CONSTRUCTION OF SIMPLE QUOTIENTS OF BERNSTEIN-ZELEVINSKY DERIVATIVES AND HIGHEST DERIVATIVE MULTISEGMENTS II: MINIMAL SEQUENCES

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Abstract. Let F be a non-Archimedean local field. For any irreducible smooth representation π of $\mathrm{GL}_n(F)$ and a multisegment $\mathfrak{m}$, we have an operation $D_{\mathfrak{m}}(\pi)$ to construct a simple quotient τ of a Bernstein-Zelevinsky derivative of π. This article continues the previous one to study the following poset

$$\mathcal{S}(\pi,\tau) := \{\mathfrak{n} : D_{\mathfrak{n}}(\pi) \cong \tau\},$$

where $\mathfrak{n}$ runs for all the multisegments. Here the partial ordering on $\mathcal{S}(\pi,\tau)$ comes from the Zelevinsky ordering. We show that the poset has a unique minimal multisegment. Along the way, we introduce two new ingredients: fine chain orderings and local minimizability.

1. Introduction

We refer the reader to [Ch22+d] for some background of Bernstein-Zelevinsky derivatives. A main goal of this article together with [Ch22+d, Ch22+b] is to give a comprehensive study on the simple quotients of Bernstein-Zelevinsky derivatives. It is also related to a branching law problem of a sign representation for affine Hecke algebras of type A (see [CS19] and [Ch22+d, Appendix]). The results for those simple quotients may be regarded as generalizing the case of ladder representations in [LM14] and generic representations [Ch21].

In the previous article [Ch22+d], we transfer some problems of studying simple quotients into combinatorial problems. This article studies the combinatorial structure of the set $\mathcal{S}(\pi,\tau)$ and a key result is the uniqueness of the minimal element in $\mathcal{S}(\pi,\tau)$. We shall explore further properties of such element in the sequel.

1.1. **Notations.** Let $G_n = \mathrm{GL}_n(F)$, the general linear group over a non-Archimedean field F. Fix a cuspidal representation ρ. We introduce basic notations:

- Let $\nu : G_n \to \mathbb{C}^\times$ be the character $\nu(g) = |\det(g)|_F$, where $|.|_F$ is the norm for F.
- For $a, b \in \mathbb{Z}$ with $b - a \in \mathbb{Z}_{\geq 0}$, we call

$$[a,b]_\rho := \left\{\nu^a\rho, \ldots, \nu^b\rho\right\} \tag{1.1}$$

  be a *segment*. We also set $[a, a-1]_\rho = \emptyset$ for $a \in \mathbb{Z}$. For a segment $\Delta = [a,b]_\rho$, we write $a(\Delta) = \nu^a\rho$ and $b(\Delta) = \nu^b\rho$. We also write:

$$[a]_\rho := [a,a]_\rho,$$

  which is called a singleton segment. We may also write $[\nu^a\rho, \nu^b\rho]$ for $[a,b]_\rho$ and write $[\nu^a\rho]$ for $[a]_\rho$. The relative length of a segment $[a,b]_\rho$ is defined as $b-a+1$, and we shall denote by $l_r([a,b]_\rho)$. The absolute length of a segment $[a,b]_\rho$ is defined as $(b-a+1)n(\rho)$, and we shall denote by $l_a([a,b]_\rho)$ as before.
- Let Seg be the set of all segments including the empty set. Let Seg_ρ be the subset of Seg containing all segments of the form $[a,b]_\rho$ for some $a, b \in \mathbb{Z}$.

- A *multisegment* is a multiset of segments. Let Mult be the set of all multisegments and let Mult_ρ be the subset of Mult containing all multisegments whose segments are in Seg_ρ. We also consider the empty set $\emptyset$ to be also in Mult.
- Two segments Δ and Δ' are said to be *linked* if $\Delta \cup \Delta'$ is still a segment, and $\Delta \not\subset \Delta'$ and $\Delta' \not\subset \Delta$. Otherwise, it is called to be not linked or unlinked.
- For $\rho_1, \rho_2 \in \mathrm{Irr}^c$, we write $\rho_2 < \rho_1$ if $\rho_1 \cong \nu^a \rho_2$ for some integer $a > 0$. For two segments Δ_1, Δ_2, we write $\Delta_1 < \Delta_2$ if Δ_1 and Δ_2 are linked and $b(\Delta_1) < b(\Delta_2)$.
- For two multisegments $\mathfrak{m}$ and $\mathfrak{n}$, we write $\mathfrak{m}+\mathfrak{n}$ to be the union of two multisegments, counting multiplicities. For a multisegment $\mathfrak{m}$ and a segment Δ, $\mathfrak{m}+\Delta = \mathfrak{m}+\{\Delta\}$ if Δ is non-empty; and $\mathfrak{m}+\Delta = \mathfrak{m}$ if Δ is empty. The notions $\mathfrak{m}-\mathfrak{n}$ and $\mathfrak{m}-\Delta$ are defined in a similar way.
- For two multisegments $\mathfrak{m}_1$ and $\mathfrak{m}_2$, write $\mathfrak{m}_2 \leq_Z \mathfrak{m}_1$ if $\mathfrak{m}_2$ can be obtained by a sequence of elementary intersection-union operations from $\mathfrak{m}_1$ (in the sense of [Ze80], see [Ch22+d]) or $\mathfrak{m}_1 = \mathfrak{m}_2$. In particular, if any pair of segments in $\mathfrak{m}$ is unlinked, then $\mathfrak{m}$ is a minimal element under $\leq_Z$. We shall equip Mult_ρ with the poset structure by $\leq_Z$.
- Let $\mathrm{Alg}(G_n)$ be the category of smooth representations of G_n. For $\pi \in \mathrm{Alg}(G_n)$, denote by $\pi^\vee$ the smooth dual of π.
- For a segment $\Delta = [a,b]_\rho$, define ${}^-\Delta = [a+1,b]_\rho$. For a multisegment $\mathfrak{m}$, define
$${}^-\mathfrak{m} = \left\{ {}^-\Delta : \Delta \in \mathfrak{m}, \Delta \text{ is not singleton } \right\} \quad \text{(counting multiplicities)}.$$
- Let $\mathrm{Irr}(G_n)$ be the set of (isomorphism classes of) irreducible smooth complex representations of G_n. Let $\mathrm{Irr} = \sqcup_n \mathrm{Irr}(G_n)$. Let Irr_ρ (resp. $\mathrm{Irr}_\rho(G_n)$) be the subset of Irr (resp. $\mathrm{Irr}_\rho(G_n)$) containing irreducible representations which are an irreducible quotient of $\nu^{a_1}\rho \times \ldots \times \nu^{a_k}\rho$, for some integers $a_1, \ldots, a_k \in \mathbb{Z}$.
- For each segment Δ, we shall denote respectively by $\mathrm{St}(\Delta)$ and $\langle \Delta \rangle$ the corresponding essentially square-integrable representation and segment representation [Ze80].
- Let $N_i \subset G_n$ (depending on n) be the unipotent radical containing matrices of the form $\begin{pmatrix} I_{n-i} & * \\ & I_i \end{pmatrix}$. For a smooth representation π of G_n, we write π_{N_i} to be its Jacquet module.
- For $\pi \in \mathrm{Irr}$, $n(\pi)$ is defined to be the number that $\pi \in \mathrm{Irr}(G_{n(\pi)})$.
- For any smooth representation π_1 of G_{n_1} and smooth representation π_2 of G_{n_2}, define $\pi_1 \times \pi_2$ to be the normalized parabolic induction.
- For any multisegment $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_k\}$ with the labellings satisfying that, for $i < j$, $\Delta_i \not< \Delta_j$. Following [Ze80, Theorem 6.1], define $\langle \mathfrak{m} \rangle$ to be the unique simple submodule of $\zeta(\mathfrak{m}) := \langle \Delta_1 \rangle \times \ldots \times \langle \Delta_k \rangle$. Define the $\mathrm{St}(\mathfrak{m})$ to be the unique simple quotient of $\lambda(\mathfrak{m}) := \mathrm{St}(\Delta_1) \times \ldots \times \mathrm{St}(\Delta_k)$.

1.2. **Main results.** For $\pi \in \mathrm{Irr}_\rho(G_n)$, there is at most one irreducible module $\tau \in \mathrm{Irr}_\rho(G_{n-i})$ such that
$$\tau \boxtimes \mathrm{St}(\Delta) \hookrightarrow \pi_{N_i}.$$
If such τ exists, we denote such τ by $D_\Delta(\pi)$. Otherwise, we set $D_\Delta(\pi) = 0$. We shall refer D_Δ to be a *St-derivative*. Let $\epsilon_\Delta(\pi)$ be the largest integer such that $(D_\Delta)^k(\pi) \neq 0$.

A sequence of segments $[a_1,b_1]_\rho, \ldots, [a_k,b_k]_\rho$ (all $a_j, b_j \in \mathbb{Z}$) is said to be in an *ascending order* if for any $i \leq j$, either $[a_i,b_i]_\rho$ and $[a_j,b_j]_\rho$ are unlinked; or $a_i < a_j$. For a multisegment $\mathfrak{n} \in \mathrm{Mult}_\rho$, which we write the segments in $\mathfrak{n}$ in an ascending order $\Delta_1, \ldots, \Delta_k$. Define
$$D_{\mathfrak{n}}(\pi) := D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi).$$

The derivative is independent of the ordering of an ascending sequence [Ch22+d]. In particular, one may choose the ordering such that $a_1 \leq \ldots \leq a_k$. We say that $\mathfrak{n}$ is *admissible* to π if $D_{\mathfrak{n}}(\pi) \neq 0$. We refer the reader to [LM16, Ch22+b, Ch22+c] (and references therein) for more theory on derivatives.

For $\pi \in \mathrm{Irr}_\rho$, denote its i-th Bernstein-Zelevinsky derivatives by $\pi^{(i)}$ (see [Ze80, Ch22+d] for precise definitions and we shall not need this in this sequel). For a simple quotient τ of $\pi^{(i)}$, define

$$\mathcal{S}(\pi, \tau) := \{\mathfrak{n} \in \mathrm{Mult}_\rho : D_{\mathfrak{n}}(\pi) \cong \tau\}.$$

The ordering $\leq_Z$ induces a partial ordering on $\mathcal{S}(\pi, \tau)$, and we shall regard it as a poset.

In [Ch22+d], we showed a combinatorial process, called *removal process*, in studying the effect of D_Δ. Two applications of removal process are given below:

Theorem 1.1. *(=Theorem 4.4) Let $\pi \in \mathrm{Irr}_\rho$. Let τ be a simple quotient of $\pi^{(i)}$ for some i. If $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{S}(\pi, \tau)$ and $\mathfrak{n}_1 \leq_Z \mathfrak{n}_2$, then any $\mathfrak{n}_3 \in \mathrm{Mult}_\rho$ satisfying $\mathfrak{n}_1 \leq_Z \mathfrak{n}_3 \leq_Z \mathfrak{n}_2$ is also in $\mathcal{S}(\pi, \tau)$.*

In other words, $\mathcal{S}(\pi, \tau)$ is *convex* in the sense of [St12, Section 3.1].

Theorem 1.2. *(=Theorem 6.4) Let $\pi \in \mathrm{Irr}_\rho$. Let τ be a simple quotient of $\pi^{(i)}$ for some i. If $\mathcal{S}(\pi, \tau) \neq \emptyset$, then $\mathcal{S}(\pi, \tau)$ has a unique minimal element with respect to $\leq_Z$.*

The condition $\mathcal{S}(\pi, \tau) \neq \emptyset$ will be removed in [Ch22+b]. There are some other criteria for a minimal sequence such as the nonoverlapping property and in terms of η_Δ in Section 9. We remark that there is no uniqueness for maximal elements in general. We give an example in Section 8.

