CONSTRUCTION OF SIMPLE QUOTIENTS OF BERNSTEIN-ZELEVINSKY DERIVATIVES AND HIGHEST DERIVATIVE MULTISEGMENTS III: PROPERTIES OF MINIMAL SEQUENCES

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Abstract. Let F be a non-Archimedean local field. For an irreducible representation π of $\mathrm{GL}_n(F)$ and a multisegment $\mathfrak{m}$, one associates a simple quotient $D_{\mathfrak{m}}(\pi)$ of a Bernstein-Zelevinsky derivative of π. In the preceding article, we showed that

$$\mathcal{S}(\pi,\tau) := \{\mathfrak{m} : D_{\mathfrak{m}}(\pi) \cong \tau\},$$

has a unique minimal element under the Zelevinsky ordering, where $\mathfrak{m}$ runs for all multisegments. The main result of this article includes commutativity and subsequent property of the minimal sequence. At the end of this article, we conjecture some module structure arising from the minimality.

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Part 1. Introduction and preliminaries

1. Introduction

1.1. The poset $\mathcal{S}(\pi,\tau)$ and the minimal sequence. Let F be a non-Archimedean local field. Let $G_n = \mathrm{GL}_n(F)$, the general linear group over F. Fix a cuspidal representation ρ throughout the whole article. All the representations we consider are smooth and over $\mathbb{C}$. Let $\mathrm{Irr}(G_n)$ be the set of irreducible representations of G_n. We shall usually not distinguish isomorphic irreducible representations. For a smooth representation π_1 of G_{n_1} and a representation π_2 of G_{n_2}, denote the normalized parabolic induction by $\pi_1 \times \pi_2$.

Let $\nu : G_n \to \mathbb{C}^\times$ be the character $\nu(g) = |\det(g)|_F$, where $|.|_F$ is the normalized absolute value for F. Let $\mathrm{Irr}_\rho(G_n)$ be the set of irreducible representations which are an irreducible constitutent of $\nu^{a_1}\rho \times \ldots \times \nu^{a_r}\rho$ for some integers $a_1, \ldots, a_r$. Let $\mathrm{Irr}_\rho = \sqcup_n \mathrm{Irr}(G_n)$.

We now define some combinatorial objects to parametrize and study representations. For $a, b \in \mathbb{Z}$ with $b - a \in \mathbb{Z}_{\geq 0}$, we call $[a,b]_\rho$ to be a *segment* (associated to ρ). We also set $[a, a-1]_\rho = \emptyset$ for $a \in \mathbb{Z}$. For a segment $\Delta = [a,b]_\rho$, we write $a(\Delta) = a$ and $b(\Delta) = b$. We also write $[a]_\rho := [a,a]_\rho$. Let Seg_ρ be the set of segments. A *multisegment* (associated to ρ) is a multiset of non-empty segments. Let Mult_ρ be the set of mulitsegments. One may refer to [Ch22+] a more general notion of multisegments, which we shall not use in this article.

For $\pi \in \mathrm{Irr}_\rho(G_n)$ and a segment $\Delta \in \mathrm{Seg}_\rho$, there is at most one irreducible module $\tau \in \mathrm{Irr}_\rho(G_{n-i})$ such that

$$\pi \hookrightarrow \tau \times \mathrm{St}(\Delta).$$

If such τ exists, we denote such τ by $D_\Delta(\pi)$. Otherwise, we set $D_\Delta(\pi) = 0$. We shall refer D_Δ to be a *derivative*.

A sequence of segments $[a_1,b_1]_\rho, \ldots, [a_k,b_k]_\rho$ (all $a_j, b_j \in \mathbb{Z}$) is said to be in an *ascending order* if for any $i \leq j$, either $[a_i,b_i]_\rho$ and $[a_j,b_j]_\rho$ are unlinked; or $a_i < a_j$. For a multisegment $\mathfrak{n} \in \mathrm{Mult}_\rho$, which we write the segments in $\mathfrak{n}$ in an ascending order $\Delta_1, \ldots, \Delta_k$. Define

$$D_\mathfrak{n}(\pi) := D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi).$$

The derivative is independent of a choice of an ascending order [Ch22+d]. In particular, one may choose an ordering such that $a_1 \leq \ldots \leq a_k$. We say that $\mathfrak{n}$ is *admissible* to π if $D_\mathfrak{n}(\pi) \neq 0$. We refer the reader to [LM16, Ch22+b, Ch22+c] for more theory on derivatives.

For $\pi \in \mathrm{Irr}_\rho$, denote its i-th Bernstein-Zelevinsky derivatives by $\pi^{(i)}$. We shall refer the reader [Ze80, Ch21, Ch22+d] for the precise definition and the main discussions and proofs will not involve the use of Bernstein-Zelevinsky derivatives. The main relation of derivatives and Bernstein-Zelevinsky derivatives is that $D_\mathfrak{n}(\pi)$ is a simple quotient of $\pi^{(i)}$, where $i = l_{abs}(\mathfrak{n})$ [Ch22+d]. The goal of this series of articles [Ch22+d, Ch22+e] is to study constructions from $D_\mathfrak{n}(\pi)$. In particular, [Ch22+e] studies the following poset:

$$\mathcal{S}(\pi,\tau) := \left\{ \mathfrak{n} \in \mathrm{Mult}_\rho : D_\mathfrak{n}(\pi) \cong \tau \right\},$$

where τ is a simple quotient of $\pi^{(i)}$. The ordering $\leq_Z$ on $\mathcal{S}(\pi,\tau)$ is the Zelevinsky ordering (see Section 4.1). We record two fundamental combinatorial structure on the set $\mathcal{S}(\pi,\tau)$:

Theorem 1.1. [Ch22+e] *We use the notatons above. Let $\mathfrak{n}_1, \mathfrak{n}_2 \in \mathcal{S}(\pi,\tau)$ with $\mathfrak{n}_1 \leq_Z \mathfrak{n}_2$. If $\mathfrak{n} \in \mathrm{Mult}_\rho$ with $\mathfrak{n}_1 \leq_Z \mathfrak{n} \leq_Z \mathfrak{n}_2$, then $\mathfrak{n} \in \mathcal{S}(\pi,\tau)$.*

Theorem 1.2. [Ch22+e] *We use the notatons above. Suppose $\mathcal{S}(\pi,\tau) \neq \emptyset$. Then $\mathcal{S}(\pi,\tau)$ has a unique $\leq_Z$-minimal element.*

For $\pi \in \mathrm{Irr}_\rho$, a multisegment $\mathfrak{n} \in \mathrm{Mult}_\rho$ is said to be *minimal* to π if $D_{\mathfrak{n}}(\pi) \neq 0$ and $\mathfrak{n}$ is $\leq_Z$-minimal in $\mathcal{S}(\pi, D_{\mathfrak{n}}(\pi))$. We shall sometimes refer such $\mathfrak{n}$ to be the *minimal multisegment* or *minimal sequence* (of derivatives).

1.2. **Main results.** The main goal of this article is to study the minimal sequence. The main results are the following subsequent and commutativity properties:

Theorem 1.3. *(Theorem 8.6) Let $\pi \in \mathrm{Irr}_\rho$. If $\mathfrak{n} \in \mathrm{Mult}_\rho$ is minimal to π, then any submultisegment $\mathfrak{n}'$ of $\mathfrak{n}$ is also minimal to π and in particular, $D_{\mathfrak{n}'}(\pi) \neq 0$.*

Theorem 1.4. *(=Theorem 9.4) Let $\pi \in \mathrm{Irr}_\rho$. If $\mathfrak{n} \in \mathrm{Mult}_\rho$ is minimal to π, then for any submultisegment $\mathfrak{n}'$ of $\mathfrak{n}$, we have:*

(1) *$\mathfrak{n} - \mathfrak{n}'$ is minimal to $D_{\mathfrak{n}'}(\pi)$; and*
(2) *$D_{\mathfrak{n}-\mathfrak{n}'} \circ D_{\mathfrak{n}'}(\pi) \cong D_{\mathfrak{n}}(\pi)$.*

The main idea of the proof for Theorem 1.4 is to reduce checking elementary intersection-union processes by Theorem 1.1. Then one uses some more basic commutativity (e.g. Proposition 5.4) to reduce to the three segment case in Definition 4.1. One important ingredient in studying the commutativity is a notion of η-invariant (see Definition 4.1), which also plays an important role in studying 'left-right' commutativity in [Ch22+c].

While our proof is largely combinatorial in nature, a motivation comes from a simple example from Lemma 10.2 and this article attempts to generalize in a larger extent. We provide more representation-theoretic aspects in latter sections to give another perspective.

By using Theorem 1.4 multiple times, we have:

Corollary 1.5. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be minimal to π. Write the segments in $\mathfrak{n}$ as $\{\Delta_1, \ldots, \Delta_r\}$ in any order. Then,*

$$D_{\Delta_r} \circ \ldots \circ D_{\Delta_1}(\pi) \cong D_{\mathfrak{n}}(\pi).$$

The commutativity and minimality also play important roles in the branching law [Ch22+b]. The uniqueness of minimality is closely related to the layer of Bernstein-Zelevinsky filtration determining a branching law [Ch22+b].

There are some further results on minimality such as construction from the removal process, which will be explored in the sequel [Ch24+]

1.3. **Discussions on applications.** For $\pi \in \mathrm{Irr}_\rho$ and $\Delta \in \mathrm{Seg}_\rho$, instead of studying $D_\Delta(\pi)$, one studies on a so-called big derivative in [Ch22+] involving some higher structures. It is shown in [Ch22+] to be useful to study a reduced decomposition [AL23] for π in the following sense:

$$\mathrm{St}(\mathfrak{p}) \times D_{\mathfrak{p}}(\pi) \twoheadrightarrow \pi \quad (\text{equivalently, } \pi \hookrightarrow D_{\mathfrak{p}}(\pi) \times \mathrm{St}(\mathfrak{p})\ ), \tag{1.1}$$

where $\mathfrak{p} = \mathfrak{mx}(\pi, \Delta)$ for some segment Δ (see (10.4) for the definition of $\mathfrak{mx}$). In Appendix B, we give a generalization to mutlisegment cases. Such reduced decomposition is also useful to study the relation between Bernstein-Zelevinsky derivatives and layers of Bernstein-Zelevinsky filtrations in [Ch22+b].

Another application is to give some inductive construction of some simple quotients of Bernstein-Zelevinsky derivatives. For example, for $\pi \in \mathrm{Irr}_\rho$ and $\Delta \in \mathrm{Seg}_\rho$, if a simple quotient of $\pi^{(i)}$ is Δ-reduced in the sense that $\mathfrak{mx}(\pi, \Delta) = \emptyset$, then one may construct such simple quotient from a simple quotient of $(D_{\mathfrak{p}}(\pi))^{(i-l)}$ via (1.1), where $l = l_{abs}(\mathfrak{p})$. The idea of this construction is closely related to the commutativity and see Proposition 15.2.

1.4. **Organization.** In first few sections, we recall some main ingredients: highest derivative multisegments in Section 2, removal processes in Section 3, and non-overlapping and intermediate segment properties in Section 4.

Sections 5 to 9 study the commutativity and subsequent property for minimal sequences. The approach is largely combinatorial using the overlapping property. Section 5 studies the two segment case while Section 7 studies the three segment case. Section 6 shows some preliminary results for general cases. Sections 8 and 9 prove the general case for the commutativity and subsequent property respectively.

Sections 10 to 13 study some representation-theoretic aspects of the minimality. Section 10 explains a representation-theoretic proof of commutativity of two segment case. Section 11 conjectures a representation-theoretic interpretation for the minimality and proves for the two segment case. Sections 12 and 13 study how the interpretation gives some applications and connections to removal processes.

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2. Highest derivative multisegments

The highest derivative multisegment is introduced in [Ch22+] as a main tool for the entire study. In this section, we first recall the definition and then prove a new realization theorem, which is of independent interests.

2.1. **Highest derivative multisegments.** A multisegment $\mathfrak{m}$ is said to be *at* $c \in \mathbb{Z}$ if any segment Δ in $\mathfrak{m}$ takes the form $[c, b]_\rho$ for some $b \geq c$. For $\pi \in \mathrm{Irr}_\rho$, define $\mathfrak{mxpt}^a(\pi, c)$ to be the maximal multisegment such that

1. for any $\Delta \in \mathfrak{mxpt}^a(\pi, c)$, $a(\Delta) = c$; and
2. $D_{\mathfrak{mxpt}^a(\pi,c)}(\pi) \neq 0$.

Here the maximality is to taken the lexicographical ordering on the $b(\Delta)$ values for all segments in $\mathfrak{mxpt}^a(\pi, c)$. See [Ch22+d] for details and examples. Define the *highest derivative multisegment* of $\pi \in \mathrm{Irr}_\rho$ to be

$$\mathfrak{hd}(\pi) := \sum_{c \in \mathbb{Z}} \mathfrak{mxpt}^a(\pi, c).$$

It is shown in [Ch22+d] that $D_{\mathfrak{hd}(\pi)}(\pi)$ is the highest derivative of π in the sense of [Ze80].

2.2. **Realization Theorem.** For $d, m \in \mathbb{Z}_{>0}$ and a cuspidal representation ρ, define

$$u_\rho(d, m) = \mathrm{St}\left(\left\{\left[-\frac{d+m-2}{2}, \frac{d-m}{2}\right]_\rho, \ldots, \left[-\frac{d-m}{2}, \frac{d+m-2}{2}\right]_\rho\right\}\right)$$

An irreducible representation π is said to be an *essentially Speh representation* if $\pi \cong \nu^c \cdot u_\rho(d, m)$ for some $c \in \mathbb{Z}$ Denote such representation by $u_\rho(c, d, m)$.

Theorem 2.1. *Let* $\mathfrak{m} \in \mathrm{Mult}_\rho$*. Then there exists* $\pi \in \mathrm{Irr}_\rho$ *such that*

$$\mathfrak{hd}(\pi) = \mathfrak{m}.$$

Proof. We label the segments in $\mathfrak{m}$ as:

$$\Delta_1, \ldots, \Delta_r$$

such that $b(\Delta_1) \leq b(\Delta_2) \leq \ldots \leq b(\Delta_r)$.

We simply let $\pi_1 = \mathrm{St}(\Delta_1)$. It is clear that $\mathfrak{hd}(\pi_1) = \{\Delta_1\}$. Now, for $i \geq 2$, we recursively define π_i to be an essentially Speh representation $u_\rho(c_i, d_i, m_i)$ such that for any $\sigma \in \mathrm{csupp}(\pi_{i-1})$, $\sigma \in \mathrm{csupp}(\pi_i)$. We just have to justify such π_i exists. To see this, we write $\Delta_i = [a_i, b_i]_\rho$ and we can first choose d_i large enough such that any representation in $\mathrm{csupp}(\pi_{i-1})$ lies in $[b_i - d_i + 1, b_i]_\rho$. (Such d_i exists by using $b(\Delta_i) \geq b(\Delta_{i-1})$.) For such fixed d_i, now we solve c_i and m_i such that $\mathfrak{hd}(u_\rho(c_i, d_i, m_i)) = \Delta_i$.

Now let

$$\pi = \pi_1 \times \ldots \times \pi_r.$$

The cuspidal conditions guarantee that π is irreducible. A similar argument in dealing with the Arthur type representation in [Ch22+d] gives that

$$\mathfrak{hd}(\pi) = \mathfrak{hd}(\pi_1) + \ldots + \mathfrak{hd}(\pi_r) = \mathfrak{m}.$$

□

Part 2. Combinatorial aspects

3. Removal processes

In this section, we recall some results in [Ch22+d].

3.1. More notations on multisegmetns. For a multisegment $\mathfrak{m}$ in Mult_ρ and an integer c, let $\mathfrak{m}[c]$ be the submultisegment of $\mathfrak{m}$ containing all the segments Δ satisfying $a(\Delta) = c$; and let $\mathfrak{m}\langle c\rangle$ be the submultisegment of $\mathfrak{m}$ containing all the segments Δ satisfying $b(\Delta) = c$.

For a multisegment $\mathfrak{m} = \{\Delta_1, \ldots, \Delta_k\}$, we also set:

$$l_{abs}(\mathfrak{m}) = l_{abs}(\Delta_1) + \ldots + l_{abs}(\Delta_k).$$

3.2. Removal process. We write $[a, b]_\rho \prec^L [a', b']_\rho$ if either $a < a'$; or $a = a'$ and $b < b'$. A segment $\Delta = [a, b]_\rho$ is said to be *admissible* to a multisegment $\mathfrak{h}$ if there exists a segment of the form $[a, c]_\rho$ in $\mathfrak{h}$ for some $c \geq b$. We now recall the *removal process*.

Definition 3.1. [Ch22+d] Let $\mathfrak{h} \in \mathrm{Mult}_\rho$ and let $\Delta = [a, b]_\rho$ be admissible to $\mathfrak{h}$. The removal process on $\mathfrak{h}$ by Δ is an algorithm to carry out the following steps:

(1) Pick a shortest segment $[a, c]_\rho$ in $\mathfrak{h}[a]$ satisfying $b \leq c$. Set $\Delta_1 = [a, b]_\rho$. Set $a_1 = a$ and $b_1 = c$.
(2) One recursively find the $\prec^L$-minimal segment $\Delta_i = [a_i, b_i]_\rho$ in $\mathfrak{h}$ such that $a_{i-1} < a_i$ and $b_i < b_{i-1}$. The process stops if one can no longer find those segments.
(3) Let $\Delta_1, \ldots, \Delta_r$ be all those segments. For $1 \leq i < r$, define $\Delta_i^{tr} = [a_{i+1}, b_i]_\rho$ and $\Delta_r^{tr} = [b_r + 1, b]_\rho$ (possibly empty).
(4) Define

$$\mathfrak{r}(\Delta, \mathfrak{h}) := \mathfrak{h} - \sum_{i=1}^{r} \Delta_i + \sum_{i=1}^{r} \Delta_i^{tr}.$$

We call $\Delta_1, \ldots, \Delta_r$ to be the *removal sequence* for $(\Delta, \mathfrak{h})$. We also define $\Upsilon(\Delta, \mathfrak{h}) = \Delta_1$, the first segment of the removal sequence. If Δ is not admissible to $\mathfrak{h}$, we set $\mathfrak{r}(\Delta, \mathfrak{h}) = \infty$, called the infinity multisegment. We also set $\mathfrak{r}(\Delta, \infty) = \infty$.

