FINITE ELEMENT APPROXIMATION OF STATIONARY FOKKER–PLANCK–KOLMOGOROV EQUATIONS WITH APPLICATION TO PERIODIC NUMERICAL HOMOGENIZATION

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Abstract. We propose and rigorously analyze a finite element method for the approximation of stationary Fokker–Planck–Kolmogorov (FPK) equations subject to periodic boundary conditions in two settings: one with weakly differentiable coefficients, and one with merely essentially bounded measurable coefficients under a Cordes-type condition. These problems arise as governing equations for the invariant measure in the homogenization of nondivergence-form equations with large drifts. In particular, the Cordes setting guarantees the existence and uniqueness of a square-integrable invariant measure. We then suggest and rigorously analyze an approximation scheme for the effective diffusion matrix in both settings, based on the finite element scheme for stationary FPK problems developed in the first part. Finally, we demonstrate the performance of the methods through numerical experiments.

1. Introduction

In the first part of this paper, we consider the numerical approximation of the stationary Fokker–Planck–Kolmogorov-type problem

$$
\begin{aligned}
-D^2 : (Au) + \nabla \cdot (bu) := -\sum_{i,j=1}^{n} \partial_{ij}^2 (a_{ij} u) + \sum_{k=1}^{n} \partial_k (b_k u) &= \nabla \cdot F \quad \text{in } Y, \\
u \text{ is } Y\text{-periodic}, &
\end{aligned}
\tag{1.1}
$$

where $Y := (0,1)^n \subset \mathbb{R}^n$ denotes the unit cell, $F \in L^2_{\mathrm{per}}(Y;\mathbb{R}^n)$, and (A,b) is a pair of given coefficients

$$
A = (a_{ij})_{1\le i,j\le n} \in L^\infty_{\mathrm{per}}(Y;\mathbb{R}^{n\times n}_{\mathrm{sym}}), \qquad b = (b_k)_{1\le k\le n} \in L^\infty_{\mathrm{per}}(Y;\mathbb{R}^n),
$$

where A is assumed to be uniformly elliptic (see (2.3)). In addition, we make one of the following two assumptions:

- Setting $\mathcal{A}$: we assume that $A \in W^{1,p}_{\mathrm{per}}(Y;\mathbb{R}^{n\times n}_{\mathrm{sym}})$ for some $p > n$.
- Setting $\mathcal{B}$: we assume that the coefficients satisfy the Cordes-type condition

$$
\exists\, \delta \in \left(\frac{n}{n+\pi^2}, 1\right] : \qquad \frac{|A|^2 + |b|^2}{(\mathrm{tr}(A))^2} \le \frac{1}{n-1+\delta} \quad \text{a.e. in } \overline{Y}, \tag{1.2}
$$

  which can be relaxed to the classical Cordes condition (1.3) if $|b| = 0$ a.e..

Date: September 11, 2024.

2010 *Mathematics Subject Classification.* 35B27, 35J15, 65N12, 65N15, 65N30.

Key words and phrases. Fokker–Planck–Kolmogorov equation, finite element methods, Cordes condition, homogenization, convergence analysis.

Here, we used the notation $|A|^2 := A : A$ and $|b|^2 := b \cdot b$. We call the condition (1.2) a Cordes-type condition as it is inspired by the classical Cordes condition

$$\exists\, \delta \in (0,1] : \qquad \frac{|A|^2}{(\mathrm{tr}(A))^2} \leq \frac{1}{n-1+\delta} \quad \text{a.e.} \tag{1.3}$$

used in the study of nondivergence-form equations $-A : D^2 v = f$, and the Cordes-type condition

$$\exists\, (\delta, \lambda) \in (0,1] \times (0,\infty) : \qquad \frac{|A|^2 + \frac{1}{2\lambda}|b|^2 + \frac{1}{\lambda^2}c^2}{(\mathrm{tr}(A) + \frac{1}{\lambda}c)^2} \leq \frac{1}{n+\delta} \quad \text{a.e.} \tag{1.4}$$

used in the study of nondivergence-form equations $-A : D^2 v - b \cdot \nabla v + cv = f$; see [15, 46, 47]. It is worth noting that (1.4) can never be satisfied when $c = 0$ since $\frac{|M|^2}{(\mathrm{tr}(M))^2} \geq \frac{1}{n}$ for any $M \in \mathbb{R}^{n\times n}$.

Problems of the form (1.1) arise naturally as the governing equation for the invariant measure, i.e., the solution to

$$-D^2 : (Ar) + \nabla \cdot (br) = 0 \quad \text{in } Y, \qquad r \text{ is } Y\text{-periodic}, \qquad \int_Y r = 1, \tag{1.5}$$

which is used to determine the effective problem in the periodic homogenization of nondivergence-form equations with large drifts, i.e.,

$$\begin{aligned} -A\left(\frac{\cdot}{\varepsilon}\right) : D^2 u_\varepsilon - \frac{1}{\varepsilon} b\left(\frac{\cdot}{\varepsilon}\right) \cdot \nabla u_\varepsilon &= f \quad \text{in } \Omega, \\ u_\varepsilon &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{1.6}$$

which is the focus of the second part of this paper. In setting $\mathcal{A}$, it is known that (1.5) has a unique positive Hölder continuous solution and that if the drift satisfies the centering condition $\int_Y rb = 0$ and f, g, Ω are sufficiently regular, then the sequence of solutions $(u_\varepsilon)_{\varepsilon>0}$ to (1.6) converges weakly in $H^1(\Omega)$ as $\varepsilon \to 0$ to the solution $\overline{u}$ of the effective problem

$$\begin{aligned} -\overline{A} : D^2 \overline{u} &= f \quad \text{in } \Omega, \\ \overline{u} &= g \quad \text{on } \partial\Omega, \end{aligned}$$

where the effective diffusion matrix $\overline{A} \in \mathbb{R}^{n\times n}$ is the symmetric positive definite matrix given by

$$\overline{A} := \int_Y r[I_n + D\chi]A[I_n + (D\chi)^{\mathrm{T}}] \tag{1.7}$$

with $\chi := (\chi_1, \ldots, \chi_n)$ and χ_j denoting the solution to

$$-A : D^2 \chi_j - b \cdot \nabla \chi_j = b_j \quad \text{in } Y, \qquad \chi_j \text{ is } Y\text{-periodic}, \qquad \int_Y \chi_j = 0 \tag{1.8}$$

for $1 \leq j \leq n$; see, e.g., [6, 16, 30]. The goal of the second part of this paper is the efficient and accurate numerical approximation of the effective diffusion matrix $\overline{A}$ from (1.7) for both settings $\mathcal{A}$ and $\mathcal{B}$.

In setting $\mathcal{A}$, we approximate the solutions to (1.5) and (1.1) by rewriting the problems in divergence-form and adapting Schatz's method, see [45], to the periodic setting to handle the resulting noncoercive variational form. Regarding the numerical approximation of (1.8), we multiply the equation by the approximation of the invariant measure r and only then rewrite the problem in divergence-form in order to overcome the low regularity of solutions to the dual problem to (1.8). Our approximation of the effective diffusion matrix (1.7) relies on a combination of the finite element schemes for r and χ_j.

In setting $\mathcal{B}$, we show that (1.5) has a unique nonnegative solution $r \in L^2_{\mathrm{per}}(Y)$, and we suggest and rigorously analyze a finite element method for its approximation which is based on the observation that r is of the form

$$r = C\frac{\mathrm{tr}(A)}{|A|^2 + |b|^2}(1 - \nabla \cdot \rho),$$

where $C > 0$ is a constant and ρ is the unique solution of a Lax–Milgram-type problem in $H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$, i.e., the subspace of $H^1_{\mathrm{per}}(Y;\mathbb{R}^n)$ consisting of functions with mean zero. Further, we suggest and rigorously analyze a finite element method for the approximation of solutions to (1.1), which is based on the observation that any solution $u \in L^2_{\mathrm{per}}(Y)$ to (1.1) is of the form

$$u = \frac{\mathrm{tr}(A)}{|A|^2 + |b|^2}(-\nabla \cdot \tilde{\rho}_0) + cr,$$

where $c \in \mathbb{R}$ is a constant and $\tilde{\rho}_0$ is the unique solution of a Lax–Milgram-type problem in $H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$. Regarding the problem (1.8), we show that under the centering condition $\int_Y rb = 0$ there exists a unique solution $\chi_j \in H^2_{\mathrm{per}}(Y)$ and we suggest and analyze a finite element method for its approximation. Our approximation of the effective diffusion matrix (1.7) relies on a combination of the finite element schemes for r and χ_j. We also discuss assumptions under which homogenization occurs in this setting.

The construction of finite element methods for FPK-type equations has received considerable attention over the past decades; we refer to [7, 8, 34, 36, 37, 41] for some of the early developments in the case of smooth coefficients. The distinguishing feature of our study in Setting $\mathcal{A}$ is the rigorous error analysis for the periodic setting. For the case of merely essentially bounded measurable coefficients, there are very few publications, including the recent primal-dual weak Galerkin approach from [39, 51]. In our study for Setting $\mathcal{B}$, we choose a different route by imposing a new Cordes-type condition, performing a suitable renormalization of the problem, and developing a simple finite element framework inspired by the prior works [19, 46, 48] on nondivergence-form problems.

Regarding the homogenization of linear elliptic equations in nondivergence-form, we refer to [14, 23, 26, 33, 48, 49] for recent developments in periodic homogenization, to [4, 31] for essential transformation procedures, and to [1, 2, 3, 24, 25] for recent developments in stochastic homogenization.

In recent years, significant progress has been made on the numerical homogenization of equations in nondivergence-form; see, e.g., [14, 18, 48] for linear equations and [20, 32, 43] for Hamilton–Jacobi–Bellman equations. Concerning the numerical homogenization of divergence-form problems with large drifts, we refer to [9, 12, 29, 38, 40, 53] and the references therein. To be best of our knowledge, we are not aware of any previous research on developing finite element methods for approximating effective diffusion matrices in the context of nondivergence-form equations with large drift terms.

Finally, we refer to [17, 21, 27, 35, 42] and the references therein for further contributions to the homogenization of convection-diffusion equations via probabilistic methods, and to [52] for a stochastic structure-preserving scheme for computing the effective diffusivity of three-dimensional periodic or chaotic flows.

We briefly explain the organization of the paper. In Section 2, we study the finite element approximation of stationary FPK equations subject to periodic boundary conditions, where Setting $\mathcal{A}$ is discussed in Section 2.1 and Setting $\mathcal{B}$ is discussed in Section 2.2, respectively. In Section 3, we extend our results to stationary FPK-type problems of the form (1.1). After that, we study the finite element approximation of the nondivergence-form problem (1.8) and the numerical approximation of the effective diffusion matrix (1.7) in Section 4, where Section 4.1 focuses on Setting $\mathcal{A}$ and Section 4.2 focuses on Setting $\mathcal{B}$. Finally, we demonstrate the theoretical results in numerical experiments provided in Section 5.

2. FEM for Stationary FPK Problems

In this section, we discuss the finite element approximation of stationary FPK equations subject to periodic boundary conditions, i.e.,

$$-D^2 : (Au) + \nabla \cdot (bu) = 0 \quad \text{in } Y, \qquad u \text{ is } Y\text{-periodic}, \tag{2.1}$$

where $Y := (0,1)^n$ denotes the unit cell in $\mathbb{R}^n$. We will always assume that

$$A \in L^\infty_{\mathrm{per}}(Y; \mathbb{R}^{n\times n}_{\mathrm{sym}}), \qquad b \in L^\infty_{\mathrm{per}}(Y; \mathbb{R}^n), \tag{2.2}$$

and that A is uniformly elliptic, i.e.,

$$\exists \lambda, \Lambda > 0 : \qquad \lambda I_n \le A \le \Lambda I_n \text{ a.e. in } \mathbb{R}^n. \tag{2.3}$$

We will consider two settings – one with an additional regularity assumption on the coefficients, and one with a Cordes-type condition on the coefficients but without any additional regularity assumptions.