1.3. **Organization of this article.** Section 2 recalls results on highest derivative multisegments and removal processes established in [Ch22+d]. Section 3 defines a notion of fine chains and fine chain orderings to facilitate comparisons with the Zelevinsky ordering. Section 4 shows the closedness property for $\mathcal{S}(\pi, \tau)$. In Section 5, we shall introduce a notion of minimizable functions, used to show the uniqueness of a minimal element in Section 6. Section 7 gives two examples of the unique minimal elements. Section 8 gives an example which uniqueness of $\leq$-maximality fails. Section 9 studies equivalent conditions for minimality in two segment cases.

2. Highest derivative multisegments and removal process

In this section, we recall some results in [Ch22+d].

2.1. **More notations on multisegments.** For an integer c, let $\mathrm{Mult}^a_{\rho,c}$ be the subset of Mult_ρ containing all multisegments $\mathfrak{m}$ such that any segment Δ in $\mathfrak{m}$ satisfies $a(\Delta) \cong \nu^c\rho$. Similarly, define $\mathrm{Mult}^b_{\rho,c}$ to be the subset of Mult_ρ containing all multisegments $\mathfrak{m}$ such that any segment Δ in $\mathfrak{m}$ satisfies $b(\Delta) \cong \nu^c\rho$. The empty sets are also considered in $\mathrm{Mult}^a_{\rho,c}$ and $\mathrm{Mult}^b_{\rho,c}$.

For a multisegment $\mathfrak{m}$ in Mult_ρ and an integer c, let $\mathfrak{m}[c]$ be the submultisegment of $\mathfrak{m}$ containing all the segments Δ satisfying $a(\Delta) \cong \nu^c\rho$; and let $\mathfrak{m}\langle c\rangle$ be the submultisegment of $\mathfrak{m}$ containing all the segments Δ satisfying $b(\Delta) \cong \nu^c\rho$.

For a multisegment $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_k\}$, we also set:

$$l_a(\mathfrak{m}) = l_a(\Delta_1) + \ldots + l_a(\Delta_k), \quad l_r(\mathfrak{m}) = l_r(\Delta_1) + \ldots + l_r(\Delta_k).$$

Here a refers to the absolute length while r refers to the relative length.

Fix an integer c. Let $\Delta_1 = [c, b_1]_\rho, \Delta_2 = [c, b_2]_\rho$ be two non-empty segments. We write $\Delta_1 \leq^a_c \Delta_2$ if $b_1 \leq b_2$, and write $\Delta_1 <^a_c \Delta_2$ if $b_1 < b_2$.

For non-empty $\mathfrak{m}_1, \mathfrak{m}_2$ in $\mathrm{Mult}^a_{\rho,c}$, label the segments in $\mathfrak{m}_1$ as: $\Delta_{1,k} \leq^a_c \ldots \leq^a_c \Delta_{1,2} \leq^a_c \Delta_{1,1}$ and label the segments in $\mathfrak{m}_2$ as: $\Delta_{2,r} \leq^a_c \ldots \leq^a_c \Delta_{2,2} \leq_c \Delta_{2,1}$. We define the lexicographical ordering: $\mathfrak{m}_1 \leq^a_c \mathfrak{m}_2$ if $k \leq r$ and, for any $i \leq k$, $\Delta_{1,i} \leq^a_c \Delta_{2,i}$. We write $\mathfrak{m}_1 <^a_c \mathfrak{m}_2$ if $\mathfrak{m}_1 \leq^a_c \mathfrak{m}_2$ and $\mathfrak{m}_1 \neq \mathfrak{m}_2$.

We also need a 'right' ordering. One can define $[a_1, c]_\rho \leq^b_c [a_2, c]_\rho$ if $a_1 < a_2$, and similarly define $[a_1, c]_\rho <^b_c [a_2, c]_\rho$. One similarly define $\leq^b_c$ on $\mathrm{Mult}^b_{\rho,c}$.

2.2. **Highest derivative multisegments.** A multisegment $\mathfrak{m}$ is said to be *at the point* $\nu^c\rho$ if any segment Δ in $\mathfrak{m}$ takes the form $[c, b]_\rho$ for some $b \geq c$. For $\pi \in \mathrm{Irr}_\rho$, define $\mathfrak{mxpt}^a(\pi, c)$ to be the maximal multisegment in $\mathrm{Mult}^a_{\rho,c}$ such that $D_{\mathfrak{mxpt}^a(\pi,c)}(\pi) \neq 0$. (see [Ch22+d] for the meaning of maximality.) Define the *highest derivative multisegment* of $\pi \in \mathrm{Irr}_\rho$ to be

$$\mathfrak{hd}(\pi) := \sum_{c \in \mathbb{Z}} \mathfrak{mxpt}^a(\pi, c).$$

It is shown in [Ch22+d] that $D_{\mathfrak{hd}(\pi)}(\pi)$ is the highest derivative of π in the sense of [Ze80].

2.3. **Removal process.** We write $[a, b]_\rho \prec^L [a', b']_\rho$ if either $a < a'$; or $a = a'$ and $b < b'$. Here L means to compare on the left value a and we avoid to further use a for confusing with previous notations. A segment $\Delta = [a, b]_\rho$ is said to be *admissible* to a multisegment $\mathfrak{h}$ if there exists a segment of the form $[a, c]_\rho$ in $\mathfrak{h}$ for some $c \geq b$. We now recall the removal process.

Definition 2.1. [Ch22+d] Let $\mathfrak{h} \in \mathrm{Mult}_\rho$ and let $\Delta = [a, b]_\rho$ be admissible to $\mathfrak{h}$. The *removal process* on $\mathfrak{h}$ by Δ is an algorithm to carry out the following steps:

(1) Pick a shortest segment $[a, c]_\rho$ in $\mathfrak{h}[a]$ satisfying $b \leq c$. Set $\Delta_1 = [a, b]_\rho$. Set $a_1 = a$ and $b_1 = c$.
(2) One recursively finds the $\prec^L$-minimal segment $\Delta_i = [a_i, b_i]_\rho$ in $\mathfrak{h}$ such that $a_{i-1} < a_i$ and $b_i < b_{i-1}$. The process stops if one can no longer find those segments.
(3) Let $\Delta_1, \ldots, \Delta_r$ be all those segments. For $1 \leq i < r$, define $\Delta_i^{tr} = [a_{i+1}, b_i]_\rho$ and $\Delta_r^{tr} = [b + 1, b_r]_\rho$ (possibly empty).
(4) Define

$$\mathfrak{r}(\Delta, \mathfrak{h}) := \mathfrak{h} - \sum_{i=1}^r \Delta_i + \sum_{i=1}^r \Delta_i^{tr}.$$

We call $\Delta_1, \ldots, \Delta_r$ to be the *removal sequence* for $(\Delta, \mathfrak{h})$. We also define $\Upsilon(\Delta, \mathfrak{h}) = \Delta_1$. If Δ is not admissible to $\mathfrak{h}$, we set $\mathfrak{r}(\Delta, \mathfrak{h}) = \infty$, called the infinity multisegment. We also set $\mathfrak{r}(\Delta, \infty) = \infty$.

Remark 2.2. A multisegment $\mathfrak{h}$ is called *generic* if any two segments in $\mathfrak{h}$ are mutually unlinked. The special feature in this case is that $\mathrm{St}(\mathfrak{h})$ is generic and $\mathfrak{hd}(\mathrm{St}(\mathfrak{h})) = \mathfrak{h}$. In such case, for $\Delta \in \mathrm{Seg}_\rho$, it is shown in [Ch21] that $D_\Delta(\pi)$ is generic. On the other hand, $\mathfrak{r}(\Delta, \mathfrak{h})$ coincides with the generic multisegment which has the same cuspidal support as $D_\Delta(\pi)$.

2.4. **Computations on removal process.** We recall the following properties for computations:

Lemma 2.3. [Ch22+d] *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$ and let $\Delta, \Delta' \in \mathrm{Seg}_\rho$ be admissible to $\mathfrak{h}$. Then*

(1) *Let $\mathfrak{h}^* = \mathfrak{h} - \Upsilon(\Delta, \mathfrak{h}) + {}^-\Upsilon(\Delta, \mathfrak{h})$. Then $\mathfrak{r}(\Delta, \mathfrak{h}) = \mathfrak{r}({}^-\Delta, \mathfrak{h}^*)$.*
(2) *Write $\Delta = [a, b]_\rho$. For any $a' < a$, $\mathfrak{r}(\Delta, \mathfrak{h})[a'] = \mathfrak{h}[a']$.*
(3) *If $\Delta \in \mathfrak{h}$, then $\mathfrak{r}(\Delta, \mathfrak{h}) = \mathfrak{h} - \Delta$.*
(4) *Suppose $a(\Delta) = a(\Delta')$. Then*

$$\Upsilon(\Delta, \mathfrak{h}) + \Upsilon(\Delta', \mathfrak{r}(\Delta, \mathfrak{h})) = \Upsilon(\Delta', \mathfrak{h}) + \Upsilon(\Delta, \mathfrak{r}(\Delta', \mathfrak{h})).$$

(5) *If* Δ, Δ' *are unlinked, then* $\mathfrak{r}(\Delta', \mathfrak{r}(\Delta, \mathfrak{h})) = \mathfrak{r}(\Delta, \mathfrak{r}(\Delta', \mathfrak{h}))$.

2.5. **Removal process for multisegments.** For $\mathfrak{h} \in \mathrm{Mult}_\rho$, and a multisegment $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_r\} \in \mathrm{Mult}_\rho$ written in an ascending order, define:

$$\mathfrak{r}(\mathfrak{m}, \mathfrak{h}) = \mathfrak{r}(\Delta_r, \ldots, \mathfrak{r}(\Delta_1, \mathfrak{h}) \ldots).$$

We say that $\mathfrak{m}$ is *admissible* to $\mathfrak{h}$ if $\mathfrak{r}(\mathfrak{m}, \mathfrak{h}) \neq \infty$.

The relation to derivatives is the following:

Theorem 2.4. [Ch22+d] *Let* $\pi \in \mathrm{Irr}_\rho$. *Let* $\mathfrak{m}, \mathfrak{m}' \in \mathrm{Mult}_\rho$ *be admissible to* π. *Then* $\mathfrak{m}, \mathfrak{m}'$ *are admissible to* $\mathfrak{hd}(\pi)$, *and furthermore,* $D_{\mathfrak{m}}(\pi) \cong D_{\mathfrak{m}'}(\pi)$ *if and only if* $\mathfrak{r}(\mathfrak{m}, \pi) = \mathfrak{r}(\mathfrak{m}', \pi)$.

2.6. **More relations to derivatives.** For $\mathfrak{h} \in \mathrm{Mult}_\rho$ and $\Delta = [a, b]_\rho \in \mathrm{Seg}_\rho$, set

$$\varepsilon_\Delta(\mathfrak{h}) = |\left\{ \widetilde{\Delta} \in \mathfrak{h}[a] : \Delta \subset \widetilde{\Delta} \right\}|.$$

Theorem 2.5. [Ch22+d, Theorem 6.20] *Let* $\Delta = [a, b]_\rho \in \mathrm{Seg}_\rho$ *be admissible to* π. *Let* $\Delta' = [a', b']_\rho \in \mathrm{Seg}_\rho$. *If either* $a' > a$; *or* Δ' *and* Δ *are unlinked, then* $\epsilon_{\Delta'}(D_\Delta(\pi) = \epsilon_{\Delta'}(\mathfrak{r}(\Delta, \pi))$.

3. Fine Chains

Recall that $\mathcal{S}(\pi, \tau)$ is defined in Section 1.2. We introduce a notion of fine chains in Definition 3.5 in order to give an effective comparison of the effect of two removal processes (Lemma 3.9).