3.3. Properties of removal process. For a segment $\Delta = [a, b]_\rho \neq \emptyset$, let ${}^-\Delta = [a+1, b]_\rho$. We recall the following properties for computations:

Lemma 3.2. [Ch22+d] *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$ and let $\Delta, \Delta' \in \mathrm{Seg}_\rho$ be admissible to $\mathfrak{h}$. Then*

(1) *Let $\mathfrak{h}^* = \mathfrak{h} - \Upsilon(\Delta, \mathfrak{h}) + {}^-\Upsilon(\Delta, \mathfrak{h})$. Then $\mathfrak{r}(\Delta, \mathfrak{h}) = \mathfrak{r}({}^-\Delta, \mathfrak{h}^*)$.*
(2) *Write $\Delta = [a, b]_\rho$. For any $a' < a$, $\mathfrak{r}(\Delta, \mathfrak{h})[a'] = \mathfrak{h}[a']$.*
(3) *If $\Delta \in \mathfrak{h}$, then $\mathfrak{r}(\Delta, \mathfrak{h}) = \mathfrak{h} - \Delta$.*

(4) *Suppose* $a(\Delta)=a(\Delta')$. *Then*

$$\Upsilon(\Delta,\mathfrak{h})+\Upsilon(\Delta',\mathfrak{r}(\Delta,\mathfrak{h}))=\Upsilon(\Delta',\mathfrak{h})+\Upsilon(\Delta,\mathfrak{r}(\Delta',\mathfrak{h})).$$

(5) *If* Δ,Δ' *are unlinked, then* $\mathfrak{r}(\Delta',\mathfrak{r}(\Delta,\mathfrak{h}))=\mathfrak{r}(\Delta,\mathfrak{r}(\Delta',\mathfrak{h}))$.

For $\mathfrak{h}\in\mathrm{Mult}_\rho$, a multisegment $\mathfrak{n}=\{\Delta_1,\ldots,\Delta_r\}\in\mathrm{Mult}_\rho$ written in an ascending order, define:

$$\mathfrak{r}(\mathfrak{n},\mathfrak{h})=\mathfrak{r}(\Delta_r,\ldots,\mathfrak{r}(\Delta_1,\mathfrak{h})\ldots).$$

We say that $\mathfrak{n}$ is *admissible* to $\mathfrak{h}$ if $\mathfrak{r}(\mathfrak{n},\mathfrak{h})\neq\infty$.

Theorem 3.3. [Ch22+d] *Let* $\pi\in\mathrm{Irr}_\rho$. *Let* $\mathfrak{m},\mathfrak{m}'\in\mathrm{Mult}_\rho$ *be admissible to* π. *Then* $\mathfrak{m},\mathfrak{m}'$ *are admissible to* $\mathfrak{hd}(\pi)$, *and furthermore,* $D_{\mathfrak{m}}(\pi)\cong D_{\mathfrak{m}'}(\pi)$ *if and only if* $\mathfrak{r}(\mathfrak{m},\pi)=\mathfrak{r}(\mathfrak{m}',\pi)$.

3.4. **More relations to derivatives.** For $\mathfrak{h}\in\mathrm{Mult}_\rho$ and $\Delta=[a,b]_\rho\in\mathrm{Seg}_\rho$, set

$$\varepsilon_\Delta(\mathfrak{h})=|\left\{\widetilde{\Delta}\in\mathfrak{h}[a]:\Delta\subset\widetilde{\Delta}\right\}|.$$

Define $\varepsilon_\Delta(\pi):=\varepsilon_\Delta(\mathfrak{hd}(\pi))$, which is equivalent to a different formulation of the same notation in [Ch22+d]. We also remark that when Δ is a singleton, this number coincides with the number defined in [Ja07, Definition 2.1.1]. In terms of derivatives, we have the following:

Theorem 3.4. [Ch22+d, Theorem 6.20] *Let* $\pi\in\mathrm{Irr}_\rho$. *Let* $\Delta=[a,b]_\rho\in\mathrm{Seg}_\rho$ *be admissible to* π. *Let* $\Delta'=[a',b']_\rho$ *be another segment. If either* $a'>a$; *or* Δ' *and* Δ *are unlinked, then*

$$\varepsilon_{\Delta'}(D_\Delta(\pi))=\varepsilon_{\Delta'}(\mathfrak{r}(\Delta,\pi)).$$

Theorem 3.5. [Ch22+d, Theorem 6.20] *Let* $\pi\in\mathrm{Irr}_\rho$. *Let* $\Delta=[a,b]_\rho\in\mathrm{Seg}_\rho$ *be admissible to* π. *Let* $\Delta'=[a',b']_\rho$ *be another segment. If* $a'<a$, *then*

$$\varepsilon_{\Delta'}(D_\Delta(\pi))\geq\varepsilon_{\Delta'}(\mathfrak{r}(\Delta,\pi))=\varepsilon_{\Delta'}(\pi).$$

We remark that the equality in Theorem 3.5 follows from Lemma 3.2(2).

4. Non-overlapping property for a sequence

In [Ch22+e], we have shown some characterizations for the miniamlity of two segment case. The goal of this section is to generalize a so-called non-overlapping property to a multisegment case. For this, we first recall some ingredients in [Ch22+e]: fine chains and local minimizability. Those ingredients are combinatorics in nature and so most statements will be formulated for Mult_ρ rather than Irr_ρ.

4.1. **Zelevinsky ordering.** Two non-empty segments $\Delta=[a,b]_\rho$ and $\Delta'=[a',b']_\rho$ in Seg_ρ are said to be *linked* if one of the following conditions holds:

(1) $a<a'\leq b+1\leq b'$; or
(2) $a'<a\leq b'+1\leq b$.

Otherwise, Δ and Δ' are called to be not linked or unlinked. For two linked segments Δ,Δ', we write $\Delta<\Delta'$ if it is in the first condition above. Otherwise, we write $\Delta'<\Delta$. For example, $[2,3]_\rho<[4,5]_\rho$

For two linked segments $\Delta=[a,b]_\rho$ and $\Delta'=[a',b']_\rho$ with $\Delta<\Delta'$, we define:

$$\Delta\cup\Delta'=[a,b']_\rho,\quad\Delta\cap\Delta'=[a',b]_\rho.$$

A multisegment $\mathfrak{n}$ is said to be obtained from $\mathfrak{m}$ by an elementary *intersection-union process* if there exists a pair of linked segments Δ,Δ' such that

$$\mathfrak{n}=\mathfrak{m}-\Delta-\Delta'+\Delta\cup\Delta'+\Delta\cap\Delta'.$$

Here the subtractions mean the (multi-)set theoretic subtraction and additions mean the (multi-)set theoretic union. We shall also use such notions for substractions and additions later. Note that $\Delta \cap \Delta'$ is possibly the empty set and in such case, we simply drop the term. For example, $\{[1,3]_\rho, [2]_\rho, [4,5]_\rho\}$ and $\{[1,2]_\rho, [2,5]_\rho\}$ are obtained from $\{[1,2]_\rho, [2,3]_\rho, [4,5]_\rho\}$ by elementary intersection-union processes.

For two multisegments $\mathfrak{m}, \mathfrak{n} \in \mathrm{Mult}_\rho$, we wrtie $\mathfrak{n} \leq_Z \mathfrak{m}$ if $\mathfrak{n}$ is obtained from $\mathfrak{m}$ be a sequence of intersection-union processes, or $\mathfrak{m} = \mathfrak{n}$. It is well-known from [Ze80] that $\leq_Z$ defines a partial ordering on Mult_ρ.

4.2. **Non-overlapping property and intermediate segment property.** We first define the η-invariant, which is crucial in studying minimality of derivatives:

Definition 4.1.
- Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\Delta = [a,b]_\rho \in \mathrm{Seg}_\rho$ be admissible to $\mathfrak{h}$. Define
$$\eta_\Delta(\mathfrak{h}) := (\varepsilon_{[a,b]_\rho}(\mathfrak{h}), \varepsilon_{[a+1,b]_\rho}(\mathfrak{h}), \ldots, \varepsilon_{[b,b]_\rho}(\mathfrak{h})).$$
- Let $\pi \in \mathrm{Irr}_\rho$. Let $\Delta \in \mathrm{Seg}_\rho$ be admissible to $\mathfrak{h}$. Define $\eta_\Delta(\pi) = \eta_\Delta(\mathfrak{hd}(\pi))$.

Definition 4.2. Let $\Delta, \Delta' \in \mathrm{Seg}_\rho$ with $\Delta < \Delta'$. Let $\mathfrak{h} \in \mathrm{Mult}_\rho$ such that Δ is admissible to $\mathfrak{h}$. Let $\widetilde{\Delta}$ be the last segment in the removal sequence for $(\Delta, \mathfrak{h})$ that contains $\nu^{-1}a(\Delta')$.

(1) We say that $(\Delta, \Delta', \mathfrak{h})$ satisfies the *non-overlapping property* if
$$\eta_{\Delta'}(D_\Delta(\pi)) = \eta_{\Delta'}(\pi).$$
(2) We say that $(\Delta, \Delta', \mathfrak{h})$ satisfies the *intermediate segment property* if there exists a segment $\widetilde{\Delta} \in \mathfrak{h}$ such that
$$a(\Delta) \leq a(\widetilde{\Delta}) < a(\Delta'), \text{ and } a(\widetilde{\Delta}) \leq b(\Delta) \leq b(\widetilde{\Delta}) < b(\Delta').$$

We remark that the original formulation of non-overlapping property in [Ch22+e] is phrased in terms of some properties among segments related to overlapping/intersection between segments. We shall use the above equivalent combinatorial formulation, which is more useful for our study later.

Proposition 4.3. [Ch22+d] *Let $\Delta, \Delta', \mathfrak{h}$ be as in Definition 4.2. We further assume that Δ' is admissible to $\mathfrak{h}$. Then the following conditions are equivalent:*

(1) $(\Delta, \Delta', \mathfrak{h})$ *satisfies the non-overlapping property.*
(2) $(\Delta, \Delta', \mathfrak{h})$ *satisfies the intermediate segment property.*
(3) $\{\Delta, \Delta'\}$ *is the minimal element in* $\mathcal{S}(\pi, \mathfrak{r}(\{\Delta, \Delta'\}, \mathfrak{h}))$.

4.3. **Fine chains.**

Definition 4.4. Let $\mathfrak{h} \in \mathrm{Mult}_\rho$ and let $\mathfrak{n} \in \mathrm{Mult}_\rho$. Let a be the smallest integer such that $\mathfrak{n}[a] \neq \emptyset$. Write $\mathfrak{n}[a] = \{\Delta_1, \ldots, \Delta_k\}$.

- Define $\mathfrak{r}_1 = \mathfrak{h}$. For $i \geq 2$, define
$$\mathfrak{r}_i := \mathfrak{r}(\Delta_{i-1}, \ldots, \mathfrak{r}(\Delta_1, \mathfrak{h}) \ldots).$$
Define
$$\mathfrak{fs}(\mathfrak{n}, \mathfrak{h}) := \{\Upsilon(\Delta_1, \mathfrak{r}_1), \ldots, \Upsilon(\Delta_k, \mathfrak{r}_k)\}.$$
- Define
$$\mathfrak{trr}(\mathfrak{n}, \mathfrak{h}) := \mathfrak{h} - \mathfrak{fs}(\mathfrak{n}, \mathfrak{h}) + {}^-(\mathfrak{fs}(\mathfrak{n}, \mathfrak{h}))$$
and
$$\mathfrak{trd}(\mathfrak{n}, \mathfrak{h}) := \mathfrak{n} - \mathfrak{n}[a] + {}^-(\mathfrak{n}[a]).$$

Definition 4.5. Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be admissible to $\mathfrak{h}$. Set $\mathfrak{h}_0 = \mathfrak{h}$ and $\mathfrak{n}_0 = \mathfrak{n}$. Define recursively

$$\mathfrak{h}_i = \mathfrak{trr}(\mathfrak{n}_{i-1}, \mathfrak{h}_{i-1}), \quad \mathfrak{n}_i = \mathfrak{trd}(\mathfrak{n}_{i-1}, \mathfrak{h}_{i-1}).$$

The *fine chain for the removal process* for $(\mathfrak{n}, \mathfrak{h})$ (or simply fine chain for $(\mathfrak{n}, \mathfrak{h})$) is the sequence

$$\mathfrak{fs}(\mathfrak{n}_0, \mathfrak{h}_0), \mathfrak{fs}(\mathfrak{n}_1, \mathfrak{h}_1), \ldots$$

The sequences $\mathfrak{h}_0, \mathfrak{h}_1, \ldots$ and $\mathfrak{n}_0, \mathfrak{n}_1, \ldots$ will also be useful later.

Lemma 4.6. *We use the notations in Definition 4.5. Then, for all i,*

$$\mathfrak{r}(\mathfrak{n}, \mathfrak{h}) = \mathfrak{r}(\mathfrak{n}_i, \mathfrak{h}_i).$$

Proof. This follows from repeated uses of Lemma 3.2(1). □

4.4. **Local minimizability.**

Definition 4.7. Let $\mathfrak{h} \in \mathrm{Mult}_\rho$ and let $\mathfrak{n} \in \mathrm{Mult}_\rho$. Let a be the smallest integer such that $\mathfrak{n}[a] \neq \emptyset$. We say that $(\mathfrak{n}, \mathfrak{h})$ is *locally minimizable* if there exists a segment Δ in $\mathfrak{n}[a+1]$ such that

$$|\left\{ \widetilde{\Delta} \in \mathfrak{n}[a] : \Delta \subset \widetilde{\Delta} \right\}| < |\left\{ \widetilde{\Delta} \in \mathfrak{fs}(\mathfrak{n}, \mathfrak{h}) : \Delta \subset \widetilde{\Delta} \right\}|.$$

Note that each $\widetilde{\Delta} \in \mathfrak{n}[a]$ satisfying $\Delta \subset \widetilde{\Delta}$ guarantees the first segment $\overline{\Delta}$ in the removal sequence satisfying $\Delta \subset \overline{\Delta}$. Roughly speaking, when the difference of two cardinalities in Definition 4.7 is non-zero, one can find a 'short' segment in $\mathfrak{n}[a]$ to do the intersection-union process which still does not change the choices of first segments in the removal processes.

Theorem 4.8. [Ch22+e] *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be admissible to $\mathfrak{h}$. Let $\mathfrak{h}_0, \mathfrak{h}_1, \ldots$ and $\mathfrak{n}_0, \mathfrak{n}_1, \ldots$ be as cosntructed in Definition 4.5. Then $\mathfrak{n}$ is minimal to $\mathfrak{h}$ if and only if $(\mathfrak{n}_i, \mathfrak{h}_i)$ is not locally minimizable for all i.*

4.5. **Non-overlapping property for a sequence.** We now generalize Proposition 4.3 to a multisegment situation, which will be used in Section 7.3. We first prove a lemma:

Lemma 4.9. *Let $\Delta = [a, b]_\rho$ and $\Delta' = [c, d]_\rho$ be two segments. Suppose $\Delta' < \Delta$. Let $\mathfrak{h} \in \mathrm{Mult}_\rho$ such that Δ' is admissible to $\mathfrak{h}$. Let $\widetilde{\Delta} = \Upsilon(\Delta', \mathfrak{h})$. If $\Delta \subset \widetilde{\Delta}$, then $\eta_\Delta(\mathfrak{r}(\Delta', \mathfrak{h})) \neq \eta_\Delta(\mathfrak{h})$.*

Proof. Let $\Delta_1, \ldots, \Delta_r$ be the removal sequence for $(\Delta', \mathfrak{h})$. Let i^*, which exists from our assumption, be the largest integer such that $b(\Delta) \leq b(\Delta_{i^*})$. Then $\Delta_{i^*}^{tr} \neq \emptyset$ contributes extra one to $\varepsilon_{\Delta_{i^*}^{tr}}$ for $\mathfrak{r}(\Delta', \mathfrak{h})$. However, the segments $\Delta_1, \ldots, \Delta_r$ and Δ_j^{tr} ($j \neq i^*$) do not contribute to $\varepsilon_{\Delta_{i^*}^{tr}}$. This implies the desired inequality. □

Proposition 4.10. *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be minimal to $\mathfrak{h}$. Let $\Delta = [a, b]_\rho$ be a segment such that $a(\Delta') < a$ and $b(\Delta') < b$ for any $\Delta' \in \mathfrak{n}$. Then $\mathfrak{n} + \Delta$ is still minimal to $\mathfrak{h}$ if and only if*

$$\eta_\Delta(\mathfrak{r}(\mathfrak{n}, \mathfrak{h})) = \eta_\Delta(\mathfrak{h}).$$

Proof. We construct a sequence of multisegments $\mathfrak{n}_0, \mathfrak{n}_1, \mathfrak{n}_2, \ldots$ and $\mathfrak{h}_0, \mathfrak{h}_1, \mathfrak{h}_2, \ldots$ as in Definition 4.5. By Lemma 4.6, we have:

$$(*) \quad \mathfrak{r}(\mathfrak{n}_0, \mathfrak{h}_0) = \mathfrak{r}(\mathfrak{n}_1, \mathfrak{h}_1) = \ldots$$

Let a_i be the smallest integer such that $\mathfrak{n}_i[a_i] \neq \emptyset$.

Let i^* be the index such that $a_{i^*} = c - 1$. If such index does not exist, it implies that $b(\widetilde{\Delta}) < c - 1$ for all $\widetilde{\Delta} \in \mathfrak{n}$. In such case, by a direct computation of removal process using Definition 3.1(3) and (4), one has $\mathfrak{r}(\mathfrak{n}, \mathfrak{h})[x] = \mathfrak{h}[x]$ for $x \geq c$. In particular,

$\eta_{\Delta}(\mathfrak{r}(\mathfrak{n},\mathfrak{h}))=\eta_{\Delta}(\mathfrak{h})$; and $\mathfrak{n}+\Delta$ is still minimal since Δ is unlinked to any segment in $\mathfrak{n}$. In other words, both conditions are automatically satisfied if such i^* does not exist.

We now assume such i^* exists. We first prove the only if direction. By the minimality condition and Theorem 4.8, $(\mathfrak{n}_{i^*}+\Delta,\mathfrak{h}_{i^*})$ is not locally minimizable. On the other hand, the hypothesis in this proposition guarantees that any segment $\widetilde{\Delta}$ in $\mathfrak{n}_{i^*}$ satisfies $\Delta\not\in\widetilde{\Delta}$ and so

$$|\left\{\widetilde{\Delta}\in\mathfrak{n}_{i^*}[a]:\Delta\subset\widetilde{\Delta}\right\}|=0.$$

The local minimizability on $(\mathfrak{n}_{i^*}+\Delta,\mathfrak{h}_{i^*})$ implies that

$$|\left\{\widetilde{\Delta}\in\mathfrak{fs}(\mathfrak{n}_{i^*}+\Delta,\mathfrak{h}_{i^*}):\Delta\subset\widetilde{\Delta}\right\}|=0.$$

In other words, for any segment $[x,y]_\rho$ in $\mathfrak{fs}(\mathfrak{n}_{i^*}+\Delta,\mathfrak{h}_{i^*})=\mathfrak{fs}(\mathfrak{n}_{i^*},\mathfrak{h}_{i^*})$ (the equaltiy follows from Definition 4.5), it must take the form $[x',y']_\rho$ for $y'\leq b-1$. Thus the segments involved or produced in the removal process (by the nesting property) cannot contribute to $\eta_{\Delta}(\mathfrak{r}(\mathfrak{n}_{i^*},\mathfrak{h}_{i^*}))$ and so

$$(**)\quad \eta_{\Delta}(\mathfrak{r}(\mathfrak{n}_{i^*},\mathfrak{h}_{i^*}))=\eta_{\Delta}(\mathfrak{h}_{i^*}).$$

Moreover, $\mathfrak{h}_{i^*}$ is obtained by truncating left points for $\nu^x\rho$ for some $x\leq c-2$. Thus, $\eta_{\Delta}(\mathfrak{h}_0)=\ldots=\eta_{\Delta}(\mathfrak{h}_{i^*})$ by Definitions 4.4 and 4.5. Combining with (*) and (**), we have the desired equation, proving the only if direction.