- Setting $\mathcal{A}$ (higher regularity): We write $(A, b) \in \mathcal{A}$ if (2.2), (2.3) hold and
$$A \in W^{1,p}_{\mathrm{per}}(Y; \mathbb{R}^{n\times n}_{\mathrm{sym}}) \text{ for some } p > n, \tag{2.4}$$
where we always assume that $p \ge 2$.
- Setting $\mathcal{B}$ (Cordes-type): We write $(A, b) \in \mathcal{B}$ if (2.2), (2.3) hold and
$$\exists\, \delta \in \left(\frac{n}{n+\pi^2}, 1\right] : \qquad \frac{|A|^2 + |b|^2}{(\mathrm{tr}(A))^2} \le \frac{1}{n-1+\delta} \quad \text{a.e. in } \mathbb{R}^n, \tag{2.5}$$
where $|A| := \sqrt{A : A}$ denotes the Frobenius norm of A.

The condition (2.5) resembles the Cordes condition; see, e.g., [15, 46, 50].

Throughout the paper, we use the notation $L^2_{\mathrm{per},0}(Y) := \{v \in L^2_{\mathrm{per}}(Y) : \int_Y v = 0\}$ and $H^k_{\mathrm{per},0}(Y) := \{v \in H^k_{\mathrm{per}}(Y) : \int_Y v = 0\}$ for $k \in \mathbb{N}$.

2.1. Setting $\mathcal{A}$.

In this section, we study the well-posedness and the finite element approximation of solutions to the FPK problem (2.1) for the case $(A, b) \in \mathcal{A}$.

2.1.1. *Well-posedness.* First, we note that when $(A,b) \in \mathcal{A}$, we can rewrite the problem (2.1) in divergence-form thanks to (2.4), i.e.,

$$-\nabla \cdot (A\nabla u + (\mathrm{div}(A) - b)u) = 0 \quad \text{in } Y, \qquad u \text{ is } Y\text{-periodic.}$$

Let us also briefly note that by (2.4) and Sobolev embeddings, we have that $A \in C^{0,\alpha}(\mathbb{R}^n; \mathbb{R}^{n\times n}_{\mathrm{sym}})$ for some $\alpha > 0$. Then, (2.1) has a unique Hölder continuous solution up to multiplicative constants. More precisely, the following result is known to hold; see, e.g., [6, 10].

Proposition 2.1 (Analysis of (2.1) in setting $\mathcal{A}$)**.** *Let $(A,b) \in \mathcal{A}$. Then, there exists a unique solution $r \in H^1_{\mathrm{per}}(Y)$ to the problem*

$$-D^2 : (Ar) + \nabla \cdot (br) = 0 \quad \text{in } Y, \qquad r \text{ is } Y\text{-periodic}, \qquad \int_Y r = 1, \tag{2.6}$$

and any solution $u \in H^1_{\mathrm{per}}(Y)$ to (2.1) is a constant multiple of r. Further, $r \in W^{1,p}_{\mathrm{per}}(Y)$ with $p > n$ as in (2.4), there holds $\inf_{\mathbb{R}^n} r > 0$, and for $g \in L^2_{\mathrm{per}}(Y)$ the problem

$$-A : D^2 v - b \cdot \nabla v = g \quad \text{in } Y, \qquad v \text{ is } Y\text{-periodic}, \qquad \int_Y v = 0 \tag{2.7}$$

admits a (unique) solution $v \in H^1_{\mathrm{per},0}(Y)$ if and only if $\int_Y gr = 0$.

Note that if $(A,b) \in \mathcal{A}$, what we mean by a solution $v \in H^1_{\mathrm{per},0}(Y)$ to (2.7) is an element v in $H^1_{\mathrm{per},0}(Y)$ that satisfies the natural weak formulation of the problem obtained by rewriting (2.7) in divergence-form.

2.1.2. *Finite element approximation of* (2.6)*.* We now discuss the finite element approximation of (2.6). The ideas are inspired by the earlier work [14].

We begin by noting that

$$\hat{r} := r - 1 \in H^1_{\mathrm{per},0}(Y) \tag{2.8}$$

is the unique solution in $H^1_{\mathrm{per},0}(Y)$ to the divergence-form problem

$$-\nabla \cdot (A\nabla \hat{r} + (\mathrm{div}(A) - b)\hat{r}) = \nabla \cdot (\mathrm{div}(A) - b) \quad \text{in } Y, \;\; \hat{r} \text{ is } Y\text{-periodic}, \quad \int_Y \hat{r} = 0.$$

More precisely, $\hat{r}$ is the unique element in $H^1_{\mathrm{per},0}(Y)$ satisfying

$$a(\hat{r}, v) = \int_Y (b - \mathrm{div}(A)) \cdot \nabla v \qquad \forall v \in H^1_{\mathrm{per},0}(Y), \tag{2.9}$$

where $a(\cdot,\cdot) : H^1_{\mathrm{per},0}(Y) \times H^1_{\mathrm{per},0}(Y) \to \mathbb{R}$ is given by

$$a(v_1, v_2) := \int_Y A\nabla v_1 \cdot \nabla v_2 + \int_Y v_1(\mathrm{div}(A) - b) \cdot \nabla v_2 \tag{2.10}$$

for $v_1, v_2 \in H^1_{\mathrm{per},0}(Y)$. Clearly, a defines a bounded bilinear form on $H^1_{\mathrm{per},0}(Y)$, i.e.,

$$|a(v_1, v_2)| \leq C_1 \|v_1\|_{H^1(Y)} \|v_2\|_{H^1(Y)} \qquad \forall v_1, v_2 \in H^1_{\mathrm{per},0}(Y) \tag{2.11}$$

for some constant $C_1 > 0$, but the bilinear form a is not coercive, making the finite element approximation of (2.9) nonstandard.

However, since $\mathrm{div}(A) - b \in L^p_{\mathrm{per}}(Y;\mathbb{R}^n)$ with $p > n$, it is easy to show using the assumed uniform ellipticity (2.3), together with Hölder, Gagliardo–Nirenberg and Young inequalities, that the following Gårding inequality holds:

$$a(v,v) \geq \frac{\lambda}{2}\|v\|^2_{H^1(Y)} - C_2\|v\|^2_{L^2(Y)} \qquad \forall v \in H^1_{\mathrm{per},0}(Y) \tag{2.12}$$

for some constant $C_2 > 0$. This enables us to prove the following result using an adaptation of Schatz's method [45].

Theorem 2.1 (Finite element approximation of (2.6)). *Let $(A,b) \in \mathcal{A}$. Let $r \in H^1_{\mathrm{per}}(Y)$ denote the unique solution to (2.6), and let $\hat{r} \in H^1_{\mathrm{per},0}(Y)$ be given by (2.8). Then, there exists a constant $C_0 > 0$ such that for any $\alpha \in (0, C_0)$ it is true that if R_h is a finite-dimensional closed linear subspace of $H^1_{\mathrm{per},0}(Y)$ with the property*

$$\inf_{v_h\in R_h} \frac{\|\psi - v_h\|_{H^1(Y)}}{\|\psi\|_{H^2(Y)}} \leq \alpha \qquad \forall \psi \in H^2_{\mathrm{per},0}(Y)\backslash\{0\}, \tag{2.13}$$

then there exists a unique $\hat{r}_h \in R_h$ such that

$$a(\hat{r}_h, v_h) = \int_Y (b - \mathrm{div}(A)) \cdot \nabla v_h \qquad \forall v_h \in R_h, \tag{2.14}$$

and setting $r_h := 1 + \hat{r}_h$, there holds

$$\|r - r_h\|_{L^2(Y)} + \alpha\|r - r_h\|_{H^1(Y)} \leq C\alpha \inf_{v_h\in R_h} \|\hat{r} - v_h\|_{H^1(Y)} \tag{2.15}$$

for some constant $C > 0$ depending only on (A,b) and n.

Proof. Let $\alpha \in (0, C_0)$, where $C_0 > 0$ will be chosen later (see (2.19)). Let R_h be a finite-dimensional closed linear subspace of $H^1_{\mathrm{per},0}(Y)$ satisfying (2.13).

Uniqueness of $\hat{r}_h$: We show that (2.14) can have at most one solution $\hat{r}_h \in R_h$. Before we start, note that in view of Proposition 2.1, there exist constants $r_0, r_1 > 0$ such that $r_0 \leq r \leq r_1$ in $\mathbb{R}^n$. Now suppose $\hat{r}_h^{(1)} \in R_h$ and $\hat{r}_h^{(2)} \in R_h$ are two solutions to (2.14), and set $z_h := \hat{r}_h^{(1)} - \hat{r}_h^{(2)}$. Noting that $z_h \in R_h \subset H^1_{\mathrm{per},0}(Y)$ and $a(z_h, v_h) = 0$ for all $v_h \in R_h$, we have by (2.12) that

$$\|z_h\|^2_{H^1(Y)} \leq \frac{2}{\lambda}\left(a(z_h,z_h) + C_2\|z_h\|^2_{L^2(Y)}\right) \leq \frac{2C_2}{\lambda}\|z_h\|_{L^2(Y)}\|z_h\|_{H^1(Y)}. \tag{2.16}$$

Since $\frac{z_h}{r} \in L^2_{\mathrm{per}}(Y)$ and $\int_Y z_h = 0$, we have by Proposition 2.1 that the problem

$$-A : D^2\psi - b\cdot\nabla\psi = \frac{z_h}{r} \quad \text{in } Y, \qquad \psi \text{ is } Y\text{-periodic}, \qquad \int_Y \psi = 0 \tag{2.17}$$

has a unique solution $\psi \in H^1_{\mathrm{per},0}(Y)$, i.e., $a(v,\psi) = \int_Y \frac{z_h}{r} v$ for any $v \in H^1_{\mathrm{per},0}(Y)$. Note that $\psi \in H^2_{\mathrm{per},0}(Y)$ and $\|\psi\|_{H^2(Y)} \leq C_3\|\frac{z_h}{r}\|_{L^2(Y)}$ for some constant $C_3 > 0$; see [22]. Then, we find that

$$\begin{aligned}\|z_h\|^2_{L^2(Y)} \leq r_1\int_Y \frac{|z_h|^2}{r} = r_1 a(z_h,\psi) &= r_1 \inf_{v_h\in R_h} a(z_h, \psi - v_h)\\ &\leq r_1 C_1 \alpha\|z_h\|_{H^1(Y)}\|\psi\|_{H^2(Y)}\\ &\leq \frac{r_1C_1C_3}{r_0}\alpha\|z_h\|_{H^1(Y)}\|z_h\|_{L^2(Y)},\end{aligned} \tag{2.18}$$

where we used the bounds on r, the properties of ψ and z_h, and (2.11). Combining this estimate with (2.16), we obtain

$$\|z_h\|_{H^1(Y)} \leq \frac{2C_2}{\lambda}\|z_h\|_{L^2(Y)} \leq \frac{2r_1 C_1 C_2 C_3}{\lambda r_0}\alpha\|z_h\|_{H^1(Y)}.$$

Let $C_0 > 0$ be chosen as

$$C_0 := \frac{\lambda r_0}{2r_1 C_1 C_2 C_3}. \tag{2.19}$$

Then, using that $\alpha < C_0$, there holds $\frac{2r_1C_1C_2C_3}{\lambda r_0}\alpha < 1$ and hence, $z_h = 0$, i.e., there is at most one solution to (2.14).

Existence of $\hat{r}_h$: As R_h is finite-dimensional, uniqueness implies existence of a solution $\hat{r}_h \in R_h$ to (2.14).