3.1. **Basic idea on the proof for Theorem 4.4.** Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. We consider two segments in the form $\Delta = [a, b]_\rho$ and $\Delta' = [a+1, b']_\rho$ for $b < b'$. In such case, let $\widetilde{\Delta} = \Delta \cup \Delta' = [a, b']_\rho$ and $\widetilde{\Delta}' = \Delta \cap \Delta' = [a+1, b]_\rho$. Let $\Omega = \Upsilon(\Delta, \mathfrak{h})$ and let $\widetilde{\Omega} = \Upsilon(\widetilde{\Delta}, \mathfrak{h})$. Let

$$\mathfrak{h}^* = \mathfrak{h} - \Omega + {}^-\Omega, \quad \widetilde{\mathfrak{h}} = \mathfrak{h} - \widetilde{\Omega} + {}^-\widetilde{\Omega}.$$

Applying Lemma 2.3, we have

$$\mathfrak{r}(\{\Delta', \Delta\}, \mathfrak{h}) = \mathfrak{r}(\left\{\Delta', {}^-\Delta\right\}, \mathfrak{h}^*)$$

and

$$\mathfrak{r}(\left\{\widetilde{\Delta}', \widetilde{\Delta}\right\}, \mathfrak{h}) = \mathfrak{r}(\left\{\widetilde{\Delta}', {}^-\widetilde{\Delta}\right\}, \widetilde{\mathfrak{h}}).$$

Note that $\{\Delta', {}^-\Delta\} = \left\{\widetilde{\Delta}', {}^-\widetilde{\Delta}\right\}$. Then the following statements are equivalent:

(1) $\mathfrak{r}(\{\Delta, \Delta'\}, \mathfrak{h}) = \mathfrak{r}(\{\widetilde{\Delta}, \widetilde{\Delta}'\}, \mathfrak{h})$;
(2) $\mathfrak{h}^* = \widetilde{\mathfrak{h}}$;
(3) $\Omega = \widetilde{\Omega}$.

The general case for the effect of intersection-union process needs some modifications for consideration and we shall focus on the condition (3). In particular, Sections 3.2 and 3.3 will improve Lemma 2.3(1) to do some multiple 'cutting-off' on starting points.

3.2. **Multiple removal of starting points.** Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$. Let a be the smallest integer such that $\mathfrak{n}[a] \neq 0$. Write $\mathfrak{n}[a] = \left\{\overline{\Delta}_1, \ldots, \overline{\Delta}_k\right\}$.

(1) Suppose $\mathfrak{n}[a]$ is admissible to $\mathfrak{h}$. Let $\mathfrak{r}_i = \mathfrak{r}(\left\{\overline{\Delta}_i, \ldots, \overline{\Delta}_1\right\}, \mathfrak{h})$ and $\mathfrak{r}_0 = \mathfrak{h}$. Define

$$\mathfrak{s}(\mathfrak{n}, \mathfrak{h}) = \left\{\Upsilon(\overline{\Delta}_1, \mathfrak{r}_0), \ldots, \Upsilon(\overline{\Delta}_k, \mathfrak{r}_{k-1})\right\}.$$

(2) Suppose $\mathfrak{n}[a]$ is not admissible to $\mathfrak{h}$. Define $\mathfrak{s}(\mathfrak{n}, \mathfrak{h}) = \emptyset$.

Lemma 3.1. *The above definition $\mathfrak{s}(\mathfrak{n},\mathfrak{h})$ is well-defined i.e. independent of the ordering for the segments in $\mathfrak{n}[a]$.*

Proof. One switches a consecutive pair of segments each time, and then applies Lemmas 2.3(4) and 2.3(5). ☐

Following from definitions,

Lemma 3.2. *With the notations as above,*

$$\mathfrak{s}(\mathfrak{n},\mathfrak{h}) = \mathfrak{s}(\mathfrak{n}[a],\mathfrak{h}) = \mathfrak{s}(\mathfrak{n}[a],\mathfrak{h}[a]).$$

We define a truncation of $\mathfrak{h}$:

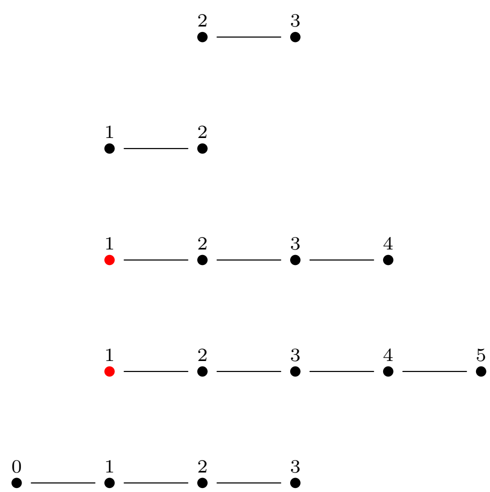
$$\mathfrak{trr}(\mathfrak{n},\mathfrak{h}) = \mathfrak{h} - \mathfrak{s}(\mathfrak{n},\mathfrak{h}) + {}^-(\mathfrak{s}(\mathfrak{n},\mathfrak{h})), \tag{3.2}$$

and a truncation of $\mathfrak{n}$:

$$\mathfrak{trd}(\mathfrak{n},\mathfrak{h}) = \mathfrak{n} - \mathfrak{n}[a] + {}^-(\mathfrak{n}[a])$$

Here we use r in $\mathfrak{trr}$ for the derivative 'resultant' multisegment and d for $\mathfrak{trd}$ for 'taking the derivative for multisegment $\mathfrak{n}$'.

Example 3.3. Let $\mathfrak{h} = \{[0,3],[1,2],[1,4],[1,5],[2,3]\}$. Let $\mathfrak{n} = \{[1,3],[1,5],[2]\}$.



The two red bullets in $\mathfrak{h}$ are 'truncated' to obtain $\mathfrak{trr}(\mathfrak{n},\mathfrak{h}) = \{[0,3],[1,2],[2,4],[2,5],[2,3]\}$. We also have $\mathfrak{s}(\mathfrak{n},\mathfrak{h}) = \{[1,4],[1,5]\}$ and $\mathfrak{trd}(\mathfrak{n},\mathfrak{h}) = \{[2,3],[2,5],[2]\}$.

Lemma 3.4. *(multiple removal of starting points, c.f. Lemma 2.3(1)) Let $\mathfrak{n},\mathfrak{h},a$ be as above. Then*

$$\mathfrak{r}(\mathfrak{n},\mathfrak{h}) = \mathfrak{r}(\mathfrak{trd}(\mathfrak{n},\mathfrak{h}),\mathfrak{trr}(\mathfrak{n},\mathfrak{h})).$$

Proof. Write $\mathfrak{n}[a] = \left\{\overline{\Delta}_1,\ldots,\overline{\Delta}_k\right\}$. Relabeling if necessary, $\overline{\Delta}_1$ is the shortest segment in $\mathfrak{n}[a]$. Let

$$\mathfrak{h}_1^* = \mathfrak{h} - \left\{\overline{\Delta}_1\right\} + \left\{{}^-\overline{\Delta}_1\right\}.$$

We observe that:

$$\begin{aligned}
\mathfrak{r}(\mathfrak{n}[a], \mathfrak{h}) &= \mathfrak{r}(\{\overline{\Delta}_2, \ldots, \overline{\Delta}_k\}, \mathfrak{r}(\overline{\Delta}_1, \mathfrak{h})) \\
&= \mathfrak{r}(\{\overline{\Delta}_2, \ldots, \overline{\Delta}_k\}, \mathfrak{r}(^-\overline{\Delta}_1, \mathfrak{h}_1^*)) \\
&= \mathfrak{r}(^-\overline{\Delta}_1, \mathfrak{r}(\{\overline{\Delta}_2, \ldots, \overline{\Delta}_k\}, \mathfrak{h}_1^*)) \\
&= \mathfrak{r}(^-\overline{\Delta}_1, \mathfrak{r}(\{^-\overline{\Delta}_2, \ldots, {}^-\overline{\Delta}_k\}, \mathfrak{trr}(\mathfrak{n}, \mathfrak{h}))) \\
&= \mathfrak{r}(^-(\mathfrak{n}[a]), \mathfrak{trr}(\mathfrak{n}, \mathfrak{h})),
\end{aligned}$$

where the second equation follows from Lemma 2.3(1), the first, third and last equations follow from Lemma 2.3(5), and the forth equation follows from the induction hypothesis (where the basic case is again Lemma 2.3(1)). (It is straightforward to check from definitions that $\mathfrak{trr}(\mathfrak{n}, \mathfrak{h}) = \mathfrak{trr}(\{\Delta_2, \ldots, \Delta_k\}, \mathfrak{h}_1^*)$.)

The lemma then follows by applying $\mathfrak{r}(\mathfrak{n} - \mathfrak{n}[a], .)$ on the first and last terms. □

3.3. **Fine chians.**

Definition 3.5. An *infinity multisegment*, denoted by ∞, is just a symbol which will be used to indicate the situation of non-admissbility. For $c \in \mathbb{Z}$, we modify the ordering $<_c^a$ on $\mathrm{Mult}_{\rho,c}^a \cup \{\infty\}$ as follows. For $\mathfrak{p}_1$, $\mathfrak{p}_2$ in $\mathrm{Mult}_{\rho,c}^a \cup \{\infty\}$, if $\mathfrak{p}_1 \neq \infty$ and $\mathfrak{p}_2 = \infty$, we also write $\mathfrak{p}_1 <_c^a \mathfrak{p}_2$. If $\mathfrak{p}_1 = \mathfrak{p}_2 = \infty$, we write $\mathfrak{p}_1 \leq_c^a \mathfrak{p}_2$.

Definition 3.6. (Collections of first segments in the removal sequence) Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$. Set $\mathfrak{n}_0 = \mathfrak{n}$ and $\mathfrak{h}_0 = \mathfrak{h}$. We recursively define:

$$\mathfrak{h}_i = \mathfrak{trr}(\mathfrak{h}_{i-1}, \mathfrak{n}_{i-1}), \quad \mathfrak{n}_i = \mathfrak{trd}(\mathfrak{h}_{i-1}, \mathfrak{n}_{i-1}).$$

The sequence of multisegments

$$\mathfrak{s}(\mathfrak{n}_0, \mathfrak{h}_0), \mathfrak{s}(\mathfrak{n}_1, \mathfrak{h}_1), \ldots$$

to be the *fine chain of removal segments* (or simply the *fine chain*) for $(\mathfrak{n}, \mathfrak{h})$. Since we usually fix $\mathfrak{h}$ and vary $\mathfrak{n}$ in our use of fine chains, we shall denote the fine chain by $\mathrm{fch}_{\mathfrak{h}}(\mathfrak{n})$. It follows from the definition that $\mathfrak{s}(\mathfrak{n}_i, \mathfrak{h}_i), \mathfrak{s}(\mathfrak{n}_{i+1}, \mathfrak{h}_{i+1}), \ldots$ is also the fine chain for $(\mathfrak{n}_i, \mathfrak{h}_i)$.

Example 3.7. Let $\mathfrak{h} = \{[0,4],[1,5]\}$.

- Let $\mathfrak{n} = \{[0,1],[1,2]\}$. Then the fine chain takes the form:
$$\{[0,4]\}, \{[1,4],[1,5]\}, \{[2,4]\}.$$
- Let $\mathfrak{n} = \{[0,2],[1]\}$. Then the fine chain is the same as the previous one.

Definition 3.8. Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. We say that two fine chains $\mathrm{fc}_{\mathfrak{h}}(\mathfrak{n})$ and $\mathrm{fc}_{\mathfrak{h}}(\mathfrak{n}')$ *coincide* if

(1) $\mathfrak{r}(\mathfrak{n}, \mathfrak{h}), \mathfrak{r}(\mathfrak{n}', \mathfrak{h}) \neq \infty$; and
(2) the two sequences $\mathrm{fc}_{\mathfrak{h}}(\mathfrak{n})$ and $\mathrm{fc}_{\mathfrak{h}}(\mathfrak{n}')$ are equal.