We now prove the if direction. By the construction of $\mathfrak{h}_i$ (Definition 4.5), we have:

$$\eta_{\Delta}(\mathfrak{h})=\eta_{\Delta}(\mathfrak{h}_{i^*})$$

Now with (*) and the hypothesis $\eta_{\Delta}(\mathfrak{h})=\eta_{\Delta}(\mathfrak{r}(\mathfrak{n},\mathfrak{h}))$, we again have that

$$\eta_{\Delta}(\mathfrak{r}(\mathfrak{n}_{i^*},\mathfrak{h}_{i^*}))=\eta_{\Delta}(\mathfrak{h}_{i^*}).$$

This condition says that $\mathfrak{fs}(\mathfrak{n}_{i^*},\mathfrak{h}_{i^*})$ cannot take the form $[c-1,y]_\rho$ for some $y\geq b$, by Lemma 4.9 and the assumption that any segment in $\mathfrak{n}$ (and so $\mathfrak{n}_{i^*}$) satisfies $b(\Delta)<b$. In other words, $(\mathfrak{n}_{i^*},\mathfrak{h}_{i^*})$ is not locally minimizable. The non-local minimizability for other pairs $(\mathfrak{n}_j,\mathfrak{h}_j)$ for $j<i^*$ follows from the minimality of $\mathfrak{n}$. Then the minimality of $\mathfrak{n}+\Delta$ to $\mathfrak{h}$ follows from Theorem 4.8. ☐

5. Two segment basic case (commutativity)

5.1. **Lemma for unlinked segments.** We recall the following first commutativity (see e.g. [Ch22+d]), which will be used later:

Lemma 5.1. *Let $\Delta,\Delta'\in\mathrm{Seg}_\rho$ be unlinked. For any $\pi\in\mathrm{Irr}_\rho$,*

$$D_{\Delta'}\circ D_{\Delta}(\pi)\cong D_{\Delta}\circ D_{\Delta'}(\pi).$$

5.2. **Intermediate segment property under a derivative.**

Lemma 5.2. *Let $\pi\in\mathrm{Irr}_\rho$. Let $\Delta,\Delta''\in\mathrm{Seg}_\rho$ be admissible to π. Let $\Delta'\in\mathrm{Seg}_\rho$. Suppose $(\Delta,\Delta',\mathfrak{hd}(\pi))$ satisfies the non-overlapping property or intermediate segment property. If $\Delta''\subset\Delta'$, then $(\Delta,\Delta',\mathfrak{hd}(D_{\Delta''}(\pi)))$ also satisfies the non-overlapping property and the intermediate segment property.*

Proof. Since the overlapping property and the intermediate segment property are equivalent by Proposition 4.3, it suffices to see that $(\Delta,\Delta',\mathfrak{hd}(D_{\Delta'}(\pi)))$ also satisfies the intermediate segment property.

Write $\Delta=[a,b]_\rho$, $\Delta'=[a',b']_\rho$ and $\Delta''=[a'',b'']_\rho$. By Theorems 3.4 and 3.5, we have that for segment $\widetilde{\Delta}$

$$(*)\quad \varepsilon_{\widetilde{\Delta}}(\mathfrak{r}(\Delta'',\pi))\leq\varepsilon_{\widetilde{\Delta}}(D_{\Delta''}(\pi)).$$

From this, one can recover $\mathfrak{hd}(\pi)[c]$ and $\mathfrak{r}(\Delta,\pi)[c]$ for each integer c. The missing part of (*) between $\mathfrak{r}(\Delta',\mathfrak{hd}(\pi))$ and $\mathfrak{hd}(D_{\Delta''}(\pi))$ is on some values $a''-1,\ldots,b''-1<b'$. Thus, one obtains $\mathfrak{hd}(D_{\Delta''}(\pi))$ by prolonging some segments in $\mathfrak{r}(\Delta'',\mathfrak{hd}(\pi))$ using (possibly some of) $a'-1,\ldots,b'-1$.

Now, by the intermediate segment property for $(\Delta,\Delta',\mathfrak{hd}(\pi))$, there exists a segment of the form $[c,d]_\rho$ satisfying:

$$a\leq c<a',\quad b\leq d<b'.$$

Now, by the above process of obtaining $\mathfrak{hd}(D_{\Delta''}(\pi))$, the segment $[c,d]_\rho$ can be prolonged to the form $[c,e]_\rho$ in $\mathfrak{r}(\Delta'',\pi)$ for some $e\leq a'-1$. Then the segment $[c,e]_\rho$ gives the desired requirement for the intermediate segment property for $(\Delta,\Delta',D_{\Delta''}(\pi))$. Thus, by Proposition 4.3, the triple satisfies the two properties. □

For a segment $\Delta=[a,b]_\rho\in\mathrm{Seg}_\rho$ and $\pi\in\mathrm{Irr}_\rho$, define:

$$|\eta|_\Delta(\pi)=\varepsilon_{[a,b]_\rho}(\pi)+\varepsilon_{[a+1,b]_\rho}(\pi)+\ldots+\varepsilon_{[b,b]_\rho}(\pi).$$

Lemma 5.3. *Let $\pi\in\mathrm{Irr}_\rho$. Let $\Delta,\Delta'\in\mathrm{Seg}_\rho$. Suppose Δ is admissible to π. Suppose $(\Delta,\Delta',\mathfrak{hd}(\pi))$ satisfies the overlapping property or the intermediate segment property. Let $\widetilde{\Delta}=\Delta\cup\Delta'$.*

$$|\eta|_{\widetilde{\Delta}}(\pi)-|\eta|_{\Delta'}(\pi)=|\eta|_{\widetilde{\Delta}}(D_\Delta(\pi))-|\eta|_{\Delta'}(D_\Delta(\pi)).$$

Proof. Write the removal sequence for (Δ,π) to be:

$$\Delta_1,\ldots,\Delta_r.$$

By the intermediate segment property, there exists a segment Δ_k such that $a(\Delta_k)<a(\Delta')$ and $b(\Delta_k)<b(\Delta')$. Let k^* be the smallest such integer. Then, with the nesting property, only $\Delta_1,\ldots,\Delta_{k^*-1}$ (among $\Delta_1,\ldots,\Delta_r$) contribute to $|\eta|_{\widetilde{\Delta}}(\pi)-|\eta|_{\Delta'}(\pi)$.

Write $\Delta=[a,b]_\rho$ and $\Delta'=[a',b']_\rho$. By Theorem 3.4, for $a\leq c\leq a'-1$

$$\varepsilon_{[c,b']_\rho}(D_\Delta(\pi))=\varepsilon_{[c,b']_\rho}(\mathfrak{r}(\Delta,\pi)).$$

Then, only $\Delta_1^{tr},\ldots,\Delta_{k^*-1}^{tr}$ (among $\Delta_1^{tr},\ldots,\Delta_r^{tr}$) can contribute to $|\eta|_{\widetilde{\Delta}}(\pi)-|\eta|_{\Delta'}(\pi)$ and so this implies the equality. □

5.3. **Commutativity and minimality for two segment case.**

Proposition 5.4. *Let $\pi\in\mathrm{Irr}_\rho$. Let $\Delta_1,\Delta_2\in\mathrm{Seg}_\rho$ be admissible to π. Suppose $\Delta_1<\Delta_2$ and $(\Delta_1,\Delta_2,\mathfrak{hd}(\pi))$ satisfies the over-lapping property, or equivalently*

$$D_{\Delta_2}\circ D_{\Delta_1}(\pi)\not\cong D_{\Delta_1\cup\Delta_2}\circ D_{\Delta_1\cap\Delta_2}(\pi).$$

Then

$$D_{\Delta_2}\circ D_{\Delta_1}(\pi)\cong D_{\Delta_1}\circ D_{\Delta_2}(\pi).$$

Proof. The equivalence of the two conditions follows from Proposition 4.3.

We shall use the notations in the proof. Indeed, in view of a criteria of commutativity in [Ch22+d, Proposition 4.12], it suffices to prove

$$D_{\Delta_1}\circ D_{\Delta_2}(\pi)\not\cong D_{\Delta_1\cup\Delta_2}\circ D_{\Delta_1\cap\Delta_2}(\pi).$$

Let $\widetilde{\Delta}=\Delta_1\cup\Delta_2$. To this end, it suffices to show that

$$(|\eta|_{\widetilde{\Delta}}-|\eta|_{\Delta_2})(D_{\Delta_1}\circ D_{\Delta_2}(\pi))=(|\eta|_{\widetilde{\Delta}}-|\eta|_{\Delta_2})(\pi). \tag{5.2}$$

Note that, by the unlinked part of Theorem 3.4,

$$(|\eta|_{\widetilde{\Delta}}-|\eta|_{\Delta_2})(D_{\Delta_2}(\pi))=(|\eta|_{\widetilde{\Delta}}-|\eta|_{\Delta_2})(\pi)$$

Now, a direct computation using the removal process on $\mathfrak{r}(\Delta_1, D_{\Delta_2}(\pi))$ that $|\eta|_{\widetilde{\Delta}}(D_{\Delta_1} \circ D_{\Delta_2}(\pi)) = |\eta|_{\widetilde{\Delta}}(D_{\Delta_2}(\pi))$.

On the other hand, by Proposition 4.3 (2), $(\Delta_1, \Delta_2, \pi)$ satisfies the intermediate segment property and so $(\Delta_1, \Delta_2, D_{\Delta_2}(\pi))$ also satisfies the intermediate segment property by Lemma 5.2. Now, by Lemma 5.3,

$$(|\eta|_{\widetilde{\Delta}} - |\eta|_{\Delta_2})(D_{\Delta_1} \circ D_{\Delta_2}(\pi)) = (|\eta|_{\widetilde{\Delta}} - |\eta|_{\Delta_2})(D_{\Delta_2}(\pi)).$$

Combining above two equations, we have (5.2) as desired. □

We shall give a proof of Proposition 5.4 from a representation-theoretic perspective (see Lemma 10.2).

6. Some preliminary resutls for subsequent and commutativity properties

6.1. Cancellative property.

Proposition 6.1. *(Cancellative property) Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{n}$ and $\mathfrak{n}'$ be multisegments with respective segments in the following respective ascending sequences:*

$$\Delta_1', \ldots, \Delta_p', \Delta_1, \ldots, \Delta_r$$

and

$$\Delta_1'', \ldots, \Delta_q'', \Delta_1, \ldots, \Delta_r.$$

Then $\mathfrak{r}(\mathfrak{n}', \pi) = \mathfrak{r}(\mathfrak{n}'', \pi)$ if and only if

$$\mathfrak{r}(\{\Delta_1', \ldots, \Delta_p'\}, \pi) = \mathfrak{r}(\{\Delta_1'', \ldots, \Delta_q''\}, \pi).$$

Proof. The if direction is straightforward. We now consider the only if direction. By Theorem 2.1, let $\pi \in \mathrm{Irr}_\rho$ such that $\mathfrak{hd}(\pi) = \mathfrak{h}$. By Theorem 3.3, we have that

$$D_{\Delta_r} \circ \ldots \circ D_{\Delta_1} \circ D_{\Delta_p'} \circ \ldots \circ D_{\Delta_1'}(\pi) \cong D_{\Delta_r} \circ \ldots \circ D_{\Delta_1} \circ D_{\Delta_q'} \circ \ldots \circ D_{\Delta_1'}(\pi).$$

For any irreducible τ and any segment Δ, denote by $I_\Delta(\tau)$ the unique irreducible submodule of $\pi \times \mathrm{St}(\Delta)$. Now, by uniqueness, $I_{\Delta_i} \circ D_{\Delta_i}(\tau) \cong \tau$ if $D_{\Delta_i}(\tau) \neq 0$ for any i and irreducible τ. Hence, we cancel the derivatives $D_{\Delta_r}, \ldots, D_{\Delta_1}$ to obtain:

$$D_{\Delta_p'} \circ \ldots \circ D_{\Delta_1'}(\pi) \cong D_{\Delta_q'} \circ \ldots \circ D_{\Delta_1'}(\pi).$$

□

6.2. First subsequent property.
We shall frequently use the following simple fact:

Lemma 6.2. [Ze80, Section 6.7] *Let $\Delta_1, \ldots, \Delta_r$ be a sequence of segments in an ascending order. SupposeΔ_k and Δ_{k+1} are linked for some k. Then*

$$\Delta_1, \ldots, \Delta_{k-1}, \Delta_k \cup \Delta_{k+1}, \Delta_k \cap \Delta_{k+1}, \Delta_{k+2}, \ldots, \Delta_r$$

is also in an ascending order.

Proposition 6.3. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{n}$ be minimal to π. We write the segments in $\mathfrak{n}$ in an ascending order*

$$\Delta_1, \ldots, \Delta_r.$$

Then, for any $s \leq r$,

1. *$\{\Delta_1, \ldots, \Delta_s\}$ is still minimal to π.*
2. *$\{\Delta_{s+1}, \ldots, \Delta_r\}$ is minimal to $\mathfrak{r}(\{\Delta_1, \ldots, \Delta_s\}, \pi)$.*

Proof. We only prove (1), and (2) can be proved similarly.

The admissibility follows from definitions (and Lemma 3.2(5)). Let

$$\mathfrak{n}' = \{\Delta_1, \ldots, \Delta_s\}.$$

We pick two linked segments Δ_i and Δ_j in $\mathfrak{n}'$ and we set

$$\mathfrak{n}'' = \mathfrak{n}' - \{\Delta_i, \Delta_j\} + \Delta_i \cup \Delta_j + \Delta_i \cap \Delta_j.$$

It suffices to show that $\mathfrak{r}(\mathfrak{n}'', \mathfrak{h}) \neq \mathfrak{r}(\mathfrak{n}, \mathfrak{h})$. To this end, we first write the segments in $\mathfrak{n}''$ in an ascending order:

$$\Delta'_1, \ldots, \Delta'_s.$$

(There are $s-1$ segments if $\Delta_i \cap \Delta_j = \emptyset$, but the below arguments could be still applied.) It follows from Lemma 6.2 that $\Delta'_1, \ldots, \Delta'_s, \Delta_{s+1}, \ldots, \Delta_r$ is still in an ascending order.

Now we return to the proof. The minimality of $\mathfrak{n}$ implies that

$$\mathfrak{r}(\{\Delta'_1, \ldots, \Delta'_s, \Delta_{s+1}, \ldots, \Delta_r\}, \pi) \neq \mathfrak{r}(\{\Delta_1, \ldots, \Delta_s, \Delta_{s+1}, \ldots, \Delta_r\}, \pi).$$

By Proposition 6.1,

$$\mathfrak{r}(\{\Delta'_1, \ldots, \Delta'_s\}, \pi) \neq \mathfrak{r}(\{\Delta_1, \ldots, \Delta_s\}, \pi),$$

as desired. □

7. Three segment basic cases

The main goal of this section is to prove the subsequent property and commutativity for three segment cases. To show the minimality, the main strategy is to use the convex property for $\mathcal{S}(\pi, \tau)$ of Theorem 1.1 and the overlapping property of Theorem 4.3.

7.1. Case: $\{\Delta_1, \Delta_3\}$ minimal to $D_{\Delta_2}(\pi)$.

Lemma 7.1. *Let $\Delta, \Delta' \in \mathrm{Seg}_\rho$ with $\Delta \subset \Delta'$. Let $\pi \in \mathrm{Irr}_\rho$ with $D_{\Delta'}(\pi) \neq 0$. Then, the following holds:*

1. *If $a(\Delta) > a(\Delta')$, then $\eta_\Delta(D_{\Delta'}(\pi)) = \eta_\Delta(\pi)$.*
2. *If $a(\Delta) = a(\Delta')$, then $\eta_\Delta(D_{\Delta'}(\pi))$ is obtained from $\eta_\Delta(\pi)$ by decreasing the coordinate $\varepsilon_\Delta(\pi)$ by 1.*

Proof. By Theorem 3.4, it suffices to compare $\eta_\Delta(\mathfrak{r}(\Delta', \pi))$ and $\eta_\Delta(\pi)$. Let the removal sequence for (Δ', π) be

$$\Delta_1, \ldots, \Delta_r.$$

For (1), we consider two cases.

(i) Suppose there does not exist an integer i^* such that $a(\Delta_{i^*}) \geq a(\Delta)$ and $a(\Delta_{i^*}) \leq b(\Delta)$. In such case, all $\Delta_1, \ldots, \Delta_r$ and $\Delta_1^{tr}, \ldots, \Delta_r^{tr}$ do contribute η_Δ. Thus we have such equality.

(ii) Suppose there exists an integer i^* such that $a(\Delta_{i^*}) \geq a(\Delta)$. Let $i^* > 1$ be the smallest such integer. Let j^* be the largest integer such that $a(\Delta_{j^*}) \leq b(\Delta)$. We have that $\Delta_{i^*}, \ldots, \Delta_{j^*}$ are all the segments in the removal sequence contributing to $\eta_\Delta(\pi)$ and $\Delta_{i^*-1}^{tr}, \ldots, \Delta_{j^*-1}^{tr}$ are all the segments in the truncated one contributing to $\eta_\Delta(\mathfrak{r}(\Delta', \pi))$. Note that, for $i^* \leq k \leq j^*$, Δ_k and Δ_{k-1}^{tr} contribute to the same coordinate $\varepsilon_{\widetilde{\Delta}}$ for some segment $\widetilde{\Delta}$. This shows the equality to two η_Δ.