Error bound: It remains to show the error bound (2.15). First, by (2.12), Galerkin orthogonality, and (2.11), we have that

$$\begin{aligned}\frac{\lambda}{2}\|\hat{r}-\hat{r}_h\|_{H^1(Y)}^2 &\leq \inf_{v_h\in R_h} a(\hat{r}-\hat{r}_h, \hat{r}-v_h) + C_2\|\hat{r}-\hat{r}_h\|_{L^2(Y)}^2 \\ &\leq \left(C_1 \inf_{v_h\in R_h}\|\hat{r}-v_h\|_{H^1(Y)} + C_2\|\hat{r}-\hat{r}_h\|_{L^2(Y)}\right)\|\hat{r}-\hat{r}_h\|_{H^1(Y)}.\end{aligned} \tag{2.20}$$

Next, by considering (2.17) with z_h replaced by $\hat{r}-\hat{r}_h$ and arguing as in (2.18) with z_h replaced by $\hat{r}-\hat{r}_h$, we find that

$$\|\hat{r}-\hat{r}_h\|_{L^2(Y)} \leq \frac{r_1C_1C_3}{r_0}\alpha\|\hat{r}-\hat{r}_h\|_{H^1(Y)}. \tag{2.21}$$

Combining (2.20) and (2.21), we obtain

$$\left(\frac{\lambda}{2} - \frac{r_1C_1C_2C_3}{r_0}\alpha\right)\|\hat{r}-\hat{r}_h\|_{H^1(Y)} \leq C_1\inf_{v_h\in R_h}\|\hat{r}-v_h\|_{H^1(Y)}.$$

Finally, noting that $\frac{\lambda}{2} - \frac{r_1C_1C_2C_3}{r_0}\alpha > 0$ by $\alpha < C_0$ and (2.19), observing that $r - r_h = \hat{r}-\hat{r}_h$, and in view of (2.21), we conclude that (2.15) holds. □

As an example, if R_h in Theorem 2.1 is chosen to be the space of continuous piecewise affine zero-mean functions on a shape-regular triangulation $\mathcal{T}_h$ of $\overline{Y}$ into triangles with longest edge $h > 0$ conforming with the requirement of periodicity, then (2.13) holds with $\alpha = \mathcal{O}(h)$.

A natural question to ask is if we can obtain a near-best approximation result in the $W^{1,p}(Y)$-norm. The following theorem answers this question positively under the additional assumption that $\mathrm{div}(A) \in L^\infty_{\mathrm{per}}(Y;\mathbb{R}^n)$, following arguments similar to [13, 44]. Note that if $(A,b) \in \mathcal{A}$ and $\mathrm{div}(A) \in L^\infty_{\mathrm{per}}(Y;\mathbb{R}^n)$, then $r \in W^{1,p}_{\mathrm{per}}(Y)$ for all $p < \infty$ by elliptic regularity theory.

Theorem 2.2 (L^p-estimates for the approximation of (2.6))**.** *Let* $p \in (1,\infty)$ *and set* $t := \frac{p}{p-1}$. *In the situation of Theorem 2.1, if additionally* $\mathrm{div}(A) \in L^\infty_{\mathrm{per}}(Y;\mathbb{R}^n)$,

$R_h \subset W^{1,\infty}_{\mathrm{per},0}(Y)$, and

$$\inf_{v_h \in R_h} \frac{\|\psi - v_h\|_{W^{1,t}(Y)}}{\|\psi\|_{W^{2,t}(Y)}} \leq \alpha \qquad \forall \psi \in W^{2,t}_{\mathrm{per},0}(Y)\backslash\{0\},$$

then, for $\alpha > 0$ sufficiently small, we have the following bound

$$\|r - r_h\|_{L^p(Y)} + \alpha\|r - r_h\|_{W^{1,p}(Y)} \leq C\alpha \inf_{v_h \in R_h} \|\hat{r} - v_h\|_{W^{1,p}(Y)} \tag{2.22}$$

for some constant $C > 0$ depending only on (A, b) and n.

Proof. First, observe that $\xi := |\hat{r} - \hat{r}_h|^{p-1}\,\mathrm{sign}(\hat{r} - \hat{r}_h) \in L^\infty_{\mathrm{per}}(Y)$ as $\hat{r}, \hat{r}_h \in L^\infty_{\mathrm{per}}(Y)$. By Proposition 2.1, there exists a unique solution $\psi \in H^1_{\mathrm{per},0}(Y)$ to the problem

$$-A : D^2\psi - b \cdot \nabla\psi = \xi - \int_Y \xi r \quad \text{in } Y, \qquad \psi \text{ is } Y\text{-periodic}, \qquad \int_Y \psi = 0.$$

Note that $\psi \in W^{2,t}_{\mathrm{per}}(Y)$ and $\|\psi\|_{W^{2,t}(Y)} \leq C\|\xi - \int_Y \xi r\|_{L^t(Y)}$ for some constant $C > 0$; see [22]. Using that $\int_Y(\hat{r} - \hat{r}_h) = 0$, we find that

$$\begin{aligned}
\|\hat{r} - \hat{r}_h\|^p_{L^p(Y)} = a(\hat{r} - \hat{r}_h, \psi) &= \inf_{v_h \in R_h} a(\hat{r} - \hat{r}_h, \psi - v_h) \\
&\lesssim \|\hat{r} - \hat{r}_h\|_{W^{1,p}(Y)} \inf_{v_h \in R_h} \|\psi - v_h\|_{W^{1,t}(Y)} \\
&\lesssim \alpha\|\hat{r} - \hat{r}_h\|_{W^{1,p}(Y)} \left\|\xi - \int_Y \xi r\right\|_{L^t(Y)} \\
&\lesssim \alpha\|\hat{r} - \hat{r}_h\|_{W^{1,p}(Y)}\|\xi\|_{L^t(Y)},
\end{aligned}$$

where the constants absorbed in "$\lesssim$" only depend on (A, b) and n. Since $\|\xi\|_{L^t(Y)} = \|\hat{r} - \hat{r}_h\|^{p-1}_{L^p(Y)}$, we deduce that

$$\|r - r_h\|_{L^p(Y)} = \|\hat{r} - \hat{r}_h\|_{L^p(Y)} \leq C\alpha\|\hat{r} - \hat{r}_h\|_{W^{1,p}(Y)} \tag{2.23}$$

for some constant $C > 0$ only depending on (A, b) and n.

Next, let $z \in H^1_{\mathrm{per},0}(Y)$ denote the unique solution to

$$-\nabla \cdot (A\nabla z) = \nabla \cdot ((\mathrm{div}(A) - b)r_h) \quad \text{in } Y, \qquad z \text{ is } Y\text{-periodic}, \qquad \int_Y z = 0,$$

and note that $z \in W^{1,p}_{\mathrm{per}}(Y)$. Observe that $\hat{r}$ solves a similar problem with r_h replaced by r. In particular,

$$\|\hat{r} - z\|_{W^{1,p}(Y)} \lesssim \|(\mathrm{div}(A) - b)(r - r_h)\|_{L^p(Y)} \lesssim \|r - r_h\|_{L^p(Y)} \lesssim \alpha\|\hat{r} - \hat{r}_h\|_{W^{1,p}(Y)}.$$

Note that $\hat{r}_h \in R_h$ is the $\tilde{a}$-orthogonal projection of z onto R_h, where $\tilde{a}(v_1, v_2) := \int_Y A\nabla v_1 \cdot \nabla v_2$. Hence, by standard finite element theory,

$$\|z - \hat{r}_h\|_{W^{1,p}(Y)} \lesssim \inf_{v_h \in R_h} \|z - v_h\|_{W^{1,p}(Y)} \lesssim \|\hat{r} - z\|_{W^{1,p}(Y)} + \inf_{v_h \in R_h} \|\hat{r} - v_h\|_{W^{1,p}(Y)}.$$

By combining with the previous estimate, we obtain

$$\|\hat{r} - \hat{r}_h\|_{W^{1,p}(Y)} \lesssim \alpha\|\hat{r} - \hat{r}_h\|_{W^{1,p}(Y)} + \inf_{v_h \in R_h} \|\hat{r} - v_h\|_{W^{1,p}(Y)}.$$

For $\alpha > 0$ sufficiently small, we can absorb the first term on the right-hand side into the left-hand side and obtain

$$\|r - r_h\|_{W^{1,p}(Y)} = \|\hat{r} - \hat{r}_h\|_{W^{1,p}(Y)} \leq C \inf_{v_h \in R_h} \|\hat{r} - v_h\|_{W^{1,p}(Y)}$$

for some constant $C > 0$. Because of (2.23), we conclude that (2.22) holds. □

2.2. **Setting $\mathcal{B}$.** In this section, we study well-posedness and the finite element approximation of solutions to the FPK problem (2.1) for the case $(A, b) \in \mathcal{B}$, i.e., we now merely assume that

$$A \in L^\infty_{\mathrm{per}}(Y; \mathbb{R}^{n\times n}_{\mathrm{sym}}), \qquad b \in L^\infty_{\mathrm{per}}(Y; \mathbb{R}^n),$$

together with uniformly ellipticity (see (2.3)), and that the Cordes-type condition

$$\exists\, \delta \in \left(\frac{n}{n+\pi^2}, 1\right] : \qquad \frac{|A|^2 + |b|^2}{(\mathrm{tr}(A))^2} \leq \frac{1}{n - 1 + \delta} \quad \text{a.e. in } \mathbb{R}^n \tag{2.24}$$

is satisfied. In this case, the problem (2.1) cannot be reformulated in divergence-form and we will seek (so-called *very weak*) solutions to (2.1) in L^2, i.e., $u \in L^2_{\mathrm{per}}(Y)$ such that

$$\int_Y uA : D^2\varphi + \int_Y ub \cdot \nabla\varphi = 0 \qquad \forall \varphi \in H^2_{\mathrm{per}}(Y).$$

Let us introduce the renormalization function

$$\gamma := \frac{\mathrm{tr}(A)}{|A|^2 + |b|^2}, \tag{2.25}$$

and the renormalized coefficients

$$\tilde{A} := \gamma A, \qquad \tilde{b} := \gamma b. \tag{2.26}$$

Noting that $\mathrm{tr}(A)$ is the sum of eigenvalues of A, and $|A|^2$ is the sum of squared eigenvalues of A, we make the following observation.

Remark 2.1 (Properties of γ)**.** *If $(A, b) \in \mathcal{B}$, then $\gamma \in L^\infty_{\mathrm{per}}(Y)$ and*

$$\gamma_0 := \frac{n\lambda}{n\Lambda^2 + \|b\|^2_{L^\infty(Y;\mathbb{R}^n)}} \leq \gamma \leq \frac{\Lambda}{\lambda^2} \quad \textit{a.e. in } \mathbb{R}^n.$$

In particular, γ is positive almost everywhere.

We then consider the renormalized FPK problem

$$-D^2 : (\tilde{A}\tilde{u}) + \nabla \cdot (\tilde{b}\tilde{u}) = 0 \quad \text{in } Y, \qquad \tilde{u} \text{ is } Y\text{-periodic}, \tag{2.27}$$

and make the following observation.

Remark 2.2 (Relationship between the original and renormalized FPK problem)**.** *Let $\mathbb{L}$ denote the set of solutions in $L^2_{\mathrm{per}}(Y)$ to the FPK problem* (2.1)*. Then,*

$$\mathbb{L} = \{\gamma\tilde{u} \,|\, \tilde{u} \in \tilde{\mathbb{L}}\},$$

where $\tilde{\mathbb{L}}$ denotes the set of solutions in $L^2_{\mathrm{per}}(Y)$ to the renormalized FPK problem (2.27)*.*

Because of Remark 2.2, we will focus our attention on the renormalized FPK problem (2.27). We will begin by analyzing the well-posedness of (2.27).