Lemma 3.9. *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\mathfrak{n}, \mathfrak{n}' \in \mathrm{Mult}_\rho$. Then $\mathfrak{r}(\mathfrak{n}, \mathfrak{h}) = \mathfrak{r}(\mathfrak{n}', \mathfrak{h}) \neq \infty$ if and only if the fine chains $\mathrm{fc}(\mathfrak{n}, \mathfrak{h})$ and $\mathrm{fc}(\mathfrak{n}', \mathfrak{h})$ coincide.*

Proof. We sketch the main idea of the proof. We write fine chains $\mathrm{fc}_{\mathfrak{h}}(\mathfrak{n})$ and $\mathrm{fc}_{\mathfrak{h}}(\mathfrak{n}')$ with notations in Definition 3.6 as:

$$\mathfrak{s}(\mathfrak{n}_0, \mathfrak{h}_0), \mathfrak{s}(\mathfrak{n}_1, \mathfrak{h}_1), \ldots$$

and

$$\mathfrak{s}(\mathfrak{n}_0', \mathfrak{h}_0'), \mathfrak{s}(\mathfrak{n}_1', \mathfrak{h}_1'), \ldots$$

with $\mathfrak{n}_0 = \mathfrak{n}$, $\mathfrak{n}_0' = \mathfrak{n}'$, $\mathfrak{h}_0 = \mathfrak{h}_0' = \mathfrak{h}$.

For the only if direction, Lemma 2.3(2) implies that $\mathfrak{h}_i = \mathfrak{h}'_i$ for all i. Then, from the construction of $\mathfrak{h}_i$ and $\mathfrak{h}'_i$, we have that $\mathfrak{s}(\mathfrak{n}_{i-1}, \mathfrak{h}_{i-1}) = \mathfrak{s}(\mathfrak{n}'_{i-1}, \mathfrak{h}'_{i-1})$. In other words, the fine chains for $(\mathfrak{n}, \mathfrak{h})$ and $(\mathfrak{n}', \mathfrak{h})$ coincide.

For the if direction, since the fine chains coincide, we must have $\mathfrak{h}_i = \mathfrak{h}'_i$ by (3.2). In particular, $\mathfrak{r}(\mathfrak{n}, \mathfrak{h}) = \mathfrak{r}(\mathfrak{n}', \mathfrak{h})$ as desired. □

3.4. **Fine chain ordering.** Multisegments $\mathfrak{n}$ and $\mathfrak{n}'$ are said to be of the *same cuspidal support* if $\cup_{\Delta \in \mathfrak{n}} \Delta = \cup_{\Delta \in \mathfrak{n}'} \Delta$ (counting multiplicities).

Definition 3.10. Let $\mathfrak{n}, \mathfrak{n}' \in \mathrm{Mult}_\rho$ be of the same cuspidal support. Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Suppse $\mathfrak{r}(\mathfrak{n}, \mathfrak{h}) \neq \infty$ and $\mathfrak{r}(\mathfrak{n}', \mathfrak{h}) \neq \infty$. Write $\mathrm{fc}_{\mathfrak{h}}(\mathfrak{n})$ as $\mathfrak{s}_1, \mathfrak{s}_2, \ldots$ and write $\mathrm{fc}_{\mathfrak{h}}(\mathfrak{n}')$ as $\mathfrak{s}'_1, \mathfrak{s}'_2, \ldots$. Similar to the notations in Definition 3.6, set inductively $\mathfrak{n}_i = \mathfrak{n}_{i-1} - \mathfrak{s}_i + {}^-\mathfrak{s}_i$ and $\mathfrak{n}'_i = \mathfrak{n}'_{i-1} - \mathfrak{s}_i + {}^-\mathfrak{s}_i$, where $\mathfrak{n}_0 = \mathfrak{n}$ and $\mathfrak{n}'_0 = \mathfrak{n}'$. Let c_i (resp. c'_i) be the smallest integer such that $\mathfrak{n}_i[c_i] \neq \emptyset$ (resp. $\mathfrak{n}_i[c'_i] \neq \emptyset$).

We define $\mathfrak{n} <^{fc} \mathfrak{n}'$, called the *fine chain ordering*, if there exists some i such that for any $j < i$, $\mathfrak{s}_j = \mathfrak{s}'_j$ and

$$\mathfrak{s}_i <^a_{c_{i-1}} \mathfrak{s}'_i.$$

We write $\mathfrak{n} \leq^{fc} \mathfrak{n}'$ if either $\mathfrak{n} <^{fc} \mathfrak{n}'$ or $\mathrm{fc}_{\mathfrak{h}}(\mathfrak{n}) = \mathrm{fc}_{\mathfrak{h}}(\mathfrak{n}')$. Note that $\leq^{fc}$ is transitive.

4. Closure under intersection-union process

The main result in this section is Theorem 4.4, which gives a combinatorial structure of $\mathcal{S}(\pi, \tau)$. Theorem 2.4 transfers the problem to study the combinatorics on the removal process.

4.1. **Effect from intersection-union process.**

Lemma 4.1. *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\mathfrak{m}_1$ be in $\mathrm{Mult}^a_{\rho,c}$. Let $\mathfrak{m}_2 \in \mathrm{Mult}^a_{\rho,c}$ be obtained from $\mathfrak{m}_1$ by replacing one segment in $\mathfrak{m}_1$ with a longer segment of the form $[c, b]_\rho$ for some b. Then*

$$\mathfrak{s}(\mathfrak{m}_1, \mathfrak{h}) \leq^a_c \mathfrak{s}(\mathfrak{m}_2, \mathfrak{h}).$$

Proof. The only difference between $\mathfrak{m}_1$ and $\mathfrak{m}_2$ is on one segment. We can arrange the segments to be the last ones in the process of obtaining $\mathfrak{s}(\mathfrak{m}_1, \mathfrak{h})$ and $\mathfrak{s}(\mathfrak{m}_2, \mathfrak{h})$ respectively. Thus the only difference between $\mathfrak{s}(\mathfrak{m}_1, \mathfrak{h})$ and $\mathfrak{s}(\mathfrak{m}_2, \mathfrak{h})$ is only one segment. The remaining follows from the definition of Υ (for picking the last segments in $\mathfrak{s}(\mathfrak{m}_1, \mathfrak{h})$ and $\mathfrak{s}(\mathfrak{m}_2, \mathfrak{h})$) and the definition of $\leq^a_c$. □

Lemma 4.2. *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Fix $\mathfrak{n} \in \mathrm{Mult}_\rho$. Let $\mathcal{N}$ be the set of all multisegments of the same cuspidal support as $\mathfrak{n}$. Then, for $\mathfrak{n}', \mathfrak{n}'' \in \mathcal{N}$,*

$$\mathfrak{n}' \leq_Z \mathfrak{n}'' \quad \Longrightarrow \quad \mathfrak{n}'' \leq^{fc} \mathfrak{n}'.$$

Proof. By the transitivity of $\leq_Z$, we reduce to the case that $\mathfrak{n}'$ is obtained from $\mathfrak{n}''$ by an elementary intersection-union operation. Let Δ_1 and Δ_2 be the two linked segments involved in the elementary intersection-union operation. Relabeling if necessary, we write:

$$\Delta_1 = [a_1, b_1]_\rho, \quad \Delta_2 = [a_2, b_2]_\rho,$$

with $a_1 < a_2$ and $b_1 < b_2$.

We again write $\mathrm{fc}_{\mathfrak{h}}(\mathfrak{n}')$ as $\mathfrak{s}'_1, \mathfrak{s}'_2, \ldots$ and write $\mathrm{fc}_{\mathfrak{h}}(\mathfrak{n}'')$ as $\mathfrak{s}''_1, \mathfrak{s}''_2, \ldots$. Similar to notations in Definition 3.6, set $\mathfrak{n}'_i = \mathfrak{n}'_{i-1} - \mathfrak{s}'_i + {}^-\mathfrak{s}'_i$ and $\mathfrak{n}''_i = \mathfrak{n}''_{i-1} - \mathfrak{s}''_i + {}^-\mathfrak{s}''_i$. Again let c_i be the smallest integer such tht $\mathfrak{n}'_i[c_i] \neq 0$. It is straighforward to see from the intersection-union operation that $\mathfrak{n}''_i[c_i]$ is obtained from $\mathfrak{n}'_i[c_i]$ by replacing a segment with a longer one (of the form $[c_i, b]_\rho$). Thus now Lemma 4.1 implies that $\mathfrak{n}'' \leq^{fc} \mathfrak{n}'$. □

Theorem 4.3. *Let $\mathfrak{n}, \mathfrak{n}' \in \mathrm{Mult}_\rho$. Suppose $\mathfrak{n}' \leq_Z \mathfrak{n}$. Let $\mathfrak{n}'' \in \mathrm{Mult}_\rho$ such that*

$$\mathfrak{n}' \leq_Z \mathfrak{n}'' \leq_Z \mathfrak{n}.$$

Then, if $\mathfrak{r}(\mathfrak{n}, \mathfrak{h}) = \mathfrak{r}(\mathfrak{n}', \mathfrak{h})$, then $\mathfrak{r}(\mathfrak{n}, \mathfrak{h}) = \mathfrak{r}(\mathfrak{n}'', \mathfrak{h})$.

Proof. If $\mathfrak{r}(\mathfrak{n}, \mathfrak{h}) \neq \mathfrak{r}(\mathfrak{n}'', \mathfrak{h})$, Lemmas 3.9 and 4.2 imply that $\mathfrak{n}'' <^{fc} \mathfrak{n}$. By Lemma 4.2 again and the transitivity of $<^{fc}$, $\mathfrak{n}' <^{fc} \mathfrak{n}$. However, Lemma 3.9 then implies $\mathfrak{r}(\mathfrak{n}', \mathfrak{h}) \neq \mathfrak{r}(\mathfrak{n}, \mathfrak{h})$, giving a contradiction. □

We translate the combinatorial statement in Theorem 4.3 to its representation-theoretic counterpart:

Theorem 4.4. *Let $\pi \in \mathrm{Irr}_\rho$ and let τ be a simple quotient of $\pi^{(i)}$. Recall that $\mathcal{S}(\pi, \tau)$ is defined in Section 1.2. Let $\mathfrak{n}, \mathfrak{n}' \in \mathcal{S}(\pi, \tau)$ with $\mathfrak{n}' \leq_Z \mathfrak{n}$. For any $\mathfrak{n}'' \in \mathrm{Mult}_\rho$ such that $\mathfrak{n}' \leq_Z \mathfrak{n}'' \leq_Z \mathfrak{n}$, we have $\mathfrak{n}'' \in \mathcal{S}(\pi, \tau)$.*

Proof. This follows from Theorem 2.4 and Theorem 4.3. □

We also have the following combinatorial consequence:

Corollary 4.5. *We use the notations in Lemma 4.2. Let*

$$\widetilde{\mathcal{N}} := \{\mathfrak{n} \in \mathcal{N} : \mathfrak{r}(\mathfrak{n}, \mathfrak{h}) \neq \infty\}.$$

We define an equivalence relation $\sim$ on $\widetilde{\mathcal{N}}$ by: $\mathfrak{n} \sim \mathfrak{n}'$ if and only if $\mathfrak{r}(\mathfrak{n}, \mathfrak{h}) = \mathfrak{r}(\mathfrak{n}', \mathfrak{h})$. Define $\preceq_Z$ on $\widetilde{\mathcal{N}}/\sim$ by: for $N, N' \in \widetilde{\mathcal{N}}/\sim$, write $N \preceq_Z N'$ if there exists $\mathfrak{n} \in N$ and $\mathfrak{n}' \in N'$ such that $\mathfrak{n} \leq_Z \mathfrak{n}'$. We similarly define the notion $\preceq^{fc}$ on $\widetilde{\mathcal{N}}$ by replacing $\leq_Z$ with $\leq^{fc}$. Then, the following holds:

- *Both $\preceq_Z$ and $\preceq^{fc}$ define a well-defined poset structure on $\widetilde{\mathcal{N}}/\sim$.*
- *The identity map on $\widetilde{\mathcal{N}}/\sim$ induces an order-reversing map between $(\widetilde{\mathcal{N}}/\sim, \preceq_Z)$ and $(\widetilde{\mathcal{N}}/\sim, \preceq^{fc})$.*

Proof. For the first bullet, the only non-evident part is the antisymmetry, which indeed follows from Lemmas lem coincide lemma and 4.2. The second bullet is a direct consequence on Lemma 4.2. □

5. Minimizability

5.1. Basic example on minimality.

Example 5.1. Let $\mathfrak{h} = \{[0,5], [3,8]\}$. Let $\mathfrak{n} = \{[0,3], [3,4]\}$. Then $\mathfrak{r}([0,3], \mathfrak{h}) = \{[4,5], [3,8]\}$ and so $\mathfrak{r}([3,4], \mathfrak{r}([0,3], \mathfrak{h})) = \{[4,8], [5]\}$. Note that the segmet $[5]$ coming from truncating the segment $[4,5]$ in $\mathfrak{r}([0,3], \mathfrak{h})$ and the segment $[4,5]$ indeed comes from tuncating the segment $[0,5]$. One wonders if one can 'combine' thes two effects. Indeed, if one could consider $\mathfrak{n}' = \{[0,4], [3]\}$, then $\mathfrak{r}([3], \mathfrak{h}) = \{[0,5], [4,8]\}$ and $\mathfrak{r}([0,4], \mathfrak{r}([3], \mathfrak{h})) = \{[5], [4,8]\}$. In the last removal process, $[5]$ is obtained directly from truncating $[0,5]$ once.

