mathfrak{h}_e}(\tau),$$

we have:

$$x_e + \text{number of segments in } \mathfrak{h}_e = \mathrm{mult}_b(\pi, e).$$

This gives (*). □

6. Operations of St-derivatives on $\mathfrak{hd}$

6.1. **Highest derivative multisegment $\mathfrak{hd}$.** For $\pi \in \mathrm{Irr}_\rho$, recall that for $c \in \mathbb{Z}$, $\mathfrak{mxpt}^a(\pi, c)$ is the maximal multisegment $\mathfrak{m}$ at $\nu^c\rho$ such that $D_{\mathfrak{m}}(\pi) \neq 0$ (Definition 4.3). Define $\mathfrak{hd}(\pi)$ to be the multisegment

$$\mathfrak{hd}(\pi) = \sum_{c \in \mathbb{Z}} \mathfrak{mxpt}^a(\pi, c),$$

which is called the *highest derivative multisegment* for π. Note that there are finitely many c such that $\mathfrak{mxpt}^a(\pi, c) \neq \emptyset$.

By definitions, we have that

$$\mathfrak{hd}(\pi)[c] = \mathfrak{mxpt}^a(\pi, c).$$

However, $\mathfrak{hd}(\pi)\langle c\rangle$ is not necessarily equal to $\mathfrak{mxpt}^b(\pi, c)$ (see Corollary 6.23 for a precise description).

6.2. **A combinatorial removal process.**

Definition 6.1. Given a multisegment $\mathfrak{h} \in \mathrm{Mult}_\rho$, a segment $\Delta = [a,b]_\rho$ is said to be *admissible* to $\mathfrak{h}$ if there exists a segment in $\mathfrak{h}$ of the form $[a,c]_\rho$ for some $c \geq b$.

Remark 6.2. Suppose $\mathfrak{h} = \mathfrak{hd}(\pi)$ for some $\pi \in \mathrm{Irr}$. Then Δ is admissible to $\mathfrak{h}$ if and only if $D_\Delta(\pi) \neq 0$. This explains the above terminology.

Definition 6.3. Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\Delta = [a,b]_\rho$ be a segment admissible to $\mathfrak{h}$. The *removal process* on $\mathfrak{h}$ by Δ is to obtain a new multisegment $\mathfrak{r}(\Delta, \mathfrak{h})$ given by the following steps:

(1) Choose a segment Δ_1 in $\mathfrak{h}$ which has shortest relative length among all segments of the form $[a,b']_\rho$ for some $b' \geq b$. (In particular, $\nu^b\rho \in \Delta_1$.)
(2) (Minimality condition and nesting condition) For $i \geq 2$, choose recursively segments $\Delta_i = [a_i, b_i]_\rho$ such that $[a_i, b_i]_\rho$ is the $\prec^L$-minimal segment (see Section 2.3) in $\mathfrak{h}$ satisfying $a_{i-1} < a_i$ and $b_{i-1} > b_i$. This step terminates when no further such segment can be found. Let Δ_r be the last segment in the process.
(3) Obtain new segments $\Delta_1^{tr}, \ldots, \Delta_r^{tr}$ defined as:
   - for $1 \leq i \leq r-1$, $\Delta_i^{tr} = [a_{i+1}, b_i]_\rho$;
   - $\Delta_r^{tr} = [b+1, b_r]_\rho$ (possibly an empty set).
(4) The new multisegment $\mathfrak{r}(\Delta, \mathfrak{h})$ is defined as:

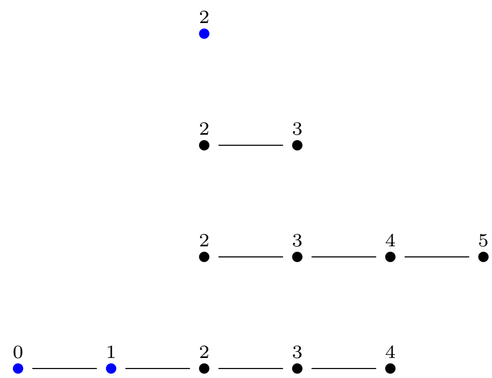
$$\mathfrak{r}(\Delta, \mathfrak{h}) = \mathfrak{h} - \sum_{i=1}^{r} \Delta_i + \sum_{i=1}^{r} \Delta_i^{tr}.$$

Remark 6.4. (a) Δ_1 in Step (1) above is guaranteed to exist by the assumption that Δ_1 is admissible to $\mathfrak{h}$.
(b) In (2), the condition that $b(\Delta_i) \cong b(\Delta_{i+1})$ indeed can be dropped, and the resulting multisegment defined in such way is the same as the way defined in Definition 6.3. Imposing such condition is more convenient for the proofs.
(c) If a segment Δ is not admissible to a multisegment $\mathfrak{h}$, we simply set $\mathfrak{r}(\Delta, \mathfrak{h}) = \infty$. Moreover, we also set $\mathfrak{r}(\Delta, \infty) = \infty$.

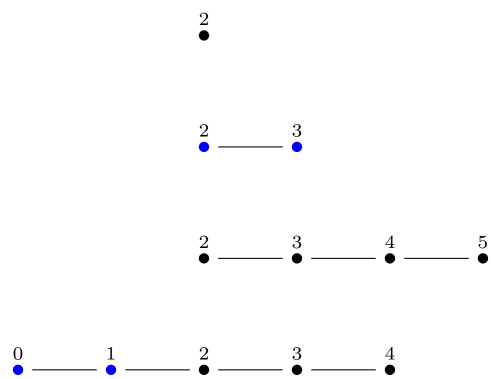
Definition 6.5. (1) In the above notation, we shall call that $\Delta_1, \ldots, \Delta_r$ form a *removal sequence* for $(\Delta, \mathfrak{h})$. The *nesting condition* refers to the condition that $\Delta_{i+1} \subsetneq \Delta_i$ for any i. The minimality condition refers to the minimal choice of Δ_i in Step (2). We shall call $\Delta_1^{tr}, \ldots, \Delta_r^{tr}$ to be the truncations of the removal sequence for $(\Delta, \mathfrak{h})$.
(2) Define $\Upsilon(\Delta, \mathfrak{h}) = \Delta_1$, the first segment in the removal sequence above.

Example 6.6. Let $\mathfrak{h} = \{[0,4],[2,5],[2,3],[2]\}$. (The blue points in the graph represent those 'removed' to give $\mathfrak{r}(\Delta, \mathfrak{h})$.)

(1) $\mathfrak{r}([0,2],\mathfrak{h}) = \{[2,4],[2,5],[2,3]\}$;

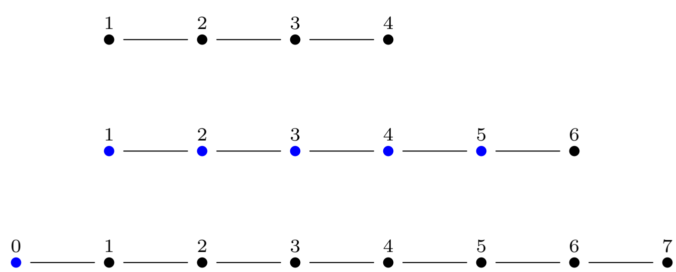


(2) $\mathfrak{r}([0,3]_\rho,\mathfrak{h}) = \{[2,4]_\rho,[2,5]_\rho,[2]_\rho\}$;



(3) $\mathfrak{r}([0,5]_\rho,\mathfrak{h})$ is not defined since $[0,5]_\rho$ is not admissible to $\mathfrak{h}$.

Example 6.7. (1) Let $\mathfrak{h} = \{[0,7],[1,4],[1,6]\}$. Let $\Delta = [0,5]$ and let $\Delta' = [1,4]$. The removal sequence for $(\Delta,\mathfrak{h})$ is $[0,7],[1,6]$. The removal sequence for $(\Delta',\mathfrak{h})$ is $[1,4]$.



The blue points represent the points removed for $\mathfrak{r}(\Delta,\mathfrak{h})$ and hence give the corresponding removal sequence for $(\Delta,\mathfrak{h})$.

(2) Let $\mathfrak{h} = \{[0,7],[1,5],[1,6]\}$. Let $\Delta = [0,5]$ and let $\Delta' = [1,4]$. The removal sequence for $(\Delta,\mathfrak{h})$ is $[0,7],[1,5]$. The removal sequence for $(\Delta',\mathfrak{h})$ is $[1,5]$.

(3) Let $\mathfrak{h} = \{[0,7],[1,5],[1,8]\}$. Let $\Delta = [0,5]$ and let $\Delta' = [1,4]$. The removal sequence for $(\Delta,\mathfrak{h})$ is $[0,7],[1,5]$, and the removal sequence for $(\Delta',\mathfrak{h})$ is $[1,5]$.

6.3. **Properties of removal process.** A simple but useful computation is the following:

Lemma 6.8. *(Removal of a cuspidal point at one time) Let* $\mathfrak{h} \in \mathrm{Mult}_\rho$*. Let* Δ *be a non-empty segment admissible to* $\mathfrak{h}$*. Let*

$$\mathfrak{h}^* = \mathfrak{h} - \{\Upsilon(\Delta, \mathfrak{h})\} + \{{}^-\Upsilon(\Delta, \mathfrak{h})\}.$$

Then

$$\mathfrak{r}(\Delta, \mathfrak{h}) = \mathfrak{r}({}^-\Delta, \mathfrak{h}^*).$$

(As a convention, $\mathfrak{r}(\emptyset, \mathfrak{h}^*) = \mathfrak{h}^*$*.)*

Proof. Let $\widetilde{\Delta}_1 = \Upsilon(\Delta, \mathfrak{h})$. Write $\Delta = [a, b]_\rho$. Suppose $\mathfrak{h}[a+1]$ has no segment $\overline{\Delta}$ satisfying $b(\Delta) \leq b(\overline{\Delta})$. Then $\Upsilon({}^-\Delta, \mathfrak{h}^*) = {}^-\widetilde{\Delta}_1$. Since the only difference between $\mathfrak{h}$ and $\mathfrak{h}^*$ is on that one segment, one checks that the remaining segments in the sequences are picked in the same way by the minimality and nesting condition. This gives $\mathfrak{r}({}^-\Delta, \mathfrak{h}^*) = \mathfrak{r}(\Delta, \mathfrak{h})$.

Suppose $\mathfrak{h}[a+1]$ has some segments $\overline{\Delta}$ satisfying $b(\Delta) \leq b(\overline{\Delta})$. Let $\overline{\Delta}^*$ be the shortest such segment. We further have two cases:

(1) Suppose $\overline{\Delta}^* \subsetneq {}^-\widetilde{\Delta}_1$.
(2) Suppose $\overline{\Delta}^* \not\subset {}^-\widetilde{\Delta}_1$ or $\overline{\Delta}^* = {}^-\widetilde{\Delta}_1$.

In Case (1), we have that $\Upsilon({}^-\Delta, \mathfrak{h}^*)$ is $\overline{\Delta}^*$, coinciding with the second segment in the removal sequence for $(\Delta, \mathfrak{h})$. By the minimality and nesting conditions, the subsequent segments in the removal sequence for $({}^-\Delta, \mathfrak{h}^*)$ are the same as those (starting from the third one) in the removal sequence for $(\Delta, \mathfrak{h})$.

In Case (2), $\Upsilon({}^-\Delta, \mathfrak{h}^*) = {}^-\widetilde{\Delta}_1$, and, by the nesting condition, the subsequent segments in the removal sequence for $({}^-\Delta, \mathfrak{h}^*)$ are those (starting from the second one) in the removal sequence for $(\Delta, \mathfrak{h})$.

In any such case, we will then obtain $\mathfrak{r}(\Delta, \mathfrak{h}) = \mathfrak{r}({}^-\Delta, \mathfrak{h}^*)$. □

We prove some further properties in Lemmas 6.9 to 6.13. One may compare with properties in the derivative side such as Lemmas 4.10 and 4.11.