2.2.1. *Well-posedness.* The key consequence of the Cordes-type condition (2.24) is captured in the following lemma.

Lemma 2.1 (Consequences of condition (2.24))**.** *Let* $(A, b) \in \mathcal{B}$*. Let* $(\tilde{A}, \tilde{b})$ *denote the pair of renormalized coefficients given by* (2.25) *and* (2.26)*. Then, the following assertions hold:*

(i) *We have the bound*

$$\left|\tilde{A} - I_n\right|^2 + \left|\tilde{b}\right|^2 \leq 1 - \delta \quad a.e.\ in\ \mathbb{R}^n.$$

(ii) *There exists a constant* $\kappa = \kappa(\delta, n) \in (0, 1]$ *such that*

$$\left\|\tilde{A} : Dw + \tilde{b} \cdot w - \nabla \cdot w\right\|_{L^2(Y)} \leq \sqrt{1-\kappa}\, \|Dw\|_{L^2(Y;\mathbb{R}^{n\times n})}$$

for any $w \in H^1_{\mathrm{per},0}(Y; \mathbb{R}^n)$*.*

Proof. Using that $\tilde{A} = \gamma A$ and $\tilde{b} = \gamma b$, we compute

$$\begin{aligned}\left|\tilde{A} - I_n\right|^2 + \left|\tilde{b}\right|^2 &= n + \gamma^2\left(|A|^2 + |b|^2\right) - 2\gamma\, \mathrm{tr}(A)\\ &= n - \gamma\, \mathrm{tr}(A)\\ &= n - \frac{(\mathrm{tr}(A))^2}{|A|^2 + |b|^2} \leq 1 - \delta,\end{aligned}$$

where we have used (2.24) in the final step. This completes the proof of (i) and it remains to show (ii). To this end, let us first note that by Poincaré's inequality (see Theorem 3.2 in [5]) there holds

$$\|w\|_{L^2(Y;\mathbb{R}^n)} \leq \frac{\sqrt{n}}{\pi}\|Dw\|_{L^2(Y;\mathbb{R}^{n\times n})} \qquad \forall w \in H^1_{\mathrm{per},0}(Y; \mathbb{R}^n). \tag{2.28}$$

Using (2.28) and (i), we find that

$$\begin{aligned}\left\|\tilde{A} : Dw + \tilde{b} \cdot w - \nabla \cdot w\right\|^2_{L^2(Y)} &= \left\|(\tilde{A} - I_n) : Dw + \tilde{b} \cdot w\right\|^2_{L^2(Y)}\\ &\leq \int_Y \left(\left|\tilde{A} - I_n\right|^2 + \left|\tilde{b}\right|^2\right)\left(|Dw|^2 + |w|^2\right)\\ &\leq (1-\delta)\left(\|Dw\|^2_{L^2(Y;\mathbb{R}^{n\times n})} + \|w\|^2_{L^2(Y;\mathbb{R}^n)}\right)\\ &\leq (1-\delta)\left(1 + \frac{n}{\pi^2}\right)\|Dw\|^2_{L^2(Y;\mathbb{R}^{n\times n})}\\ &= (1-\kappa)\|Dw\|^2_{L^2(Y;\mathbb{R}^{n\times n})}\end{aligned}$$

for any $w \in H^1_{\mathrm{per},0}(Y; \mathbb{R}^n)$, where $\kappa := (\delta - \frac{n}{n+\pi^2})\frac{n+\pi^2}{\pi^2}$. Note that $\kappa \in (0, 1]$ as $\frac{n}{n+\pi^2} < \delta \leq 1$. □

Remark 2.3. *If* $|b| = 0$ *a.e., then Lemma 2.1 holds with* $\kappa = \delta$*, and we can relax the range of* δ *in* (2.24) *to* $\delta \in (0, 1]$*.*

Remark 2.4 (Another bound). *In the situation of Lemma 2.1 there holds*

$$\left\| \tilde{A} : D^2\varphi + \tilde{b}\cdot\nabla\varphi - \Delta\varphi \right\|_{L^2(Y)} \leq \sqrt{1-\kappa}\,\|\Delta\varphi\|_{L^2(Y)} \qquad \forall \varphi \in H^2_{\mathrm{per}}(Y),$$

where $\kappa \in (0,1]$ is the constant from Lemma 2.1(ii). Indeed, this inequality follows from choosing $w = \nabla\varphi \in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$ in Lemma 2.1(ii) and using the fact that that $\|D^2\varphi\|_{L^2(Y)} = \|\Delta\varphi\|_{L^2(Y)}$ for any $\varphi \in H^2_{\mathrm{per}}(Y)$.

Next, let us observe that any $\tilde{u} \in L^2_{\mathrm{per}}(Y)$ can be written as $\tilde{u} = c - \Delta\psi_c$ for some unique $c \in \mathbb{R}$ (namely $c = \int_Y \tilde{u}$) and unique $\psi_c \in H^2_{\mathrm{per},0}(Y)$. Inserting this ansatz into (2.27) leads to the problem of finding $c \in \mathbb{R}$ and $\psi_c \in H^2_{\mathrm{per},0}(Y)$ such that

$$B_1(\psi_c,\varphi) = c\int_Y \left(\tilde{A} : D^2\varphi + \tilde{b}\cdot\nabla\varphi\right) \qquad \forall\varphi \in H^2_{\mathrm{per},0}(Y), \tag{2.29}$$

where $B_1(\cdot,\cdot) : H^2_{\mathrm{per},0}(Y) \times H^2_{\mathrm{per},0}(Y) \to \mathbb{R}$ is the bilinear form defined by

$$B_1(\varphi_1,\varphi_2) := \int_Y \Delta\varphi_1 \left(\tilde{A} : D^2\varphi_2 + \tilde{b}\cdot\nabla\varphi_2\right). \tag{2.30}$$

We can show the following result:

Theorem 2.3 (Analysis of (2.1) in setting $\mathcal{B}$). *Let $(A,b) \in \mathcal{B}$. Let γ denote the renormalization function and $(\tilde{A},\tilde{b})$ be the pair of renormalized coefficients given by* (2.25), (2.26)*. Further, let $B_1 : H^2_{\mathrm{per},0}(Y) \times H^2_{\mathrm{per},0}(Y) \to \mathbb{R}$ denote the bilinear form defined in* (2.30)*.*

(i) *For any $c \in \mathbb{R}$ there exists a unique solution $\psi_c \in H^2_{\mathrm{per},0}(Y)$ to* (2.29)*, and we have that $\psi_c = c\psi_1$.*

(ii) *The function $\tilde{r} := 1 - \Delta\psi_1$ is the unique solution in $L^2_{\mathrm{per}}(Y)$ to the problem*

$$-D^2 : \left(\tilde{A}\tilde{r}\right) + \nabla\cdot\left(\tilde{b}\tilde{r}\right) = 0 \quad \text{in } Y, \qquad \tilde{r} \text{ is } Y\text{-periodic}, \qquad \int_Y \tilde{r} = 1. \tag{2.31}$$

Further, we have that $\tilde{r} \geq 0$ a.e. in $\mathbb{R}^n$.

(iii) *The set $\mathbb{L}$ of all solutions in $L^2_{\mathrm{per}}(Y)$ to the FPK problem* (2.1) *is given by $\mathbb{L} = \{c\gamma\tilde{r} \mid c \in \mathbb{R}\}$.*

Proof. (i) We show that the Lax–Milgram theorem applies. Clearly, B_1 is a bounded bilinear form on $H^2_{\mathrm{per},0}(Y)$ and the right-hand side in (2.29) defines a bounded linear functional on $H^2_{\mathrm{per},0}(Y)$. It remains to show that B_1 is coercive on $H^2_{\mathrm{per},0}(Y)$. By Remark 2.4, we have for any $\varphi \in H^2_{\mathrm{per},0}(Y)$ that

$$\begin{aligned} B_1(\varphi,\varphi) &= \|\Delta\varphi\|^2_{L^2(Y)} + \int_Y \Delta\varphi\left(\tilde{A} : D^2\varphi + \tilde{b}\cdot\nabla\varphi - \Delta\varphi\right) \\ &\geq (1-\sqrt{1-\kappa})\|\Delta\varphi\|^2_{L^2(Y)}, \end{aligned}$$

where $\kappa \in (0,1]$ is the constant from Lemma 2.1. The proof is concluded by observing that $1-\sqrt{1-\kappa} > 0$ and that $\|\varphi\|_{H^2(Y)} \leq C\|\Delta\varphi\|_{L^2(Y)}$ for any $\varphi \in H^2_{\mathrm{per},0}(Y)$, where $C > 0$ is a constant. The fact that $\psi_c = c\psi_1$ follows from the linearity of B_1 in its first argument and the uniqueness of ψ_c.

(ii) Clearly, $\tilde{r} := 1 - \Delta\psi_1 \in L^2_{\mathrm{per}}(Y)$ is a solution to (2.31). Conversely, any solution $\tilde{r} \in L^2_{\mathrm{per}}(Y)$ to (2.31) can be written as $\tilde{r} = 1 - \Delta\psi$ for some unique

$\psi \in H^2_{\mathrm{per},0}(Y)$. Clearly, ψ must satisfy (2.29) with $c = 1$ and hence, by (i), $\psi = \psi_1$. We now show that $\tilde{r} \geq 0$ a.e. in $\mathbb{R}^n$, using a mollification argument similar to [11]. For $k \in \mathbb{N}$, we set

$$\tilde{A}_{(k)} := (\tilde{a}_{ij} * w_k)_{1\leq i,j\leq n} \in C^\infty_{\mathrm{per}}(Y; \mathbb{R}^{n\times n}_{\mathrm{sym}}), \qquad \tilde{b}_{(k)} := (\tilde{b}_i * w_k)_{1\leq i\leq n} \in C^\infty_{\mathrm{per}}(Y; \mathbb{R}^n),$$

where $w_k := k^n w(k\,\cdot)$ for some $w \in C^\infty_c(\mathbb{R}^n)$ with $w \geq 0$ in $\mathbb{R}^n$ and $\int_{\mathbb{R}^n} w = 1$.

Note that $\tilde{A}_{(k)}$ is uniformly elliptic. In particular, there exists a positive solution $\tilde{r}_k \in C^\infty_{\mathrm{per}}(Y; (0,\infty))$ to $-D^2 : (\tilde{A}_{(k)}\tilde{r}_k) + \nabla \cdot (\tilde{b}_{(k)}\tilde{r}_k) = 0$ in Y with $\int_Y \tilde{r}_k = 1$; see, e.g., [6]. Let $\psi_k \in H^2_{\mathrm{per},0}(Y)$ be such that $\Delta\psi_k = 1 - \tilde{r}_k$. We are going to show that $\|\Delta\psi_k\|_{L^2(Y)}$ is uniformly bounded. To this end, first note that

$$\begin{aligned}|I_n - \tilde{A}_{(k)}(y)|^2 + |\tilde{b}_{(k)}(y)|^2 &= \left|\int_{\mathbb{R}^n} [I_n - \tilde{A}(y-\cdot)]w_k\right|^2 + \left|\int_{\mathbb{R}^n} \tilde{b}(y-\cdot)w_k\right|^2 \\ &\leq \int_{\mathbb{R}^n} \left(|I_n - \tilde{A}(y-\cdot)|^2 + |\tilde{b}(y-\cdot)|^2\right) w_k \leq 1-\delta\end{aligned}$$

for any $y \in \mathbb{R}^n$ by Lemma 2.1(i), and hence, the bound from Remark 2.4 still holds when $(\tilde{A},\tilde{b})$ is replaced by $(\tilde{A}_{(k)},\tilde{b}_{(k)})$. Then, we find that

$$\begin{aligned}(1-\sqrt{1-\kappa})\|\Delta\psi_k\|^2_{L^2(Y)} &\leq \int_Y \Delta\psi_k \left(\tilde{A}_{(k)} : D^2\psi_k + \tilde{b}_{(k)}\cdot\nabla\psi_k\right) \\ &= \int_Y \left(\tilde{A}_{(k)} : D^2\psi_k + \tilde{b}_{(k)}\cdot\nabla\psi_k\right) \leq C\|\Delta\psi_k\|_{L^2(Y)}\end{aligned}$$

for some constant $C = C(n,\lambda,\Lambda,\|b\|_{L^\infty(Y)}) > 0$ independent of k, where we have used that $|\tilde{b}_{(k)}| \leq \|\tilde{b}\|_{L^\infty(Y;\mathbb{R}^n)}$ and $|\tilde{A}_{(k)}| \leq \sqrt{n}\frac{\Lambda}{\lambda}$ in $\mathbb{R}^n$ (as $|\tilde{A}| = \gamma|A| \leq \sqrt{n}\frac{\Lambda}{\lambda}$ a.e. in $\mathbb{R}^n$). In particular, $\|\Delta\psi_k\|_{L^2(Y)}$ is uniformly bounded, and hence, $\|\tilde{r}_k\|_{L^2(Y)} = \|1-\Delta\psi_k\|_{L^2(Y)}$ is uniformly bounded.