For (2), it is similar, but i^* in above notation becomes 1. Again, for $2 \leq k \leq j^*$, Δ_k and Δ_{k-1}^{tr} contribute to the same ε. The term Δ_1 explains $\varepsilon_\Delta(\pi)$ is decreased by 1 to obtain $\varepsilon_\Delta(D_{\Delta'}(\pi))$. □

Lemma 7.2. *Let $\mathfrak{m} = \{\Delta_1, \Delta_2, \Delta_3\} \in \mathrm{Mult}_\rho$ in an ascending order. Let $\pi \in \mathrm{Irr}_\rho$ be such that $\mathfrak{m}$ is minimal to π. Then $\{\Delta_1, \Delta_3\}$ is also minimal to $D_{\Delta_2}(\pi)$.*

Proof. By Proposition 5.4 (for the linked case between Δ_1 and Δ_2) and Proposition 5.1 (for the unlinked case between Δ_1 and Δ_2), we have that

$$D_{\Delta_3} \circ D_{\Delta_2} \circ D_{\Delta_1}(\pi) \cong D_{\Delta_3} \circ D_{\Delta_1} \circ D_{\Delta_2}(\pi).$$

It is automatic if Δ_1 and Δ_3 are unlinked. So we shall assume that Δ_1 and Δ_3 are linked. There are three possibilities:

- Δ_2 is unlinked to Δ_1 (and so $\{\Delta_2, \Delta_1, \Delta_3\}$ is still in an ascending order). Then the minimality of $\mathfrak{m}$ and Proposition 6.3 implies this case.
- Δ_2 is unlinked to Δ_3. We consider following possibilities:
  (i) $\Delta_3 \subset \Delta_2$. Then $\{\Delta_1, \Delta_3\}$ is also minimal to π by Lemma 5.1 and Proposition 6.3. Hence, $\eta_{\Delta_3}(D_{\Delta_1}(\pi)) = \eta_{\Delta_3}(\pi)$. On the other hand, if $a(\Delta_3) > a(\Delta_2)$, by Lemma 7.1(1),
  $$\eta_{\Delta_3}(D_{\Delta_2}(\pi)) = \eta_{\Delta_3}(\pi)$$
  and
  $$\eta_{\Delta_3}(D_{\Delta_2} \circ D_{\Delta_1}(\pi)) = \eta_{\Delta_3}(D_{\Delta_1}(\pi)).$$
  Combining two equations, we have
  $$\eta_{\Delta_3}(D_{\Delta_2} \circ D_{\Delta_1}(\pi)) = \eta_{\Delta_3}(D_{\Delta_2}(\pi))$$
  and so, by Proposition 5.4 and Proposition 5.1 again,
  $$\eta_{\Delta_3}(D_{\Delta_1} \circ D_{\Delta_2}(\pi)) = \eta_{\Delta_3}(D_{\Delta_2}(\pi)).$$
  Thus, we have the minimality by Proposition 4.3.
  When $a(\Delta_3) \cong a(\Delta_2)$, the argument is similar. The only difference, by Lemma 7.1(2), is that $\eta_{\Delta_3}(D_{\Delta_2}(\pi))$ (resp. $\eta_{\Delta_3}(D_{\Delta_2} \circ D_{\Delta_1}(\pi))$) is obtained from $\eta_{\Delta_3}(\pi)$ (resp. $\eta_{\Delta_3}(D_{\Delta_1}(\pi))$) by decreasing the $\varepsilon_{\Delta_3}(\pi)$ (resp. $\varepsilon_{\Delta_3}(D_{\Delta_1}(\pi))$) by 1.
  (ii) $\Delta_2 \subset \Delta_3$. In such case, one first has that $(\Delta_1, \Delta_3, \mathfrak{hd}(\pi))$ satisfies the non-overlapping property by the minimality of $\{\Delta_1, \Delta_3\}$ to π. Then, by Lemma 5.2 to show that $(\Delta_1, \Delta_3, \mathfrak{hd}(D_{\Delta_2}(\pi)))$ still satisfies the non-overlapping property. Thus, $\{\Delta_1, \Delta_3\}$ is minimal to $D_{\Delta_2}(\pi)$.
  (iii) $b(\Delta_2) < a(\Delta_3)$. Then the ascending order and linkedness between Δ_1 and Δ_3 also give that Δ_1 and Δ_2 are not linked. This goes back to the above bullet.
  (iv) $b(\Delta_3) < a(\Delta_2)$. Then the ascending order and linkedness between Δ_1 and Δ_3 also give that Δ_1 and Δ_2 are not linked. This goes back to the above bullet.
- $\Delta_1 < \Delta_2 < \Delta_3$. If $\{\Delta_1, \Delta_3\}$ is not minimal to $D_{\Delta_2}(\pi)$, then
  $$D_{\Delta_1 \cup \Delta_3} \circ D_{\Delta_1 \cap \Delta_3} \circ D_{\Delta_2}(\pi) \cong D_{\Delta_3} \circ D_{\Delta_1} \circ D_{\Delta_2}(\pi) \cong D_{\Delta_3} \circ D_{\Delta_2} \circ D_{\Delta_1}(\pi).$$
  This contradicts the minimality of $\mathfrak{m}$.

□

7.2. **Case: $\{\Delta_2, \Delta_3\}$ minimal to π.**

Lemma 7.3. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\Delta_1, \Delta_2, \Delta_3$ be segments in an ascending order. If $\{\Delta_1, \Delta_2, \Delta_3\}$ is minimal to π, then $\{\Delta_2, \Delta_3\}$ is also minimal to π.*

Proof. When Δ_2 and Δ_3 are unlinked, there is nothing to prove. We assume that Δ_2 and Δ_3 are linked and so $\Delta_2 < \Delta_3$. We consider the following cases:

- $b(\Delta_3) \leq b(\Delta_1)$. Then Δ_1 is unlinked to both Δ_2 and Δ_3. Then the minimality of $\{\Delta_2, \Delta_3\}$ to π follows from the minimality of $\{\Delta_1, \Delta_2, \Delta_3\}$ to π by Proposition 6.3.

- $b(\Delta_1) < b(\Delta_3)$ and $a(\Delta_1) < a(\Delta_3)$. Then $\{\Delta_1, \Delta_2\}$ with Δ_3 is in the situation of Proposition 4.10. Then $\eta_{\Delta_3}(D_{\{\Delta_1,\Delta_2\}}(\pi)) = \eta_{\Delta_3}(\pi)$. On the other hand, we have $\eta_{\Delta_3}(D_{\Delta_1} \circ D_{\Delta_2}(\pi)) = \eta_{\Delta_3}(D_{\Delta_2}(\pi))$ by Lemma 7.2 with Proposition 4.3(1)⇔(3). Thus, we have $\eta_{\Delta_3}(\pi) = \eta_{\Delta_3}(D_{\Delta_2}(\pi))$. This implies $\{\Delta_2, \Delta_3\}$ is minimal to π by Proposition 4.3(1)⇔(3).
- $b(\Delta_1) < b(\Delta_3)$ and $a(\Delta_3) \leq a(\Delta_1)$. In particular, Δ_1 is unlinked to Δ_3. Since we are assuming $\Delta_1, \Delta_2, \Delta_3$ are in ascending order, and we are assuming that Δ_2 and Δ_3 are linked, we also have that Δ_1 and Δ_2 are unlinked. Then the minimality of $\{\Delta_2, \Delta_3\}$ to π follows from the minimality of $\{\Delta_1, \Delta_2, \Delta_3\}$ to π.

□

7.3. **Case: $\{\Delta_1, \Delta_3\}$ minimal to π.**

Lemma 7.4. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\Delta_1, \Delta_2, \Delta_3$ be segments in an ascending order. If $\{\Delta_1, \Delta_2, \Delta_3\}$ is minimal to π, then $\{\Delta_1, \Delta_3\}$ is minimal to π.*

Proof. We may assume Δ_1 and Δ_3 are linked. Otherwise, there is nothing to prove. We consider the following cases.

- $\Delta_1 < \Delta_2 < \Delta_3$. By Proposition 5.4, we have
$$D_{\{\Delta_1,\Delta_3,\Delta_2\}}(\pi) \cong D_{\Delta_2} \circ D_{\{\Delta_1,\Delta_3\}}(\pi).$$
If the minimality does not hold, then we have
$$D_{\{\Delta_1,\Delta_2,\Delta_3\}}(\pi) \cong D_{\{\Delta_2,\Delta_1\cup\Delta_3,\Delta_1\cap\Delta_3\}}(\pi)$$
since $\Delta_1 \cup \Delta_3, \Delta_1 \cap \Delta_3, \Delta_2$ are still in an ascending order. This contradicts to the minimality of $\{\Delta_1, \Delta_2, \Delta_3\}$ to π.
- Δ_2 and Δ_3 are not linked. Then we can switch the labellings of Δ_2 and Δ_3, which gives the minimality of $\{\Delta_1, \Delta_3\}$ to π by Proposition 6.3.
- Δ_1 and Δ_2 are not linked. In this case, we can switch the labellings for Δ_1 and Δ_2 by using linkedness. Then the result follows from Lemma 7.3.

□

7.4. **Case: $\{\Delta_1, \Delta_2\}$ minimal to $D_{\Delta_3}(\pi)$.** We now need some inputs from representation theory to prove a combinatorics result. Let $N_i \subset G_n$ (depending on n) be the unipotent radical containing matrices of the form $\begin{pmatrix} I_{n-i} & * \\ & I_i \end{pmatrix}$. For a smooth representation π of G_n, we write π_{N_i} to be its Jacquet module.

Lemma 7.5. *Let $\Delta = [a,b]_\rho, \Delta' = [a',b']_\rho$ be two segments such that $\Delta < \Delta'$. Let $\omega = \mathrm{St}(\{\Delta, \Delta'\})$. Let $\mathfrak{m}$ be a multisegment whose segments $\Delta'' = [a'', b'']_\rho$ satisfy that $b'' = b$ and $a'' < a$. Then $\mathrm{St}(\mathfrak{m}) \times \omega$ is irreducible and*
$$\mathrm{St}(\mathfrak{m}) \times \omega \cong \omega \times \mathrm{St}(\mathfrak{m}).$$

Proof. It is well-known that the second assertion implies the first one. We only have to prove the first one. Since $\mathrm{St}(\mathfrak{m})$ can be written as $\times_{\Delta\in\mathfrak{m}}\mathrm{St}(\Delta)$, it reduces to the case that $\mathfrak{m}$ contains only one segment and so now we consider $\mathfrak{m} = \left\{\widetilde{\Delta}\right\}$.

We analyse possible composition factors of
$$\mathrm{St}(\widetilde{\Delta}) \times \omega.$$
Since we know that a composition factor of $\mathrm{St}(\widetilde{\Delta}) \times \omega$ is also a composition factor of $\mathrm{St}(\widetilde{\Delta}) \times \mathrm{St}(\Delta) \times \mathrm{St}(\Delta')$, the possible composition factors are
$$\mathrm{St}(\left\{\widetilde{\Delta}, \Delta, \Delta'\right\}), \mathrm{St}(\widetilde{\Delta} \cup \Delta' + \widetilde{\Delta} \cap \Delta' + \Delta), \mathrm{St}(\Delta \cup \Delta' + \Delta \cap \Delta' + \widetilde{\Delta}).$$

We denote the three representations π_1, π_2, π_3 respectively.

Thus it suffices to show that the last two composition factors cannot appear in $\mathrm{St}(\widetilde{\Delta}) \times \omega$. We first consider π_2. Note that π_2 is generic. However, ω is not generic and so $\mathrm{St}(\widetilde{\Delta}) \times \omega$ cannot contains a generic composition factor and so π_2 cannot appear in $\mathrm{St}(\widetilde{\Delta}) \times \omega$.

We now consider π_3. Let $l = l_{abs}(\Delta \cup \Delta')$. Then $(\pi_3)_{N_l}$ has the composition factor $\mathrm{St}(\Delta) \boxtimes \mathrm{St}(\Delta \cup \Delta')$. Now we consider composition factors in $(\mathrm{St}(\widetilde{\Delta}) \times \omega)_{N_l}$. If $(\mathrm{St}(\widetilde{\Delta}) \times \omega)_{N_l}$ contains the factor $\mathrm{St}(\Delta) \boxtimes \mathrm{St}(\Delta \cup \Delta')$, a simple composition factor is a simple composition factor in

$$\mathrm{St}(\Delta) \times \omega_1 \boxtimes \omega_2,$$

where $\omega_1 \boxtimes \omega_2$ is a simple composition factor in ω_{N_l}. However, the possibilities of those composition factors are well-known and it is impossible for ω_2 to be the factor $\mathrm{St}(\Delta \cup \Delta')$. □

Lemma 7.6. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\Delta_1, \Delta_2, \Delta_3$ be segments satisfying $\Delta_1 < \Delta_2 < \Delta_3$. Suppose $\{\Delta_1, \Delta_2, \Delta_3\}$ is minimal to π. Then $D_{\Delta_3}(\pi)$. Let $\widetilde{\Delta} = \Delta_1 \cup \Delta_2$. Then the followings hold:*

1. *$|\eta|_{\widetilde{\Delta}}(\pi) - |\eta|_{\Delta_2}(\pi) = |\eta|_{\widetilde{\Delta}}(D_{\Delta_3}(\pi)) - |\eta|_{\Delta_2}(D_{\Delta_3}(\pi))$;*
2. *If $(\Delta_1, \Delta_2, \pi)$ satisfies the intermediate segment property, then $(\Delta_1, \Delta_2, D_{\Delta_3}(\pi))$ also satisfies the intermediate segment property.*

Here the subtraction in (1) means the subtraction entry-wise.

Proof. We have shown in Lemma 7.3 that $\{\Delta_2, \Delta_3\}$ is minimal to π. Thus, we have

$$D_{\Delta_3} \circ D_{\Delta_2}(\pi) \not\cong D_{\Delta_2 \cup \Delta_3} \circ D_{\Delta_2 \cap \Delta_3}(\pi).$$

By a standard argument in [Ch22+d], we have that π is the unique simple submodule of

$$D_{\Delta_3} \circ D_{\Delta_2}(\pi) \times \mathrm{St}(\{\Delta_2, \Delta_3\}).$$

Write $\Delta_2 = [a_2, b_2]_\rho$. Let

$$\mathfrak{m} = \sum_{c < a_2} \varepsilon_{[c,b_2]_\rho}(D_{\Delta_3}(\pi)) \cdot [c, b_2]_\rho,$$

and

$$\mathfrak{p} = \sum_{c < a_2} \varepsilon_{[c,b_2]_\rho}(\pi) \cdot [c, b_2]_\rho.$$

Thus, we have:

$$D_{\Delta_3} \circ D_{\Delta_2}(\pi) \hookrightarrow D_{\mathfrak{m}} \circ D_{\Delta_3} \circ D_{\Delta_2}(\pi) \times \mathrm{St}(\mathfrak{m})$$

and so

$$\pi \hookrightarrow D_{\Delta_2} \circ D_{\Delta_2}(\pi) \times \mathrm{St}(\{\Delta_2, \Delta_3\}) \hookrightarrow D_{\mathfrak{m}} \circ D_{\Delta_3} \circ D_{\Delta_2}(\pi) \times \mathrm{St}(\mathfrak{m}) \times \mathrm{St}(\{\Delta_2, \Delta_3\}).$$

By Lemma 7.5,

$$\pi \hookrightarrow D_{\mathfrak{m}} \circ D_{\Delta_3} \circ D_{\Delta_2}(\pi) \times \mathrm{St}(\{\Delta_2, \Delta_3\}) \times \mathrm{St}(\mathfrak{m}).$$

This implies that $D_{\mathfrak{m}}(\pi) \neq 0$ and so $\mathfrak{m}$ is a submultisegment of $\mathfrak{p}$. On the other hand, using Lemma 3.2 and Theorem 3.5, we also have that $\mathfrak{p}$ is a submultisegment of $\mathfrak{m}$. Hence, $\mathfrak{m} = \mathfrak{p}$. Translating to η-invariants, we obtain (1).

We now prove (2). Write $\Delta_1 = [a_1, b_1]_\rho$. Suppose $(\Delta_1, \Delta_2, \pi)$ satisfies the intermediate segment property. Then there exists a segment $[a, b]_\rho$ in $\mathfrak{hd}(\pi)$ satisfying that $a_1 \leq a < a_2$ and $b_1 \leq b < b_2$. Then,

$$\varepsilon_{[a,b]_\rho}(\pi) > \varepsilon_{[a,b_2]_\rho}(\pi)$$

By (1), we have that

$$\varepsilon_{[a,b_2]_\rho}(\pi) = \varepsilon_{[a,b_2]_\rho}(D_{\Delta_3}(\pi).$$

By Theorem 3.5,

$$\varepsilon_{[a,b]_\rho}(D_{\Delta_3}(\pi)) \geq \varepsilon_{[a,b]_\rho}(\pi).$$

Combining the above equalties and inequalities, we have:

$$\varepsilon_{[a,b]_\rho}(D_{\Delta_3}(\pi)) > \varepsilon_{[a,b_2]_\rho}(D_{\Delta}(\pi)).$$

This implies that there exists a segment $[a, b']_\rho$ in $\mathfrak{hd}(D_{\Delta_3}(\pi))$ with $b' < b_2$. Thus $(\Delta_1, \Delta_2, D_{\Delta_3}(\pi))$ satisfies the intermediate segment property. □

Lemma 7.7. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\Delta_1, \Delta_2, \Delta_3$ be segments in an ascending order. If $\{\Delta_1, \Delta_2, \Delta_3\}$ is minimal to π, then $\{\Delta_1, \Delta_2\}$ is also minimal to $D_{\Delta_3}(\pi)$.*

Proof. If Δ_1 and Δ_2 are unlinked, then there is nothing to prove. If Δ_2 and Δ_3 are unlinked, then we use Lemma 5.1 to transfer to Lemma 7.2.

The remaining case is that Δ_1 and Δ_2 are linked, and Δ_2 and Δ_3 are linked. In other words, $\Delta_1 < \Delta_2 < \Delta_3$. This case follows from Proposition 4.3(2)⇔(3) and Lemma 7.6. □

7.5. Case: $\{\Delta_1, \Delta_2\}$ minimal to π.

Lemma 7.8. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\Delta_1, \Delta_2, \Delta_3$ be segments in an ascending order. If $\{\Delta_1, \Delta_2, \Delta_3\}$ is minimal to π, then $\{\Delta_1, \Delta_2\}$ is also minimal to π.*

The above lemma is a special case of Proposition 6.3(1).

7.6. Case: $\{\Delta_2, \Delta_3\}$ minimal to $D_{\Delta_1}(\pi)$.

Lemma 7.9. *lem basic subsequent property 23 1 Let $\pi \in \mathrm{Irr}_\rho$. Let $\Delta_1, \Delta_2, \Delta_3$ be segments in an ascending order. If $\{\Delta_1, \Delta_2, \Delta_3\}$ is minimal to π, then $\{\Delta_1, \Delta_2\}$ is also minimal to $D_{\Delta_1}(\pi)$.*

The above lemma is again a special case of Proposition 6.3(2).

8. Subsequent property of minimal sequence

8.1. Consecutive pairs.

Definition 8.1. Let $\mathfrak{m} \in \mathrm{Mult}_\rho$. Two segments Δ_1 and Δ_2 in $\mathfrak{m}$ are said to be *consecutive* in $\mathfrak{m}$ if

- $\Delta_1 < \Delta_2$ i.e. Δ_1 and Δ_2 are linked with $a(\Delta_1) < a(\Delta_2)$
- there is no other segment Δ' in $\mathfrak{m}$ such that

  $$a(\Delta_1) \leq a(\Delta') \leq a(\Delta_2), \quad b(\Delta_1) \leq b(\Delta') \leq b(\Delta_2)$$

  and Δ' is linked to either Δ_1 or Δ_2.