Lemma 6.9. *(No effect on previous segments) Let* $\mathfrak{h} \in \mathrm{Mult}_\rho$*. Let* $\Delta = [a, b]_\rho \in \mathrm{Seg}_\rho$ *be admissible to* $\mathfrak{h}$*. Then for any* $a' < a$*,* $\mathfrak{h}[a'] = \mathfrak{r}(\Delta, \mathfrak{h})[a']$*.*

Proof. This follows directly form Definition 6.3 since those segments do not involve in the removal process. □

Lemma 6.10. *(Removing a whole segment in* $\mathfrak{h}$*) Let* $\mathfrak{h} \in \mathrm{Mult}_\rho$*. Let* $\Delta \in \mathfrak{h}$*. Then*

$$\mathfrak{r}(\Delta, \mathfrak{h}) = \mathfrak{h} - \Delta.$$

Proof. Write $\Delta = [a, b]_\rho$. Note that $\Upsilon(\Delta, \mathfrak{h}) = \Delta$. The nesting property guarantees that there is no other segments in the removal sequence for $(\Delta, \mathfrak{h})$. □

Lemma 6.11. *Let* $\mathfrak{h} \in \mathrm{Mult}_\rho$*. Let* $\Delta, \Delta' \in \mathrm{Seg}_\rho$ *with* $a(\Delta) \cong a(\Delta')$*. Then*

$$\{\Upsilon(\Delta, \mathfrak{h}) + \Upsilon(\Delta', \mathfrak{r}(\Delta, \mathfrak{h}))\} = \{\Upsilon(\Delta', \mathfrak{h}) + \Upsilon(\Delta, \mathfrak{r}(\Delta', \mathfrak{h}))\}.$$

The above lemma is a straightforward exercise from definitions and we omit a proof.

Lemma 6.12. *(Removal sequence involving the largest end point) Let* $\mathfrak{h} \in \mathrm{Mult}_\rho$*. Let* $\Delta \in \mathrm{Seg}_\rho$ *be non-empty and admissible to* $\mathfrak{h}$*. Let* c *be the largest integer such that* $\mathfrak{h}\langle c\rangle \neq 0$*. If one of the segments in the removal sequence for* $(\mathfrak{n}, \mathfrak{h})$ *is in* $\mathfrak{h}\langle c\rangle$*, then* $\Upsilon(\Delta, \mathfrak{h}) \in \mathfrak{h}\langle c\rangle$*.*

Proof. This follows from the nesting property in the removal sequence. □

Lemma 6.13. *(Commutativity for unlinked segments) Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\Delta, \Delta' \in \mathrm{Seg}_\rho$ be unlinked segments. Suppose $\mathfrak{r}(\Delta, \mathfrak{h}) \neq 0$ and $\mathfrak{r}(\Delta', \mathfrak{r}(\Delta, \mathfrak{h})) \neq 0$. Then*

$$\mathfrak{r}(\Delta', \mathfrak{r}(\Delta, \mathfrak{h})) = \mathfrak{r}(\Delta, \mathfrak{r}(\Delta', \mathfrak{h})).$$

Proof. We shall prove by an induction on the sum of lengths of all segments in $\mathfrak{h}$. For induction purpose, we also allow Δ, Δ' to be an empty set and in such case, it is trivial. We now assume both are not empty sets. By switching the labellings if necessary, we may and shall assume that $a(\Delta') \geq a(\Delta)$. Let $\widetilde{\Delta}_1 = \Upsilon(\Delta, \mathfrak{h})$.

Case 1: $a(\Delta) \not\cong a(\Delta'), \nu^{-1}a(\Delta')$. Hence $a(\Delta') > a(\Delta)$. Now we consider

$$\mathfrak{h}^* = \mathfrak{h} - \left\{\widetilde{\Delta}_1\right\} + \left\{{}^-\widetilde{\Delta}_1\right\}.$$

and so, by Lemma 6.8, $\mathfrak{r}(\Delta, \mathfrak{h}) = \mathfrak{r}({}^-\Delta, \mathfrak{h}^*)$. Thus,

$$(*) \quad \mathfrak{r}(\Delta', \mathfrak{r}(\Delta, \mathfrak{h})) = \mathfrak{r}(\Delta', \mathfrak{r}({}^-\Delta, \mathfrak{h}^*)).$$

Now using Lemma 6.9 and the assumption in this specific case, we still have $\Upsilon(\Delta, \mathfrak{r}(\Delta', \mathfrak{h})) = \widetilde{\Delta}_1$. Hence, we have that: by Lemma 6.8 again,

$$\mathfrak{r}(\Delta, \mathfrak{r}(\Delta', \mathfrak{h})) = \mathfrak{r}({}^-\Delta, \mathfrak{r}(\Delta', \mathfrak{h}^*)), \tag{6.9}$$

where we also use $\mathfrak{r}(\Delta', \mathfrak{h}) - \left\{\widetilde{\Delta}_1\right\} + \left\{{}^-\widetilde{\Delta}_1\right\} = \mathfrak{r}(\Delta', \mathfrak{h}^*)$ by Lemma 6.9.

Now,

$$\mathfrak{r}(\Delta', \mathfrak{r}(\Delta, \mathfrak{h})) = \mathfrak{r}(\Delta', \mathfrak{r}({}^-\Delta, \mathfrak{h}^*)) = \mathfrak{r}({}^-\Delta, \mathfrak{r}(\Delta', \mathfrak{h}^*)) = \mathfrak{r}(\Delta, \mathfrak{r}(\Delta', \mathfrak{h})),$$

where the middle equality follows from the inductive case. Hence, we are done.

Case 2: $a(\Delta) \cong a(\Delta')$. Let

$$\mathfrak{h}^{**} = \mathfrak{h} - \Upsilon(\Delta, \mathfrak{h}) - \Upsilon(\Delta', \mathfrak{r}(\Delta, \mathfrak{h})) + {}^-\Upsilon(\Delta, \mathfrak{h}) + {}^-\Upsilon(\Delta', \mathfrak{r}(\Delta, \mathfrak{h})).$$

We use Lemma 6.8 twice and combine with Lemma 6.11 to obtain:

$$\mathfrak{r}(\Delta', \mathfrak{r}(\Delta, \mathfrak{h})) = \mathfrak{r}({}^-\Delta', \mathfrak{r}({}^-\Delta, \mathfrak{h}^{**}))$$

and

$$\mathfrak{r}(\Delta, \mathfrak{r}(\Delta', \mathfrak{h})) = \mathfrak{r}({}^-\Delta, \mathfrak{r}({}^-\Delta', \mathfrak{h}^{**})).$$

Then the equality follows from the induction.

Case 3: $a(\Delta) \cong \nu^{-1}a(\Delta')$. We further divide into two more cases:

(1) There is a segment $\widehat{\Delta}$ in $\mathfrak{h}$ such that $a(\widehat{\Delta}) \cong \nu a(\Delta)$ and $\Delta' \subset \widehat{\Delta} \subsetneq {}^-\widetilde{\Delta}_1$. In such case, one has the equalities (*) and (6.9) as in Case (1).

(2) There is no such segment in the above case. Let $\overline{\Delta}_1 = \Upsilon(\Delta', \mathfrak{h})$. Let $\mathfrak{h}^* = \mathfrak{h} - \{\overline{\Delta}_1\} + \{{}^-\overline{\Delta}_1\}$. Then, by Lemma 6.8, we have that

$$\mathfrak{r}(\Delta', \mathfrak{h}) = \mathfrak{r}({}^-\Delta', \mathfrak{h}^*).$$

Hence,

$$(*) \quad \mathfrak{r}(\Delta, \mathfrak{r}(\Delta', \mathfrak{h})) = \mathfrak{r}(\Delta, \mathfrak{r}({}^-\Delta', \mathfrak{h}^*)).$$

On the other hand, by the nesting property for the removal sequence for $(\Delta, \mathfrak{h})$ and the assumption in this case, $\overline{\Delta}_1$ cannot be involved in the removal sequence for $(\Delta, \mathfrak{h})$. Hence,

$$\mathfrak{r}(\Delta, \mathfrak{h}) - \{\overline{\Delta}_1\} + \{{}^-\overline{\Delta}_1\} = \mathfrak{r}(\Delta, \mathfrak{h}^*).$$

By using the assumption in this case, we still have that

$$\Upsilon(\Delta', \mathfrak{r}(\Delta, \mathfrak{h}^*)) = \overline{\Delta}_1.$$

Thus, by Lemma 6.8 again,

$$(**) \quad \mathfrak{r}(\Delta', \mathfrak{r}(\Delta, \mathfrak{h})) = \mathfrak{r}({}^-\Delta', \mathfrak{r}(\Delta, \mathfrak{h}^*)).$$

Now, using (*) and (**) and induction case, we have that:

$$\mathfrak{r}(\Delta', \mathfrak{r}(\Delta, \mathfrak{h})) = \mathfrak{r}(\Delta, \mathfrak{r}(\Delta', \mathfrak{h})).$$

□

6.4. **Derivative resultant multisegments.**

Definition 6.14. (1) Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. A multisegment $\mathfrak{n}$ is said to be *admissible* to $\mathfrak{h}$, if we label the segments $\Delta_1, \ldots, \Delta_k$ in $\mathfrak{n}$ in an ascending order, Δ_i is admissible to

$$\mathfrak{r}(\Delta_{i-1}, \ldots \mathfrak{r}(\Delta_1, \mathfrak{h})$$

for all $i = 1, \ldots, k$. By Lemma 6.13, it is independent of a choice of an ascending order.

(2) Using the notations in (1), we also write

$$\mathfrak{r}(\mathfrak{n}, \mathfrak{h}) = \mathfrak{r}(\{\Delta_1, \ldots, \Delta_k\}, \mathfrak{h})$$

for $\mathfrak{r}(\Delta_k, \mathfrak{r}(\Delta_{k-1}, \ldots \mathfrak{r}(\Delta_1, \mathfrak{h}) \ldots)$. We also consider $\emptyset$ to be admissible to any $\mathfrak{h} \in \mathrm{Mult}_\rho$.

For any multisegment $\mathfrak{n}$ admissible to $\mathfrak{h}$, we say $\mathfrak{r}(\mathfrak{n}, \mathfrak{h})$ to be a *derivative resultant multisegment* for $\mathfrak{h}$.

6.5. **Shrinking derivatives.** The following lemma is more technical, but the original motivation comes from the representation-theoretic counterpart for the removal process e.g. Theorem 7.2. One should also compare with some reduction techniques in [LM22, Ch22+, Ch22+d].

Lemma 6.15. *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\mathfrak{n}$ be a multisegment admissible to $\mathfrak{h}$. Let $c \in \mathbb{Z}$ such that, for any $i \geq 1$, $\nu^{c+i}\rho$ is not in any segment of $\mathfrak{n}$. Let $\mathfrak{s} = \mathfrak{r}(\mathfrak{n}, \mathfrak{h})$. Recall that $\mathfrak{h}\langle c\rangle$ (resp. $\mathfrak{s}\langle c\rangle$) is the submultisegment of $\mathfrak{h}$ (resp. $\mathfrak{s}$) containing all the segments Δ in $\mathfrak{h}$ (resp. $\mathfrak{s}$) satisfying $b(\Delta) \cong \nu^c\rho$. Furthermore, we assume that*

$$\mathfrak{s}\langle e\rangle = \mathfrak{h}\langle e\rangle$$

for any $e \geq c+1$. Then

$$\mathfrak{s} - \mathfrak{s}\langle c\rangle + \mathfrak{h}\langle c\rangle$$

is also a derivative resultant multisegment for $\mathfrak{h}$.

Proof. We shall prove by an induction on $l_a(\mathfrak{h})$. We write the segments in $\mathfrak{n}$ in the following ascending order:

$$(*) \quad \Delta_1 \preceq^L \ldots \preceq^L \Delta_p.$$

When $l_a(\mathfrak{h}) = 0$, it is clear.