Therefore, there exists an $\tilde{r}_0 \in L^2_{\mathrm{per}}(Y)$ such that, upon passing to a subsequence (not indicated), $\tilde{r}_k \rightharpoonup \tilde{r}_0$ weakly in $L^2(Y)$. There holds $\int_Y \tilde{r}_0 = 1$ and, using that $\tilde{A}_{(k)} \to \tilde{A}$ in the $L^2(Y;\mathbb{R}^{n\times n})$-norm and $\tilde{b}_{(k)} \to \tilde{b}$ in the $L^2(Y;\mathbb{R}^n)$-norm as $k\to\infty$, we have for any $\varphi \in C^\infty_{\mathrm{per}}(Y)$ that

$$\int_Y \tilde{r}_0(\tilde{A} : D^2\varphi + \tilde{b}\cdot\nabla\varphi) = \lim_{k\to\infty}\int_Y \tilde{r}_k(\tilde{A}_{(k)} : D^2\varphi + \tilde{b}_{(k)}\cdot\nabla\varphi) = 0,$$

i.e., $\tilde{r}_0 \in L^2_{\mathrm{per}}(Y)$ is a solution to (2.31). Since $\tilde{r}$ is the unique solution to (2.31) in $L^2_{\mathrm{per}}(Y)$, and since $\tilde{r}_k > 0$ in $\mathbb{R}^n$, we conclude that $\tilde{r} = \tilde{r}_0 \geq 0$ a.e. in $\mathbb{R}^n$.

(iii) In view of our discussion above Theorem 2.3, and the results (i)–(ii), it is easy to confirm that the set $\tilde{\mathbb{L}}$ of all solutions in $L^2_{\mathrm{per}}(Y)$ to the renormalized FPK problem (2.27) is given by

$$\tilde{\mathbb{L}} = \{c - \Delta\psi_c \,|\, c\in\mathbb{R}\} = \{c(1-\Delta\psi_1)\,|\,c\in\mathbb{R}\} = \{c\tilde{r}\,|\,c\in\mathbb{R}\}.$$

By Remark 2.2, it follows that $\mathbb{L} = \{\gamma\tilde{u}\,|\,\tilde{u}\in\tilde{\mathbb{L}}\} = \{c\gamma\tilde{r}\,|\,c\in\mathbb{R}\}$. □

We immediately obtain the following consequences:

Corollary 2.1 (Invariant measure in setting $\mathcal{B}$). *For any $(A,b)\in\mathcal{B}$, there exists a unique solution $r\in L^2_{\mathrm{per}}(Y)$ to*

$$-D^2:(Ar)+\nabla\cdot(br)=0 \quad \text{in } Y, \qquad r \text{ is } Y\text{-periodic}, \qquad \int_Y r = 1, \tag{2.32}$$

and there holds $r\geq 0$ a.e. in $\mathbb{R}^n$. Further, we can obtain that

$$r=\frac{1}{\int_Y \gamma\tilde{r}}\gamma\tilde{r}$$

with γ defined in (2.25) *and $\tilde{r}$ defined in Theorem 2.3(ii).*

2.2.2. *Finite element approximation of* (2.31). Let us recall from Theorem 2.3 that $u\in L^2_{\mathrm{per}}(Y)$ is a solution to (2.1), if and only if $u=c\gamma\tilde{r}$ for some $c\in\mathbb{R}$. Hence, once we know the unique solution $\tilde{r}\in L^2_{\mathrm{per}}(Y)$ to (2.31), we know all solutions to (2.1). We now discuss the finite element approximation of $\tilde{r}$, inspired by ideas from [48].

One could approximate $\tilde{r}$ via an $H^2_{\mathrm{per},0}(Y)$-conforming finite element method for the approximation of $\psi_1\in H^2_{\mathrm{per},0}(Y)$ from Theorem 2.3(i). More conveniently, we can avoid using H^2-conforming methods by using the following scheme based on an $H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$-conforming finite element method to approximate an auxiliary function $\rho\in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$ satisfying $\nabla\cdot\rho=\Delta\psi_1$. To this end, we introduce the bilinear form $B_2(\cdot,\cdot):H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)\times H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)\to\mathbb{R}$ given by

$$B_2(v,w):=\int_Y(\nabla\cdot v)(\tilde{A}:Dw+\tilde{b}\cdot w)+\frac{1}{2}\int_Y(Dv-(Dv)^{\mathrm{T}}):(Dw-(Dw)^{\mathrm{T}})$$

for $v,w\in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$. Note that $B_2(\nabla\varphi_1,\nabla\varphi_2)=B_1(\varphi_1,\varphi_2)$ for any $\varphi_1,\varphi_2\in H^2_{\mathrm{per},0}(Y)$. The second integral in the definition of B_2 is added to make B_2 coercive.

Lemma 2.2 (Characterization of $\tilde{r}$). *Let $(A,b)\in\mathcal{B}$, and let $(\tilde{A},\tilde{b})$ be the pair of renormalized coefficients given by* (2.25), (2.26). *Let $B_2:H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)\times H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)\to\mathbb{R}$ be defined as above. Then, B_2 is a bounded and coercive bilinear form on $H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$. In particular, there exists a unique $\rho\in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$ such that*

$$B_2(\rho,w)=\int_Y\left(\tilde{A}:Dw+\tilde{b}\cdot w\right) \qquad \forall w\in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n), \tag{2.33}$$

and the unique solution $\tilde{r}\in L^2_{\mathrm{per}}(Y)$ to (2.31) *is given by $\tilde{r}=1-\nabla\cdot\rho$.*

Proof. First, we show that B_2 is coercive. Using that for any $w\in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$, we have that

$$\|\nabla\cdot w\|^2_{L^2(Y)}+\frac{1}{2}\|Dw-(Dw)^{\mathrm{T}}\|^2_{L^2(Y;\mathbb{R}^{n\times n})}=\|Dw\|^2_{L^2(Y;\mathbb{R}^{n\times n})}, \tag{2.34}$$

and using Lemma 2.1(ii), we find that for any $w\in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$ there holds

$$\begin{aligned}B_2(w,w)&=\|Dw\|^2_{L^2(Y;\mathbb{R}^{n\times n})}+\int_Y(\nabla\cdot w)\left(\tilde{A}:Dw+\tilde{b}\cdot w-\nabla\cdot w\right)\\&\geq(1-\sqrt{1-\kappa})\|Dw\|^2_{L^2(Y;\mathbb{R}^{n\times n})},\end{aligned}$$

where $\kappa = \kappa(\delta, n) \in (0,1]$ is the constant from Lemma 2.1. By noting that $1 - \sqrt{1-\kappa} > 0$ and that $w \mapsto \|Dw\|_{L^2(Y;\mathbb{R}^{n\times n})}$ defines a norm on $H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$, this concludes the proof of coercivity of B_2.

Next, we show boundedness of B_2. Using that by (2.25), (2.26) and Remark 2.1, we can get that

$$\left|\tilde{A}\right|^2 + \left|\tilde{b}\right|^2 = \gamma^2(|A|^2 + |b|^2) = \gamma \operatorname{tr}(A) \le \frac{\Lambda^2}{\lambda^2} n \quad \text{a.e. in } \mathbb{R}^n,$$

and using (2.28) and (2.34), we have for any $w_1, w_2 \in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$ that

$$\begin{aligned}|B_2(w_1,w_2)| &\le \frac{\Lambda}{\lambda}\sqrt{n}\,\|\nabla\cdot w_1\|_{L^2(Y)}\sqrt{\|Dw_2\|^2_{L^2(Y;\mathbb{R}^{n\times n})} + \|w_2\|^2_{L^2(Y;\mathbb{R}^n)}} \\ &\quad + \|Dw_1\|_{L^2(Y;\mathbb{R}^{n\times n})}\|Dw_2\|_{L^2(Y;\mathbb{R}^{n\times n})} \\ &\le \left(1 + \frac{\Lambda}{\lambda}\sqrt{n}\sqrt{1+\frac{n}{\pi^2}}\right)\|Dw_1\|_{L^2(Y;\mathbb{R}^{n\times n})}\|Dw_2\|_{L^2(Y;\mathbb{R}^{n\times n})},\end{aligned}$$

which concludes the proof of the boundedness of B_2.

Since the right-hand side in (2.33) defines a bounded linear functional on $H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$, we have by the Lax–Milgram theorem that (2.33) has a unique solution $\rho \in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$. Finally, noting that $1-\nabla\cdot\rho \in L^2_{\mathrm{per}}(Y)$, that $\int_Y (1-\nabla\cdot\rho) = 1$, and that for any $\varphi \in H^2_{\mathrm{per},0}(Y)$, there holds

$$\int_Y (1-\nabla\cdot\rho)\left(\tilde{A} : D^2\varphi + \tilde{b}\cdot\nabla\varphi\right) = \int_Y \left(\tilde{A} : D^2\varphi + \tilde{b}\cdot\nabla\varphi\right) - B_2(\rho,\nabla\varphi) = 0$$

by (2.33) with $w = \nabla\varphi$, we conclude that $\tilde{r} = 1 - \nabla\cdot\rho$ by uniqueness of $\tilde{r}$. □

In view of Lemma 2.2, we immediately obtain the following simple method to approximate $\tilde{r}$.

Theorem 2.4 (Finite element approximation of (2.31))**.** *Suppose that the assumptions of Lemma 2.2 hold. Further, let P_h be a closed linear subspace of $H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$. Then, there exists a unique $\rho_h \in P_h$ such that*

$$B_2(\rho_h, w_h) = \int_Y \left(\tilde{A} : Dw_h + \tilde{b}\cdot w_h\right) \qquad \forall w_h \in P_h,$$

and setting $\tilde{r}_h := 1 - \nabla\cdot\rho_h \in L^2_{\mathrm{per}}(Y)$, there holds

$$\|\tilde{r} - \tilde{r}_h\|_{L^2(Y)} \le C \inf_{w_h\in P_h} \|D(\rho - w_h)\|_{L^2(Y;\mathbb{R}^{n\times n})} \tag{2.35}$$

for some constant $C = C(\lambda,\Lambda,\delta,n) > 0$.