For convenience, we define a multisegment analogue of $\mathcal{S}(\pi, \tau)$. For $\mathfrak{h}, \mathfrak{p} \in \mathrm{Mult}_\rho$,

$$\mathcal{S}'(\mathfrak{h}, \mathfrak{p}) = \{\mathfrak{m} \in \mathrm{Mult}_\rho : \mathfrak{r}(\mathfrak{m}, \mathfrak{h}) = \mathfrak{p}\}.$$

The above example shows that $\mathfrak{n}$ is not $\leq_Z$-minimal in $\mathcal{S}'(\mathfrak{h}, \mathfrak{r}(\mathfrak{n}, \mathfrak{h}))$. The intuition in Example 5.1 will be formulated properly in Section 9.1, but we shall first deal with more general multisegments (rather than only two segments).

5.2. **Local minimizability.** We now define minimizability in Definition 5.2 to show the uniqueness for the $\leq_Z$-minimal element in $\mathcal{S}(\pi,\tau)$ in Theorem 6.4.

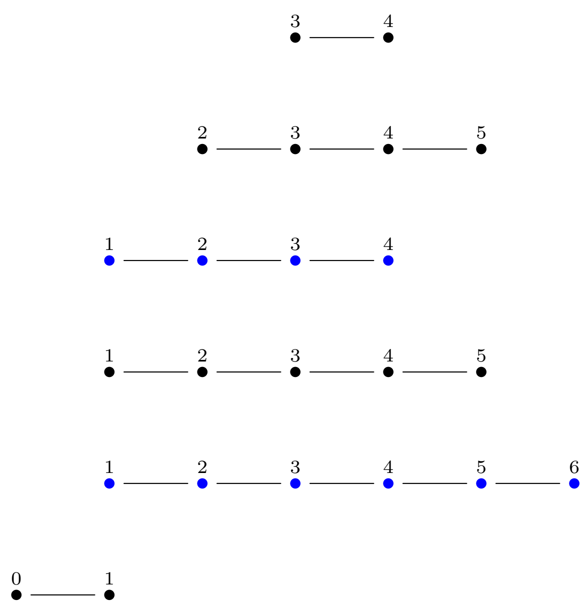
Definition 5.2. Let $\mathfrak{h} \in \mathrm{Mult}_\rho$ and let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be admissible to $\mathfrak{h}$. Let a be smallest integer such that $\mathfrak{n}[a] \neq \emptyset$. We say that $(\mathfrak{n},\mathfrak{h})$ is *locally minimizable* if there exists a segment $\overline{\Delta}$ in $\mathfrak{n}[a+1]$ such that the following holds:

$$|\left\{\Delta \in \mathfrak{n}[a] : \overline{\Delta} \subset \Delta\right\}| < |\left\{\Delta \in \mathfrak{s}(\mathfrak{n},\mathfrak{h}) : \overline{\Delta} \subset \Delta\right\}|.$$

We emphasis that the non-strict inequality $\leq$ always hold.

Remark 5.3. We give more explanations on Definition 5.2. As suggested from the terminology, those locally minimizable $(\mathfrak{n},\mathfrak{h})$ is to find some $\mathfrak{n}' <_Z \mathfrak{n}$ such that $\mathfrak{r}(\mathfrak{n}',\mathfrak{h}) = \mathfrak{r}(\mathfrak{n},\mathfrak{h})$. For instance, if all segments Δ in $\mathfrak{n}[a]$ satisfy $\overline{\Delta} \subset \Delta$, the removal process guarantees that any Δ in $\mathfrak{s}(\mathfrak{n},\mathfrak{h})$ also satisfies $\overline{\Delta} \subset \Delta$. Hence, the inequality in Definition 5.2 is not satisifed. On the other hand, all segments in $\mathfrak{n}[a]$ is not linked to $\overline{\Delta}$ and so there is no intersection-union operation for segments in $\mathfrak{n}[a]$ and $\overline{\Delta}$.

Example 5.4. Let $\mathfrak{h} = \{[0,1],[1,4],[1,5],[1,6],[2,5],[3,4]\}$, let $\mathfrak{n} = \{[1,3],[1,6],[2,4]\}$ and let $\mathfrak{n}' = \{[1,3],[1,6],[2,5]\}$.



The blue points represent $\mathfrak{s}(\mathfrak{n},\mathfrak{h})$ and $\mathfrak{s}(\mathfrak{n}',\mathfrak{h})$. Note that

$$|\{\Delta \in \mathfrak{n}[1] : [2,4] \subset \Delta\}| = 1, \quad |\{\Delta \in \mathfrak{s}(\mathfrak{n},\mathfrak{h}) : [2,4] \subset \Delta\}| = 2$$

and so $(\mathfrak{n},\mathfrak{h})$ is locally minimizable. On the other hand,

$$|\{\Delta \in \mathfrak{n}'[1] : [2,4] \subset \Delta\}| = 1, \quad |\{\Delta \in \mathfrak{s}(\mathfrak{n}',\mathfrak{h}) : [2,4] \subset \Delta\}| = 1.$$

Hence $(\mathfrak{n}',\mathfrak{h})$ is not locally minimizable.

Lemma 5.5. *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$ and let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be admissible to $\mathfrak{h}$. Let $\mathfrak{n}' = \mathfrak{trd}(\mathfrak{n},\mathfrak{h})$ and $\mathfrak{h}' = \mathfrak{trr}(\mathfrak{n},\mathfrak{h})$. Let a be the smallest integer such that $\mathfrak{n}'[a] \neq \emptyset$. Fix some $c > a+1$. Fix a segment $\Delta' \in \mathrm{Seg}_\rho^{a,c}$. Suppose*

$$(*) \quad |\left\{\Delta \in \mathfrak{n}' : \overline{\Delta} \subset \Delta\right\}| < |\left\{\Delta \in \mathfrak{s}(\mathfrak{n}',\mathfrak{h}') : \overline{\Delta} \subset \Delta\right\}|.$$

- *There exists a segment* $\widetilde{\Delta}$ *in* $\mathfrak{n}[a]+\mathfrak{n}[a+1]$ *such that*
$$\mathfrak{s}(\mathfrak{n},\mathfrak{h})=\mathfrak{s}(\widetilde{\mathfrak{n}},\mathfrak{h}),\quad \mathfrak{s}(\mathfrak{n}',\mathfrak{h}')=\mathfrak{s}(\widetilde{\mathfrak{n}}',\mathfrak{h}'),$$
where $\widetilde{\mathfrak{n}}$ *is obtained from* $\mathfrak{n}$ *by an elementary intersection-union process between* $\widetilde{\Delta}$ *and* $\overline{\Delta}$, *and* $\widetilde{\mathfrak{n}}'=\mathfrak{trd}(\mathfrak{n},\mathfrak{h})$.
- *Furthermore, if* $|\left\{\Delta\in\mathfrak{n}[a]:\overline{\Delta}\subset\Delta\right\}| = |\left\{\Delta\in\mathfrak{s}(\mathfrak{n},\mathfrak{h}):\overline{\Delta}\subset\Delta\right\}|$, *then the segment* $\widetilde{\Delta}$ *can be chosen in* $\mathfrak{n}[a+1]$.

Proof. Let
$${}^-(\mathfrak{n}[a])=\left\{{}^-\Delta:\Delta\in\mathfrak{n}[a],\Delta\neq[a]\right\}.$$
Let $r=|{}^-(\mathfrak{n}[a])|$ and let $s=|\mathfrak{n}[a+1]|$. We arrange the segments in $\mathfrak{n}'$ as follows: the first r segments are those in ${}^-(\mathfrak{n}[a])$, and the remaining segments are those in $\mathfrak{n}[a+1]$. To facilitate discussions, the first r segments are labelled as
$$\Delta_1,\ldots,\Delta_r$$
and the remaining segments are:
$$\widetilde{\Delta}_1,\ldots,\widetilde{\Delta}_s$$
We also set
$$\Lambda_i=\Upsilon(\Delta_i,\mathfrak{r}(\{\Delta_1,\ldots,\Delta_{i-1}\},\mathfrak{h}'),$$
$$\widetilde{\Lambda}_i=\Upsilon(\widetilde{\Delta}_i,\mathfrak{r}(\left\{\widetilde{\Delta}_1,\ldots,\widetilde{\Delta}_{i-1},\Delta_1,\ldots,\Delta_r\right\},\mathfrak{h}').$$

Case 1:
$$|\left\{\Delta\in\mathfrak{n}[a]:\overline{\Delta}\subset\Delta\right\}|=|\left\{\Delta\in\mathfrak{s}(\mathfrak{n},\mathfrak{h}):\overline{\Delta}\subset\Delta\right\}|.$$
This condition and the nesting property implies that for the first r segments Δ_i, if $\overline{\Delta}\not\subset\Delta_i$, then
$$\overline{\Delta}\not\subset\Lambda_i.$$
Thus condition (*) implies that there exists a segment $\widetilde{\Delta}_i$ such that $\overline{\Delta}\not\subset\widetilde{\Delta}_i$ and $\overline{\Delta}\subset\Lambda_i$. Now we do the intersection-union operation on $\overline{\Delta}$ and $\widetilde{\Delta}_i$ to obtain $\widetilde{\mathfrak{n}}$ from $\mathfrak{n}$. Then $\mathfrak{n}[a]=\widetilde{\mathfrak{n}}[a]$ and so $\mathfrak{s}(\mathfrak{n},\mathfrak{h})=\mathfrak{s}(\widetilde{\mathfrak{n}},\mathfrak{h})$. And, $\mathfrak{n}'$ and $\widetilde{\mathfrak{n}}':=\mathfrak{trd}(\widetilde{\mathfrak{n}},\mathfrak{h}')$ are only differed by $\widetilde{\Delta}_i$ and $\widetilde{\Delta}_i\cup\overline{\Delta}$. However, if we impose the same ordering in computing $\mathfrak{s}(\widetilde{\mathfrak{n}}',\mathfrak{h}')$, it is straightforward to use $\overline{\Delta}\subset\Lambda_i$ to see that
$$\mathfrak{s}(\mathfrak{n}',\mathfrak{h}')=\mathfrak{s}(\widetilde{\mathfrak{n}}',\mathfrak{h}').$$

Case 2:
$$|\left\{\Delta\in\mathfrak{n}[a]:\overline{\Delta}\subset\Delta\right\}|<|\left\{\Delta\in\mathfrak{s}(\mathfrak{n},\mathfrak{h}):\overline{\Delta}\subset\Delta\right\}|.$$
Now, by (*), there exists a segment $\widetilde{\Delta}=\Delta_i$ or $\widetilde{\Delta}_i$ in $\mathfrak{n}'$ such that $\overline{\Delta}\not\subset\widetilde{\Delta}$ and $\overline{\Delta}\subset\Lambda$, where $\Lambda=\Lambda_i$ or $\widetilde{\Lambda}_i$ according to $\widetilde{\Delta}$.

If $\widetilde{\Delta}=\widetilde{\Delta}_i$ for some i, then the intersection-union operation is done between $\widetilde{\Delta}$ and $\overline{\Delta}$. The argument is similar to Case 1 and we omit the details.