Let $\widetilde{\Delta} = \Upsilon(\Delta_1, \mathfrak{h})$. We first consider the case that $\widetilde{\Delta} \notin \mathfrak{h}\langle c\rangle$. Note that, by the definition of $\mathfrak{r}$,

$$\mathfrak{r}(\mathfrak{n} - \Delta_1, \mathfrak{r}(\Delta_1, \mathfrak{h})) = \mathfrak{r}(\mathfrak{n}, \mathfrak{h}).$$

Let $\mathfrak{h}' = \mathfrak{r}(\Delta_1, \mathfrak{h})$. The assumptions in the lemma imply that $\widetilde{\Delta} \in \mathfrak{h}\langle e\rangle$ for some $e < c$ and then the nesting property with $\widetilde{\Delta} \notin \mathfrak{h}\langle c\rangle$ imply that $\mathfrak{h}\langle c\rangle = \mathfrak{h}'\langle c\rangle$. Now induction hypothesis with (*) gives that

$$\mathfrak{s} - \mathfrak{s}\langle c\rangle + \mathfrak{h}\langle c\rangle = \mathfrak{s} - \mathfrak{s}\langle c\rangle + \mathfrak{h}'\langle c\rangle$$

is still a derivative resultant multisegment i.e. for some multisegment $\widetilde{\mathfrak{n}}$,

$$\mathfrak{r}(\widetilde{\mathfrak{n}}, \mathfrak{h}') = \mathfrak{s} - \mathfrak{s}\langle c\rangle + \mathfrak{h}\langle c\rangle.$$

It remains to observe from (*) that we still have

$$\mathfrak{r}(\widetilde{\mathfrak{n}}+\Delta_1,\mathfrak{h})=\mathfrak{r}(\widetilde{\mathfrak{n}},\mathfrak{h}').$$

We now consider the case that $\widetilde{\Delta}\in\mathfrak{h}\langle c'\rangle$ for some $c'\geq c$. Let

$$\mathfrak{h}^*=\mathfrak{h}-\left\{\widetilde{\Delta}\right\}+\left\{{}^-\widetilde{\Delta}\right\},$$

$$\mathfrak{n}^*=\mathfrak{n}-\{\Delta_1\}+\left\{{}^-\Delta_1\right\}.$$

By Lemma 6.8,

$$\mathfrak{r}(\Delta_1,\mathfrak{h})=\mathfrak{r}({}^-\Delta_1,\mathfrak{h}^*)$$

and so, by Lemma 6.13,

$$(**)\quad \mathfrak{r}(\mathfrak{n},\mathfrak{h})=\mathfrak{r}(\mathfrak{n}^*,\mathfrak{h}^*).$$

Now, by induction hypothesis, we have that $\mathfrak{s}-\mathfrak{s}\langle c\rangle+\mathfrak{h}^*\langle c\rangle$ is a derivative resultant multisegment, say there exists a multisegment $\widetilde{\mathfrak{n}}$ such that

$$\mathfrak{r}(\widetilde{\mathfrak{n}},\mathfrak{h}^*)=\mathfrak{s}-\mathfrak{s}\langle c\rangle+\mathfrak{h}^*\langle c\rangle. \tag{6.10}$$

We now claim:
Claim: $\mathfrak{r}(\widetilde{\mathfrak{n}},\mathfrak{h}^*)[a]=\mathfrak{h}^*[a]$, where a satisfies $a(\widetilde{\Delta})\cong\nu^a\rho$. (This claim means that after shrinking, no more segments in $\mathfrak{h}^*[a]$ are involved.)
Proof of claim: In the removal sequences for $(\widetilde{\mathfrak{n}},\mathfrak{h}^*)$, all the first segments involved are $\widetilde{\Delta}$ by the minimality in the removal process and the assumption that $\mathfrak{s}\langle e\rangle=\mathfrak{h}\langle e\rangle$ for all $e\geq c+1$. Thus, we have that,

$$\mathfrak{s}=\mathfrak{r}(\mathfrak{n},\mathfrak{h})=\mathfrak{r}(\mathfrak{n}-\mathfrak{n}[a]+{}^-(\mathfrak{n}[a]),\mathfrak{h}-\left\{\widetilde{\Delta},\ldots,\widetilde{\Delta}\right\}+\left\{{}^-\widetilde{\Delta},\ldots,{}^-\widetilde{\Delta}\right\}),$$

where both $\widetilde{\Delta}$ and ${}^-\widetilde{\Delta}$ appear $|\mathfrak{n}[a]|$-times. Thus, by Lemma 6.9,

$$\mathfrak{s}[a]=\mathfrak{h}[a]-\overbrace{\{\widetilde{\Delta},\ldots,\widetilde{\Delta}\}}^{|\mathfrak{n}[a]|\text{-times}}=\mathfrak{h}^*[a]-\overbrace{\{\widetilde{\Delta},\ldots,\widetilde{\Delta}\}}^{|\mathfrak{n}[a]|-1\text{ times}}.$$

Hence, $\mathfrak{s}[a]-\mathfrak{s}[a]\langle c\rangle=\mathfrak{h}[a]-\mathfrak{h}[a]\langle c\rangle=\mathfrak{h}^*[a]-\mathfrak{h}^*[a]\langle c\rangle$ and so $(\mathfrak{s}-\mathfrak{s}\langle c\rangle+\mathfrak{h}^*\langle c\rangle)[a]=\mathfrak{h}^*[a]$. This proves the claim by (6.10).

Now the claim with our minimal choice of Δ_1, we must have that

$$(**)\quad \widetilde{\mathfrak{n}}[d]=\emptyset,\ \text{for any integer } d\leq a.$$

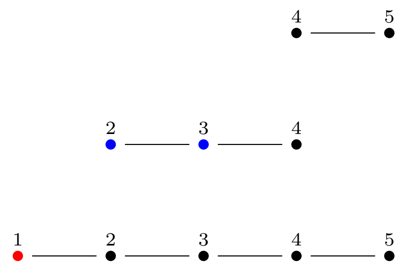
Then one observes that $\mathfrak{r}(\widetilde{\mathfrak{n}},\mathfrak{h})=\mathfrak{r}(\widetilde{\mathfrak{n}},\mathfrak{h}^*)+\left\{\widetilde{\Delta}\right\}-\left\{{}^-\widetilde{\Delta}\right\}$ by using Lemma 6.9, (**) and the fact that any segment in the removal sequence for $(\widetilde{\mathfrak{n}},\mathfrak{h}^*)$ does not involve ${}^-\widetilde{\Delta}$ (using (6.10)). Hence,

$$\mathfrak{r}(\widetilde{\mathfrak{n}},\mathfrak{h})=\mathfrak{s}-\mathfrak{s}\langle c\rangle+\mathfrak{h}^*\langle c\rangle+\left\{\widetilde{\Delta}\right\}-\left\{{}^-\widetilde{\Delta}\right\}=\mathfrak{s}-\mathfrak{s}\langle c\rangle+\mathfrak{h}\langle c\rangle$$

is still a derivative resultant multisegment as desired. □

Example 6.16. Let $\mathfrak{h}=\{[1,5],[2,4],[4,5]\}$.

(1) Let $\Delta = [1,3]$. Then $\mathfrak{r}(\Delta, \mathfrak{h}) = \{[2,5],[4],[4,5]\}$. Then $\{[1,5],[4],[4,5]\}$ is also a derivative resultant multisegment by Lemma 6.15. One can also check that $\mathfrak{r}([2,3],\mathfrak{h}) = \{[1,5],[4],[4,5]\}$.



Here those red and blue points are removed to obtain $\mathfrak{r}([1,3],\mathfrak{h})$, and the red point is added to obtain a new derivative resultant multisegment.

(2) Let $\mathfrak{n} = \{[1,3],[2]\}$. Then $\mathfrak{r}(\mathfrak{n},\mathfrak{h}) = \{[3,5],[4],[4,5]\}$. Then $\{[1,5],[4],[4,5]\}$ is a derivative resultant multisegment.

Lemma 6.17. *We use notations in the previous lemma. Let $\mathfrak{n}' \in \mathrm{Mult}_\rho$ such that*

$$\mathfrak{r}(\mathfrak{n}',\mathfrak{h}) = \mathfrak{s} - \mathfrak{s}\langle c\rangle + \mathfrak{h}\langle c\rangle.$$

Then

$$\mathfrak{r}(\mathfrak{h}\langle c\rangle + \mathfrak{n}',\mathfrak{h}) = \mathfrak{s} - \mathfrak{s}\langle c\rangle.$$

Proof. Note that

$$\mathfrak{r}(\mathfrak{h}\langle c\rangle + \mathfrak{n}',\mathfrak{h}) = \mathfrak{r}(\mathfrak{h}\langle c\rangle, \mathfrak{r}(\mathfrak{n}',\mathfrak{h})) = \mathfrak{r}(\mathfrak{h}\langle c\rangle, \mathfrak{s} - \mathfrak{s}\langle c\rangle + \mathfrak{h}\langle c\rangle).$$

Then the lemma follows from Lemma 6.10. □

6.6. **Effect of St-derivatives.** We shall now compute the effect of St-derivatives on the invariant ε_Δ. We need a preparation lemma first, which allows one to do the induction. For a segment $\Delta = [a,b]_\rho$, define ${}^+\Delta = [a-1,b]_\rho$.

Lemma 6.18. *Let $\mathfrak{m} \in \mathrm{Mult}_\rho$ and let $\pi = \langle \mathfrak{m}\rangle$. Let c be an integer such that $\mathfrak{m}\langle c\rangle \neq \emptyset$ and let $k = \varepsilon_c(\pi)$. Let $\widetilde{\pi} = D_c^k(\pi)$. For $\Delta = [a,b]_\rho$ with $b > 0$,*

(1) *if $a > c+1$, then*
$$\varepsilon_\Delta(\widetilde{\pi}) = \varepsilon_\Delta(\pi);$$
(2) *if $a = c+1$, then*
$$\varepsilon_\Delta(\widetilde{\pi}) = \varepsilon_\Delta(\pi) + \varepsilon_{+\Delta}(\pi)$$

Proof. We consider (1). Suppose $c \neq b$. Otherwise, it is easier (see Section 4.4). Let $l = \varepsilon_\Delta(\pi)$. Then we have

$$(\nu^c\rho)^{\times k} \times \widetilde{\pi} \twoheadrightarrow \pi \hookrightarrow \pi' \times (\mathrm{St}(\Delta))^{\times l}$$

By geometric lemma and comparing cuspidal support, we have a non-zero composition factor on π'_{N_i} of the form $\tau \boxtimes (\mathrm{St}(\Delta))^{\times l}$, where i is equal to l times the absolute length of Δ and $\tau \in \mathrm{Irr}$, and hence $\widetilde{\pi}_{N_i}$ also has a quotient of such form $\tau \boxtimes (\mathrm{St}(\Delta))^{\times l}$. Thus $l \leq \varepsilon_\Delta(\widetilde{\pi})$.

Let $l' = \varepsilon_\Delta(\widetilde{\pi})$. We have an embedding:

$$\pi \hookrightarrow \pi' \times (\nu^c\rho)^{\times k} \hookrightarrow \pi'' \times \mathrm{St}(\Delta)^{\times l'} \times (\nu^c\rho)^{\times k} \cong \pi'' \times (\nu^c\rho)^{\times k} \times \mathrm{St}(\Delta)^{\times l'},$$

where the last isomorphism follows from [Ze80, Theorem 4.2]. Hence, by Frobenius reciprocity, $l = \varepsilon_\Delta(\pi) \geq l'$. Thus $l = l'$ and this proves (1).