Proof. This follows immediately from the statement and proof of Lemma 2.2 together with a standard Galerkin orthogonality argument. The constant $C > 0$ in (2.35) can be taken as $C := \frac{1+(\Lambda/\lambda)\sqrt{n}\sqrt{1+(n/\pi^2)}}{1-\sqrt{1-\kappa}}$, where $\kappa = \kappa(\delta,n) \in (0,1]$ is the constant from Lemma 2.1. □

3. Remarks on Nonhomogeneous Stationary FPK-type Problems

In this section, we briefly discuss the finite element approximation of nonhomogeneous stationary FPK-type problems subject to periodic boundary conditions, as the ideas from the previous section can be straightforwardly extended to this problem class. For $F \in L^2_{\mathrm{per}}(Y;\mathbb{R}^n)$, let us consider the problem

$$-D^2 : (Au) + \nabla \cdot (bu) = \nabla \cdot F \quad \text{in } Y, \qquad u \text{ is } Y\text{-periodic}, \tag{3.1}$$

where $Y := (0,1)^n$ denotes the unit cell in $\mathbb{R}^n$, and $(A,b) \in \mathcal{A}$ or $(A,b) \in \mathcal{B}$. Recall the settings $\mathcal{A}$ (higher regularity) and $\mathcal{B}$ (Cordes-type) from Section 2.

3.1. Setting $\mathcal{A}$.

First, we note that when $(A,b) \in \mathcal{A}$, we can rewrite the problem (3.1) in divergence-form thanks to (2.4), i.e.,

$$-\nabla \cdot (A\nabla u + (\mathrm{div}(A) - b)u) = \nabla \cdot F \quad \text{in } Y, \qquad u \text{ is } Y\text{-periodic}.$$

Regarding the uniqueness of solutions, we know from Proposition 2.1 that if a solution $u \in H^1_{\mathrm{per}}(Y)$ to (3.1) exists, then it is unique up to the addition of constant multiples of r, where r denotes the unique solution to (2.6). Regarding the existence of solutions to (3.1), the following result is known in Setting $\mathcal{A}$; see, e.g., [22].

Proposition 3.1 (Well-posedness of (3.1) in setting $\mathcal{A}$)**.** *Let $(A,b) \in \mathcal{A}$ and $F \in L^2_{\mathrm{per}}(Y;\mathbb{R}^n)$. Then, there exists a solution $u \in H^1_{\mathrm{per}}(Y)$ to (3.1), and u is unique up to the addition of a constant multiple of the unique solution r to (2.6).*

To have a unique solution, let us now restrict our attention to the unique solution $u_0 \in H^1_{\mathrm{per}}(Y)$ to the following problem

$$-\nabla \cdot (A\nabla u_0 + (\mathrm{div}(A) - b)u_0) = \nabla \cdot F \;\text{ in } Y, \;\; u_0 \text{ is } Y\text{-periodic}, \;\; \int_Y u_0 = 0, \tag{3.2}$$

whose existence and uniqueness follows from Proposition 3.1. Then, arguing as in Section 2.1.2, we obtain the following approximation result, the proof of which is omitted.

Theorem 3.1 (Finite element approximation of (3.2))**.** *Let $(A,b) \in \mathcal{A}$ and $F \in L^2_{\mathrm{per}}(Y;\mathbb{R}^n)$. Let $u_0 \in H^1_{\mathrm{per},0}(Y)$ denote the unique solution to (3.2), and let $a : H^1_{\mathrm{per},0}(Y) \times H^1_{\mathrm{per},0}(Y) \to \mathbb{R}$ denote the bilinear form from (2.10). Then, there exists a constant $C_0 > 0$ such that for any $\alpha \in (0, C_0)$ it is true that if R_h is a finite-dimensional closed linear subspace of $H^1_{\mathrm{per},0}(Y)$ with the property*

$$\inf_{v_h \in R_h} \frac{\|\psi - v_h\|_{H^1(Y)}}{\|\psi\|_{H^2(Y)}} \leq \alpha \qquad \forall \psi \in H^2_{\mathrm{per},0}(Y)\backslash\{0\},$$

then there exists a unique $u_h \in R_h$ such that $a(u_h, v_h) = -\int_Y F \cdot \nabla v_h$ for all $v_h \in R_h$, and there holds

$$\|u_0 - u_h\|_{L^2(Y)} + \alpha \|u_0 - u_h\|_{H^1(Y)} \leq C\alpha \inf_{v_h \in R_h} \|u_0 - v_h\|_{H^1(Y)}$$

for some constant $C > 0$ depending only on (A,b) and n.

If desired, L^p approximation estimates can be derived similarly to Theorem 2.2 under suitable regularity assumptions on F.

3.2. **Setting $\mathcal{B}$.** When $(A,b)\in\mathcal{B}$, we consider the renormalized problem

$$-D^2:(\tilde{A}\tilde{u})+\nabla\cdot(\tilde{b}\tilde{u})=\nabla\cdot F \quad \text{in } Y, \qquad \tilde{u} \text{ is } Y\text{-periodic}, \tag{3.3}$$

where the pair of renormalized coefficients $(\tilde{A},\tilde{b})$ is given by (2.25), (2.26). Similarly to Remark 2.2, it is clear that $u\in L^2_{\mathrm{per}}(Y)$ is a solution to the original problem (3.1) if and only if there is a solution $\tilde{u}\in L^2_{\mathrm{per}}(Y)$ to (3.3) such that $u=\gamma\tilde{u}$, where γ is defined in (2.25). Hence, from now on we focus on the renormalized problem (3.3).

Regarding uniqueness of solutions, we know from Theorem 2.3 that if a solution $\tilde{u}\in L^2_{\mathrm{per}}(Y)$ to (3.3) exists, then it is unique up to the addition of a constant multiple of $\tilde{r}$, where $\tilde{r}\in L^2_{\mathrm{per}}(Y)$ denotes the unique solution to (2.31). In order to have at most one solution, we therefore focus our attention on the more restricted problem

$$-D^2:(\tilde{A}\tilde{u}_0)+\nabla\cdot(\tilde{b}\tilde{u}_0)=\nabla\cdot F \quad \text{in } Y, \qquad \tilde{u}_0 \text{ is } Y\text{-periodic}, \qquad \int_Y \tilde{u}_0=0, \tag{3.4}$$

which indeed has a unique solution, as summarized in the following theorem.

Theorem 3.2 (Well-posedness of (3.3))**.** *Let $(A,b)\in\mathcal{B}$ and $F\in L^2_{\mathrm{per}}(Y;\mathbb{R}^n)$. Let $(\tilde{A},\tilde{b})$ be the pair of renormalized coefficients given by* (2.25), (2.26)*, and let $\tilde{r}\in L^2_{\mathrm{per}}(Y)$ denote the unique solution to* (2.31)*. Further, let $B_2: H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)\times H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)\to\mathbb{R}$ denote the bilinear form from Lemma 2.2. Then, the following assertions hold.*

(i) *There exists a unique $\tilde{\rho}_0\in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$ such that $B_2(\tilde{\rho}_0,w)=-\int_Y F\cdot w$ for all $w\in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$.*
(ii) *The function $\tilde{u}_0:=-\nabla\cdot\tilde{\rho}_0$ is the unique solution in $L^2_{\mathrm{per}}(Y)$ to* (3.4).
(iii) *A function $\tilde{u}\in L^2_{\mathrm{per}}(Y)$ is a solution to* (3.3) *if and only if $\tilde{u}=\tilde{u}_0+c\tilde{r}$ for some constant $c\in\mathbb{R}$.*

We omit the proof as the results follow immediately from the previous discussion and the coercivity of B_2 on $H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$. Then, arguing as in Section 2.2.2, we obtain the following approximation result, the proof of which is omitted.

Theorem 3.3 (Finite element approximation of (3.4))**.** *Suppose that the assumptions of Theorem 3.2 hold. Further, let P_h be a closed linear subspace of $H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$. Then, there exists a unique $\tilde{\rho}_h\in P_h$ such that $B_2(\tilde{\rho}_h,w_h)=-\int_Y F\cdot w_h$ for all $w_h\in P_h$, and setting $\tilde{u}_h:=-\nabla\cdot\tilde{\rho}_h\in L^2_{\mathrm{per}}(Y)$, there holds*

$$\|\tilde{u}_0-\tilde{u}_h\|_{L^2(Y)}\le C\inf_{w_h\in P_h}\|D(\tilde{\rho}_0-w_h)\|_{L^2(Y;\mathbb{R}^{n\times n})}$$

for some constant $C=C(\lambda,\Lambda,\delta,n)>0$.

4. Numerical Approximation of Effective Diffusion Matrices

4.1. **Setting $\mathcal{A}$.** First, let us discuss the case $(A,b)\in\mathcal{A}$. We denote the invariant measure by r, i.e., the unique solution to the periodic FPK problem (2.6) given by Proposition 2.1. As usual, we write $Y:=(0,1)^n$, and we introduce the notations $\langle\varphi\rangle:=\int_Y\varphi r$ and $\varphi^\varepsilon:=\varphi(\frac{\cdot}{\varepsilon})$ for any Y-periodic function φ and $\varepsilon>0$.

4.1.1. *The homogenization result.* When $(A,b)\in\mathcal{A}$, it is known (see, e.g., [31]) that if the drift b satisfies the centering condition

$$\langle b\rangle = 0, \tag{4.1}$$

then for any bounded domain $\Omega\subset\mathbb{R}^n$ with $\partial\Omega\in C^{1,1}$, $f\in L^2(\Omega)$, and $g\in H^2(\Omega)$, the sequence of solutions $(u_\varepsilon)_{\varepsilon>0}\in H^2(\Omega)$ to

$$\begin{aligned} -A^\varepsilon : D^2u_\varepsilon - \varepsilon^{-1}\, b^\varepsilon\cdot\nabla u_\varepsilon &= f \quad \text{in } \Omega,\\ u_\varepsilon &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{4.2}$$

converges weakly in $H^1(\Omega)$ as $\varepsilon\to 0$ to the solution $\overline{u}$ of the homogenized problem

$$\begin{aligned} -\overline{A} : D^2\overline{u} &= f \quad \text{in } \Omega,\\ \overline{u} &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{4.3}$$

where the effective diffusion matrix $\overline{A}\in\mathbb{R}^{n\times n}$ is the symmetric positive definite matrix given by

$$\overline{A} := \left\langle [I_n + D\chi]A[I_n+(D\chi)^{\mathrm{T}}]\right\rangle, \qquad \chi = (\chi_1,\dots,\chi_n), \tag{4.4}$$

with $\chi_j\in H^1_{\mathrm{per}}(Y)$, $1\le j\le n$, denoting the solution to

$$-A : D^2\chi_j - b\cdot\nabla\chi_j = b_j \quad \text{in } Y, \qquad \chi_j \text{ is } Y\text{-periodic}, \qquad \int_Y \chi_j = 0, \tag{4.5}$$

whose existence and uniqueness are guaranteed by Proposition 2.1.

We make it our goal to approximate the effective diffusion matrix $\overline{A}$, a quantity that only depends on (A,b) and is, in particular, independent of Ω, f, g.

4.1.2. *Approximation of the effective diffusion matrix.* First, we recall that for $(A,b)\in\mathcal{A}$, we have that $r\in W^{1,p}_{\mathrm{per}}(Y)$ with $p>n$ as in (2.4). In particular, r is Hölder continuous. We also recall from Section 2 that we have constructed a finite element method for the approximation of r in the $H^1(Y)$-norm (see Theorem 2.1) and in the $W^{1,p}(Y)$-norm (assuming additionally that $\operatorname{div}(A)\in L^\infty$; see Theorem 2.2). In particular, by Sobolev embedding, we can produce a sequence of approximations $(r_h)_{h>0}$ with $\|r-r_h\|_{L^\infty(Y)}\to 0$.