(The last linkedness condition guarantees that $\Delta' \neq \Delta_1 \cap \Delta_2$ and $\Delta' \neq \Delta_1 \cup \Delta_2$.)

Example 8.2.
- Let $\mathfrak{h} = \{[0,3], [1,4], [2,5]\}$. Then $[0,3], [1,4]$ form a pair of consecutive segments. Similarly, $[1,4], [2,5]$ also form a pair of consecutive segments, but $[0,3], [2,5]$ do not form a pair of consecutive segments.
- Let $\mathfrak{h} = \{[0,4], [1,2], [2,5]\}$. Then $[0,4], [2,5]$ form a pair of consecutive segments; and $[1,2], [2,5]$ also form a pair of consecutive segments.
- Let $\mathfrak{h} = \{[0,3], [1,3], [2,4], [2,5]\}$. Then $[1,3], [2,4]$ form a pair of consecutive segments, while $[0,3], [2,4]$ do not form a pair of consecutive segments.

The terminology of consecutive segments is suggested by its property in the intersection-union process.

Lemma 8.3. *Let Δ_1, Δ_2 be linked segments with $\Delta_1 < \Delta_2$. Suppose there exists a segment Δ' satisfying the conditions in the second bullet of Definition 8.1. Then, if Δ' is linked to Δ_i ($i = 1, 2$), then*

$$\{\Delta_1 \cap \Delta_2, \Delta_1 \cup \Delta_2, \Delta'\} \leq_Z \{\Delta_i \cap \Delta', \Delta_i \cup \Delta', \Delta_j\} \leq_Z \{\Delta_1, \Delta_2, \Delta'\}.$$

Here j is the index other than i i.e. $j \in \{1, 2\} - \{i\}$.

The above lemma follows from a direct checking and we omit the details. A simple combinatorics give the following:

Lemma 8.4. (1) *Let $\mathfrak{m}, \mathfrak{m}' \in \mathrm{Mult}_\rho$ such that $\mathfrak{m}' \leq_Z \mathfrak{m}$ with $\mathfrak{m}' \neq \mathfrak{m}$. Then there exists a pair of consecutive segments Δ, Δ' in $\mathfrak{m}$ such that for the multisegment $\mathfrak{m}''$ obtained from $\mathfrak{m}$ by the elementary intersection-union process involving Δ and Δ', $\mathfrak{m}' \leq_Z \mathfrak{m}'' \leq_Z \mathfrak{m}$.*

(2) *Let $\mathfrak{m} \in \mathrm{Mult}_\rho$. Let Δ, Δ' be a pair of consecutive segments in $\mathfrak{m}$ with $\Delta < \Delta'$. Let $\mathfrak{m}'$ be the submultisegment of $\mathfrak{m} - \Delta - \Delta'$ that contains all the segments $\widetilde{\Delta}$ with $a(\Delta') \leq a(\widetilde{\Delta})$ or $b(\Delta') \leq b(\widetilde{\Delta})$. Write the segments in $\mathfrak{m} - \mathfrak{m}'$ in an ascending order: $\Delta_1, \ldots, \Delta_r$ and write the segments in $\mathfrak{m}'$ in an ascending order: $\Delta'_1, \ldots, \Delta'_s$. Then the sequence:*

$$\Delta_1, \ldots, \Delta_r, \Delta, \Delta', \Delta'_1, \ldots, \Delta'_s$$

is ascending.

Proof. For (1), it suffices to show for $\mathfrak{m}'$ obtained by a pair of elementary intersection-union operation involving $\overline{\Delta}$ and $\overline{\Delta}'$. If the segments involved in the operation are consecutive, then the statement is immediate. Otherwise, there exists a segment $\widetilde{\Delta}$ such that $\widetilde{\Delta}$ is linked to either $\overline{\Delta}$ or $\overline{\Delta}'$, and produce a multisegment $\widetilde{\mathfrak{m}}$ such that $\mathfrak{m}' \leq_Z \widetilde{\mathfrak{m}} \leq_Z \mathfrak{m}''$. We repeat the process if such linked pair is still not consecutive. Note that if $\widetilde{\Delta}$ is linked to $\overline{\Delta}$ (resp. $\overline{\Delta}'$), we must have $\widetilde{\Delta} \cap \overline{\Delta}$ (resp. $\widetilde{\Delta} \cap \overline{\Delta}'$) strictly longer than $\overline{\Delta} \cap \overline{\Delta}'$, and hence after repeating the process several times, we obtain desired consecutive segments.

For (2), it is a direct check from the definition of an ascending order. ☐

8.2. **Minimality under commutativity (second basic case).**

Lemma 8.5. *Let $\pi \in \mathrm{Irr}$. Let $\Delta, \Delta_1, \Delta_2, \ldots, \Delta_r$ be in an ascending order and minimal to π. Then $\{\Delta, \Delta_{k+1}, \ldots, \Delta_r\}$ is also minimal to $D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi) \neq 0$, and*

$$D_{\Delta_r} \circ \ldots \circ D_{\Delta_{k+1}} \circ D_\Delta \circ D_{\Delta_k} \circ \ldots D_{\Delta_1}(\pi) \cong D_{\Delta_r} \circ \ldots \circ D_{\Delta_1} \circ D_\Delta(\pi).$$

Proof. By induction, it suffices to show when $k = 1$. The case that Δ and Δ_1 are unlinked is easy by Lemma 5.1 and Proposition 6.3(1). Suppose $\{\Delta, \Delta_2, \ldots, \Delta_r\}$ is not minimal to $D_{\Delta_1}(\pi)$ to arrive a contradiction. By Theorem 1.1, there exists a pair of consecutive segments $\Delta' < \Delta''$ such that the multisegment obtained from the intersection-union operation of those two segments Δ', Δ'' gives the same derivative on π.

Note that $\{\Delta_2, \ldots, \Delta_r\}$ is minimal to $D_{\Delta_1} \circ D_\Delta(\pi) \cong D_\Delta \circ D_{\Delta_1}(\pi)$ (the last isomorphism by Proposition 5.4). Thus the possible cases could be that one of Δ', Δ'' is Δ and so $\Delta' = \Delta$. Let

$$\mathfrak{n} = \{\Delta_i : a(\Delta_i) \leq a(\Delta) \quad \text{or} \quad b(\Delta_i) \leq b(\Delta)\}$$

By the ascending arrangement, any Δ_i is unlinked to Δ and Δ_1 by using the ascending property for the sequence $\Delta, \Delta_1, \ldots, \Delta_r$ (and $\Delta < \Delta_1$). (In particular, Δ_1 is not in $\mathfrak{n}$.)

Let $\mathfrak{m} = \{\Delta_2, \ldots, \Delta_r\}$. Now, since we chose Δ and Δ'' are consecutive, we can arrange and relabel the segments in an ascending order:

$$\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_l, \Delta, \Delta'', \widetilde{\Delta}_{k+1}, \ldots, \widetilde{\Delta}_{r-1},$$

where all $\widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_l$ are all elements in $\mathfrak{n}$ and $\widetilde{\Delta}_{k+1}, \ldots, \widetilde{\Delta}_{r-1}$ are all elements in $\mathfrak{m}-\mathfrak{n}$.
Case 1: $\mathfrak{n}=\emptyset$. We still have that $\Delta, \Delta_1, \Delta'', \widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_{r-1}$ form an ascending order and is minimal to π. In particular, we have $\Delta, \Delta_1, \Delta''$ is minimal to π by the cancellative property. Thus

$$D_{\Delta''} \circ D_{\Delta} \circ D_{\Delta_1}(\pi) \not\cong D_{\Delta'' \cup \Delta} \circ D_{\Delta'' \cap \Delta} \circ D_{\Delta_1}(\pi)$$

by Lemma 7.2 and Proposition 5.4. Since

$$\Delta'' \cap \Delta, \Delta'' \cup \Delta, \widetilde{\Delta}_1, \ldots, \widetilde{\Delta}_{r-1}$$

still form an ascending order (Lemma 6.2), applying $D_{\widetilde{\Delta}_1}, \ldots, D_{\widetilde{\Delta}_{r-1}}$ gives different derivatives on π and so this gives a contradiction.

Case 2: $\mathfrak{n} \neq \emptyset$. This implies that $\Delta+\mathfrak{m}-\mathfrak{n}$ is not minimal to $D_{\mathfrak{n}} \circ D_{\Delta_1}(\pi)$. However, we have that, by using unlinkedness discussed in the second paragraph,

$$D_{\mathfrak{n}} \circ D_{\Delta_1}(\pi) \cong D_{\Delta_1} \circ D_{\mathfrak{n}}(\pi).$$

and $\Delta+\Delta_1+\mathfrak{m}-\mathfrak{n}$ is minimal to $D_{\mathfrak{n}}(\pi)$ by Proposition 6.3. However, from Case 1, we have that $\Delta+\mathfrak{m}-\mathfrak{n}$ is minimal to

$$D_{\Delta_1} \circ D_{\mathfrak{n}}(\pi) (\cong D_{\mathfrak{n}} \circ D_{\Delta_1}(\pi)).$$

This gives a contradiction. □

8.3. **Minimality of a subsequent sequence.**

Theorem 8.6. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be minimal to π. Then any submultisegment of $\mathfrak{n}$ is also minimal to π.*

Proof. By an induction, it suffices to show the minimality for $\mathfrak{n}'=\mathfrak{n}-\Delta$ for some segment Δ in $\mathfrak{n}$. By Theorem 1.1 and Lemma 8.4, it reduces to check that for any multisegment $\mathfrak{m}'$ obtained from $\mathfrak{n}'$ by an elementary intersection-union operations involving two consecutive segments,

$$D_{\mathfrak{m}'}(\pi) \neq D_{\mathfrak{n}'}(\pi).$$

Denote such two consecutive segments by $\widetilde{\Delta} < \widetilde{\Delta}'$.

Let $\mathfrak{m}$ be obtained by the intersection-union process from $\mathfrak{n}$ involving $\widetilde{\Delta}$ and $\widetilde{\Delta}'$. Then $\mathfrak{m}'=\mathfrak{m}-\Delta$.

We consider following possibilities:

(1) Case 1: Suppose $\widetilde{\Delta}$ and $\widetilde{\Delta}'$ still form a pair of consecutive segments in $\mathfrak{n}$. Then we write the segments in $\mathfrak{n}$ as in Lemma 8.4 (with obvious notation replacement):

$$\Delta_1, \ldots, \Delta_r, \widetilde{\Delta}, \widetilde{\Delta}', \Delta_1', \ldots, \Delta_s'.$$

(a) Case 1(a): Δ appears in one of $\Delta_1', \ldots, \Delta_s'$. If $D_{\mathfrak{n}-\Delta}(\pi)=D_{\mathfrak{m}-\Delta}(\pi)$, then the cancellative property (Proposition 6.1) and Lemma 8.4 (also see Proposition 6.3) imply that

$$D_{\left\{\Delta_1, \ldots, \Delta_r, \widetilde{\Delta}, \widetilde{\Delta}'\right\}}(\pi) \not\cong D_{\left\{\Delta_1, \ldots, \Delta_r, \widetilde{\Delta} \cup \widetilde{\Delta}', \widetilde{\Delta} \cap \widetilde{\Delta}'\right\}}(\pi).$$

However, this implies $D_{\mathfrak{n}}(\pi) \not\cong D_{\mathfrak{m}}(\pi)$ by applying $D_{\Delta_1'}, \ldots, D_{\Delta_s'}$ with D_{Δ} omitted.

(b) Case 1(b): Δ appears in one of $\Delta_1, \ldots, \Delta_r$. Let

$$\mathfrak{p}=\left\{\Delta_1, \ldots, \Delta_r, \widetilde{\Delta}, \widetilde{\Delta}'\right\}-\Delta$$

and let $\mathfrak{q}$ be obtained from $\mathfrak{p}$ by an elementary intersection-union operation on $\widetilde{\Delta}$ and $\widetilde{\Delta}'$. By the cancellative property (Proposition 6.1), it suffices to show that $D_{\mathfrak{p}}(\pi) \not\cong D_{\mathfrak{q}}(\pi)$.

Let $\tau = D_{\mathfrak{p}-\widetilde{\Delta}-\widetilde{\Delta}'-\Delta}(\pi)$. By repeatedly using Lemma 8.5, we have:

$$D_{\mathfrak{p}+\Delta}(\pi) \cong D_{\widetilde{\Delta}'} \circ D_{\widetilde{\Delta}} \circ D_{\Delta}(\tau)$$

and $\left\{\Delta, \widetilde{\Delta}, \widetilde{\Delta}'\right\}$ is minimal to τ. Now by Lemma 7.3, we have that

$$D_{\mathfrak{p}}(\pi) = D_{\widetilde{\Delta}'} \circ D_{\widetilde{\Delta}}(\tau) \not\cong D_{\widetilde{\Delta}\cup\widetilde{\Delta}'} \circ D_{\widetilde{\Delta}\cap\widetilde{\Delta}'}(\tau) = D_{\mathfrak{q}}(\pi)$$

as desired.

(2) Case 2: $\widetilde{\Delta}$ and $\widetilde{\Delta}'$ do not form a consecutive pair. We then first write the segments in $\mathfrak{n}'$ as in Lemma 8.4:

$$\Delta_1, \ldots, \Delta_r, \widetilde{\Delta}, \widetilde{\Delta}', \Delta_1', \ldots, \Delta_s'.$$

Since the pair is not consecutive, the segment Δ must take the form as in the second bullet of Definition 8.1. Then one can still check that

$$\Delta_1, \ldots, \Delta_r, \widetilde{\Delta}, \Delta, \widetilde{\Delta}', \Delta_1', \ldots, \Delta_s'$$

is an ascending sequence. Now let $\tau = D_{\{\Delta_1,\ldots,\Delta_r\}}(\pi)$. By the cancellative property (Proposition 6.1), it suffices to show that

$$(*) \quad D_{\left\{\widetilde{\Delta},\widetilde{\Delta}'\right\}}(\tau) \neq D_{\left\{\widetilde{\Delta}\cap\widetilde{\Delta}',\widetilde{\Delta}\cup\widetilde{\Delta}'\right\}}(\tau).$$

This follows from the basic case of Lemma 7.4.

□

9. Commutativity and minimality

In this section, we study the commutativity for a minimal sequence.

9.1. Commutativity and minimality.

Lemma 9.1. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{m} \in \mathrm{Mult}_\rho$ minimal to π. Let $\Delta \in \mathfrak{m}$. Then $D_{\mathfrak{m}-\Delta} \circ D_\Delta(\pi) \cong D_{\mathfrak{m}}(\pi)$.*

Proof. We write the segments in $\mathfrak{m} - \Delta$ in an ascending order: $\Delta_1, \ldots, \Delta_r$. By Proposition 6.3(1), we can reduce to the case that $\Delta_1, \ldots, \Delta_r, \Delta$ still form an ascending order. By Propositions 6.3(2) and the basic case (Proposition 5.4),

$$D_{\mathfrak{m}}(\pi) \cong D_\Delta \circ D_{\mathfrak{m}-\Delta}(\pi) \cong D_{\Delta_r} \circ D_\Delta \circ D_{\mathfrak{m}-\Delta-\Delta_r}(\pi).$$

By Theorem 8.6, $\mathfrak{m}-\Delta_r$ is still minimal to π. We now inductively obtain the statement. □

We first study a special case of commutativity and minimality, and we shall prove a full version in Theorem 9.3.

Lemma 9.2. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{m} \in \mathrm{Mult}_\rho$ be minimal to π. Let c (resp. d) be the largest integer such that $\mathfrak{m}[c] \neq 0$ (resp. $\mathfrak{m}\langle d\rangle \neq 0$). Let $\Delta \in \mathfrak{m}[c]$ or $\in \mathfrak{m}\langle d\rangle$. Then $\mathfrak{m} - \Delta$ is minimal to $D_\Delta(\pi)$.*

Proof. The condition in the lemma guarantees that Δ can be arranged in the last one for an ascending order for $\mathfrak{m}$.

Suppose $\mathfrak{m} - \Delta$ is not minimal to $D_\Delta(\pi)$. Let $\mathfrak{m}' = \mathfrak{m} - \Delta$. By Theorem 1.1 and Lemma 8.4, it suffices to show that for any pair $\widetilde{\Delta} < \widetilde{\Delta}'$ of consecutive segments,

$$\mathfrak{n}' := \mathfrak{m}' - \left\{\widetilde{\Delta}, \widetilde{\Delta}'\right\} + \widetilde{\Delta} \cup \widetilde{\Delta}' + \widetilde{\Delta} \cap \widetilde{\Delta}'$$

does not give the same derivative on $D_\Delta(\pi)$ i.e.

$$(*) \quad D_{\mathfrak{n}'} \circ D_\Delta(\pi) \not\cong D_{\mathfrak{m}'} \circ D_\Delta(\pi).$$

Now we arrange and relabel the segments in $\mathfrak{m}'$ as in Lemma 8.4:

$$\Delta_1, \ldots, \Delta_k, \widetilde{\Delta}, \widetilde{\Delta}', \Delta_{k+3}, \ldots, \Delta_r,$$

which is in an ascending order. By Proposition 6.3, in order to show (*), it suffices to show that $\left\{\Delta_1, \ldots, \Delta_k, \widetilde{\Delta}, \widetilde{\Delta}'\right\}$ is still minimal to $D_\Delta(\pi)$. Let

$$\tau = D_{\Delta_k} \circ \ldots \circ D_{\Delta_1}(\pi), \text{ and } \tau' = D_{\Delta_k} \circ \ldots D_{\Delta_1} \circ D_\Delta(\pi).$$

By Lemma 6.2, it suffices to prove

$$D_{\widetilde{\Delta}'} \circ D_{\widetilde{\Delta}}(\tau') \not\cong D_{\widetilde{\Delta}' \cup \widetilde{\Delta}} \circ D_{\widetilde{\Delta}' \cap \widetilde{\Delta}}(\tau'). \tag{9.3}$$

Let $\mathfrak{p} = \{\Delta_1, \ldots, \Delta_k\}$. To this end, by the subsequent property (Theorem 8.6), we have that $\mathfrak{p} + \widetilde{\Delta}' + \widetilde{\Delta} + \Delta$ is minimal to π. Thus, we also have that $\left\{\Delta, \widetilde{\Delta}', \widetilde{\Delta}\right\}$ is minimal to $\tau = D_{\mathfrak{p}}(\pi)$ by Proposition 6.3. Now, Lemma 7.7 implies:

$$D_{\widetilde{\Delta}'} \circ D_{\widetilde{\Delta}} \circ D_\Delta(\tau) \not\cong D_{\widetilde{\Delta}' \cup \widetilde{\Delta}} \circ D_{\widetilde{\Delta}' \cap \widetilde{\Delta}} \circ D_\Delta(\tau).$$

On the other hand, by the subsequent property (Theorem 8.6), which gives that $\{\Delta_1, \ldots, \Delta_k, \Delta\}$ is minimal to π. Then combining with Lemma 9.1, we have $D_\Delta(\tau) \cong \tau'$. Combining, we have the desired non-isomorphism (9.3). □

9.2. **Minimality on commutated sequence: general case.**

Theorem 9.3. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be minimal to π. Let $\Delta \in \mathfrak{n}$. Then $\mathfrak{n} - \Delta$ is minimal to $D_\Delta(\pi)$ and*

$$D_{\mathfrak{n}-\Delta} \circ D_\Delta(\pi) \cong D_{\mathfrak{n}}(\pi).$$

Proof. We first prove the second assertion. We write the segments in $\mathfrak{n}$ in an ascending order:

$$\Delta_1, \ldots, \Delta_k, \Delta, \Delta_{k+1}, \ldots, \Delta_r$$

with $b(\Delta_1) \leq \ldots \leq b(\Delta) \leq \ldots \leq b(\Delta_r)$. Then, by Proposition 6.3, $\{\Delta_1, \ldots, \Delta_k, \Delta\}$ is still minimal to $\mathfrak{n}$. Then, the second assertion follows from Lemma 9.2.