We now consider (2). We shall write $\mathfrak{h}_c = \mathfrak{mxpt}^a(\pi, c)$ and $\mathfrak{h}_{c+1} = \mathfrak{mxpt}^a(\pi, c+1)$ (see Definition 4.3). Note that by the definition of a maximal multisegment and ε_c, $\mathfrak{h}_c$ has exactly k-segments. By Lemma 5.1, we have an embedding:

$$\pi \hookrightarrow \omega \times \mathrm{St}(\mathfrak{h}_{c+1}) \times \mathrm{St}(\mathfrak{h}_c)$$

and so we also have:

$$\pi \hookrightarrow \omega \times \mathrm{St}(\mathfrak{h}_{c+1} + {}^{-}(\mathfrak{h}_c)) \times (\nu^c\rho)^{\times k}.$$

(see Section 3.7 for the notation ${}^{-}(\mathfrak{h}_c)$). Thus, by Corollary 3.8,

$$\widetilde{\pi} \hookrightarrow \omega \times \mathrm{St}(\mathfrak{h}_{c+1} + {}^{-}(\mathfrak{h}_c)).$$

Then

$$(*) \quad \varepsilon_\Delta(\widetilde{\pi}) \geq \varepsilon_\Delta(\pi) + \varepsilon_{+\Delta}(\pi).$$

To prove the opposite inequality, suppose it fails for some $\Delta = [c+1, b]_\rho$. Let $l = \varepsilon_\Delta(\widetilde{\pi})$. Then we can write

$$\pi \hookrightarrow D^l_\Delta(\widetilde{\pi}) \times (\mathrm{St}(\Delta))^{\times l} \times (\nu^c\rho)^{\times k},$$

which implies that

$$\pi \hookrightarrow D^l_\Delta(\widetilde{\pi}) \times \omega,$$

for some composition factor ω in $\mathrm{St}(\Delta)^{\times l} \times (\nu^c\rho)^{\times k}$. Since the only possible segments appearing in the multisegment for ω include $[c, b]_\rho$, $[c+1, b]_\rho$ or $[c]_\rho$, we must have:

$$\pi \hookrightarrow D^l_\Delta(\widetilde{\pi}) \times (\nu^c\rho)^r \times \mathrm{St}([c, b]_\rho)^s \times \mathrm{St}([c+1, b]_\rho)^t,$$

where $s + t = \varepsilon_\Delta(\widetilde{\pi}) = l$. Hence, by our assumption, we have either:

$$s > \varepsilon_\Delta(\pi) \quad \text{ or } \quad t > \varepsilon_{+\Delta}(\pi).$$

However, one applies Frobenius reciprocity and obtains a contradiction to the definition of $\varepsilon_\Delta(\pi)$ or $\varepsilon_{+\Delta}(\pi)$, as desired. $\square$

The following is a key property that allows one to deduce that the derivative resultant multisegments 'matching' the effect of St-derivatives by an induction.

Lemma 6.19. *Let $\mathfrak{m} \in \mathrm{Mult}_\rho$ and let $\pi = \langle \mathfrak{m} \rangle$. Let c be an integer such that $\varepsilon_c(\pi) \neq 0$. Let $\Delta = [c, b]_\rho$ for some b. Let $\widetilde{\pi} = D^k_c(\pi)$, where $k = \varepsilon_c(\pi)$. Then*

$$\pi \hookrightarrow \widetilde{\pi} \times (\nu^c\rho)^k.$$

Then

$$D_\Delta(\pi) \hookrightarrow D_{-\Delta}(\widetilde{\pi}) \times (\nu^c\rho)^{k-1}.$$

Proof. A simple application of geometric lemma and comparison of cuspidal support gives that:

$$D_\Delta(\pi) \boxtimes \mathrm{St}(\Delta) \hookrightarrow \mathrm{Ind}^{G_{n'} \times G_{n''}}_{P_{n'-r,r} \times P_{n''-s,s}} (\pi_{N_r} \times ((\nu^c\rho)^k)_{N_{n(\rho)}})^\phi,$$

where $n' = n(\widetilde{\pi}) - l_a({}^{-}\Delta)$, $n'' = (k-1) \cdot n(\rho), r = l_a({}^{-}\Delta), s = n(\rho)$ and ϕ is a twist to get a $G_{n'-r} \times G_r \times G_{n''-s} \times G_s$-representation.

Now apply Frobenius reciprocity on the second factor and then apply the adjointness of the tensor product to get

$$(*) \quad D_\Delta(\pi) \hookrightarrow \mathrm{Hom}_{G_r}(\mathrm{St}({}^{-}\Delta), \widetilde{\pi}_{N_r}) \times (\nu^c\rho)^{\times(k-1)},$$

where $r = l_a({}^{-}\Delta)$. (Note that for the form of the second factor in (*), one may use [Ch22+, Lemma 9.1 or Theorem 9.4] while one may also simply use the fact that all composition factors in $((\nu^c\rho)^{\times k})_{N_{n(\rho)}}$ are isomorphic to $(\nu^c\rho)^{\times(k-1)} \boxtimes (\nu^c\rho)$.)

By Lemma 4.10 and Proposition 4.4, we have that $\varepsilon_c(D_\Delta(\pi)) = k-1$. Hence, Corollary 3.8 and (*) give

$$D_c^{k-1} \circ D_\Delta(\pi) \hookrightarrow \mathrm{Hom}_{G_r}(\mathrm{St}(^-\Delta), \widetilde{\pi}_{N_r})$$

and hence $D_c^{k-1} \circ D_\Delta(\pi) \cong D_{-\Delta}(\widetilde{\pi})$, which gives the lemma. □

We now study the change of $\mathfrak{mxpt}^a$ under the derivatives. The proof is based on two special cases: Lemma 6.18 and Proposition 4.4.

Theorem 6.20. *Let $\mathfrak{m} \in \mathrm{Mult}_\rho$ and let $\Delta = [a,b]_\rho$ be a segment. Let $\pi = \langle \mathfrak{m} \rangle$ and let $\mathfrak{h} = \mathfrak{hd}(\pi)$.*

(1) *Suppose Δ is not admissible to $\mathfrak{h}$. Then $D_\Delta(\pi) = 0$.*
(2) *Suppose Δ is admissible to $\mathfrak{h}$. Let $\pi' = D_\Delta(\pi)$. Then*
   - *For any $c \geq a$,*
   $$\mathfrak{mxpt}^a(\pi', c) = \mathfrak{r}(\Delta, \pi)[c].$$
   (See Section 2.2 for the notations.)
   - *For any $c < a$, $\mathfrak{mxpt}^a(\pi', c)$ satisfies*
   $$\mathfrak{h}[c] \leq_c^a \mathfrak{mxpt}^a(\pi', c).$$
   - *For any Δ' unlinked to Δ and $a(\Delta') \not\cong a(\Delta)$,*
   $$\varepsilon_{\Delta'}(\pi') = \varepsilon_{\Delta'}(\pi).$$

Proof. (1) follows from definitions. The second bullet of (2) can be proved by a similar manner as the first case in the proof of Lemma 4.11 and we omit the details.

We shall prove the first bullet of (2) by an induction on $n(\pi)$. When $n(\pi) = 0, 1$, it is trivial. Let $\Delta = [a,b]_\rho$ be an admissible segment for π. Let $\widetilde{\pi} = D_a^k(\pi)$, where $k = \varepsilon_a(\pi)$. Now we have that:

$$(*) \quad \pi \hookrightarrow \widetilde{\pi} \times (\nu^a\rho)^{\times k}$$

By Lemma 6.19,

$$D_\Delta(\pi) \hookrightarrow D_{-\Delta}(\widetilde{\pi}) \times (\nu^a\rho)^{\times(k-1)}.$$

Now we consider two cases:

(i) Suppose $c \geq a+2$. In such case, $D_\Delta \circ D_a^{k-1}(\pi) \cong D_a^{k-1} \circ D_\Delta(\pi)$. Let $\mathfrak{n}$ be a multisegment at $\nu^c\rho$. Then

$$D_\mathfrak{n} \circ D_\Delta \circ D_a^{k-1}(\pi) \neq 0 \quad \Leftrightarrow \quad D_a^{k-1} \circ D_\mathfrak{n} \circ D_\Delta(\pi) \neq 0 \quad \Leftrightarrow D_\mathfrak{n} \circ D_\Delta(\pi) \neq 0,$$

where the first 'if and only if' condition follows by applying Lemma 4.10 twice, and the second 'if and only if' condition follows by an analogous statement for Lemma 4.11 in the linked case. Now one deduces the maximal multisegment at $\nu^c\rho$ ($c \geq a$) for $D_\Delta(\pi)$ as follows: First,

$$\mathfrak{mxpt}^a(D_{\Delta^-}(\widetilde{\pi}), c) = \mathfrak{r}(\Delta^-, \widetilde{\pi})[c] = \mathfrak{r}(\Delta, \pi)[c],$$

where the first equality follows from the inductive case and the second equality follows from using Lemma 6.18 and the definition of removal process. Then, by Lemma 6.18 (with Lemma 6.19), we have that $\mathfrak{mxpt}^a(D_{\Delta^-}(\widetilde{\pi}), c) = \mathfrak{mxpt}^a(D_\Delta(\pi), c)$. Combining all, we have

$$\mathfrak{mxpt}^a(D_\Delta(\pi), c) = \mathfrak{r}(\Delta, \pi)[c]$$

as desired.

(ii) Suppose $c=a+1$. Let

$$\mathfrak{s}=\mathfrak{mxpt}^a(\pi,a)-\Delta_0,$$

where Δ_0 is the shortest segment in $\mathfrak{mxpt}^a(\pi,a)$ such that $\Delta\subset\Delta_0$.

By Lemma 6.18,

$$\mathfrak{n}:=\mathfrak{mxpt}^a(\widetilde{\pi},a+1)={}^-(\mathfrak{mxpt}^a(\pi,a))+\mathfrak{mxpt}^a(\pi,a+1).$$

By Proposition 4.4,

$$D_{\mathfrak{s}}\circ D_{\widetilde{\Delta}}(\pi)\neq 0$$

and now by repeated uses of Lemma 6.19,

$$D_{\mathfrak{s}}\circ D_{\Delta}(\pi)\cong D_{-\mathfrak{s}}\circ D_{-\Delta}(\widetilde{\pi}).$$

Hence, by the inductive case,

$$\mathfrak{mxpt}^a(D_{\mathfrak{s}}\circ D_{\Delta}(\pi),a+1)=\mathfrak{mxpt}^a(D_{-\mathfrak{s}}\circ D_{-\Delta}(\widetilde{\pi}),a+1)$$

is precisely

$$\mathfrak{r}({}^-\mathfrak{s}+{}^-\Delta,\widetilde{\pi})[a+1]=\mathfrak{r}(\mathfrak{s}+\Delta,\pi)[a+1].$$

The last equality follows from the rules of removal process and Lemma 6.18.

Indeed we also have

$$\mathfrak{r}(\mathfrak{s}+\Delta,\pi)[a+1]=\mathfrak{r}(\mathfrak{s},\mathfrak{r}(\Delta,\pi))[a+1]=\mathfrak{r}(\Delta,\pi)[a+1],$$

where the second equality follows from that applying $D_{\Delta'}$ for each $\Delta'\in\mathfrak{s}$ will simply remove the segment Δ' in $\mathfrak{r}(\Delta,\pi)$ by Lemma 6.10.

Since $\mathfrak{s}=\mathfrak{mxpt}^a(D_{\Delta}(\pi),a)$, Lemma 5.1 implies that

$$\mathfrak{mxpt}^a(D_{\Delta}(\pi),a+1)=\mathfrak{mxpt}^a(D_{\mathfrak{s}}\circ D_{\Delta}(\pi),a+1).$$

Now, combining all the equations, we have that:

$$\mathfrak{mxpt}^a(D_{\Delta}(\pi),a+1)=\mathfrak{r}(\Delta,\pi)[a+1].$$

(iii) Suppose $c=a$. Then $\mathfrak{mxpt}^a(\pi',a)$ follows from Proposition 4.4 and Lemma 4.2.

We now prove the third bullet of (2). The inequality

$$\varepsilon_{\Delta'}(\pi')\leq\varepsilon_{\Delta'}(\pi)$$

follows from Lemma 4.10. The opposite inequality follows from an application of geometric lemma. We omit the details. □

Definition 6.21. Let $\pi\in\mathrm{Irr}_\rho$. Let $\mathfrak{n}\in\mathrm{Mult}_\rho$. We say that $\mathfrak{n}$ is *admissible* to π if, for writing segments $\Delta_1,\ldots,\Delta_r$ in $\mathfrak{n}$ in an ascending order, $D_{\Delta_r}\circ\ldots\circ D_{\Delta_1}(\pi)\neq 0$. By Lemma 4.10, the admissibility is independent of a choice of an ascending sequence.