In view of the definition (4.4) of the effective diffusion matrix, it is thus sufficient for us to approximate the solution χ_j to (4.5) in the $H^1(Y)$-norm. The key difficulty is that for the dual problem

$$-D^2 : (A\phi) + \nabla\cdot(b\phi) = z \quad \text{in } Y, \qquad \phi \text{ is } Y\text{-periodic}, \qquad \int_Y \phi = 0$$

with $z\in L^2_{\mathrm{per},0}(Y)$, we generally cannot expect that $\phi\in H^2_{\mathrm{per}}(Y)$ and hence, a duality argument similar to the one in the proof of Theorem 2.1 does not work. E.g., when $n=1$, $b=0$, and $z(y)=\sin(2\pi y)$, then $\phi(y)=\frac{\sin(2\pi y)+c}{4\pi^2 A(y)}$ for some $c\in\mathbb{R}$. This simple example shows that lack of regularity of A limits the regularity of ϕ.

To overcome this difficulty, we note that the solution χ_j to (4.5) belongs to $H^2_{\mathrm{per}}(Y)$, and thus we can multiply the equation (4.5) by the positive Hölder continuous function r from Proposition 2.1 without changing the set of solutions:

$$-rA : D^2\chi_j - rb\cdot\nabla\chi_j = rb_j \quad \text{in } Y, \qquad \chi_j \text{ is } Y\text{-periodic}, \qquad \int_Y \chi_j = 0.$$

Since $(A,b)\in\mathcal{A}$, introducing $\beta := rb - \operatorname{div}(rA) \in L^p_{\mathrm{per}}(Y;\mathbb{R}^n)$, $p>n$, we can rewrite the equation in divergence-form as

$$-\nabla\cdot(rA\nabla\chi_j) - \beta\cdot\nabla\chi_j = rb_j \quad \text{in } Y, \qquad \chi_j \text{ is } Y\text{-periodic}, \qquad \int_Y \chi_j = 0. \tag{4.6}$$

Now, for $z\in L^2_{\mathrm{per},0}(Y)$, we can rewrite the dual problem

$$-D^2 : (rA\psi) + \nabla\cdot(rb\psi) = z \quad \text{in } Y, \qquad \psi \text{ is } Y\text{-periodic}, \qquad \int_Y \psi = 0, \tag{4.7}$$

using that $\nabla\cdot\beta = 0$ weakly in Y by definition of r (see Proposition 2.1), as

$$-\nabla\cdot(rA\nabla\psi) + \beta\cdot\nabla\psi = z \quad \text{in } Y, \qquad \psi \text{ is } Y\text{-periodic}, \qquad \int_Y \psi = 0. \tag{4.8}$$

Since $\beta\in L^p_{\mathrm{per}}(Y;\mathbb{R}^n)$ with $p>n$, we have that $\psi\in H^2_{\mathrm{per}}(Y)$ and $\|\psi\|_{H^2(Y)}\leq C\|z\|_{L^2(Y)}$ for some constant $C>0$.

Noting that the underlying variational formulation of (4.6) still obeys a Gårding inequality as well as boundedness, a duality argument similar to the one given in the proof of Theorem 2.1 leads to an error bound for the approximation of χ_j based on (4.6) in the case when r is known.

Now, in general, r is not known and we need to incorporate into our numerical method a finite element method for the approximation of r from Section 2.1.2. We choose piecewise affine finite elements for simplicity. To this end, consider a shape-regular triangulation $\mathcal{T}_h$ of $\overline{Y}$ into simplices with maximal overall edge-length $h>0$ that is consistent with the requirement of periodicity.

Theorem 4.1 (Approximation of χ_j in setting $\mathcal{A}$)**.** *Let $(A,b)\in\mathcal{A}$, $j\in\{1,\dots,n\}$, and let $\chi_j\in H^1_{\mathrm{per},0}(Y)$ denote the unique solution to (4.5). Let R_h denote the finite-dimensional subspace of $H^1_{\mathrm{per},0}(Y)$ consisting of continuous Y-periodic piecewise affine functions on $\mathcal{T}_h$ with zero mean over Y. Let $(r_h)_{h>0}\subset W^{1,p}_{\mathrm{per}}(Y)$ with $p>n$ as in (2.4) be such that $e_h := \|r-r_h\|_{W^{1,p}(Y)}\to 0$ as $h\to 0$, where $r\in H^1_{\mathrm{per}}(Y)$ denotes the unique solution to (2.6). Then, for $h>0$ sufficiently small, there exists a unique $\chi_{j,h}\in R_h$ such that*

$$\int_Y r_hA\nabla\chi_{j,h}\cdot\nabla v_h - \int_Y v_h(r_hb - \operatorname{div}(r_hA))\cdot\nabla\chi_{j,h} = \int_Y r_hb_jv_h \quad \forall v_h\in R_h. \tag{4.9}$$

Furthermore, the following error bounds hold:

$$\|\chi_j-\chi_{j,h}\|_{H^1(Y)} \lesssim h\|\chi_j\|_{H^2(Y)} + \sqrt{e_h}(1+\|\chi_j\|_{H^2(Y)}), \tag{4.10}$$

and

$$\|\chi_j-\chi_{j,h}\|_{L^2(Y)} \lesssim (h+e_h)\|\chi_j-\chi_{j,h}\|_{H^1(Y)} + e_h(1+\|\chi_j\|_{H^1(Y)}), \tag{4.11}$$

where the constant absorbed in " $\lesssim$ " only depends on (A,b) and n.

Proof. We set $\beta_h := r_h b - \mathrm{div}(r_h A)$ and $\beta := rb - \mathrm{div}(rA)$. We introduce

$$a_h(q,v) := \int_Y r_h A\nabla q \cdot \nabla v - \int_Y v\beta_h \cdot \nabla q, \qquad l_h(v) := \int_Y r_h b_j v,$$
$$a_0(q,v) := \int_Y rA\nabla q \cdot \nabla v - \int_Y v\beta \cdot \nabla q, \qquad l_0(v) := \int_Y r b_j v$$

for $q, v \in H^1_{\mathrm{per},0}(Y)$. Then, (4.9) reads $a_h(\chi_{j,h}, v_h) = l_h(v_h)$ for all $v_h \in R_h$, and we have from (4.6) that $a_0(\chi_j, v) = l_0(v)$ for all $v \in H^1_{\mathrm{per},0}(Y)$. Using the assumed uniform ellipticity of A, positivity of r, and the fact that $\beta \in L^p_{\mathrm{per}}(Y;\mathbb{R}^n)$ with $p > n$, it is easily seen that we have the Gårding inequality

$$a_0(v,v) \geq \frac{\lambda \inf_{\mathbb{R}^n} r}{2}\|v\|^2_{H^1(Y)} - c_1\|v\|^2_{L^2(Y)} \qquad \forall v \in H^1_{\mathrm{per},0}(Y)$$

for some constant $c_1 > 0$. Using that $\|r_h - r\|_{L^\infty(Y)} + \|\beta_h - \beta\|_{L^p(Y;\mathbb{R}^n)} \lesssim e_h$ and the Sobolev embedding $H^1(Y) \hookrightarrow L^{\frac{2p}{p-2}}(Y)$ as $p > n$, we also have that

$$|a_h(q,v) - a_0(q,v)| \lesssim e_h\|q\|_{H^1(Y)}\|v\|_{H^1(Y)} \qquad \forall q, v \in H^1_{\mathrm{per},0}(Y).$$

Uniqueness of $\chi_{j,h}$: Suppose that $\chi^{(1)}_{j,h}, \chi^{(2)}_{j,h} \in R_h$ are two solutions to (4.9), and set $z_h := \chi^{(1)}_{j,h} - \chi^{(2)}_{j,h}$. Noting that since $z_h \in R_h \subset H^1_{\mathrm{per},0}(Y)$ there holds $a_h(z_h, v_h) = 0$ for all $v_h \in R_h$, we find that

$$\begin{aligned}\|z_h\|^2_{H^1(Y)} &\lesssim a_0(z_h,z_h) + \|z_h\|^2_{L^2(Y)} \\ &\lesssim a_h(z_h,z_h) + \|z_h\|^2_{L^2(Y)} + e_h\|z_h\|^2_{H^1(Y)} \lesssim \|z_h\|^2_{L^2(Y)} + e_h\|z_h\|^2_{H^1(Y)}.\end{aligned}$$

In particular, as $e_h \to 0$, we have for $h > 0$ sufficiently small that

$$\|z_h\|_{H^1(Y)} \lesssim \|z_h\|_{L^2(Y)}. \tag{4.12}$$

Let $\psi \in H^1_{\mathrm{per},0}(Y)$ denote the unique solution to the dual problem (4.7) with $z = z_h$, or equivalently, (4.8) with $z = z_h$ (existence and uniqueness of ψ follow from Proposition 3.1, noting $(rA, rb) \in \mathcal{A}$ and $z_h \in L^2_{\mathrm{per},0}(Y)$). Then, in view of the discussion following (4.7), we have that $\psi \in H^2_{\mathrm{per},0}(Y)$ and $\|\psi\|_{H^2(Y)} \lesssim \|z_h\|_{L^2(Y)}$. Denoting the piecewise linear quasi-interpolant of ψ by $\mathcal{I}_h\psi$, we introduce $\psi_{\mathcal{I}} := \mathcal{I}_h\psi - \int_Y \mathcal{I}_h\psi = \mathcal{I}_h\psi + \int_Y(\psi - \mathcal{I}_h\psi) \in R_h$. Then, using that $\|\psi - \psi_{\mathcal{I}}\|_{H^1(Y)} \lesssim h\|\psi\|_{H^2(Y)} \lesssim h\|z_h\|_{L^2(Y)}$, we obtain that (recall that $\nabla \cdot \beta = 0$ weakly)

$$\begin{aligned}\|z_h\|^2_{L^2(Y)} = a_0(z_h,\psi) &\lesssim a_h(z_h,\psi) + e_h\|z_h\|_{H^1(Y)}\|\psi\|_{H^1(Y)} \\ &\lesssim a_h(z_h,\psi - \psi_{\mathcal{I}}) + e_h\|z_h\|_{H^1(Y)}\|z_h\|_{L^2(Y)} \\ &\lesssim a_0(z_h,\psi - \psi_{\mathcal{I}}) + e_h\|z_h\|_{H^1(Y)}\|z_h\|_{L^2(Y)} \\ &\lesssim \|z_h\|_{H^1(Y)}\|\psi - \psi_{\mathcal{I}}\|_{H^1(Y)} + e_h\|z_h\|_{H^1(Y)}\|z_h\|_{L^2(Y)} \\ &\lesssim (h + e_h)\|z_h\|_{H^1(Y)}\|z_h\|_{L^2(Y)}.\end{aligned}$$

Combining this with (4.12) yields $z_h = 0$ for $h > 0$ sufficiently small, i.e., there is at most one solution to (4.9).

Existence of $\chi_{j,h}$: As R_h is finite-dimensional, uniqueness implies existence of a solution $\chi_{j,h} \in R_h$ to (4.9).