We now prove the first assertion. By Theorem 1.1 and Lemma 8.4, it suffices to prove for a consecutive pair of segments. Let $\widetilde{\Delta}, \widetilde{\Delta}'$ be a pair of consecutive segments in $\mathfrak{n} - \Delta$. Let

$$\mathfrak{n}' = \mathfrak{n} - \Delta - \left\{\widetilde{\Delta}, \widetilde{\Delta}'\right\} + \widetilde{\Delta} \cup \widetilde{\Delta}' + \widetilde{\Delta} \cap \widetilde{\Delta}'.$$

We consider the following two cases:

(1) $\widetilde{\Delta}$ and $\widetilde{\Delta}'$ still form a pair of consecutive segments in $\mathfrak{n}$. We arrange and relabel the segments as:

$$\Delta_1, \ldots \Delta_p, \widetilde{\Delta}, \widetilde{\Delta}', \Delta_{p+1}, \ldots, \Delta_r$$

with $\Delta_{p+1}, \ldots, \Delta_r$ to be all the segments with $a(\Delta_t) \geq a(\widetilde{\Delta}')$ or $b(\Delta_t) \geq b(\widetilde{\Delta}')$. Similar to Lemma 8.4, one has that

$$\Delta_1, \ldots, \Delta_p, \widetilde{\Delta} \cup \widetilde{\Delta}', \widetilde{\Delta} \cap \widetilde{\Delta}', \Delta_{p+1}, \ldots, \Delta_r$$

form an ascending sequence.

(i) Suppose Δ appears in one of $\Delta_1, \ldots, \Delta_p$, say Δ_i. Let $\tau = D_{\Delta_p} \circ \ldots \circ D_{\Delta_{i+1}} \circ D_{\Delta_{i-1}} \circ \ldots \circ D_{\Delta_1} \circ D_{\Delta}(\pi)$. In such case, the proved second assertion gives that

$$\tau \cong D_{\Delta_p} \circ \ldots \circ D_{\Delta_1}(\pi).$$

The isomorphism with the minimality of $\left\{\Delta_1, \ldots, \Delta_p, \widetilde{\Delta}, \widetilde{\Delta}'\right\}$ to π and the discussion on the ascending sequence above gives that

$$D_{\widetilde{\Delta}'} \circ D_{\widetilde{\Delta}}(\tau) \not\cong D_{\widetilde{\Delta}' \cup \widetilde{\Delta}} \circ D_{\widetilde{\Delta}' \cap \widetilde{\Delta}}(\tau).$$

Applying $D_{\Delta_{p+1}}, \ldots, D_{\Delta_r}$, we have that

$$D_{\mathfrak{n}-\Delta} \circ D_{\Delta}(\pi) \not\cong D_{\mathfrak{n}'} \circ D_{\Delta}(\pi).$$

(ii) Suppose Δ appears in one of $\Delta_{p+1}, \ldots, \Delta_r$, say Δ_j. By rearranging and relabeling the segments in $\Delta_{p+1}, \ldots, \Delta_r$ if necessary, we assume $b(\Delta_{p+1}) \leq \ldots \leq b(\Delta_r)$ if $b(\widetilde{\Delta}') \leq b(\Delta_j)$ and assume $a(\Delta_{p+1}) \leq \ldots \leq a(\Delta_r)$ if $a(\widetilde{\Delta}') \leq a(\Delta_j)$. Let

$$\mathfrak{n}_j = \left\{\Delta_1, \ldots, \Delta_p, \widetilde{\Delta}, \widetilde{\Delta}', \Delta_{p+1}, \ldots, \Delta_j = \Delta\right\},$$

$$\mathfrak{n}_j' = \left\{\Delta_1, \ldots, \Delta_p, \widetilde{\Delta} \cup \widetilde{\Delta}', \widetilde{\Delta} \cap \widetilde{\Delta}', \Delta_{p+1}, \ldots, \Delta_j = \Delta\right\}.$$

Then, by Lemma 9.2,

$$D_{\mathfrak{n}_j'-\Delta} \circ D_{\Delta}(\pi) \not\cong D_{\mathfrak{n}_j-\Delta} \circ D_{\Delta}(\pi)$$

Thus, applying the derivatives $D_{\Delta_{j+1}}, \ldots, D_{\Delta_r}$, we have $D_{\mathfrak{n}'} \circ D_{\Delta}(\pi) \not\cong D_{\mathfrak{n}-\Delta} \circ D_{\Delta}(\pi)$ as desired.

(2) $\widetilde{\Delta}$ and $\widetilde{\Delta}'$ do not form a consecutive pair in $\mathfrak{n}$. One uses similar argument in Theorem 8.6 to reduce to three segment case. Then one reduces to a basic case (see similar discussions in the proof of Theorem 8.6), that is Lemma 7.2.

□

9.3. **General form of commutativity and minimality.**

Theorem 9.4. *Let $\pi \in \mathrm{Irr}_{\rho}$. Let $\mathfrak{n} \in \mathrm{Mult}_{\rho}$ be minimal to π. Let $\mathfrak{n}'$ be a submultisegment of $\mathfrak{n}$. Then $D_{\mathfrak{n}-\mathfrak{n}'} \circ D_{\mathfrak{n}'}(\pi) \cong D_{\mathfrak{n}}(\pi)$ and $\mathfrak{n} - \mathfrak{n}'$ is minimal to $D_{\mathfrak{n}'}(\pi)$.*

Proof. This follows by repeatedly using Theorem 9.3. □

Part 3. Representation-theoretic aspects

10. η-invariant and commutativity

We shall first discuss the representation-theoretic intrepretation for η-invariants. Then we explain how to relate with the commutativity. The representation-theoretic approach provides different techniques, which are of independent interests.

10.1. **Representation-theoretic counterpart of η_{Δ}.** Let $\Delta = [a, b]_{\rho}$. For $\pi \in \mathrm{Irr}$, let

$$\mathfrak{mx}(\pi, \Delta) = \sum_{a \leq a' \leq b} \varepsilon_{[a', b]_{\rho}}(\pi) \cdot [a', b]_{\rho}, \tag{10.4}$$

which means that $[a', b]_{\rho}$ appears with multiplicity $\varepsilon_{[a', b]_{\rho}}(\pi)$ in $\mathfrak{mx}(\pi, \Delta)$. This is the multisegment analogue of the η-invariant.

The following is the key property:

Lemma 10.1. [Ch22+] *Let $l = l_{abs}(\mathfrak{mx}(\pi, \Delta))$. Let $\mathfrak{m} = \mathfrak{mx}(\pi, \Delta)$. Then $D_{\mathfrak{m}}(\pi) \boxtimes \mathrm{St}(\mathfrak{m})$ is a direct summand in π_{N_l}.*

10.2. **Commutativity.** Here we explain how to view the commutativity of Proposition 5.4 from the perspective of Lemma 10.1. We first show the following new lemma, using Lemma 10.1:

Lemma 10.2. *Let $\pi \in \mathrm{Irr}_\rho(G_n)$. Let $\Delta_1, \Delta_2 \in \mathrm{Seg}_\rho$ be admissible to π. Suppose $\Delta_1 < \Delta_2$. Suppose $(\Delta_1, \Delta_2, \pi)$ satisfies the non-overlapping property. Let $\mathfrak{m} = \mathfrak{mx}(\pi, \Delta_2)$. Then $D_{\Delta_1} \circ D_{\mathfrak{m}}(\pi) \cong D_{\mathfrak{m}} \circ D_{\Delta_1}(\pi)$.*

Proof. Let $l = l_{abs}(\Delta_1)$ and let $N = N_l$. We have that:

$$\pi \hookrightarrow D_{\mathfrak{m}}(\pi) \times \mathrm{St}(\mathfrak{m}).$$

Then

$$D_{\Delta_1}(\pi) \boxtimes \mathrm{St}(\Delta_1) \hookrightarrow \pi_N \hookrightarrow (D_{\mathfrak{m}}(\pi) \times \mathrm{St}(\mathfrak{m}))_N.$$

An anslysis on the layers in geometric lemma, we have that

$$(*) \quad D_{\Delta_1}(\pi) \boxtimes \mathrm{St}(\Delta_1) \hookrightarrow D_{\mathfrak{m}}(\pi)_{N_l} \dot{\times}^1 \mathrm{St}(\mathfrak{m}),$$

where $\dot{\times}^1$ means that the induction to a $G_{n-l} \times G_l$-representation. By the non-overlapping property and Lemma 10.1, $D_{\mathfrak{m}} \circ D_{\Delta_1}(\pi) \boxtimes \mathrm{St}(\mathfrak{m})$ is a direct summand in $D_{\Delta_1}(\pi)_{N_{l'}}$, where $l' = l_{abs}(\mathfrak{m})$. Furthermore, no other composition factors in $D_{\Delta_1}(\pi)_{N_{l'}}$take the form $\tau \boxtimes \mathrm{St}(\mathfrak{m})$. Now, via Frobenius reciprocity on the map in (*), we obtain a non-zero map from $D_{\Delta_1}(\pi)_{N_{l'}} \boxtimes \mathrm{St}(\Delta_1)$ to $(D_{\mathfrak{m}}(\pi)_{N_l} \boxtimes \mathrm{St}(\mathfrak{m}))^{\phi}$, where ϕ is a twisting sending a $G_{n-l-l'} \times G_l \times G_{l'}$-representation to a $G_{n-l-l'} \times G_{l'} \times G_l$-representation. Then

$$D_{\mathfrak{m}} \circ D_{\Delta_1}(\pi) \boxtimes \mathrm{St}(\Delta_1) \hookrightarrow D_{\mathfrak{m}}(\pi)_{N_l}.$$

Thus, we have $D_{\mathfrak{m}} \circ D_{\Delta_1}(\pi) \cong D_{\Delta_1} \circ D_{\mathfrak{m}}(\pi)$. □

Lemma 10.2 can also be deduced from Proposition 5.4 and Lemma 5.2. On the other hand, one can also give another proof for Proposition 5.4 by using Lemmas 5.2 and 10.2.

11. Conjectural interpretation for minimal sequences

11.1. **Minimality for two segments.**

Proposition 11.1. *Let $\pi \in \mathrm{Irr}_\rho$. Let Δ_1, Δ_2 be a pair of linked segments with $\Delta_1 < \Delta_2$. Suppose*

$$D_{\Delta_1 \cup \Delta_2} \circ D_{\Delta_1 \cap \Delta_2}(\pi) \ncong D_{\Delta_2} \circ D_{\Delta_1}(\pi).$$

Then the unique map

$$D_{\Delta_2} \circ D_{\Delta_1}(\pi) \boxtimes (\mathrm{St}(\Delta_1) \times \mathrm{St}(\Delta_2)) \to \pi_N,$$

where $N = N_{l_{abs}(\Delta_1)+l_{abs}(\Delta_2)}$, is injective.

Proof. Suppose the map is not injective. Since $D_{\Delta_2} \circ D_{\Delta_1}(\pi) \boxtimes (\mathrm{St}(\Delta_1) \times \mathrm{St}(\Delta_2))$ is indecomposable and has length 2, the image of the map can only be isomorphic to $D_{\Delta_2} \circ D_{\Delta_1}(\pi) \boxtimes (\mathrm{St}(\Delta_1 \cup \Delta_2) \times \mathrm{St}(\Delta_1 \cap \Delta_2))$. This implies that

$$D_{\Delta_2} \circ D_{\Delta_1}(\pi) \boxtimes (\mathrm{St}(\Delta_1 \cup \Delta_2) \times \mathrm{St}(\Delta_1 \cap \Delta_2))$$

is a submodule of π_N. Then, applying Frobenius reciprocity, we have that π is the unique submodule of $D_{\Delta_2} \circ D_{\Delta_1}(\pi) \times \mathrm{St}(\Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2)$ (see [LM16, Ch22+b]).

Recall that $\mathrm{St}(\Delta_1 \cup \Delta_2 + \Delta_1 \cap \Delta_2) \cong \mathrm{St}(\Delta_1 \cup \Delta_2) \times \mathrm{St}(\Delta_1 \cap \Delta_2)$. Let τ be the unique submodule of $D_{\Delta_2} \circ D_{\Delta_1}(\pi) \times \mathrm{St}(\Delta_1 \cup \Delta_2)$ so that

$$(*) \quad D_{\Delta_1 \cup \Delta_2}(\tau) \cong D_{\Delta_2} \circ D_{\Delta_1}(\pi).$$

Then the uniqueness of submodule above also forces that

$$\pi \hookrightarrow \tau \times \mathrm{St}(\Delta_1 \cap \Delta_2)$$

and so $D_{\Delta_1\cap\Delta_2}(\pi)\cong\tau$. Combinbing with (*), we have:

$$D_{\Delta_1\cup\Delta_2}\circ D_{\Delta_1\cap\Delta_2}(\pi)\cong D_{\Delta_2}\circ D_{\Delta_1}(\pi),$$

giving a contradiction. □

In the Appendix, we shall prove a converse of Proposition 11.1.

11.2. **A representation-theoretic interpretation of minimal sequences.**

Definition 11.2. For a multisegment $\mathfrak{h}=\{\Delta_1,\ldots,\Delta_r\}\in\mathrm{Mult}_\rho$ labelled in an ascending order, define

$$\widetilde{\lambda}(\mathfrak{h}):=\mathrm{St}(\Delta_1)\times\ldots\times\mathrm{St}(\Delta_r).$$

We shall call it a *co-standard representation*. Sometimes $\lambda(\mathfrak{h})$ is used for standard modules and so we prefer to use $\widetilde{\lambda}$ here.

We conjecture that Proposition 11.1 can be generalized to general minimal multisegments.

Conjecture 11.3. *Let $\pi\in\mathrm{Irr}_\rho$. Let $\mathfrak{n}\in\mathrm{Mult}_\rho$ be minimal to π. Then the unique non-zero map*

$$D_{\mathfrak{n}}(\pi)\boxtimes\widetilde{\lambda}(\mathfrak{n})\to\pi_{N_l},$$

where $l=l_{abs}(\Delta_1)+\ldots+l_{abs}(\Delta_r)$, is injective.

We remark that the uniqueness of the non-zero map in Conjecture 11.3 follows from the uniqueness of simple quotients, shown in [Ch22+d].

12. Some applications on the embedding model

In this section, we shall discuss applications of the embedding model arising from the minimality in Proposition 11.1.

Lemma 12.1. *Let $\Delta',\Delta'',\Delta'''\in\mathrm{Seg}_\rho$. Suppose $\Delta'<\Delta''$. Let $\tau=D_{\Delta''}\circ D_{\Delta'}(\pi)$ and let $l'=l_{abs}(\Delta'),l''=l_{abs}(\Delta''),l'''=l_{abs}(\Delta''')$. Suppose the followings hold:*

- $\dim\mathrm{Hom}_{G_{n-l'-l''}\times G_{l'+l''}}(\tau\boxtimes(\mathrm{St}(\Delta')\times\mathrm{St}(\Delta''')),(\kappa\times\mathrm{St}(\Delta'''))_{N_{l'+l''}})\leq 1$;
- *The non-zero map in the first bullet factors through the natural embedding:*

  $$\kappa_{N_{l'+l''}}\dot{\times}^1\mathrm{St}(\Delta'')\hookrightarrow(\kappa\times\mathrm{St}(\Delta'''))_{N_{l'+l''}}$$

  from the bottom layer in the geometric lemma. Here $\dot{\times}^1$ again denotes a parabolic induction from a $G_{n-l'-l''-l'''}\times G_{l'''}\times G_{l'+l''}$-representation to a $G_{n-l'-l''}\times G_{l'+l''}$-representation.
- $D_{\Delta'''}\circ D_{\Delta''}\circ D_{\Delta'}(\pi)\cong D_{\Delta''}\circ D_{\Delta'''}\circ D_{\Delta'}(\pi)$.

Then, if $\{\Delta',\Delta''\}$ is minimal to π, then $\{\Delta',\Delta''\}$ is minimal to $D_{\Delta'''}(\pi)$.

Proof. Suppose $\{\Delta',\Delta''\}$ is not minimal to $D_{\Delta'''}(\pi)$. Let $i=l'+l''$. Let $\lambda_1=\mathrm{St}(\Delta'\cup\Delta'')\times\mathrm{St}(\Delta'\cap\Delta'')$. Let $i=l_{abs}(\Delta')+l_{abs}(\Delta'')$. Then, by Proposition 11.1,

$$D_{\Delta''}\circ D_{\Delta'}\circ D_{\Delta'''}(\pi)\boxtimes\lambda_1\hookrightarrow D_{\Delta'''}(\pi)_{N_i}.$$

On the other hand, we have the following embedding:

$$\pi\hookrightarrow D_{\Delta'''}(\pi)\times\mathrm{St}(\Delta''').$$

Let $\omega=D_{\Delta''}\circ D_{\Delta'}\circ D_{\Delta'''}(\pi)\times\mathrm{St}(\Delta''')$. Via the bottom layer in the geometric lemma, we have an embedding:

$$\omega\boxtimes\lambda_1\hookrightarrow(D_{\Delta'''}(\pi)\times\mathrm{St}(\Delta'''))_{N_i}.$$

By the third bullet of the hypothesis, $D_{\Delta''} \circ D_{\Delta'}(\pi)$ is a submodule of ω. Combining above, we have that:

$$D_{\Delta''} \circ D_{\Delta'}(\pi) \boxtimes \lambda_1 \hookrightarrow (D_{\Delta'''}(\pi) \times \mathrm{St}(\Delta'''))_{N_i}.$$

Let $\lambda_2 = \mathrm{St}(\Delta') \times \mathrm{St}(\Delta'')$. Then, by the minimality of $\{\Delta', \Delta''\}$ to π and Proposition 11.1, we have

$$D_{\Delta''} \circ D_{\Delta'}(\pi) \boxtimes \lambda_2 \hookrightarrow \pi_{N_i}.$$

This induces another embedding:

$$D_{\Delta''} \circ D_{\Delta'}(\pi) \boxtimes \lambda_2 \hookrightarrow (D_{\Delta'''}(\pi) \times \mathrm{St}(\Delta'''))_{N_i}.$$

Since λ_1 and λ_2 have no isomorphic submodules, the above two embeddings give:

$$D_{\Delta''} \circ D_{\Delta'}(\pi) \boxtimes (\lambda_1 \oplus \lambda_2) \hookrightarrow (D_{\Delta'''}(\pi) \times \mathrm{St}(\Delta'''))_{N_i}.$$

However, this induces two non-zero maps from $D_{\Delta''} \circ D_{\Delta'}(\pi) \boxtimes \lambda_1$ to $(D_{\Delta'''}(\pi) \times \mathrm{St}(\Delta'''))_{N_i}$, which are not scalar multiple of each other. This contradicts to the first bullet. ☐

Lemma 12.1 provides another strategy for checking minimality in some three segment cases in Section 7 e.g. the second bullet case in the proof of Lemma 7.2 and the linked case in the proof of Lemma 7.7. Checking the second bullet usually involves analysis on the layers arising from the geometric lemma while checking the third bullet usually uses some known commutativity from minimality (Proposition 5.4) in some other cases. Checking the first case requires some inputs of multiplicity theorems from [AGRS10] and [Ch23].