Corollary 6.22. *Let $\pi\in\mathrm{Irr}_\rho$. Let $\mathfrak{n}\in\mathrm{Mult}_\rho$. Then $\mathfrak{n}$ is admissible to π if and only if $\mathfrak{n}$ is admissible to $\mathfrak{hd}(\pi)$.*

Proof. Write $\Delta_k=[a_k,b_k]_\rho$. We shall assume that $a_1\leq\ldots\leq a_k$. We consider the if direction. Let k be the smallest integer such that Δ_k is not admissible to

$$\mathfrak{r}(\{\Delta_1,\ldots,\Delta_{k-1}\},\mathfrak{hd}(\pi)).$$

We have that

$$(*)\quad \mathfrak{mxpt}^a(D_{\Delta_{k-1}}\circ\ldots\circ D_{\Delta_1}(\pi),a_k)=\mathfrak{r}(\{\Delta_1,\ldots,\Delta_{k-1}\},\mathfrak{hd}(\pi))[a_k]$$

by Theorem 6.20(2). Thus the admissibility of Δ_k to $\mathfrak{r}(\{\Delta_1,\ldots,\Delta_{k-1}\},\mathfrak{hd}(\pi))$ implies that

$$\{\Delta_k\}\leq_{a_k}\mathfrak{mxpt}^a(D_{\Delta_{k-1}}\circ\ldots\circ D_{\Delta_1}(\pi),a_k)$$

by (*). Hence $D_{\Delta_k}\circ D_{\Delta_{k-1}}\circ\ldots\circ D_{\Delta_1}(\pi)\neq 0$.

The only if direction is similar by using (*) and we omit the details. $\square$

6.7. **Multisegment at a right point (revised).**

Corollary 6.23. *Let $\pi \in \mathrm{Irr}_\rho$. Fix an integer c. Then*

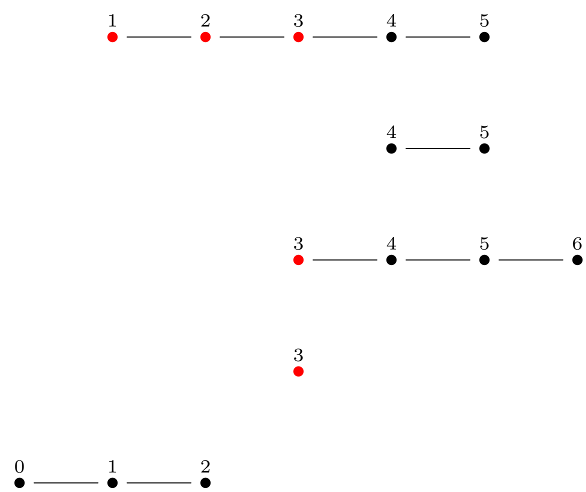
$$\mathfrak{mxpt}^b(\pi, c) = \sum_{[a,b]_\rho \in \mathfrak{hd}(\pi), a \leq c \leq b} [a, c]_\rho .$$

Proof. Let $\mathfrak{k} = \sum_{[a,b]_\rho \in \mathfrak{hd}(\pi), a \leq c \leq b} [a, c]_\rho$. By Theorem 6.20(2) several times, we see that $D_{\mathfrak{k}}(\pi) \neq 0$. On the other hand, $\mathfrak{r}(\mathfrak{k}, \mathfrak{hd}(\pi))$ does not have any segment Δ such that $\nu^c\rho \in \Delta$. Hence, by Theorem 6.20(2) again, $\mathfrak{k}$ is maximal and so $\mathfrak{mxpt}^b(\pi, c) = \mathfrak{k}$. $\square$

More generally, by Theorem 6.20(2), we have:

Corollary 6.24. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{m} \in \mathrm{Mult}^b_{\rho,c}$ for some c. Then $D_{\mathfrak{m}}(\pi) \neq 0$ if and only if $\mathfrak{m}$ is a submultisegment of $\mathfrak{mxpt}^b(\pi, c)$.*

Example 6.25. Let $\pi \in \mathrm{Irr}_\rho$ with $\mathfrak{hd}(\pi)$ taking the form:



The red points contribute to $\mathfrak{mxpt}^b(\pi, 3)$ to give $\{[1, 3], [3], [3]\}$.

7. Isomorphic simple quotients of Bernstein-Zelevinsky derivatives

7.1. **Complementary sequence of St-derivatives.** For $\pi \in \mathrm{Irr}_\rho$ and $\mathfrak{n} \in \mathrm{Mult}_\rho$, we shall write:

$$\mathfrak{r}(\mathfrak{n}, \pi) := \mathfrak{r}(\mathfrak{n}, \mathfrak{hd}(\pi)),$$

where the latter term is defined in Definition 6.3. We prove a main property of the derivative resultant multisegment.

Theorem 7.1. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be admissible to π. Let $\mathfrak{m} = \mathfrak{r}(\mathfrak{n}, \pi)$. Then*

$$D_{\mathfrak{m}} \circ D_{\mathfrak{n}}(\pi) \cong \pi^- .$$

Proof. Write $\mathfrak{n} = \{\Delta_1, \ldots, \Delta_k\}$ in an ascending order. Let $\omega = D_{\mathfrak{n}}(\pi)$. We shall prove by a backward induction on the sum of the lengths of all those $\Delta_1, \ldots, \Delta_k$. Since the sequence is admissible, the sum must be not greater than the level of π (by Proposition 3.17). If the sum is equal to the level of π, then $\omega \cong \pi^-$ by the irreducibility of the highest derivative of π. In this case, $\mathfrak{r}(\mathfrak{n}, \pi) = \emptyset$ and so we have such case.

Let c^* be the smallest integer such that $\mathfrak{r}(\{\Delta_1, \ldots, \Delta_k\}, \pi)[c^*] \neq 0$. In other words, $\nu^{c^*}\rho$ is isomorphic to a $\leq$-minimal element in

$$\{a(\Delta') : \Delta' \in \mathfrak{r}(\{\Delta_1, \ldots, \Delta_k\}, \pi)\}.$$

We shall choose the ascending order of the sequence such that

$$a(\Delta_1) \leq \ldots \leq a(\Delta_k).$$

Let $r \leq k$ such that $\Delta_1, \ldots, \Delta_r$ be all the segments such that $\nu^{c^*}\rho \geq a(\Delta_i)$ for $i = 1, \ldots, r$. We rearrange the segments $\Delta_{\delta(r+1)}, \ldots, \Delta_{\delta(k)}$ so that

$$b(\Delta_{\delta(k)}) \geq \ldots \geq b(\Delta_{\delta(r+1)}),$$

where δ is a permutation on $\{r+1, \ldots, k\}$.

Let $\tau = D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi)$. Note that the sequence $\Delta_{\delta(r+1)}, \ldots, \Delta_{\delta(k)}$ can be obtained from $\Delta_{r+1}, \ldots, \Delta_k$ by repeatedly switching two adjacent unlinked segments. Hence, by Lemma 4.10,

$$(*) \quad D_{\Delta_{\delta(k)}} \circ \ldots \circ D_{\Delta_{\delta(r+1)}}(\tau) \cong D_{\Delta_k} \circ \ldots \circ D_{\Delta_{r+1}}(\tau) \cong \omega.$$

Now we let $\widetilde{\Delta}$ be the longest segment in $\mathfrak{r}(\{\Delta_1, \ldots, \Delta_k\}, \pi)[c^*]$. We claim that: for $i \geq 2$,

$$D_{\widetilde{\Delta}} \circ D_{\Delta_{\delta(i)}} \circ \ldots \circ D_{\Delta_{\delta(1)}}(\tau) \cong D_{\Delta_{\delta(i)}} \circ D_{\widetilde{\Delta}} \circ D_{\Delta_{\delta(i-1)}} \circ \ldots \circ D_{\Delta_{\delta(1)}}(\tau),$$

and for $i = 1$,

$$D_{\widetilde{\Delta}} \circ D_{\Delta_{\delta(1)}}(\tau) \cong D_{\Delta_{\delta(1)}} \circ D_{\widetilde{\Delta}}(\tau).$$

Suppose the claim holds for the meanwhile. We have a new ascending sequence of segments,

$$(7.11) \qquad \Delta_1, \ldots, \Delta_r, \widetilde{\Delta}, \Delta_{r+1}, \ldots, \Delta_k,$$

which is admissible since the composition of their corresponding derivatives is non-zero by (*) and the claim. Now one applies induction hypothesis to obtain that

$$D_{\mathfrak{r}(\mathfrak{n}+\widetilde{\Delta}, \pi)} \circ D_{\mathfrak{n}+\widetilde{\Delta}}(\pi) \cong \pi^-.$$

By Lemma 6.9, $\mathfrak{r}(\mathfrak{n}, \pi) = \mathfrak{r}(\mathfrak{n} + \widetilde{\Delta}, \pi) + \widetilde{\Delta}$. Thus, by the claim, we have that

$$D_{\mathfrak{r}(\mathfrak{n}, \pi)}(\omega) \cong \pi^-.$$

It remains to prove the claim. Indeed, it will follow from Lemma 4.10 or Lemma 4.14 if we could check those conditions in the lemma. Use the notations in the claim. If $\widetilde{\Delta}$ and $\Delta_{\delta(i)}$ are unlinked, then we use Lemma 4.10 and we are done. Now suppose $\widetilde{\Delta}$ and $\Delta_{\delta(i)}$ are linked. Note that, by Lemma 6.9,

- $\mathfrak{r}(\{\Delta_1, \ldots, \Delta_i\}, \pi)[c^*] = \mathfrak{r}(\{\Delta_1, \ldots, \Delta_r, \Delta_{\delta(r+1)}, \ldots, \Delta_{\delta(i)}\}, \pi)[c^*]$.

This implies that, by Theorem 6.20,

$$(+) \quad D_{\widetilde{\Delta}}(\kappa) \neq 0,$$

where

$$\kappa = D_{\Delta_{\delta(i-1)}} \circ \ldots \circ D_{\Delta_{\delta(r+1)}} \circ D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi).$$

Now let $\Delta' = \Delta_{\delta(i)} \cup \widetilde{\Delta}$ and we have to check that $D_{\Delta'}(\kappa) = 0$. Note that $b(\Delta') > b(\widetilde{\Delta})$. Thus, we have:

(1) by the maximality of our choice on $\widetilde{\Delta}$ in $\mathfrak{r}(\{\Delta_1, \ldots, \Delta_k\}, \pi)$ and the above bullet,
$$D_{\Delta'}(\tau) = 0$$
(2) our arrangement on $\Delta_{\delta(p)}$s gives that Δ' and $\Delta_{\delta(x)}$ are unlinked for $x = 1, \ldots, i-1$. (Here we also use that $a(\Delta_{\delta(x)}) > \nu^{c_*}\rho$. See the third paragraph of the proof.)

Hence,

$$(++) \quad D_{\Delta'}(\kappa) = D_{\Delta'} \circ D_{\Delta_{\delta(i-1)}} \circ \ldots \circ D_{\Delta_{\delta(1)}}(\tau) = D_{\Delta_{\delta(i-1)}} \circ \ldots \circ D_{\Delta_{\delta(1)}} \circ D_{\Delta'}(\tau) = 0,$$

where the second equality follows from (2) above with Lemma 4.10 and the last equality follows from (1) above.

Since $D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi) \neq 0$, we also have

$$(+++) \quad D_{\Delta_{\delta(i)}}(\kappa) \neq 0.$$

Hence, the conditions $(+), (++), (+++)$ guarantee conditions in Lemma 4.14 and this completes the proof of the claim. □

7.2. **Isomorphic simple quotients under St-derivatives.** We now prove a main result using the highest derivative multisegment and the removal process to determine when two sequences of St-derivatives give rise to isomorphic simple quotients of a Bernstein-Zelevinsky derivative. The strategy for 'if' direction below of Theorem 7.2 below is that we use Theorem 7.1 to construct isomorphic modules by taking the same sequence of St-derivatives. The strategy for 'only if' direction is to find some St-derivatives that kill one, but not another one. However, in order to do so, we need to do it on some other derivatives via some constructions.