We now consider the case that $\widetilde{\Delta}=\Delta_i$ for some i. For convenience, set ${}^+[a+1,c]_\rho=[a,c]_\rho$ for any c. Note that all ${}^+\Delta_k$ $(k=1,\ldots r)$ constitute all the non-singleton segments in $\mathfrak{n}[a]$. We can use the ordering ${}^+\Delta_1,\ldots,{}^+\Delta_r$ (with other singleton segments at the end) to compute $\mathfrak{s}(\mathfrak{n},\mathfrak{h})$; and similarly use that ordering with ${}^+\Delta_i$ replaced by ${}^+\Delta_i\cup\overline{\Delta}$ to compute $\mathfrak{s}(\widetilde{\mathfrak{n}},\mathfrak{h})$. The only difference is to compute the first segments for ${}^+\Delta_i$ and ${}^+\Delta_i\cup\overline{\Delta}$, but we can still guarantee that choices for first segments (for computing $\mathfrak{s}(\mathfrak{n},\mathfrak{h})$ and $\mathfrak{s}(\mathfrak{n}',\mathfrak{h}')$) still coincide by using the nesting property of the removal process and the condition $\overline{\Delta}\subset\Lambda$. Hence, $\mathfrak{s}(\mathfrak{n},\mathfrak{h})=\mathfrak{s}(\widetilde{\mathfrak{n}},\mathfrak{h})$. Computing $\mathfrak{s}(\mathfrak{n}',\mathfrak{h}')=\mathfrak{s}(\widetilde{\mathfrak{n}}',\mathfrak{h}')$ is again similar since the only difference between $\mathfrak{n}'$ and $\widetilde{\mathfrak{n}}'$ is Δ_i and $\Delta_i\cup\overline{\Delta}$. $\square$

6. Uniqueness of minimality in $\mathcal{S}(\pi,\tau)$

The terminology of minimizability is suggested by the following lemma:

Lemma 6.1. *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$ and let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be admissible to $\mathfrak{h}$. Let the fine chain $\mathrm{fc}(\mathfrak{n},\mathfrak{h})$ take the form*

$$\mathfrak{s}(\mathfrak{n}_0,\mathfrak{h}_0),\mathfrak{s}(\mathfrak{n}_1,\mathfrak{h}_1),\ldots$$

as in Definition 3.6. If $(\mathfrak{n}_j,\mathfrak{h}_j)$ is not locally minimizable for any j, then there is no multisegment $\mathfrak{n}'$ such that $\mathfrak{n}' \lneq_Z \mathfrak{n}$ and $\mathfrak{r}(\mathfrak{n},\mathfrak{h})=\mathfrak{r}(\mathfrak{n}',\mathfrak{h})$.

Proof. Suppose there is a multisegment $\mathfrak{n}' \lneq_Z \mathfrak{n}$ and $\mathfrak{r}(\mathfrak{n},\mathfrak{h})=\mathfrak{r}(\mathfrak{n}',\mathfrak{h})$. Then, by Theorem 4.4, we may take $\mathfrak{n}'$ to be obtained from $\mathfrak{n}$ by an elementary intersection-union process. Let $\widetilde{\Delta}=[\widetilde{a},\widetilde{b}]_\rho, \overline{\Delta}=[\overline{a},\overline{b}]_\rho$ in $\mathfrak{n}$ be the segments involved in the intersection-union process, and switching labeling if necessary, we may assume that $\widetilde{a}<\overline{a}$.

We similarly obtain the fine chain

$$\mathfrak{s}(\mathfrak{n}_0',\mathfrak{h}_0'),\mathfrak{s}(\mathfrak{n}_1',\mathfrak{h}_1'),\ldots,$$

for $(\mathfrak{n}',\mathfrak{h})$. We consider j such that $\overline{a}-1$ is the smallest integer c such that $\mathfrak{n}_j[c]\neq\emptyset$. (Such j exists by using the condition that $\widetilde{\Delta}$ and $\overline{\Delta}$ are linked.)

Then, in $\mathfrak{n}_j$, we have a segment $[\overline{a}-1,\widetilde{b}]_\rho$ coming by truncating $\widetilde{\Delta}$. If we replace $[\overline{a}-1,\widetilde{b}]_\rho$ in $\mathfrak{n}_j$ by $[\overline{a}-1,\overline{b}]_\rho$, this gives $\mathfrak{n}_j'$.

Now

$$|\left\{\Delta\in\mathfrak{n}_j:\widetilde{\Delta}\subset\Delta\right\}|<|\left\{\Delta\in\mathfrak{n}_j':\widetilde{\Delta}\subset\Delta\right\}|\leq|\left\{\Delta\in\mathfrak{s}(\mathfrak{n}_j',\mathfrak{h}_j):\widetilde{\Delta}\subset\Delta\right\}|,$$

where the first strict inequality comes from $[\overline{a}-1,\overline{b}]_\rho$. But by Lemma 3.9, two fine chains coincide and in particular $\mathfrak{s}(\mathfrak{n}_j,\mathfrak{h}_j)=\mathfrak{s}(\mathfrak{n}_j',\mathfrak{h}_j)$. Hence, we now have that $(\mathfrak{n}_j,\mathfrak{h}_j)$ is locally minimizable by Definition 5.2 as desired. ☐

We now prove the converse of Lemma 6.1:

Lemma 6.2. *We keep using the notations in Lemma 6.1. If $(\mathfrak{n}_j,\mathfrak{h}_j)$ is locally minimizable for some j, then there is a multisegment $\mathfrak{n}'$ such that $\mathfrak{n}' \lneq_Z \mathfrak{n}$ and $\mathfrak{r}(\mathfrak{n},\mathfrak{h})=\mathfrak{r}(\mathfrak{n}',\mathfrak{h}')$.*

Proof. We pick any j such that $(\mathfrak{n}_j,\mathfrak{h}_j)$ is locally minimizable. Let a be the smallest integer such that $\mathfrak{n}_j[a]\neq\emptyset$. The below argument is similar if $j=1$ and so we assume $j>1$ for convenience of the stated form of Lemma 5.5.

Note that $\mathfrak{n}_j[a]={}^-(\mathfrak{n}_{j-1}[a-1])+\mathfrak{n}[a]$. (Here we have $\mathfrak{n}_{j-1}[a]=\mathfrak{n}[a]$.) The local minimizability condition implies that we can use the first bullet of Lemma 5.5 (set $c=a+1$ in our case) with respect to a certain segment in $\mathfrak{n}[a+1]$, denoted by $\overline{\Delta}$. Then, that lemma implies that we can find a segment $\widetilde{\Delta}$ in ${}^-(\mathfrak{n}_{j-1}[a-1])+\mathfrak{n}[a]$ satisfying the required properties in the lemma.

The first case is that $\widetilde{\Delta}$ comes from $\mathfrak{n}[a]$. In this case, let $\widetilde{\mathfrak{n}}$ be the multisegment obtained by the intersection-union operation of the segments $\widetilde{\Delta}$ and $\overline{\Delta}$. Then it is straightforward from definitions that

$$\mathfrak{s}(\mathfrak{n}_0,\mathfrak{h}_0)=\mathfrak{s}(\widetilde{\mathfrak{n}}_0,\mathfrak{h}_0),\ldots,\mathfrak{s}(\mathfrak{n}_{j-1},\mathfrak{h}_{j-1})=\mathfrak{s}(\widetilde{\mathfrak{n}}_{j-1},\mathfrak{h}_{j-1}),$$

where $\mathfrak{h}_0=\mathfrak{h}$, $\mathfrak{s}(\widetilde{\mathfrak{n}}_k,\mathfrak{h}_k)$ are the first $j-1$ terms of $\mathrm{fc}(\widetilde{\mathfrak{n}},\mathfrak{h})$. However, $\mathfrak{s}(\mathfrak{n}_j,\mathfrak{h}_j)=\mathfrak{s}(\widetilde{\mathfrak{n}}_j,\mathfrak{h}_j)$ is guaranted by Lemma 5.5. But then $\mathfrak{n}_{j+1}=\widetilde{\mathfrak{n}}_{j+1}$ and so the reamining terms in two fine chains also agree. Hence, two fine chains coincide and so $\mathfrak{r}(\mathfrak{n},\mathfrak{h})=\mathfrak{r}(\widetilde{\mathfrak{n}},\mathfrak{h})$ by Lemma 3.9.

The second case is that $\widetilde{\Delta}$ cannot come from $\mathfrak{n}[a]$. In such case, the second bullet of Lemma 5.5 implies that we have

$$|\left\{\Delta\in\mathfrak{n}_{j-1}[a-1]:\overline{\Delta}\subset\Delta\right\}|<|\left\{\Delta\in\mathfrak{s}(\mathfrak{n}_{j-1}[a-1],\mathfrak{h}):\overline{\Delta}\subset\Delta\right\}|.$$

But then, we can apply Lemma 5.5 again to find another segment $\widetilde{\widetilde{\widetilde{\Delta}}}$. If such segment $\widetilde{\widetilde{\widetilde{\Delta}}}$ can be found in $\mathfrak{n}_{j-2}[a-1]$, then we repeat the similar argument of the first case above. Otherwise, we apply Lemma 5.5 again. In those cases, the coincidence of the fine chains are guaranteed by Lemma 5.5. Hence, we also have $\mathfrak{r}(\mathfrak{n},\mathfrak{h})=\mathfrak{r}(\widetilde{\mathfrak{n}},\mathfrak{h})$ by Lemma 3.9. (Strictly speaking in terms of the way in stating Lemma 5.5, one has to trace the proof to see that the choices for $\widetilde{\Delta}$ and $\widetilde{\widetilde{\Delta}}$ in each step can be made to agree after truncating the point $[a-1]_\rho$. Since there is no new idea on that, we avoid further notation complications.) $\square$

We explain the main idea of the proof for Proposition 6.3 below, which is inductive in nature. One first picks two minimal multisegments $\mathfrak{n}$ and $\mathfrak{n}'$ in $\mathcal{S}'(\mathfrak{h},\mathfrak{p})$. One then finds $\prec^L$-minimal segments $\widetilde{\Delta}$ and $\widetilde{\Delta}'$ in $\mathfrak{n}$ and $\mathfrak{n}'$ respectively. If $\widetilde{\Delta}=\widetilde{\Delta}'$, then one uses induction to argue $\mathfrak{n}-\widetilde{\Delta}=\mathfrak{n}'-\widetilde{\Delta}$. If $\widetilde{\Delta}\neq\widetilde{\Delta}'$, then one first reduces to the case that $\widetilde{\Delta}\subsetneq\widetilde{\Delta}'$. Then one applies induction hypothesis to show $\widetilde{\Delta}'$ is also in $\mathfrak{n}$. Then one shows that $\widetilde{\Delta}$ and $\widetilde{\Delta}'$ in $\mathfrak{n}$ give rise the local minimizability.

Proposition 6.3. *Let $\mathfrak{h}\in \mathrm{Mult}_\rho$. Then there exists a unique minimal element in $\mathcal{S}'(\mathfrak{h},\mathfrak{p})$ if $\mathcal{S}'(\mathfrak{h},\mathfrak{p})\neq\emptyset$.*

Proof. Let $\mathfrak{n},\mathfrak{n}'$ be two minimal multisegments in $\mathcal{S}(\mathfrak{h},\mathfrak{p})$. Let a be the smallest integer such that $\mathfrak{n}[a]\neq 0$. Then, by a comparison on cuspidal representations, a is also the smallest integer such that $\mathfrak{n}'[a]\neq 0$.

Suppose $\mathfrak{n}[a]\cap\mathfrak{n}'[a]\neq\emptyset$. Let $\widetilde{\Delta}\in\mathfrak{n}[a]\cap\mathfrak{n}'[a]$. Then we consider $\mathcal{S}(D_{\widetilde{\Delta}}(\pi),\tau)$. The minimality for $\mathfrak{n}$ and $\mathfrak{n}'$ also guarantees that $\mathfrak{n}-\widetilde{\Delta}$ and $\mathfrak{n}'-\widetilde{\Delta}$ are also minimal in $\mathcal{S}(\mathfrak{r}(\widetilde{\Delta},\mathfrak{h}),\mathfrak{p})$. Thus, by induction, we have that $\mathfrak{n}-\widetilde{\Delta}=\mathfrak{n}'-\widetilde{\Delta}$ and so $\mathfrak{n}=\mathfrak{n}'$.