13. Embedding model, minimality and removal process

13.1. **Combintorial preparations.** Let S_n be the symmetric group permuting the integers $\{1, \ldots, n\}$.

Definition 13.1. Let $w \in S_n$. For $1 \leq k \leq n$ and $1 \leq l \leq n$, define

$$w[k, l] := |\{a : 1 \leq a \leq k, w(a) \geq l\}|.$$

We shall write $\leq_B$ to be the Bruhat ordering on S_n i.e. $w' \leq_B w$ if and only if w' is a subword of a reduced expression of w. We write $w' <_B w$ if $w' \leq_B w$ and $w' \neq w$.

Proposition 13.2. [BB05, Theorem 2.1.5] *Let $w, w' \in S_n$. Then the following statements are equivalent:*

(1) *$w' \leq_B w$;*
(2) *$w'[k, l] \leq w[k, l]$ for any k, l.*

We shall now discuss a special situation. Let $S^{n-i,i}$ be the set of minimal representatives in the cosets in $S_n/(S_{n-i} \times S_i)$. It is well-known that $w \in S^{n-i,i}$ if and only if

- $w(k) < w(l)$ if $k, l \in \{1, \ldots, n-i\}$;
- $w(k) < w(l)$ if $k, l \in \{n-i+1, \ldots, n\}$.

Proposition 13.3. *Let $w, w' \in S^{n-i,i}$. The following statements are equivalent:*

(1) *$w' \leq_B w$;*
(2) *$w'(k) < w(k)$ for all $k = 1, \ldots, n-i$;*
(3) *$w'(k) > w(k)$ for all $k = n-i+1, \ldots, n$.*

Proof. Note that $w' \leq_B w$ if and only if $w'^{-1} \leq_B w^{-1}$. We first prove (2)⇒(1).

Let x_l be the smallest integer in $\{1, \ldots, n-i\}$ such that $w^{-1}(x_l) \geq l$ if it exists and let $x_l = n-i+1$ if such integer does not exist. We have the following formulae:

(i) If $k \leq n-i$,

$$w[k, l] = \max\{0, k - x + 1\}.$$

(ii) If $k > n-i$, let $k' = k-(n-i)$. Then

$$w[k,l] = \max\{0, k'-(l-x)\} + (n-i) - x + 1$$

Note that when $l \leq k'$, the formula becomes $w'^{-1}[k,l] = k-l+1$.

We similarly have the formula for $w'[k,l]$ and we shall replace the respective x_l's by x_l''s in the above discussions. Now, by using (2), we have that $x_l \leq x_l'$ for all l. Thus, if we are in Case (i), it is clear that $w'[k,l] \leq w[k,l]$. If we are now in Case (ii), we only have to additionaly note that $k-l+1$ is always a lower bound for $w[k,l]$. By Proposition 13.2, we then have (1).

We now prove (1)⇒(2). Suppose (2) does not hold to arrive a contradiction. Then there exists a smallest k^* in $\{1, \ldots, n-i\}$ such that $w'(k^*) > w^{-1}(k^*)$. Let $l^* = w'(k^*)$. Then $w'[k^*, l^*] = 1$, but $w^{-1}[k^*, l^*] = 0$.

We now prove (1)⇔(3). Let $\iota : S_n \to S_n$ be the involution given by $i \leftrightarrow n-i+1$ ($i = 1, \ldots n$). This induces a natural bijection between $(S_{n-i} \times S_i) \setminus S_n$ and $(S_i \times S_{n-i}) \setminus S_n$. Then (3) follows from translation under the bijection and the proved equivalence of (1) and (2). □

13.2. **Embedding model and removal process.**

Proposition 13.4. *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $l = l_{abs}(\Delta)$. Then there exists an embedding*

$$\widetilde{\lambda}(\mathfrak{r}(\Delta, \mathfrak{h})) \boxtimes \mathrm{St}(\Delta) \hookrightarrow \widetilde{\lambda}(\mathfrak{h})_{N_l}.$$

Proof. Write the segments in $\mathfrak{m}$ as $\Delta_i = [a_i, b_i]_\rho$ ($i = 1, \ldots, r$). We arrange the segments in $\mathfrak{m}$ satisfying:

- $b_1 \leq b_2 \leq \ldots \leq b_r$;
- if $b_i = b_{i+1}$, then $a_{i+1} \geq a_i$.

We now apply the geometric lemma on $\widetilde{\lambda}(\mathfrak{m})_{N_l}$. We first write the segments as:

$$\Delta_1 = [a_1, b_1]_\rho, \ldots, \Delta_r = [a_r, b_r]_\rho.$$

and $\Delta = [a, b]_\rho$. Set

$$\Delta_k^{+,l_k} = [a_k + l_k, b_k]_\rho, \quad \Delta_k^{-,l_k} = [a_k, a_k + l_k - 1]_\rho$$

Those are possiby empty sets.

Then the layers arising from the geometric lemma takes the form:

$$(*) \quad (\mathrm{St}(\Delta_1^{+,l_1}) \times \ldots \times \mathrm{St}(\Delta_r^{+,l_r})) \boxtimes (\mathrm{St}(\Delta_1^{-,l_1}) \times \ldots \times \mathrm{St}(\Delta_r^{-,l_r}),$$

where $l_1, \ldots, l_r$ run for all integers such that $l_1 + \ldots + l_r = l/\deg(\rho)$. We now describe the underlying element in $(S_{t_1} \times \ldots \times S_{t_r}) \setminus S_n/(S_{n-i} \times S_i)$ corresponding to the geometric lemma. Let $d = \deg(\rho)$. Let $t_k = l_{abs}(\Delta_k^{+,l_k})$. The assignment takes the form:

$$n-(j-1) \mapsto t_1 + \ldots + t_{r-1} + t_r - (j-1) \quad \text{for } j = 1, \ldots, l_r d\ ,$$
$$(n-l_r)-(j-1) \mapsto t_1 + \ldots + t_{r-1} - (j-1) \quad \text{for } j = 1, \ldots, l_{r-1} d,$$
$$\vdots$$
$$(n-l_2-\ldots-l_r)-(j-1) \mapsto t_1 - (j-1) \quad \text{for } j = 1, \ldots, l_1 d\ .$$

The assignment for $x < n - l_1 d - l_2 d - \ldots - l_r d$ is then uniquely determined by using the properties of elements in $S_n/(S_{n-i} \times S_i)$ stated before Proposition 13.2.

We now consider a specific layer from the geometric lemma. The segments are chosen in the order as the sequence removal process labelled as:

$$\Delta_{p_1}, \ldots, \Delta_{p_e}. \tag{13.5}$$

For those indexes, if there is more than one segment for Δ_{p_i}, we always choose the one of smaller index p_i. By using the nesting property and our arrangement of segments in $\mathfrak{m}$, we have that

$$p_1 > p_2 > \ldots > p_e.$$

If $k \neq p_i$ for some $i = 1, \ldots, e$, then we set $\widetilde{l}_k = 0$. If $k = p_i$ for some $i = 1, \ldots, e-1$, then we set $\widetilde{l}_k = a_{p_{i-1}} - a_{p_i}$. If $k = p_e$, then we set $\widetilde{l}_k = b - a_{p_e}$.

We now compare study layers of the geometric lemma of the form (*) such that, as sets,

$$\Delta_1^{+,l_1} \cup \ldots \cup \Delta_r^{+,l_r} = \Delta.$$

For such layer, we say it is in *standard order* if

$$a(\Delta_1^{+,l_x}) \leq a(\Delta_r^{+,l_y})$$

for any $x < y$ with $l_x, l_y \neq 0$.

We now analyse some behaviours of two cases.

(1) Case 1: $\Delta_1^{+,l_1}, \ldots, \Delta_r^{+,l_r}$ are in standard order. Let $q_1 < \ldots < q_k$ be all the indexes such that $l_{q_x} \neq 0$. We first prove the following claim:

Claim 1: The sequence $\Delta_{q_1}, \ldots, \Delta_{q_k}$ satisfies the nesting property i.e.

$$\Delta_{q_1} \supset \ldots \supset \Delta_{q_k}.$$

Proof of Claim 1: Suppose the sequence does not satisfy the nesting property. Since the nesting property is not transitive, we have that a pair $\Delta_{q_x}, \Delta_{q_{x+1}}$ does not satisfy the nesting property. Due to the arrangement of the segments in $\mathfrak{m}$, we must have that $b_{q_x} < b_{q_{x+1}}$ and so the violation of the nestng property implies that

$$a_{q_x} < a_{q_{x+1}}.$$

However, this then contradicts that the sequence is in standard order.

As a consequence of Claim 1, we also ahve that $l_{q_i} = a_{q_{i-1}} - a_{q_i}$.

Let w^* be the fixed element in $(S_{t_1} \times \ldots \times S_{t_r}) \setminus S_n/(S_{n-i} \times S_i)$ associated to (13.5) and let w be the element in $(S_{t_1} \times \ldots \times S_{t_r}) \setminus S_n/(S_{n-i} \times S_i)$ associated to any layer in standard order in the sense defined above with $w \neq w^*$.

Claim 2: $w \not\leq_B w'$.

Proof of Claim 2: The strategy is to apply Proposition 13.3. Let i^* be the largest integer such that $p_{i^*} = q_{i^*}$. Set $i^* = 0$ if such integer does not exist i.e. $p_1 \neq q_1$.

If $i_* = r$, then q_{i^*+1} is also not defined. Otherwise, by the nesting property shown in Claim 1, we obtain a contradiction to choices of segments in the removal sequence for $(\Delta, \mathfrak{h})$. This implies that $w' = w$, giving a contradiction.

Thus $i^* < r$. Now we compare $\Delta_{p_{i^*+1}}$ and $\Delta_{q_{i^*+1}}$. By using Definition 3.1(2) and Claim 1, we must have that

$$a_{p_{i^*+1}} \leq a_{q_{i^*+1}},$$

or q_{i^*+1} is not defined.

We further divide into two subcases.

- $a_{p^*+1} = a_{q_{i^*+1}}$. But now, we must have that $b_{q_{i^*+1}} \geq b_{p^*+1}$ by the nesting property in Claim 1 again. Then, from our arrangements and choices, $q_{i^*+1} > p_{i^*+1}$. As noted from above that $l_{q_1} = \widetilde{l}_{p_1}, \ldots, l_{q_{i^*}} = \widetilde{l}_{p_{i^*}}$, one checks that

$$w(n - l_1 - \ldots - l_{i^*} - 1) > w^*(n - l_1 - \ldots - l_{i^*} - 1)$$

and so $w \not\leq_B w'$ by Proposition 13.3.

- $a_{p^*+1} > a_{q_{i^*}+1}$ or q_{i^*+1} is not defined. In such case, $l_{q_{i^*}} > \widetilde{l}_{p_{i^*}}$. Hence we also have that:

$$w(n - l_1 - \ldots - l_{i^*} - 1) > w^*(n - l_1 - \ldots - l_{i^*} - 1)$$

and so again $w \not\leq_B w'$ by Proposition 13.3.

(2) Case 2: $\Delta_1^{+,l_1}, \ldots, \Delta_r^{+,l_r}$ are not in standard order.

Let $\omega_{l_1,\ldots,l_r} = \mathrm{St}(\Delta_1^{-,l_1}) \times \ldots \times \mathrm{St}(\Delta_r^{-,l_r})$ and similarly let $\omega_{\widetilde{l}_1,\ldots,\widetilde{l}_r}$.

Claim 3: Let $\widetilde{l} = \widetilde{l}_1 + \ldots + \widetilde{l}_r$. For all j,

$$\mathrm{Ext}^j_{G_{n-\widetilde{l}} \times G_{\widetilde{l}}}(\omega_{\widetilde{l}_1,\ldots,\widetilde{l}_r} \boxtimes \mathrm{St}(\Delta), \omega_{l_1,\ldots,l_r} \boxtimes \mathrm{St}(\Delta_1^{+,l_1}) \times \ldots \times \mathrm{St}(\Delta_r^{+,l_r})) = 0$$

Proof of Claim 3: We apply Frobenius reciprocity on the second factor. Then the Jacquet module of $\mathrm{St}(\Delta)$ takes the form

$$\mathrm{St}(\Delta_1') \boxtimes \ldots \boxtimes \mathrm{St}(\Delta_r')$$

with $\Delta_1', \ldots, \Delta_r'$ in standard order. Since $\Delta_1^{+,l_1}, \ldots, \Delta_r^{+,l_r}$ are not in standard order, an argument on comparing cuspidal support gives

$$\mathrm{Ext}^j_{G_{\widetilde{l}}}(\mathrm{St}(\Delta), \mathrm{St}(\Delta_1') \boxtimes \ldots \boxtimes \mathrm{St}(\Delta_r')) = 0$$

for all j. Then Künneth formula then gives the claim.

We now go back to the proof. The element $w^* \in (S_{t_1} \times \ldots \times S_{t_r}) \setminus S_n/(S_{n-i} \times S_i)$ is defined as above. For each $w \in (S_{t_1} \times \ldots \times S_{t_r}) \setminus S_n/(S_{n-i} \times S_i)$, let $\kappa(w)$ be the associated layer taking the form (*). Note that $\omega_{\widetilde{l}_1,\ldots,\widetilde{l}_r} \cong \widetilde{\lambda}(\mathfrak{r}(\Delta, \mathfrak{h}))$.

Let $\kappa_{\leq w^*}$ (resp. $\kappa_{<w^*}$) be the submodule of $\widetilde{\lambda}(\mathfrak{h})_{N_l}$ that consists of only the layers associated to $w' \in (S_{t_1} \times \ldots \times S_{t_r}) \setminus S_n/(S_{n-i} \times S_i)$ satisfying $w' \leq_B w^*$ (resp. $w' <_B w^*$). We have the short exact sequence:

$$0 \to \kappa_{<w^*} \to \kappa_{\leq w^*} \to \kappa(w^*) \to 0.$$

On the other hand, there is an embedding

$$\omega_{\widetilde{l}_1,\ldots,\widetilde{l}_r} \boxtimes \mathrm{St}(\Delta) \hookrightarrow \kappa(w^*)$$

Hence, we have a submodule κ' of $\kappa_{\leq w^*}$ admitting a short exact sequence:

$$0 \to \kappa_{<w^*} \to \kappa' \to \omega_{\widetilde{l}_1,\ldots,\widetilde{l}_r} \boxtimes \mathrm{St}(\Delta) \to 0.$$

However, for $w' < w^*$, we can conclude that

$$\mathrm{Ext}^j_{G_{n-l} \times G_l}((\mathfrak{r}(\Delta, \mathfrak{h})) \boxtimes \mathrm{St}(\Delta), \kappa(w')) = 0)$$

for all j. This follows from a standard cuspdial support argument if the G_l-part of $\kappa(w')$ does not have the same cuspidal support as $\mathrm{St}(\Delta)$, and follows from Claim 2 and Claim 3 otherwise. In other words, we have:

$$\kappa' \cong \kappa_{<w^*} \oplus \widetilde{\lambda}(\mathfrak{r}(\Delta, \mathfrak{h})) \boxtimes \mathrm{St}(\Delta).$$

This then gives the following desired embedding

$$(\mathfrak{r}(\Delta, \mathfrak{h})) \boxtimes \mathrm{St}(\Delta) \hookrightarrow \kappa' \hookrightarrow \kappa \hookrightarrow \lambda(\widetilde{\mathfrak{h}})_{N_l}.$$

$\square$

13.3. **Conjectures.** For $n_1 + \ldots + n_s = n$, define $P_{n_1,\ldots,n_s}$ to be the parabolic subgroup of G_n generated by the matrices $\mathrm{diag}(g_1, \ldots, g_s)$ (each $g_i \in G_{n_i}$) and upper triangular matrices. Let $N_{n_1,\ldots,n_s}$ be the unipotent radical of $P_{n_1,\ldots,n_s}$.

We end with some conjectures for the embedding model, which are possibly used to interpret some results in this article from representation-theoretic viewpoint:

Conjecture 13.5. *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. The embedding in Proposition 13.4 is unique.*

We remark that Conjecture 13.5 is not a mere consequence of the multiplicity one theorem for standard representations in [Ch23].

Conjecture 13.6. *Let $\mathfrak{h} \in \mathrm{Mult}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be minimal to $\mathfrak{h}$. Let $l = l_{abs}(\mathfrak{n})$. Then there exists a unique embedding:*

$$\widetilde{\lambda}(\mathfrak{r}(\mathfrak{n}, \mathfrak{h})) \boxtimes \widetilde{\lambda}(\mathfrak{n}) \hookrightarrow \widetilde{\lambda}(\mathfrak{h})_{N_l}.$$

Proposition 13.4 is a special case of Conjecture 13.6.