Theorem 7.2. *Let $\Delta_1, \ldots, \Delta_k$ and $\Delta'_1, \ldots, \Delta'_l$ be two ascending admissible sequences of segments. Then*

$$D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi) \cong D_{\Delta'_l} \circ \ldots \circ D_{\Delta'_1}(\pi)$$

if and only if

$$\mathfrak{r}(\{\Delta_1, \ldots, \Delta_k\}, \pi) = \mathfrak{r}(\{\Delta'_1, \ldots, \Delta'_l\}, \pi).$$

Proof. Let

$$\omega_1 = D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi), \quad \omega_2 = D_{\Delta'_l} \circ \ldots \circ D_{\Delta'_1}(\pi).$$

For if direction, we write $\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_r$ to be all the segments in $\mathfrak{r}(\{\Delta_1, \ldots, \Delta_k\}, \pi)$ as an ascending sequence. It follows from Theorem 7.1 that

$$D_{\widetilde{\Delta}_r} \circ \ldots \circ D_{\widetilde{\Delta}_1}(\omega_1) \cong D_{\widetilde{\Delta}_r} \circ \ldots \circ D_{\widetilde{\Delta}_1}(\omega_2) \cong \pi^-.$$

Hence, both ω_1 and ω_2 are isomorphic to

$$I_{\widetilde{\Delta}_1} \circ \ldots \circ I_{\widetilde{\Delta}_r}(\pi^-),$$

where $I_{\widetilde{\Delta}}(\tau)$ (for $\tau \in \mathrm{Irr}$) denotes the unique irreducible submodule of $\tau \times \mathrm{St}(\widetilde{\Delta})$ (see Lemma 3.4, and for more discussions on 'integrals', see [LM16, Ch22+, Ch22+c]). In particular, they are isomorphic.

We now consider the only if direction. We denote by

$$\mathfrak{r}_1 = \mathfrak{r}(\{\Delta_1, \ldots, \Delta_k\}, \pi), \quad \mathfrak{r}_2 = \mathfrak{r}(\{\Delta'_1, \ldots, \Delta'_l\}, \pi).$$

For $p = 1, 2$, let $\mathfrak{r}_p\langle c\rangle$ be the sub-multisegment of $\mathfrak{r}_p$ containing all the segments Δ satisfying $b(\Delta) \cong \nu^c\rho$. Suppose c^* is the smallest integer such that

$$\mathfrak{r}_1\langle c^*\rangle \neq \mathfrak{r}_2\langle c^*\rangle.$$

Let $\overline{c}$ be the largest integer such that $\nu^{\overline{c}}\rho \in \mathrm{csupp}(\pi)$. Let $c_i = \overline{c} - i$ for $i \geq 0$, and let z be the integer such that $c_z = c^*$. Set

$$\omega_{1,0} = D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi), \quad \omega_{2,0} = D_{\Delta'_l} \circ \ldots \circ D_{\Delta'_1}(\pi).$$

For $p = 1, 2$, we inductively, for each c_i $(i = 0, \ldots, z)$, define representations:

$$\kappa_{1,i} = D_{\mathfrak{p}_{i-1}}(\omega_{1,i-1}), \quad \kappa_{2,i} = D_{\mathfrak{p}_{i-1}}(\omega_{2,i-1}),$$

where $\mathfrak{p}_{i-1} = \mathfrak{r}_1\langle c_{i-1}\rangle = \mathfrak{r}_2\langle c_{i-1}\rangle$ (possibly empty and the equality follows from our choice of c^*), and

$$\omega_{1,i}, \quad \omega_{2,i}$$

are the representations with the derivative resultant multisegment obtained in Lemma 6.15 (we are in the special case that $\mathfrak{s}\langle c\rangle = 0$ in the notation of Lemma 6.15) i.e. by Lemma 6.17 and the (proved) if direction of this theorem, $\omega_{1,i}$ and $\omega_{2,i}$ are those satisfying that

$$\kappa_{1,i} \cong D_{\mathfrak{q}_{i-1}}(\omega_{1,i}), \quad \kappa_{2,i} \cong D_{\mathfrak{q}_{i-1}}(\omega_{2,i}),$$

where $\mathfrak{q}_{i-1} = \mathfrak{h}\langle c_{i-1}\rangle$.

We have either

$$D_{\mathfrak{r}_2\langle c^*\rangle}(\kappa_{1,c^*}) = 0, \quad \text{ or } D_{\mathfrak{r}_1\langle c^*\rangle}(\kappa_{2,c^*}) = 0.$$

This implies that $\kappa_{1,c^*} \not\cong \kappa_{2,c^*}$, and so we obtain that, for $p = 1, 2$,

$$I_{\mathfrak{q}_{z-1}} \circ D_{\mathfrak{p}_{z-1}} \circ \ldots \circ I_{\mathfrak{q}_0} \circ D_{\mathfrak{p}_0}(\omega_{p,0}) \cong \kappa_{p,c^*}.$$

Here $I_{\mathfrak{q}_{i-1}}(\kappa_{p,c^*})$ is the unique simple submodule of $\kappa_{p,c^*} \times \mathrm{St}(\mathfrak{q}_{i-1})$ (see Lemma 3.4). Hence, we must have $\omega_{1,0} \not\cong \omega_{2,0}$. □

7.3. **Comparing with ρ-derivatives.** We give a simple quotient of a Bernstein-Zelevinsky derivative that cannot be obtained from an ascending sequence of derivatives of cuspidal representations.

Example 7.3. Let $\mathfrak{m} = \{[0,1]_\rho, [1]_\rho, [1,2]_\rho\}$ and let $\pi = \langle\mathfrak{m}\rangle$. Then $\mathfrak{hd}(\pi) = \{[1]_\rho, [1,2]_\rho\}$. Hence, there is a simple quotient of $\pi^{(2)}$ obtained by applying $D_{[1,2]_\rho}$ and its derivative resultant multisegment is $\{[1]_\rho\}$. But $D_{[2]_\rho} \circ D_{[1]_\rho}(\pi) = 0$.

8. Examples of highest derivative multisegments

8.1. **Generic representations.** An irreducible representation π of G_n is said to be generic if $\pi^{(n)} \neq 0$. According to [Ze80], for $\mathfrak{m} \in \mathrm{Mult}_\rho$, $\langle\mathfrak{m}\rangle$ is generic if and only if all the segments are singletons. Equivalently, $\langle\mathfrak{m}\rangle \cong \mathrm{St}(\mathfrak{n})$ for a multisegment $\mathfrak{n}$ whose all segments are unlinked. One can compute $\mathfrak{n}$, for example, by the Mœglin-Waldspurger algorithm [MW86]. In this case, $\mathfrak{hd}(\pi) = \mathfrak{n}$ (e.g. use (2.4)).

8.2. **Arthur representations.** We write

$$\Delta_\rho(d) = [-(d-1)/2, (d-1)/2]_\rho.$$

Let

$$u_\rho(d,m) = \langle \left\{ \nu^{(m-1)/2}\Delta_\rho(d), \ldots, \nu^{-(m-1)/2}\Delta_\rho(d) \right\} \rangle.$$

Let $Y(u_\rho(d,m)) = \nu^{(d-m)/2}\rho$. The representations $u_\rho(d,m)$ are so-called Speh representations. For each Speh representation $u_\rho(d,m)$, it associates with a segment

$$\Delta(u_\rho(d,m)) := [(d-m)/2, (d+m-2)/2]_\rho.$$

It follows from [LM16] (also see [CS19]) that

$$\mathfrak{hd}(u_\rho(d,m)) = \{\Delta(u_\rho(d,m))\}.$$

Proposition 8.1. *Let π be a Arthur type representation in Irr_ρ i.e.*

$$\pi = \pi_1 \times \ldots \times \pi_r,$$

where each π_a is a Speh representation. Then

$$\mathfrak{hd}(\pi) = \Delta(\pi_1) + \ldots + \Delta(\pi_r).$$

Proof. To compute the lower bound of the multisegment at a point $\nu^c\rho$ (see Proposition 4.4), we rearrange [Ta86] the Speh representations such that

$$\pi_1 \times \ldots \times \pi_k \times \pi_{k+1} \times \ldots \times \pi_r$$

satisfying that $Y(\pi_1) \cong \ldots \cong Y(\pi_k) \cong \nu^c\rho$ and, for $i = k+1, \ldots, r$, $Y(\pi_i) \not\cong \nu^c\rho$.

One has that a composition factor of the form $(\pi_1^- \times \ldots \times \pi_k^-) \boxtimes (\mathrm{St}(\Delta(\pi_1)) \times \ldots \times \mathrm{St}(\Delta(\pi_k))$ is in

$$(\pi_1 \times \ldots \times \pi_k)_{N_i}.$$

Now, using geometric lemma,

$$((\pi_1 \times \ldots \times \pi_k)_{N_i} \times (\pi_{k+1} \times \ldots \times \pi_r))^{\phi} \hookrightarrow (\pi_1 \times \ldots \times \pi_r)_{N_i},$$

where ϕ is a twist so that the resulting representation is a $G_{n-i} \times G_i$-representation. Here $n = n(\pi)$. Combining with the previous paragraph, we have that

$$\tau \boxtimes (\mathrm{St}(\Delta(\pi_1)) \times \ldots \times \mathrm{St}(\Delta(\pi_k)) \hookrightarrow \pi_{N_i}.$$

Hence, the maximal multisegment of π at the point $\nu^{(d-m)/2}\rho$ is

$$\mathfrak{a}_{(d-m)/2} \geq \Delta(\pi_1) + \ldots + \Delta(\pi_k).$$

We obtain the lower bound for each multisegment at each point. We conclude that the lower bound is also the upper bound by the level of π and using Theorem 5.2. $\square$

8.3. **Ladder representations.** As we saw above, the highest derivative multisegment for a Speh representation is simply a segment. Let $\pi \in \mathrm{Irr}_\rho$ be a ladder representation. Then its associated multisegment $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_r\} \in \mathrm{Mult}_\rho$ satisfies the property:

$$a(\Delta_1) < \ldots < a(\Delta_r), \quad b(\Delta_1) < \ldots < b(\Delta_r).$$

Note that there is a unique multisegment $\mathfrak{n}$ such that $\cup_{\Delta \in \mathfrak{n}} \Delta = \{b(\Delta_1), \ldots, b(\Delta_r)\}$ and its segments are mutually unlinked. We have that $\mathfrak{hd}(\pi) = \mathfrak{n}$.

8.4. **$\square$-irreducible representations.** An irreducible representation π of G_n is said to be $\square$-irreducible if $\pi \times \pi$ is still irreducible [LM18]. For progress on characterizing such classes of modules, see, for example [Le03, GLS11, KKKO18, LM18] and references therein.

Proposition 8.2. *Let $\pi \in \mathrm{Irr}$. Suppose π is $\square$-irreducible. Then*

$$\mathfrak{hd}(\pi \times \pi) = \mathfrak{hd}(\pi) + \mathfrak{hd}(\pi).$$

Proof. One can, for example, deduce from Proposition 4.9. We omit the details. $\square$

9. Appendix: Bernstein-Zelevinsky derivatives for affine Hecke algebras

9.1. **BZ functor.** In this section, we explain how the results in this paper can be formulated in the affine Hecke algebra setting and we first give some background. We mainly follow [CS19], but we remark that we only need the Iwahori case to transfer results to the affine Hecke algebras using the Borel-Casselman's equivalence [Bo76], in which earlier work of Barbasch-Moy and Reeder [BM94, Re02] shows that generic irreducible representations correspond to modules containing the sign module of the finite Hecke algebra, and is later used to study the unitary dual problem by Barbasch-Ciubotaru [BC08]. Using the idea of finite Hecke algebra modules in characterizing modules also goes back to earlier work of Rogawski [Ro85].