Now suppose $\mathfrak{n}[a]\cap\mathfrak{n}'[a]=\emptyset$ to obtain a contradiction. Let Δ and Δ' be the shortest segment in $\mathfrak{n}[a]$ and $\mathfrak{n}'[a]$ respectively. Switching labeling if necessary, we may assume that $\Delta\subsetneq\Delta'$. Then, by Lemma 3.4,

$$\mathfrak{r}(\mathfrak{n},\mathfrak{h})=\mathfrak{r}(\mathfrak{trd}(\mathfrak{n},\mathfrak{h}),\mathfrak{trr}(\mathfrak{n},\mathfrak{h})),\quad \mathfrak{r}(\mathfrak{n}',\mathfrak{h})=\mathfrak{r}(\mathfrak{trd}(\mathfrak{n}',\mathfrak{h}),\mathfrak{trr}(\mathfrak{n}',\mathfrak{h})).$$

By Lemma 3.9,

$$\mathfrak{trr}(\mathfrak{n},\mathfrak{h})=\mathfrak{trr}(\mathfrak{n}',\mathfrak{h})$$

By Lemma 6.2, $(\mathfrak{trr}(\mathfrak{n},\mathfrak{h}),\mathfrak{trd}(\mathfrak{n},\mathfrak{h}))$ and the terms from the fine chains are not locally minimizable. Similarly, this also holds for $(\mathfrak{trr}(\mathfrak{n}',\mathfrak{h}),\mathfrak{trd}(\mathfrak{n},\mathfrak{h}))$. However, Lemma 6.1 implies that both $\mathfrak{trd}(\mathfrak{n},\mathfrak{h})$ and $\mathfrak{trd}(\mathfrak{n}',\mathfrak{h})$ are minimal in $\mathcal{S}(\mathfrak{trr}(\mathfrak{n},\mathfrak{h}),\mathfrak{p})=\mathcal{S}(\mathfrak{trr}(\mathfrak{n}',\mathfrak{h}),\mathfrak{p})$. Hence, by induction,

$$\mathfrak{trd}(\mathfrak{n},\mathfrak{h})=\mathfrak{trd}(\mathfrak{n}',\mathfrak{h}).$$

But then, the disjointness assumption implies that ${}^-\Delta'\in\mathfrak{n}$. But ${}^-\Delta'\not\subset\Delta$ and ${}^-\Delta'\subset\Upsilon(\Delta,\mathfrak{h})$. This implies

$$|\left\{\widetilde{\Delta}\in\mathfrak{n}[a]:{}^-\Delta'\subset\widetilde{\Delta}\right\}|<|\left\{\widetilde{\Delta}\in\mathfrak{s}(\mathfrak{n},\mathfrak{h}):{}^-\Delta'\subset\widetilde{\Delta}\right\}|.$$

Hence, $(\mathfrak{n},\mathfrak{h})$ is locally minimizable. This contradicts to Lemma 6.2. $\square$

Theorem 6.4. *Let $\pi\in\mathrm{Irr}_\rho$ and let τ be a simple quotient of $\pi^{(i)}$ for some i. Then $\mathcal{S}(\pi,\tau)$ has a unique minimal element if $\mathcal{S}(\pi,\tau)\neq\emptyset$. Here the minimality is with respect to $\leq_Z$.*

Proof. This follows from Proposition 6.3 and Theorem 2.4. $\square$

7. Examples of minimality

7.1. Minimality for the highest derivative multisegment.

Theorem 7.1. *Let $\pi \in \mathrm{Irr}_\rho$. Then $\mathfrak{hd}(\pi)$ is minimal in $\mathcal{S}(\pi, \pi^-)$.*

Proof. It is shown in [Ch22+d] that $D_{\mathfrak{hd}(\pi)}(\pi) \cong \pi^-$. It remains to prove that $\mathfrak{hd}(\pi)$ is minimal in $\mathcal{S}(\pi, \pi^-)$. Theorem 4.4 reduces to show that if $\mathfrak{n}$ is a multisegment obtained by an elementary intersection-union process from $\mathfrak{hd}(\pi)$, then $D_{\mathfrak{n}}(\pi) = 0$.

Let $\Delta_1 = [a_1, b_1]_\rho, \Delta_2 = [a_2, b_2]_\rho$ be two linked segments in $\mathfrak{hd}(\pi)$. Relabeling if necessary, we assume that $a_1 < a_2$. Define

$$\mathfrak{n} = \mathfrak{hd}(\pi) - \{\Delta_1, \Delta_2\} + \Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2.$$

Then, $\mathfrak{n}[e] = \mathfrak{hd}(\pi)[e]$ for any $e < a_1$ and $\mathfrak{n}[a_1] \not\leq_{a_1} \mathfrak{hd}(\pi)[a_1]$. Hence, by Theorems 2.4 and 2.5,

$$D_{\mathfrak{n}[a_1]} \dots D_{\mathfrak{n}[c]}(\pi) = D_{\mathfrak{n}[a_1]} \circ D_{\mathfrak{hd}(\pi)[a_1-1]} \circ \dots \circ D_{\mathfrak{hd}(\pi)[c]}(\pi) = 0,$$

and so $\mathfrak{n} \notin \mathcal{S}(\pi, \pi^-)$. Here c is the smallest integer such that $\mathfrak{hd}(\pi)[c] \neq 0$. ☐

7.2. Minimal multisegment for the generic case.

Proposition 7.2. *Let $\pi \in \mathrm{Irr}_\rho$ be generic. Let τ be a (generic) simple quotient of $\pi^{(i)}$ for some i. Then the minimal multisegment in $\mathcal{S}(\pi, \tau)$ is generic i.e. any two segments in the minimal multisegment are unlinked.*

One may prove the above proposition by some analysis of derivative resultant multisegments. We shall give another proof using the following lemma:

Lemma 7.3. *Let $\pi \in \mathrm{Irr}_\rho(G_n)$. For any i, and for any irreducible submodule $\tau_1 \boxtimes \tau_2$ of π_{N_i} as $G_{n-i} \times G_i$-representations, both τ_1 and τ_2 are generic.*

Proof. Argued as in [Ch21, Corollary 2.6], we have that a simple quotient of π_{N_i} takes the form $\tau \boxtimes \omega$ for some generic $\tau \in \mathrm{Irr}(G_{n-i})$. Hence it remains to show ω is also generic. We consider

$$\pi_{N_i} \twoheadrightarrow \tau \boxtimes \omega$$

and taking the twisted Jacquet functor on the G_{n-i}-parts yields that

$${}^{(n-i)}\pi \twoheadrightarrow \omega.$$

Now using [Ch21, Corollary 2.6] for left derivatives, we have that ω is also generic as desired. ☐

Proof of Proposition 7.2. Let $\pi \in \mathrm{Irr}_\rho$ be generic and let τ be a simple (generic) quotient of $\pi^{(i)}$ (see [Ch21, Corollary 2.6]). Then, π_{N_i} has a simple quotient of the form $\tau \boxtimes \omega$ for some $\omega \in \mathrm{Irr}_\rho(G_i)$. By Lemma 7.3, ω is also generic and hence $\omega \cong \mathrm{St}(\Delta_1) \times \dots \times \mathrm{St}(\Delta_k)$ for some mutually unlinked segments $\Delta_1, \dots, \Delta_k$. Now, π is the unique submodule of $\tau \times \mathrm{St}(\Delta_1) \times \dots \times \mathrm{St}(\Delta_k)$. By a standard argument, we have that:

$$D_{\Delta_1} \circ \dots \circ D_{\Delta_k}(\pi) \cong \tau.$$

Hence, $\{\Delta_1, \dots, \Delta_k\} \in \mathcal{S}(\pi, \tau)$. The minimality of $\{\Delta_1, \dots, \Delta_k\}$ is automatic since any generic multisegment is minimal in Mult_ρ with respect to $\leq_Z$. Now the statement follows from the uniqueness in Theorem 6.4.

8. Non-uniqueness of maximal elements in $\mathcal{S}(\pi, \tau)$

8.1. **Highest derivative multisegments.** Let $\pi \in \mathrm{Irr}_\rho$. Then $\mathcal{S}(\pi, \pi^-)$ contains a unique maximal multisegment, and such multisegment has all segments to be singletons. Combining with Theorem 4.4, one can describe all multisegments in $\mathcal{S}(\pi, \pi^-)$.

As mentioned in [Ch22+d], in general, derivatives of cuspidal representations are not enough for constructing all simple quotients of Bernstein-Zelvinsky derivatives, and so multisegments whose segments are singleton may not be in the set $\mathcal{S}(\pi, \tau)$ in general.

8.2. **Failure of uniqueness of maximality.** We give an example to show that in general, there is no uniqueness for $\leq_Z$-maximal elements in $\mathcal{S}(\pi, \tau)$.

Let

$$\mathfrak{h} = \{[0,3],[0,1],[1,2],[1,2],[2],[3]\}.$$

Let $\mathfrak{n} = \{[0,3],[1,2]\}$. Then

$$\mathfrak{r} := \mathfrak{r}(\mathfrak{n}, \mathfrak{h}) = \{[0,1],[1,2],[2],[3]\} = \mathfrak{h} - \mathfrak{n}.$$

We claim that

$$\mathcal{S}(\pi, \mathfrak{r}) = \{\mathfrak{n}, \{[0,3],[1],[2]\}, \{[0,2],[1,3]\}\}.$$

It is direct to check that the three elements are in $\mathcal{S}(\pi, \tau)$, and the last two elements are both maximal.

To see that there are no more elements, we first observe that any multisegment $\mathfrak{n}'$ in $\mathcal{S}(\pi, \mathfrak{r})$ has only one segment $\widetilde{\Delta}$ with $a(\widetilde{\Delta}) = \nu^0$. By considering the first segment in the removal sequence $\mathfrak{r}(\widetilde{\Delta}, \mathfrak{h})$, we note that $[0],[0,1] \notin \mathcal{S}(\pi, \mathfrak{r})$. In other words, $[0,2]$ or $[0,3]$ in $\mathcal{S}(\pi, \mathfrak{r})$. It remains to check that the following three elements:

$$\{[0,2],[1],[2],[3]\}, \{[0,2],[1,2],[3]\}, \{[0,2],[1],[2,3]\}$$

are not in $\mathcal{S}(\pi, \mathfrak{r})$.

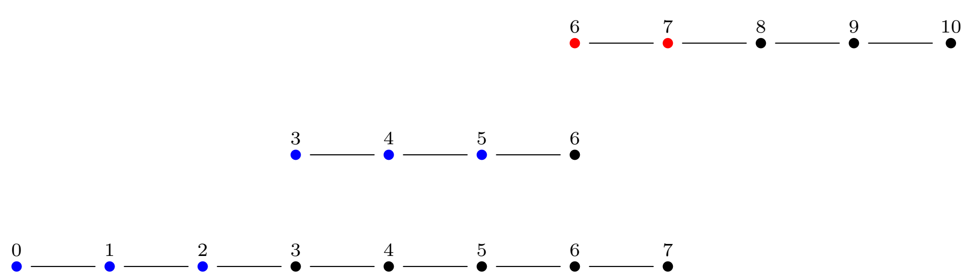
9. Minimality for two segment case

In this section, we study the minimality for two segment case.