Conjecture 13.7. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be minimal to π. Let $\mathfrak{h} = \mathfrak{hd}(\pi)$. Let $l_1 = l_{abs}(\mathfrak{n})$ and let $l_2 = l_{abs}(\mathfrak{hd}(\pi)) - l_{abs}(\mathfrak{n})$. Suppose Conjectures 11.3 and 13.6 hold. We have the following diagram of maps:*

$$\begin{array}{ccccc} D_{\mathfrak{h}}(\pi) \boxtimes \widetilde{\lambda}(\mathfrak{r}(\mathfrak{n}, \pi)) \boxtimes \widetilde{\lambda}(\mathfrak{n}) & \overset{\iota_1}{\hookrightarrow} & D_{\mathfrak{h}}(\pi) \boxtimes \widetilde{\lambda}(\mathfrak{h})_{N_{l_2,l_1}} & \overset{\iota_2}{\hookrightarrow} & \pi_{N_{n-l_1-l_2,l_2,l_1}} \\ & & & & \uparrow \iota_3 \\ & & & & D_{\mathfrak{n}}(\pi)_{N_{n-l_1-l_2,l_2}} \boxtimes \widetilde{\lambda}(\mathfrak{n}) \end{array},$$

where

- *ι_1 is the map in Conjecture 13.6.*
- *ι_2 is the map induced from the unique embedding $D_{\mathfrak{h}}(\pi) \boxtimes \widetilde{\lambda}(\mathfrak{h})$ in Conjecture 11.3*
- *ι_3 is the map induced from the unique embedding $D_{\mathfrak{n}}(\pi) \boxtimes \widetilde{\lambda}(\mathfrak{n}) \hookrightarrow \pi_{n-l_1,l_1}$ in Conjecture 11.3.*

Then $\iota_2 \circ \iota_1$ factors through ι_3.

Part 4. **Appendices**

14. Appendix A: Non-isomorphic derivatives

We prove a converse of Proposition 11.1 in this appendix. For a ladder representation or a generic representation σ of G_k, let $I_\sigma(\pi)$ be the unique submodule of $\pi \times \sigma$ (see [LM16], also see [Ch22+, Ch22+d]).

14.1. **Non-isomorphic integrals.**

Proposition 14.1. *Let $\pi \in \mathrm{Irr}_\rho$. Let Δ_1, Δ_2 be two linked segments. Let $\Delta_1' = \Delta_1 \cup \Delta_2$ and let $\Delta_2' = \Delta_1 \cap \Delta_2$ (possibly the empty set). Let $\sigma = \mathrm{St}(\{\Delta_1, \Delta_2\})$ and let $\sigma' = \mathrm{St}(\Delta_1' + \Delta_2')$. Then $I_\sigma(\pi) \not\cong I_{\sigma'}(\pi)$.*

Proof. We shall use the invariant $\mathfrak{mx}(., \Delta)$ to distinguish the two representations. Write $\Delta_1 = [a_1, b_1]_\rho$ and $\Delta_2 = [a_2, b_2]_\rho$. Switching labels if necessary, we may and shall assume $a_1 < a_2$. We shall prove by an induction on the sum $l_{abs}(\Delta_1) + l_{abs}(\Delta_2)$. When the sum is 2, the argument is similar to the cases below and we omit the details.

Case 1: $a_1 \neq a_2 - 1$. Let $k = \varepsilon_{[a_1]_\rho}(\sigma)$. We have that $\varepsilon_{[a_1]_\rho}(I_\sigma(\pi)), \varepsilon_{[a_2]_\rho}(I_{\sigma'}(\pi)) = \varepsilon_{[a_1]_\rho}(\pi) + 1$. Let $\kappa = \mathrm{St}(\{{}^-\Delta_1, \Delta_2\})$ and let $\kappa' = \mathrm{St}(\{{}^-\Delta_1', \Delta_2'\}$. We write $D_{a_1} = D_{[a_1]_\rho}$. Furthermore,

$$D_{a_1}^{k+1}(I_\sigma(\pi)) = I_\kappa(D_{a_1}^k(\pi)), \quad D_{a_1}^{k+1}(I_{\sigma'}(\pi)) = I_{\kappa'}(D_{a_1}^k(\pi)).$$

Then, by induction, we have that $I_\kappa(D_{a_1}^k(\pi)) \not\cong I_{\kappa'}(D_{a_1}^k(\pi))$ as desired.
Case 2: $a_1 = a_2 - 1$. In this case, let $\mathfrak{m} = \mathfrak{mx}(\pi, \Delta_1)$. Then

$$\pi \hookrightarrow D_{\mathfrak{m}}(\pi) \times \mathrm{St}(\mathfrak{m})$$

Now let $\widetilde{\Delta} = [a_2, b_1]_\rho = \Delta_1 \cap \Delta_2$ and let $\widetilde{\sigma} = D_{\widetilde{\Delta}}(\sigma)$. We have that

$$\begin{aligned} I_\sigma(\pi) &\hookrightarrow \pi \times \sigma \\ &\hookrightarrow D_{\mathfrak{m}}(\pi) \times \mathrm{St}(\mathfrak{m}) \times \sigma \\ &\cong D_{\mathfrak{m}}(\pi) \times \sigma \times \mathrm{St}(\mathfrak{m}) \\ &\hookrightarrow D_{\mathfrak{m}}(\pi) \times \widetilde{\sigma} \times \mathrm{St}(\widetilde{\Delta} + \mathfrak{m}), \end{aligned}$$

where the isomorphism in the third line follows from $\mathrm{St}(\Delta') \times \sigma \cong \sigma \times \mathrm{St}(\Delta')$ for any Δ_1-saturated segment Δ' (see e.g. [MW86, Lemme II 10.1]).

Since $\mathfrak{mx}(D_{\mathfrak{m}}(\pi), \Delta_1) = \emptyset$ and $\mathfrak{mx}(\widetilde{\sigma}, \Delta_1) = \emptyset$, we have that

$$\mathfrak{mx}(I_\sigma(\pi), \Delta_1) = \mathfrak{m} + \widetilde{\Delta}$$

The last equality follows from an application on the geometric lemma (see details from the proof of [Ch22+, Proposition 11.1]).

Similarly, let $\widetilde{\sigma}' = D_{\Delta_1} \circ D_{\widetilde{\Delta}}(\pi)$. We also have that:

$$\begin{aligned} I_{\sigma'}(\pi) &\hookrightarrow \pi \times \sigma' \\ &\hookrightarrow D_{\mathfrak{m}}(\pi) \times \mathrm{St}(\mathfrak{m}) \times \sigma' \\ &\cong D_{\mathfrak{m}}(\pi) \times \sigma' \times \mathrm{St}(\mathfrak{m}) \\ &\hookrightarrow D_{\mathfrak{m}}(\pi) \times \widetilde{\sigma}' \times \mathrm{St}(\Delta_1) \times \mathrm{St}(\widetilde{\Delta}) \times \mathrm{St}(\mathfrak{m}) \\ &\hookrightarrow D_{\mathfrak{m}}(\pi) \times \widetilde{\sigma}' \times \mathrm{St}(\mathfrak{m} + \Delta_1 + \widetilde{\Delta}) \end{aligned}$$

Again, $\mathfrak{mx}(D_{\mathfrak{m}}(\pi), \Delta_1) = \emptyset$ and $\mathfrak{mx}(\widetilde{\sigma}', \Delta_1) = \emptyset$. Thus,

$$\mathfrak{mx}(I_{\sigma'}(\pi), \Delta_1) = \mathfrak{m} + \Delta_1 + \widetilde{\Delta}.$$

Thus, comparing the invariant $\mathfrak{mx}(., \Delta_1)$, we have $I_\sigma(\pi) \not\cong I_{\sigma'}(\pi)$. ☐

It is an interesting to investigate if an analogue of Proposition 14.1 can be obtained if one replaces essentially square-integrable representations by other interesting representations such as Speh representations and ladder representations. The composition factors for parabolically induced from Speh representations and ladder representations are studied in [Ta15] and [Gu21], and so it is possible to develop a parallel theory from those via above approach.

14.2. **Consequences.** We similarly define those notions for derivatives for ladder representations (also see e.g. [Ch22+c]). If there exists $\omega \in \mathrm{Irr}(G_{n-k})$ such that $\omega \boxtimes \sigma \hookrightarrow \pi_{N_k}$ for σ defined in the above lemma, then denote such ω by $D_\sigma(\pi)$. Otherwise, set $D_\sigma(\pi) = 0$.

Corollary 14.2. *We use the notations in Proposition 14.1. Then $D_\sigma(\pi) \not\cong D_{\sigma'}(\pi)$ if both terms are non-zero.*

Proof. Let $\pi' = D_\sigma(\pi)$. Then $I_\sigma(\pi') \not\cong I_{\sigma'}(\pi')$ and so $\pi \not\cong I_{\sigma'} \circ D_\sigma(\pi)$. Applying $D_{\sigma'}$ on both sides, we obtain the corollary. ☐

Corollary 14.3. *Let $\Delta_1, \Delta_2 \in \mathrm{Seg}_\rho$ such that $\Delta_1 < \Delta_2$. Let $\pi \in \mathrm{Irr}_\rho$. Suppose $D_{\Delta_2} \circ D_{\Delta_1}(\pi) \neq 0$. If the non-zero map*

$$D_{\Delta_2} \circ D_{\Delta_1}(\pi) \boxtimes (\mathrm{St}(\Delta_1) \times \mathrm{St}(\Delta_2)) \to \pi_{N_{l_a(\Delta_1)+l_a(\Delta_2)}}$$

is injective, then $\{\Delta_1, \Delta_2\}$ is minimal to π.

Proof. If the map is injective, then $D_{\Delta_2} \circ D_{\Delta_1}(\pi) \boxtimes \mathrm{St}(\{\Delta_1 + \Delta_2\})$ is a submodule of $\pi_{N_{l_a(\Delta_1)+l_a(\Delta_2)}}$. This implies that

$$D_{\Delta_2} \circ D_{\Delta_1}(\pi) \cong D_{\mathrm{St}(\Delta_1+\Delta_2)}(\pi).$$

Then the corollary follows from Corollary 14.2. □

15. Appendix B: Applications

15.1. Minimality under Δ-reduced condition.

Corollary 15.1. *Let $\pi \in \mathrm{Irr}_\rho$. Let $\mathfrak{n} \in \mathrm{Mult}_\rho$ be minimal to π. Let Δ be a segment. Suppose $b(\Delta') < b(\Delta)$ for any segment $\Delta' \in \mathfrak{n}$ with $b(\Delta') \neq b(\Delta)$. If $\eta_\Delta(D_\mathfrak{n}(\pi)) = 0$, then $\mathfrak{mx}(\pi, \Delta) \subseteq \mathfrak{n}$.*

Proof. Let $\Delta = [a, b]_\rho$. Let $\mathfrak{p} = \mathfrak{mx}(\pi, \Delta)$. Let $\mathfrak{p}'$ be all the segments Δ' in $\mathfrak{n}$ such that $b(\Delta') = b$ and $\Delta' \subset \Delta$.

We first prove the following claim:
Claim: $|\mathfrak{p}'| \geq |\mathfrak{p}|$.

Proof of Claim: We also let $\mathfrak{p}''$ be all the segments Δ' in $\mathfrak{n}$ such that $b(\Delta') = b$ and $\Delta \subsetneq \Delta'$. Note that

$$\eta_\Delta(D_{\mathfrak{p}''} \circ D_{\mathfrak{n}-\mathfrak{p}''}(\pi)) = \eta_\Delta(D_{\mathfrak{n}-\mathfrak{p}''}(\pi)).$$

Hence, we also have $\eta_\Delta(D_{\mathfrak{n}-\mathfrak{p}''}(\pi)) = 0$. But then, by counting the multiplicity at $b(\Delta)$, we must have that $|\mathfrak{p}'| \geq |\mathfrak{p}|$. This proves the claim.

If $\mathfrak{p} \neq \mathfrak{p}'$, then Claim 1 implies that there exists $a \leq c \leq b$ such that the number of segments $[c, b]_\rho$ in $\mathfrak{p}'$ is strictly greater than $\varepsilon_{[c,b]_\rho}(\pi)$. Let c^* be such smallest integer. Now let $\widetilde{\mathfrak{n}}$ be the submultisegment of $\mathfrak{n}$ whose segments Δ' satisfy $a(\Delta') < a$. By the subsequent property, $\widetilde{\mathfrak{n}} + [c^*, b]_\rho$ and $\widetilde{\mathfrak{n}} + \mathfrak{p}'$ are still minimal to π. Now, by Proposition 4.10 and the minimality of $\widetilde{\mathfrak{n}} + [c^*, b]_\rho$, $\varepsilon_{[c^*,b]_\rho}(\mathfrak{r}(\widetilde{\mathfrak{n}}, \pi)) = \varepsilon_{[c^*,b]_\rho}(\pi)$. But, by the admissibility of $\widetilde{\mathfrak{n}} + \mathfrak{p}'$, $\varepsilon_{[c^*,b]_\rho}(\mathfrak{r}(\widetilde{\mathfrak{n}}, \pi)) > \varepsilon_{[c^*,b]_\rho}(\pi)$. This gives a contradiction. Thus we must have $\mathfrak{p}' = \mathfrak{p}$. □

15.2. **Generalized reduced decomposition.** The notion of reduced decomposition is introduced in [AL23, Section 7] for a segment Δ.

We now describe a generalization of reduced decompositin for multisegments. Let $\pi \in \mathrm{Irr}_\rho(G_n)$. Let $\mathfrak{n}$ be minimal to π. Let b be the largest $b(\Delta)$ among all segments Δ in $\mathfrak{n}$. We then choose the longest segment $\Delta_1 \in \mathfrak{n}$ such that $b(\Delta_1) = b$. Let $\mathfrak{p}_1 = \mathfrak{mx}(D_\mathfrak{n}(\pi), \Delta)$. We now set $\mathfrak{m}_1' = \mathfrak{n} + \mathfrak{p}_1$. In general, $\mathfrak{n}_1$ is not minimal to π and so one find the minimal element, denoted by $\mathfrak{m}_1$, in $\mathcal{S}(\pi, D_{\mathfrak{m}_1}(\pi))$. Then, by Corollary 15.1, $\mathfrak{m}_1 = \mathfrak{q}_1 + \mathfrak{n}_2$ for some multisegment $\mathfrak{n}_2$ and $\mathfrak{q}_1 = \mathfrak{mx}(\pi, \Delta)$. Thus, from commutativity, we now have that $D_{\mathfrak{m}_1}(\pi) \cong D_{\mathfrak{n}_2} \circ D_{\mathfrak{q}_1}(\pi)$. Thus, one may consider that $\pi \hookrightarrow D_{\mathfrak{q}_1}(\pi) \times \mathrm{St}(\mathfrak{q}_1)$ is the step for the reduction. One can repeat the same process for $\mathfrak{n}_2$ and then repeatedly to obtain a sequence of triples $(\mathfrak{p}_1, \mathfrak{q}_1, \mathfrak{n}_1), \ldots, (\mathfrak{p}_r, \mathfrak{q}_r, \mathfrak{n}_r)$ until the process terminates. Then we obtain a kind of reduced decompositon for π with respect to $\mathfrak{n}$ as follows:

$$\pi \hookrightarrow (D_{\mathfrak{q}_r} \circ \cdots \circ D_{\mathfrak{q}_1})(\pi) \times \mathrm{St}(\mathfrak{q}_r) \times \ldots \times \mathrm{St}(\mathfrak{q}_1).$$

Let $l=l_{abs}(\mathfrak{n})$. One may expect there is a natural map:

$$D_{\mathfrak{n}}(\pi)\boxtimes \mathrm{St}(\mathfrak{n}) \rightarrow (D_{\mathfrak{q}_r}\circ \cdots \circ D_{\mathfrak{q}_1})(\pi)\dot{\times}^1(\mathrm{St}(\mathfrak{q}_r)\times \ldots \times \mathrm{St}(\mathfrak{q}_1))_{N_l},$$

where $\dot{\times}^1$ is a parabolic induction from a $G_{n_1}\times G_{n_2}\times G_l$-representation to a$G_{n_1+n_2}\times G_l$ representation. Here $n_1=n-l_{abs}(\mathfrak{q}_1)-\ldots-l_{abs}(\mathfrak{q}_r)$ and $n_2=l_{abs}(\mathfrak{p}_1)+\ldots+l_{abs}(\mathfrak{p}_r)$.

15.3. **An inductive construction of simple quotients of Bernstein-Zelevinsky derivatives.** Let $\pi\in \mathrm{Irr}_{\rho}$. Let $\Delta\in \mathrm{Seg}_{\rho}$ and let $\mathfrak{n}\in \mathrm{Mult}_{\rho}$. Let $\mathfrak{p}=\mathfrak{mx}(\pi,\Delta)$. Then $D_{\mathfrak{n}}\circ D_{\mathfrak{p}}(\pi)$ is a simple quotient of the $l_{abs}(\mathfrak{n}+\mathfrak{p})$-th Bernstein-Zelevinsky derivative of $\mathrm{St}(\mathfrak{p})\times D_{\mathfrak{p}}(\pi)$. Then one may ask if $D_{\mathfrak{n}}\circ D_{\mathfrak{p}}(\pi)$ is also a simple quotient of the $l_{abs}(\mathfrak{n}+\mathfrak{p})$-th Bernstein-Zelevinsky derivative of π. In a special case, we have the following criteria using commutativity:

Proposition 15.2. *We use the set-up mentioned above.*

1. *Suppose $\mathfrak{n}$ is minimal to $D_{\mathfrak{p}}(\pi)$. Suppose further that for any segment $\Delta'\in\mathfrak{n}$, $b(\Delta')<b(\Delta)$. If $D_{\mathfrak{n}}\circ D_{\mathfrak{p}}(\pi)\cong D_{\mathfrak{m}}(\pi)$ for some $\mathfrak{m}\in \mathrm{Mult}_{\rho}$, then $\mathfrak{n}+\mathfrak{p}$ is minimal to π.*
2. *Suppose $\mathfrak{m}$ is minimal to π. Suppose further that for any segment $\Delta'\in\mathfrak{m}$, $b(\Delta')\leq b(\Delta)$. Then $D_{\mathfrak{m}}(\pi)\cong D_{\mathfrak{n}}\circ D_{\mathfrak{p}}(\pi)$ and $\mathfrak{m}=\mathfrak{n}+\mathfrak{p}$ for some multisegment $\mathfrak{n}$ minimal to $D_{\mathfrak{p}}(\pi)$.*

Proof. We first consider (1). Suppose $D_{\mathfrak{n}}\circ D_{\mathfrak{p}}(\pi)\cong D_{\mathfrak{m}}(\pi)$ for some $\mathfrak{m}\in\mathrm{Mult}_{\rho}$. Without loss of generality, we may assume that $\mathfrak{m}$ is minimal to π. Then, by Corollary 15.1, $\mathfrak{m}=\mathfrak{p}+\mathfrak{n}'$. By the subsequent property and commutativty property, $\mathfrak{n}'$ is also minimal to $D_{\mathfrak{p}}(\pi)$ and $D_{\mathfrak{n}'}\circ D_{\mathfrak{p}}(\pi)\cong D_{\mathfrak{m}}(\pi)$. By Theorem 1.2, we then have that $\mathfrak{n}=\mathfrak{n}'$ and this implies (1).

We now consider (2). By Corollary 15.1, $\mathfrak{p}\subset\mathfrak{m}$ and so $\mathfrak{m}=\mathfrak{p}+\mathfrak{n}$ for some multisegment $\mathfrak{n}$. By the subsequent property and commutativity property (Theorem 1.4), we also have that $\mathfrak{n}$ is minimal to $D_{\mathfrak{p}}(\pi)$. ☐

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