A key to formulate the Bernstein-Zelevinsky derivative in [CS19] is using an explicit affine Hecke algebra structure of the Iwahori component of the Gelfand-Graev representation in [CS18], and such expression is also obtained in Brubaker-Buciumas-Bump- Friedberg [BBBF18]. Our realization of the Gelfand-Graev representation is obtained via viewing the affine Hecke algebra as the convolution algebra on Iwahori-biinvariant functions of G_n, and there is an alternate approach of describing affine Hecke algebra in terms of an endomorphism algebra due to Heiermann [He11]. For Hecke algebras arising from other Bernstein components, see the work of Waldspurger [Wa86] and Bushnell-Kutzko [BK93], and also the work of Sécherre-Stevens [SS12] for inner forms of general linear groups.

Let $q \in \mathbb{C}^\times$. Assume that q is not a root of unity. The affine Hecke algebra $\mathcal{H}_n := \mathcal{H}_n(q)$ is defined as an associative algebra over $\mathbb{C}$ with the generators $T_1, \ldots, T_{n-1}$ and $\theta_1^{\pm 1}, \ldots, \theta_n^{\pm 1}$ satisfying the following relations:

(1) $T_k T_{k+1} T_k = T_{k+1} T_k T_{k+1}$ for $k = 1, \ldots, n-2$;
(2) $(T_k + 1)(T_k - q) = 0$ for all k;
(3) $\theta_k \theta_l = \theta_l \theta_k$ for all k, l;
(4) $T_k \theta_k - \theta_{k+1} T_k = (q-1)\theta_k$ for all k;
(5) $T_k \theta_l = \theta_l T_k$ if $l \neq k, k+1$.

The subalgebra, denoted $\mathcal{H}_{S_n}$, generated by $T_1, \ldots, T_{n-1}$ is isomorphic to the finite Hecke algebra attached to the symmetric group S_n. For $k = 1, \ldots, n-1$, define s_k to be the transposition between k and $k-1$. For $w \in S_n$ with a reduced expression $w = s_{k_1} \ldots s_{k_r}$, define $T_w = T_{k_1} \ldots T_{k_r}$. It is well-known that T_w is independent of a choice of a reduced expression of w.

We now define the analogous Bernstein-Zelevinsky functor for $\mathcal{H}_n$ [CS19]. There is a natural embedding from $\mathcal{H}_{n-i} \otimes \mathcal{H}_i$ to $\mathcal{H}_n$ explicitly given by: for $k = 1, \ldots, n-i-1$, $m(T_k \otimes 1) = T_k$; for $k = 1, \ldots, i$, $m(1 \otimes T_k) = T_{n-i+k}$; for $k = 1, \ldots, n-i$, $m(\theta_k \otimes 1) = \theta_k$; for $k = 1, \ldots, i$, $m(1 \otimes \theta_k) = \theta_{n-i+k}$. Define the sign projector:

$$\mathbf{sgn}_i = \frac{1}{\sum_{w \in S_i} (1/q)^{l(w)}} \sum_{w \in S_n} (-1/q)^{l(w)} T_w \in \mathcal{H}_{S_i},$$

Let $\mathbf{S}_i^n = m(1 \otimes \mathbf{sgn}_i)$. The i-th Bernstein-Zelevinsky derivative for an $\mathcal{H}_n$-module σ is defined as:

$$\mathbf{BZ}_i(\sigma) = \mathbf{S}_i^n(\sigma).$$

The following result for $i = 1$ is covered by [GV01] by using ρ-derivatives. We remark that [GV01] covers other cases such as the works of Kleshchev and Brundan [Kl95] and [Br98] for some positive characteristic algebras. See a survey of Kleshchev [Kl10] for an overview of this problem. We remark that the branching law has deep connections with the theory of crystal bases. For instance, the decomposition matrix for restriction coincides with the coefficients of crystal bases in certain way, see the work of Lascoux-Leclerc-Thibon [LLT96] and Ariki [Ar96] and even the development for other classical types by Enomoto-Kashiwara, Miemietz, Varagnolo-Vasserot, Shan-Varagnolo-Vasserot [EK08, Mie08, VV11, SVV11]. The following result generalizes part of [GV01] and opens up some possibilities of connections with crystal theory:

Theorem 9.1. *Let σ be an irreducible $\mathcal{H}_n$-module. Then the socle and cosocle of $\mathbf{BZ}_i(\sigma)$ are multiplicity-free.*

Proof. We first assume that q is a prime power. We choose F to be a p-adic field with $|\mathcal{O}/\omega\mathcal{O}| = q$, where $\mathcal{O}$ is the ring of integers in F and ω is the uniformizer. Then, by [CS19, Theorem 4.2] and Lemma 3.11, we have the multiplicity-free result in such case. Note that in the case that σ has a real central character (see [CS19, Section 5.2], also see [OS10, Section 2]), we can further obtain that for the corresponding graded Hecke algebra $\mathbb{H}_n$, its analogous Bernstein-Zelevinsky derivative $\mathbf{gBZ}_i(\widetilde{\sigma})$ also has multiplicity-free socle and cosocle, where $\widetilde{\sigma}$ is the corresponding module under Lusztig's second reduction [Lu89] (see [CS19, Theorem 6.3]). Here $\mathbf{gBZ}_i$ is defined in [CS19, Section 6.3]. Thus this implies that $\mathbf{gBZ}_i(\widetilde{\sigma}')$ has multiplicity-free socle and cosocle for any simple module $\widetilde{\sigma}'$ of $\mathbb{H}_n$ of real central character. (We remark that, by a rescaling argument, $\mathbb{H}_n$ can be defined as an associative algebra over $\mathbb{C}$ generated by the group algebra $\mathbb{C}[S_n]$ and the polynomial ring $S(\mathbb{C}^n)$ subject to some relations independent of q.)

We now consider arbitrary q (which is not of root of unity). In such case, by using Lusztig's first reduction [Lu89] (see [CS19, Section 5.2]) and [CS19, Theorem 5.3], we can transfer to the problem of some affine Hecke algebra modules $\mathcal{H}_{n_1} \otimes \ldots \otimes \mathcal{H}_{n_k}$ with $n_1 + \ldots + n_k = n$, and in which, we can apply Lusztig's second reduction. Now the result follows from the graded Hecke algebra case in previous paragraph. $\square$

For a given segment $\Delta = [a, b]$ for $b - a \in \mathbb{Z}_{\geq 0}$, the Steinberg module $\mathrm{St}_{\mathcal{H}}(\Delta)$ of $\mathcal{H}_{b-a+1}$ is the 1-dimensional module $\mathbb{C}v$ determined by: for all k,

$$T_k.v = -v, \quad \theta_k.v = q^{a+k-1}v.$$

For the St-derivatives, for an $\mathcal{H}_n$-module σ and a given segment Δ, one defines $D_\Delta(\sigma)$ to be either zero or the unique $\mathcal{H}_{n-i}$-module τ such that

$$\tau \boxtimes \mathrm{St}_{\mathcal{H}}(\Delta) \hookrightarrow \sigma|_{\mathcal{H}_{n-i} \otimes \mathcal{H}_i}.$$

Now one can define analogously the terminology of highest derivative segments, derivative resultant segments to formulate and prove the corresponding statements.

9.2. **Left BZ functor.** We define an automorphism $\zeta = \zeta_n$ on $\mathcal{H}_n$ determined by:

$$\zeta(\theta_k) = \theta_{n-k+1}^{-1}, \quad \zeta(T_k) = T_{n-k}$$

for any k. Note that ζ will send the relation (4) for the affine Hecke algebra to

$$T_{n-k}\theta_{n-k+1}^{-1} - \theta_{n-k}^{-1}T_{n-k} = (q-1)\theta_{n-k+1}^{-1},$$

which is equivalent to $\theta_{n-k}T_{n-k} - T_{n-k}\theta_{n-k+1} = (q-1)\theta_{n-k}$.

The left Bernstein-Zelevinsky functor ${}_i\mathbf{BZ}$ in the spirit of [CS21, Ch21] is defined as:

$${}_i\mathbf{BZ}(\sigma) = \zeta_{n-i}(\mathbf{BZ}_i(\zeta_n(\sigma))).$$

For any $\mathcal{H}_n$-module σ and $s \in \mathbb{C}$, we can define $\chi^s \otimes \sigma$ as

$$T_k \cdot_{\chi^s \otimes \sigma} v = T_k \cdot_\sigma v, \quad \theta_k \cdot_{\chi^s \otimes \sigma} v = q^s \theta_k \cdot_\sigma v.$$

We define the shifted Bernstein-Zelevinsky functors as:

$$\mathbf{BZ}_{[i]}(\sigma) = \chi^{-1/2} \otimes \mathbf{BZ}_i(\sigma),$$

$${}_{[i]}\mathbf{BZ}(\sigma) = \chi^{1/2} \otimes {}_i\mathbf{BZ}(\sigma).$$

As shown in [CS21] and the argument in Theorem 9.1, we have the following asymmetry property:

Theorem 9.2. *Let σ be an irreducible $\mathcal{H}_n$-module. Suppose i is not the level of σ i.e. not the largest integer such that $\mathbf{BZ}_i(\sigma) \neq 0$. If $\mathbf{BZ}_{[i]}(\sigma) \neq 0$ and ${}_{[i]}\mathbf{BZ}(\sigma) \neq 0$, then any simple quotient (resp. submodule) of $\mathbf{BZ}_{[i]}(\sigma)$ is not isomorphic to that of ${}_{[i]}\mathbf{BZ}(\sigma)$.*

References

[AL22] A. Aizenbud and E. Lapid, A binary operation on irreducible components of Lusztig's nilpotent varieties I: definition and properties, to appear in Pure and Applied Mathematics Quarterly.

[AGRS10] A. Aizenbud, D. Gourevitch, S. Rallis and G. Schiffmann, Multiplicity one theorems, Ann. of Math. (2) **172** (2010), no. 2, 1407-1434.

[Ar96] S. Ariki, On the decomposition numbers of the Hecke algebra of G(M, 1, n), J. Math. Kyoto Univ. 36 (1996), 789-808.

[AM20] H. Atobe, A. Mínguez, The explicit Zelevinsky-Aubert duality, Compositio. Math. 159(2) (2023), 380-418. doi:10.1112/S0010437X22007904

[BM94] Barbasch, D., Moy, A.: Whittaker models with an Iwahori-fixed vector. Contemp. Math. 177, 101-105 (1994)

[BC08] Barbasch, D., Ciubotaru, D. (2009). Whittaker unitary dual of affine graded Hecke algebras of type E. Compositio Mathematica, 145(6), 1563-1616. doi:10.1112/S0010437X09004230

[Be92] J. Bernstein. Represenations of p-adic groups. Harvard University, 1992. Lectures by Joseph Bernstein. Written by Karl E. Rumelhart.

[BZ76] I. N. Bernstein and A. V. Zelevinsky, Representations of the group $\mathrm{GL}(n, F)$, where F is a non-archimedean local field, Russian Math. Surveys 31:3 (1976), 1-68.

[BZ77] I. N. Bernstein and A. V. Zelevinsky, *Induced representations of reductive p-adic groups*, I, Ann. Sci. Ecole Norm. Sup. **10** (1977), 441-472.