9.1. **Non-overlapping Property.**

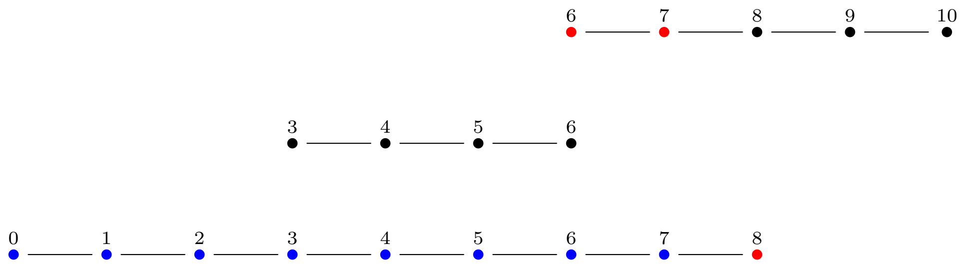
Definition 9.1. Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\Delta \in \mathrm{Seg}_\rho$ be admissible to $\mathfrak{h}$. Let $\Delta' \in \mathrm{Seg}_\rho$ linked to Δ with $\Delta' > \Delta$. We say that the triple $(\Delta, \Delta', \mathfrak{h})$ satisfies the *non-overlapping property* if for the shortest segment $\overline{\Delta}$ in the removal sequence for $(\Delta, \mathfrak{h})$ that contains $\nu^{-1}a(\Delta')$, we have $\Delta' \not\subset \overline{\Delta}$. (For later application, we do not impose the condition that Δ' is admissible to $\mathfrak{h}$.)

Example 9.2. (1) Let $\mathfrak{h} = \{[0,7],[3,6],[6,10]\}$. Let $\Delta = [0,5]$ and let $\Delta' = [6,7]$. Then $(\Delta, \Delta', \mathfrak{h})$ satisfies the non-overlapping property.



The blue points are those points removed by applying $\mathfrak{r}(\Delta, .)$ while the red points are those points removed by applying $\mathfrak{r}(\Delta', .)$. Note that the shortest segment in the removal sequence containing $[5]$ is $[3,6]$, which does not contain $[6,7]$.

(2) Let $\mathfrak{h} = \{[0,8],[3,6],[6,10]\}$. Let $\Delta = [0,7]$ and let $\Delta' = [6,8]$. Then $(\Delta, \Delta', \mathfrak{h})$ does not satisfy the non-overlapping property. The graph for carrying out the removal sequence looks like:



In the graph above, the segment $[0,8]$ contains $[5]$, and $[6,8] \subset [0,8]$.

Lemma 9.3. *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\Delta \in \mathrm{Seg}_\rho$ be admissible to $\mathfrak{h}$. Let $\Delta' \in \mathrm{Seg}_\rho$ be admissible to $\mathfrak{r}(\Delta, \mathfrak{h})$. Suppose Δ' is linked to Δ with $\Delta' > \Delta$. Then $(\Delta, \Delta', \mathfrak{h})$ does not satisfy the non-overlapping property if and only if*

$$\mathfrak{r}(\{\Delta \cap \Delta', \Delta \cup \Delta'\}, \mathfrak{h}) = \mathfrak{r}(\{\Delta, \Delta'\}, \mathfrak{h}).$$

Proof. Let $\mathfrak{n} = \{\Delta, \Delta'\}$. Suppose $\mathfrak{r}(\{\Delta \cap \Delta', \Delta \cup \Delta'\}, \mathfrak{h}) \neq \mathfrak{r}(\{\Delta, \Delta'\}, \mathfrak{h})$. Lemma 2.3(1) and the nesting property in the removal process reduces to the case that $a(\Delta) \cong \nu^{-1}a(\Delta')$. Now, showing not satisfying non-overlapping property is simply a reformulation of locally minimizability by Lemma 6.2.

Suppose $\mathfrak{r}(\{\Delta \cap \Delta', \Delta \cup \Delta'\}, \mathfrak{h}) = \mathfrak{r}(\{\Delta, \Delta'\}, \mathfrak{h})$. By Lemma 2.3(1), it again reduces to $a(\Delta) \cong \nu^{-1}a(\Delta')$. It then follows from Lemma 6.1 that $(\{\Delta, \Delta'\}, \mathfrak{h})$ is locally minimizable and so this gives the non-overlapping property. $\square$

9.2. Intermediate segment property.

Definition 9.4. Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\Delta \in \mathrm{Seg}_\rho$ be admissible to $\mathfrak{h}$. Let $\Delta' \in \mathrm{Seg}_\rho$ linked to Δ with $\Delta' > \Delta$. We say that the triple $(\Delta, \Delta', \mathfrak{h})$ satisfies the *intermediate segment property* if there exists a segment $\widetilde{\Delta}$ in $\mathfrak{h}$ such that

$$a(\Delta) \leq a(\widetilde{\Delta}) < a(\Delta'), \text{ and } b(\Delta) \leq b(\widetilde{\Delta}) < b(\Delta'). \tag{9.3}$$

9.3. **Criteria in terms of η-invariants.** Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. For a segment $\Delta = [a,b]_\rho$ admissible to $\mathfrak{h}$, note that, by Theorem 2.5, $\varepsilon_\Delta(\mathfrak{h}\mathfrak{d}(\pi)) = \varepsilon_\Delta(\pi)$.) Let

$$\eta_\Delta(\mathfrak{h}) = (\varepsilon_{[a,b]_\rho}(\mathfrak{h}), \varepsilon_{[a+1,b]_\rho}(\mathfrak{h}), \ldots, \varepsilon_{[b,b]_\rho}(\mathfrak{h})). \tag{9.4}$$

The η-invariant defined above plays an important role in defining a notion of *generalized GGP relevant pairs* in [Ch22+b].

Proposition 9.5. *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\Delta \in \mathrm{Seg}_\rho$ be admissible to $\mathfrak{h}$. Let $\Delta' \in \mathrm{Seg}_\rho$ be linked to Δ with $\Delta' > \Delta$. Then the following conditions are equivalent:*

(1) *The triple $(\Delta, \Delta', \mathfrak{h})$ satisfies the non-overlapping property.*
(2) $\eta_{\Delta'}(\mathfrak{h}) = \eta_{\Delta'}(\mathfrak{r}(\Delta, \mathfrak{h}))$.
(3) *The triple $(\Delta, \Delta', \mathfrak{h})$ satisfies the intermediate segment property.*

Proof. We first prove (3) implies (2). Suppose (3) holds. We denote by

$$\Delta_1, \ldots, \Delta_r$$

the removal sequence for $(\Delta, \mathfrak{h})$. Using (3) and (4) of the removal process in Definition 2.1, those $\Delta_1, \ldots, \Delta_r$ in $\mathfrak{h}$ are replaced by their truncations, denoted by

$$\Delta_1^{tr}, \ldots, \Delta_r^{tr}.$$

By using the intermediate segment property and the minimality condition in the removal process, there exists a segment of the form (9.3) in the removal sequence for $(\Delta, \mathfrak{h})$. Let i^* be the smallest index such that Δ_{i^*} satisfies (*). Note that, by considering $a(\Delta_j)$,

$$\Delta_1, \ldots, \Delta_{i^*-1}$$

do not contribute to $\eta_{\Delta'}(\mathfrak{h})$ by definitions. By the definition of truncation and (9.3) for Δ_{i^*}, we have that $\Delta_1^{tr}, \ldots, \Delta_{i^*-1}^{tr}$ also do not contribute to $\eta_{\Delta'}(\mathfrak{r}(\Delta, \mathfrak{h}))$. From the choice of Δ_{i^*} and the nesting property, we also have that, by considering $b(\Delta_j)$,

$$\Delta_{i^*}, \ldots, \Delta_r$$

do not contribute to $\eta_{\Delta'}(\mathfrak{h})$, and similarly, $\Delta_{i^*}^{tr}, \ldots, \Delta_r^{tr}$ do not contribute to $\eta_{\Delta'}(\mathfrak{r}(\Delta, \mathfrak{h}))$. Thus, we have that

$$\eta_{\Delta'}(\mathfrak{h}) = \eta_{\Delta'}(\mathfrak{r}(\Delta, \mathfrak{h})).$$

We now prove (2) implies (1). Again write $\Delta' = [a', b']_\rho$. Suppose $(\Delta, \Delta', \mathfrak{h})$ does not satisfy the nonoverlapping property. Again, denote by

$$\Delta_1, \ldots, \Delta_r$$

the removal sequence for $(\Delta, \mathfrak{h})$. Let Δ_l be the shortest segment in the removal sequence containing $\nu^{-1}a(\Delta')$. Note that

$$\Delta_1, \ldots, \Delta_{l-1}, \Delta_l$$

do not contribute to $\eta_{\Delta'}(\mathfrak{h})$ (by considering $a(\Delta_i)$) and similarly,

$$\Delta_1^{tr}, \ldots, \Delta_{l-1}^{tr}$$

do not contribute to $\eta_{\Delta'}(\mathfrak{r}(\Delta, \mathfrak{h}))$. However, Δ_l^{tr} contributes to $\eta_{\Delta'}(\mathfrak{r}(\Delta, \mathfrak{h}))$. This causes a difference of 1 in the coordinate $\varepsilon_{\Delta'}$ for $\eta_{\Delta'}(\mathfrak{h})$ and $\eta_{\Delta'}(\mathfrak{r}(\Delta, \mathfrak{h}))$.

It remains to see the following claim:

Claim: For $k > l$, Δ_k contributes to $\eta_{\Delta'}(\mathfrak{h})$ if and only if Δ_k^{tr} contributes to $\eta_{\Delta'}(\mathfrak{r}(\Delta, \mathfrak{h}))$.

Proof of claim: If Δ_k does not contribute to $\eta_{\Delta'}(\mathfrak{h})$, then $b(\Delta_k) < b(\Delta')$ and so $b(\Delta_k^{tr}) < b(\Delta')$ (or Δ_k^{tr} is dropped or a empty set). This implies that Δ_k^{tr} does not contribute to $\eta_{\Delta'}(\mathfrak{r}(\Delta, \mathfrak{h}))$.

On the other hand, if Δ_k contributes to $\eta_{\Delta'}(\mathfrak{h})$, then $b(\Delta_k) \geq b(\Delta')$. Note that Δ_k^{tr} is non-empty by using $\Delta < \Delta'$. Thus we also have $b(\Delta_k^{tr}) \geq b(\Delta')$. We also have that Δ_k^{tr} contributes to $\eta_{\Delta'}(\mathfrak{r}(\Delta, \mathfrak{h}))$. This completes proving the claim.

Note that (1) $\Rightarrow$ (3) follows from the segment involved in the definition of overlapping property. Thus, we also have (1) $\Rightarrow$ (2).

We now consider (3) $\Rightarrow$ (1). Among those segments in $\mathfrak{h}$ satisfying (9.3), we pick the $\prec^L$-minimal one $\widetilde{\Delta}^*$ (see Section 2.3 for $\prec^L$). Note that such segment also satisfies $\nu^{-1}a(\Delta') \in \widetilde{\Delta}^*$ and $\Delta' \not\subset \widetilde{\Delta}^*$. Now (3) implies that at least one segment in the removal sequence for $(\Delta, \mathfrak{h})$ contains a segment of the form (9.3), and so one uses the nesting property in the removal sequence to show the non-overlapping property. $\square$

Example 9.6. Let $\mathfrak{h} = \{[0, 5], [3, 8]\}$.

- Let $\Delta = [0,3]$ and let $\Delta' = [3,6]$. In such case, $\eta_{\Delta'}(\mathfrak{h}) = (1,0,0,0)$. Similarly, $\eta_{\Delta'}(\mathfrak{r}(\Delta,\mathfrak{h})) = \eta_{\Delta'}(\{[3,8],[4,5]\}) = (1,0,0,0)$.
- Let $\Delta = [0,3]$ and let $\Delta' = [3,4]$. In such case, $\eta_{\Delta'}(\mathfrak{h}) = (1,0)$. And $\eta_{\Delta'}(\mathfrak{r}(\Delta,\mathfrak{h})) = \eta_{\Delta'}(\{[3,8],[4,5]\}) = (1,1)$.

A consequence of Proposition 9.5 is the following:

Corollary 9.7. *Suppose the triple $(\Delta, \Delta', \mathfrak{h})$ satisfies the non-overlapping property. Then, for any Δ'-saturated segment $\widetilde{\Delta}$ linked to Δ, the triple $(\Delta, \widetilde{\Delta}, \mathfrak{h})$ also satisfies the non-overlapping property.*

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