[Bo76] Borel, A. Admissible representations of a semi-simple group over a local field with vectors fixed under an iwahori subgroup. Invent Math 35, 233-259 (1976). https://doi.org/10.1007/BF01390139

[BBBF18] Brubaker, B., Buciumas, V., Bump, D. et al. Hecke modules from metaplectic ice. Sel. Math. New Ser. 24, 2523-2570 (2018). https://doi.org/10.1007/s00029-017-0372-0

[Br98] Brundan, J. (1998). Modular branching rules and the Mullineux map for Hecke algebras of type A. Proceedings of the London Mathematical Society, 77(3), 551-581. doi:10.1112/S0024611598000562

[BK93] C.J. Bushnell and P.C. Kutzko, The admissible dual of GL(N) via compact open subgroups, Annals Math. Studies, Princeton Univ. Press (1993)

[Ca95] W. Casselman, Introduction to the theory of admissible representations of p-adic reductive groups, 1995

[CaSh98] William Casselman and Freydoon Shahidi, On irreducibility of standard modules for generic representations, Ann. Sci. École Norm. Sup. (4) 31 (1998), no. 4, 561-589 (English, with English and French summaries). DOI https://doi.org/10.1016/S0012-9593(98)80107-9

[Ch21] K.Y. Chan, Homological branching law for $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$: projectivity and indecomposability, Invent. math. (2021). https://doi.org/10.1007/s00222-021-01033-5

[Ch22] K.Y. Chan, Restriction for general linear groups: The local non-tempered Gan-Gross-Prasad conjecture (non-Archimedean case), Crelles Journal, vol. 2022, no. 783, 2022, pp. 49-94. https://doi.org/10.1515/crelle-2021-0066

[Ch23] K.Y. Chan, Ext-multiplicity theorem for standard representations of $(\mathrm{GL}_{n+1}, \mathrm{GL}_n)$, Math. Z. 303, 45 (2023). https://doi.org/10.1007/s00209-022-03198-y

[Ch22+] K.Y. Chan, On the product functor on inner forms of general linear group over a non-Archimedean local field, arXiv:2206.10928 (2022)

[Ch22+b] K.Y. Chan, Quotient branching law for p-adic $(\mathrm{GL}_{n+1}, \mathrm{GL}_n)$ I: generalized Gan-Gross-Prasad relevant pairs, arXiv:2212.05919 (2022)

[Ch22+c] K.Y. Chan, On commutations of derivatives and integrals of $\square$-irreducible representations for p-adic GL, arXiv:2210.17249 (2022)

[Ch22+d] K.Y. Chan, Construction of Bernstein-Zelevinsky derivatives and highest derivative mutlisegments II, preprint.

[CS18] Kei Yuen Chan and Gordan Savin, Iwahori component of the Gelfand-Graev representation, Math. Z. 288 (2018), no. 1-2, 125-133. MR 3774407, DOI 10.1007/s00209-017-1882-3

[CS19] K.Y. Chan and G. Savin, Bernstein-Zelevinsky derivatives: a Hecke algebra approach, International Mathematics Research Notices, Volume 2019, Issue 3, February 2019, Pages 731-760, https://doi.org/10.1093/imrn/rnx138

[CS21] K.Y. Chan and G. Savin, A vanishing Ext-branching theorem for $(\mathrm{GL}_{n+1}(F), \mathrm{GL}_n(F))$, Duke Math Journal, 2021, 170 (10), 2237-2261. https://doi.org/10.1215/00127094-2021-0028

[CPS17] Cogdell J.W., Piatetski-Shapiro I.I. (2017) Derivatives and L-Functions for GLn. In: Cogdell J., Kim JL., Zhu CB. (eds) Representation Theory, Number Theory, and Invariant Theory. Progress in Mathematics, vol 323. Birkhäuser, Cham. https://doi.org/10.1007/978-3-319-59728-7-5

[De16] T. Deng, Parabolic induction and geometry of orbital varieties for GL(n), arXiv:1603:06387, PhD thesis, Université Sorbinne Paris Cité (2016)

[GGP20] W.T. Gan, B.H. Gross, and D. Prasad, Branching laws for classical groups: the non-tempered case, Compositio Mathematica, 156(11) (2020), 2298-2367. doi:10.1112/S0010437X20007496

[GLS11] Christof Geiß, Bernard Leclerc, and Jan Schröer, Kac-Moody groups and cluster algebras, Adv. Math. 228 (2011), no. 1, 329-433.

[GV01] Grojnowski, I., Vazirani, M. Strong multiplicity one theorems for affine Hecke algebras of type A. Transformation Groups 6, 143-155 (2001). https://doi.org/10.1007/BF01597133

[Gu22] M. Gurevich, On restriction of unitarizable representations of general linear groups and the non-generic local Gan-Gross-Prasad conjecture. J. Eur. Math. Soc. 24 (2022), no. 1, pp. 265-302, DOI 10.4171/JEMS/1093

[Gu21] M. Gurevich, Graded Specht modules as Bernstein-Zelevinsky derivatives of the RSK model, to appear in IMRN, arXiv:2110.11381

[He11] Heiermann, V. Opérateurs d'entrelacement et algèbres de Hecke avec paramètres d'un groupe réductif p-adique: le cas des groupes classiques. Sel. Math. New Ser. 17, 713-756 (2011). https://doi.org/10.1007/s00029-011-0056-0

[Ja07] C. Jantzen, Jacquet modules of p-adic general linear groups, Represent. Theory **11** (2007), 45-83 https://doi.org/10/1090/S1088-4165-0700316-0

[JS83] H. Jacquet and J. Shalika, The Whittaker models of induced representations, Pacific Journal of Mathematics 109 (1983), 1, 107-120.

[JK22] Y. Jo, M. Krishnamurthy, The Langlands-Shahidi method for pairs via types and cover, to appear in Representation Theory

[KKKO18] S.-J. Kang, M. Kashiwara, M. Kim and S.-J. Oh, J. Amer. Math. Soc. 31 (2018), 349-426, doi.org/10.1090/jams/895

[EK08] N. Enomoto, M. Kashiwara, Symmetric Crystals and LLT-Ariki type conjectures for the affine Hecke algebras of type B. Combinatorial representation theory and related topics RIMS Kôkyûroku Besstsu, B8, Res. Inst. Math. Sci. (RIMS), Kyoto, pp. 1-20 (2008)

[Kl95] A.S. Kleshchev, Branching rules for modular representations of symmetric groups. II. J. reine angew. Math. 459 (1995), 163 212.

[Kl10] Alexander Kleshchev, Representation Theory of symmetric groups and related Hecke algebras, Bull. Amer. Math. Soc. 47 (2010), 419-481,https://doi.org/10.1090/S0273-0979-09-01277-4

[LM14] E. Lapid and A. Mínguez. On a determinantal formula of Tadić. Amer. J. Math. 136 (2014): 111-142.

[LM16] E. Lapid, A. Mínguez, *On parabolic induction on inner forms of the general linear group over a non-Archimedean local field*, Sel. Math. New Ser. (2016) 22, 2347-2400.

[LM18] E. Lapid, A. Mínguez, Geometric conditions for $\square$-irreducibility of certain representations of the general linear group over a non-archimedean local field, Advances in Mathematics, Volume 339, 2018, Pages 113-190, https://doi.org/10.1016/j.aim.2018.09.027.

[LM22] E. Lapid, Mínguez, A binary operation on irreducible components of Lusztig's nilpotent varieties II: applications and conjectures for representations of GLn over a non-archimedean local field, to appear in Pure and Applied Mathematics Quarterly.

[LLT96] A. Lascoux, B. Leclerc, and J.-Y. Thibon, Hecke algebras at roots of unity and crystal bases of quantum affine algebras, Comm. Math. Phys. 181 (1996), 205-263.

[Le03] B. Leclerc, Imaginary vectors in the dual canonical basis of Uq(n), Transform. Groups 8 (2003), no. 1, 95-104.

[Lu89] G. Lusztig, Affine Hecke algebras and their graded versions. J. Amer. Math. Soc. 2 (1989): 599-635.

[Lu91] G. Lusztig, Quivers, perverse sheaves, and quantized enveloping algebras J. Amer. Math. Soc. 4 (1991), 365-421

[Mie08] V. Miemietz, On Representations of Affine Hecke Algebras of Type B . Algebr Represent Theory 11, 369 (2008), doi.org/10.1007/s10468-008-9086-5

[MW86] C. Mœglin, J.-L. Waldspurger, Sur l'involution de Zelevinski. J. Reine Angew. Math. 372, 136-177 (1986), doi.org/10.1515/crll.1986.372.136

[MW12] C. Mœglin and J.-L. Waldspurger, La conjecture locale de Gross-Prasad pour les groupes speciaux orthogonaux: le cas general, Sur les conjectures de Gross et Prasad. II, Astersque. No. **247** (2012), 167-216.

[Mi08] A. Mínguez, Correspondance de Howe explicite : paires duales de type II, Annales scientifiques de l'École Normale Supérieure, Série 4, Tome 41 (2008) no. 5, pp. 717-741, doi.org/10.24033/asens.2080

[Mi09] A. Mínguez, Surl'irréducitibilité d'une induite parabolique, 2009, no. 629, Crelle's Journal, doi.org/10.1515/CRELLE.2009.028

[Ma13] Nadir Matringe, Essential Whittaker functions for GL(n), Documenta Math. 18 (2013) 1191-1214

[Of18] O. Offen, On symplectic periods and restriction to SL(2n), Math. Z. 294, 1521-1552 (2020), doi.org/10.1007/s00209-019-02390-x

[OS10] Eric Opdam. Maarten Solleveld. Discrete series characters for affine Hecke algebras and their formal degrees. Acta Math. 205 (1) 105 - 187, 2010, doi.org/10.1007/s11511-010-0052-9

[Pr18] D. Prasad, An Ext-analogue of branching laws, ICM proceedings 2018.

[Ra07] Raghuram, A. (2007). A Künneth Theorem for p-Adic Groups. Canadian Mathematical Bulletin, 50(3), 440-446, doi:10.4153/CMB-2007-043-5

[Re02] M. Reeder, Isogenies of Hecke algebras and a Langlands correspondence for ramified principal series representations, Represent. Theory 6 (2002), 101-126, DOI: https://doi.org/10.1090/S1088-4165-02-00167-X

[Ro85] J.D. Rogawski, On modules over the Hecke algebra of a p-adic group. Invent Math 79, 443-465 (1985). https://doi.org/10.1007/BF01388516

[SVV11] P. Shan, M. Varagnolo, E. Vasserot, Canonical bases and affine Hecke algebras of type D, Advances in Mathematics, Volume 227, Issue 1, 2011, Pages 267-291, doi.org/10.1016/j.aim.2011.01.024.

[SV17] V. Sécherre, C. G. Venketasubramanian. Modular representations of GL(n) distinguished by GL(n-1) over a p-adic field. IMRN, 2017 (14), pp.4435-4492. 10.1093/imrn/rnw150.

[SS12] V. Sécherre, S. Stevens, Smooth Representations of GLm(D) VI: Semisimple Types, International Mathematics Research Notices, Volume 2012, Issue 13, 2012, Pages 2994-3039, https://doi.org/10.1093/imrn/rnr122

[Ta87] M. Tadić: Unitary representations of GL(n), derivatives in the non-Archimedean case, V. Mathematikertreffen Zagreb-Graz (Mariatrost/Graz, 1986), Ber. Math.-Statist. Sekt. Forschungsgesellsch. Joanneum, vol. 274, Forschungszentrum Graz, Graz, pp. Ber. No. 281, 19 (1987)

[Ta86] M. Tadić, On the classification of irreducible unitary representations of GL(n) and the conjectures of Bernstein and Zelevinsky, Ann. Sci. École Norm. Sup., **19** (1986), 335-382.

[VV11] M. Varagnolo, E. Vasserot, Canonical bases and affine Hecke algebras of type B. Invent. math. 185, 593-693 (2011). https://doi.org/10.1007/s00222-011-0314-y

[Ve13] Venketasubramanian, C.G. On representations of GL(n) distinguished by GL(n-1) over a p-adic field. Isr. J. Math. 194, 1-44 (2013). doi.org/10.1007/s11856-012-0152-7

[Wa86] Waldspurger, J.-L.. Algèbres de Hecke et induites de représentations cuspidales, pour GL (N). Journal für die reine und angewandte Mathematik 370 (1986): 127-191.

[Xu17] B. Xu, On Mœglin's parametrization of Arthur packets for p-adic quasisplit Sp(N) and SO(N). Canad. J. Math. 69 (2017), no. 4, 890-960.

[Ze80] A. Zelevinsky, *Induced representations of reductive p-adic groups II*, Ann. Sci. Ecole Norm. Sup. **13** (1980), 154-210.

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