Error bound: Let $\Psi \in H^1_{\mathrm{per},0}(Y)$ denote the unique solution to the dual problem (4.7) (or equivalently (4.8)) with $z := \chi_j - \chi_{j,h} \in H^1_{\mathrm{per},0}(Y)$. Note that $\Psi \in H^2_{\mathrm{per},0}(Y)$ and $\|\Psi\|_{H^2(Y)} \lesssim \|z\|_{L^2(Y)}$. Writing $\Psi_{\mathcal{I}} := \mathcal{I}_h\Psi - \int_Y \mathcal{I}_h\Psi \in R_h$, we obtain

$$
\begin{aligned}
&\|z\|^2_{L^2(Y)} = a_0(z,\Psi) \\
&\lesssim a_h(z,\Psi) + e_h\|z\|_{H^1(Y)}\|\Psi\|_{H^1(Y)} \\
&\lesssim a_h(z,\Psi-\Psi_{\mathcal{I}}) + [l_0 - l_h](\Psi_{\mathcal{I}}) + [a_h - a_0](\chi_j,\Psi_{\mathcal{I}}) + e_h\|z\|_{H^1(Y)}\|z\|_{L^2(Y)} \\
&\lesssim a_0(z,\Psi-\Psi_{\mathcal{I}}) + [l_0 - l_h](\Psi_{\mathcal{I}}) + [a_h - a_0](\chi_j,\Psi_{\mathcal{I}}) + e_h\|z\|_{H^1(Y)}\|z\|_{L^2(Y)} \\
&\lesssim (h+e_h)\|z\|_{H^1(Y)}\|z\|_{L^2(Y)} + [l_0 - l_h](\Psi_{\mathcal{I}}) + [a_h - a_0](\chi_j,\Psi_{\mathcal{I}}) \\
&\lesssim (h+e_h)\|z\|_{H^1(Y)}\|z\|_{L^2(Y)} + e_h\|z\|_{L^2(Y)} + e_h\|\chi_j\|_{H^1(Y)}\|z\|_{L^2(Y)}
\end{aligned}
$$

for $h > 0$ sufficiently small, which implies (4.11). Writing $\chi_{j,\mathcal{I}} := \mathcal{I}_h\chi_j - \int_Y \mathcal{I}_h\chi_j \in R_h$ and $\eta_h := \chi_{j,\mathcal{I}} - \chi_{j,h} = z - (\chi_j - \chi_{j,\mathcal{I}}) \in R_h$, we have that

$$
\begin{aligned}
&\|z\|^2_{H^1(Y)} \lesssim a_0(z,z) + \|z\|^2_{L^2} \\
&\lesssim a_0(z,\chi_j - \chi_{j,\mathcal{I}}) + \|z\|^2_{L^2} + a_0(z,\eta_h) \\
&\lesssim h\|\chi_j\|_{H^2}\|z\|_{H^1} + \|z\|^2_{L^2} + a_h(z,\eta_h) + e_h\|\eta_h\|_{H^1}\|z\|_{H^1} \\
&\lesssim h\|\chi_j\|_{H^2}\|z\|_{H^1} + \|z\|^2_{L^2} + [l_0 - l_h](\eta_h) + [a_h - a_0](\chi_j,\eta_h) + e_h\|\eta_h\|_{H^1}\|z\|_{H^1} \\
&\lesssim h\|\chi_j\|_{H^2}\|z\|_{H^1} + \|z\|^2_{L^2} + e_h\|\eta_h\|_{H^1}(1 + \|\chi_j\|_{H^1} + \|z\|_{H^1}) \\
&\lesssim h\|\chi_j\|_{H^2}\|z\|_{H^1} + \|z\|^2_{L^2} + e_h(h\|\chi_j\|_{H^2} + \|z\|_{H^1})(1 + \|\chi_j\|_{H^1} + \|z\|_{H^1}) \\
&\lesssim h\|\chi_j\|_{H^2}\|z\|_{H^1} + \|z\|^2_{L^2} + e_h(1 + \|\chi_j\|^2_{H^2} + \|z\|^2_{H^1}),
\end{aligned}
$$

where we wrote $\|\cdot\|_{L^2} = \|\cdot\|_{L^2(Y)}$ and $\|\cdot\|_{H^1} = \|\cdot\|_{H^1(Y)}$. Using (4.11), we obtain

$$
\begin{aligned}
\|z\|^2_{H^1(Y)} &\lesssim h^2\|\chi_j\|^2_{H^2(Y)} + \|z\|^2_{L^2(Y)} + e_h(1 + \|\chi_j\|^2_{H^2(Y)}) \\
&\lesssim h^2\|\chi_j\|^2_{H^2(Y)} + (h+e_h)^2\|z\|^2_{H^1(Y)} + e_h(1 + \|\chi_j\|^2_{H^2(Y)})
\end{aligned}
$$

for $h > 0$ sufficiently small. Absorbing the second term on the right-hand side into the left-hand side, we conclude that (4.10) holds. ☐

We are now in a situation to state a result concerning the approximation of the effective diffusion matrix.

Corollary 4.1 (Approximation of $\overline{A}$ in setting $\mathcal{A}$)**.** *Suppose that the assumptions of Theorem 4.1 hold, and let $\overline{A}$ be given by (4.4). Then, by introducing*

$$
\overline{A}_h := \int_Y r_h[I_n + D\chi_h]A[I_n + (D\chi_h)^{\mathrm{T}}], \qquad \chi_h := (\chi_{1,h},\dots,\chi_{n,h}),
$$

we have for $h > 0$ sufficiently small that

$$
\left|\overline{A} - \overline{A}_h\right| \lesssim \left(h\|\chi\|_{H^2(Y;\mathbb{R}^n)} + \sqrt{e_h}\left(1 + \|\chi\|_{H^2(Y;\mathbb{R}^n)}\right)\right)\left(1 + \|D\chi\|_{L^2(Y;\mathbb{R}^{n\times n})}\right),
$$

where the constant absorbed in "$\lesssim$" only depends on (A,b) and n.

Proof. Writing $d_h := \|D(\chi-\chi_h)\|_{L^2(Y;\mathbb{R}^{n\times n})}$, we have by the triangle inequality that $\left|\overline{A}-\overline{A}_h\right| \leq T_1+T_2+T_3$, where

$$\begin{aligned}
T_1 &:= \left|\left\langle [I_n + D\chi]\, A\, (D(\chi-\chi_h))^{\mathrm{T}} \right\rangle\right| \lesssim d_h \left(1+\|D\chi\|_{L^2(Y;\mathbb{R}^{n\times n})}\right), \\
T_2 &:= \left|\left\langle (D(\chi-\chi_h))\, A \left[I_n + (D\chi_h)^{\mathrm{T}}\right] \right\rangle\right| \lesssim d_h \left(1+\|D\chi_h\|_{L^2(Y;\mathbb{R}^{n\times n})}\right), \\
T_3 &:= \left|\int_Y (r-r_h)\, [I_n + D\chi_h]\, A \left[I_n + (D\chi_h)^{\mathrm{T}}\right]\right| \lesssim e_h \left(1+\|D\chi_h\|_{L^2(Y;\mathbb{R}^{n\times n})}\right)^2.
\end{aligned}$$

We have used that $A \in L^\infty(Y;\mathbb{R}^{n\times n})$, $r \in L^\infty(Y)$, and $\|r-r_h\|_{L^\infty(Y)} \lesssim e_h$ by Sobolev embedding. By (4.10), we deduce for $h>0$ sufficiently small that

$$\begin{aligned}
\left|\overline{A}-\overline{A}_h\right| &\lesssim \left(d_h + e_h(1+\|D\chi\|_{L^2(Y;\mathbb{R}^{n\times n})})\right)\left(1+\|D\chi\|_{L^2(Y;\mathbb{R}^{n\times n})}\right) \\
&\lesssim \left(h\|\chi\|_{H^2(Y;\mathbb{R}^n)} + \sqrt{e_h}\left(1+\|\chi\|_{H^2(Y;\mathbb{R}^n)}\right)\right)\left(1+\|D\chi\|_{L^2(Y;\mathbb{R}^{n\times n})}\right),
\end{aligned}$$

as required. □

4.2. **Setting $\mathcal{B}$.** We now discuss the case $(A,b)\in\mathcal{B}$, and we write $(\tilde{A},\tilde{b})$ to denote the pair of renormalized coefficients defined by (2.25), (2.26). We denote the unique solution to the periodic FPK problem (2.31) by $\tilde{r}$ (given by Theorem 2.3). Recall from Corollary 2.1 that the invariant measure r, i.e., the unique solution to (2.32) in $L^2_{\mathrm{per}}(Y)$, is given by

$$r = \frac{1}{\int_Y \gamma\tilde{r}}\gamma\tilde{r}. \tag{4.13}$$

We assume the centering condition

$$\langle b\rangle = 0, \tag{4.14}$$

where we use the notation $\langle\varphi\rangle := \int_Y \varphi r$. Let us discuss the well-posedness and finite element approximation of the problem (4.5) in this setting. Using techniques similar to Section 2.2, we can show the existence and uniqueness of a solution $\chi_j \in H^2_{\mathrm{per},0}(Y)$ to (4.5), as well as construct a simple finite element scheme for its approximation.

Theorem 4.2 (Well-posedness and finite element approximation of (4.5))**.** *Let $(A,b)\in\mathcal{B}$. Let $(\tilde{A},\tilde{b})$ denote the pair of renormalized coefficients given by* (2.25), (2.26), *and suppose that* (4.14) *holds. Further, let $B_1 : H^2_{\mathrm{per},0}(Y)\times H^2_{\mathrm{per},0}(Y)\to\mathbb{R}$ denote the bilinear form defined in* (2.30), *and let $B_2 : H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)\times H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)\to\mathbb{R}$ denote the bilinear form from Lemma 2.2. Then, the following assertions hold.*

(i) *There exists a unique $\chi_j \in H^2_{\mathrm{per},0}(Y)$ such that $B_1(\varphi,\chi_j) = -\int_Y \tilde{b}_j\Delta\varphi$ for all $\varphi\in H^2_{\mathrm{per},0}(Y)$. Further, χ_j is the unique solution to* (4.5) *in $H^2_{\mathrm{per},0}(Y)$.*

(ii) *The function $\xi_j := \nabla\chi_j$ is the unique element $\xi_j \in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$ that satisfies $B_2(w,\xi_j) = -\int_Y \tilde{b}_j\nabla\cdot w$ for all $w\in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$. Further, for any closed linear subspace P_h of $H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$, there exists a unique $\xi_{j,h}\in P_h$ such that $B_2(w_h,\xi_{j,h}) = -\int_Y \tilde{b}_j\nabla\cdot w_h$ for all $w_h\in P_h$, and we have the bound*

$$\|\nabla\chi_j - \xi_{j,h}\|_{H^1(Y;\mathbb{R}^n)} \leq C \inf_{w_h\in P_h}\|D(\nabla\chi_j - w_h)\|_{L^2(Y;\mathbb{R}^{n\times n})}$$

for some constant $C = C(\lambda,\Lambda,\delta,n)>0$.

Proof. (i) By the proof of Theorem 2.1, we know that B_1 is a coercive bounded bilinear form on $H^2_{\mathrm{per},0}(Y)$. Hence, the first part of (i) follows from the Lax–Milgram theorem. For the second part of (i), it is clear that if $\chi_j \in H^2_{\mathrm{per},0}(Y)$ solves (4.5), then $B_1(\varphi,\chi_j) = \int_Y \tilde{b}_j(-\Delta\varphi)$ for all $\varphi \in H^2_{\mathrm{per},0}(Y)$. For the converse, suppose $\chi_j \in H^2_{\mathrm{per},0}(Y)$ satisfies $B_1(\varphi,\chi_j) = \int_Y \tilde{b}_j(-\Delta\varphi)$ for all $\varphi \in H^2_{\mathrm{per},0}(Y)$, and let $\Phi \in L^2_{\mathrm{per}}(Y)$. Then, there exists a unique $\varphi \in H^2_{\mathrm{per},0}(Y)$ such that $\Phi = c\tilde{r} - \Delta\varphi$ with $c := \int_Y \Phi$. In view of (2.31) and (4.14), it follows that

$$\int_Y \left(-\tilde{A} : D^2\chi_j - \tilde{b}\cdot\nabla\chi_j\right)\Phi = B_1(\varphi,\chi_j) = \int_Y \tilde{b}_j(-\Delta\varphi) = \int_Y \tilde{b}_j\Phi,$$

where we used that $\int_Y \tilde{b}_j\tilde{r} = \langle b_j\rangle \int_Y \gamma\tilde{r} = 0$. Since $\Phi \in L^2_{\mathrm{per}}(Y)$ was arbitrary and $\gamma > 0$ a.e., we obtain that χ_j solves (4.5).

(ii) By (i) and the definition of B_2, we have that $\xi_j := \nabla\chi_j \in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$ and $B_2(w,\xi_j) = \int_Y \tilde{b}_j(-\nabla\cdot w)$ for all $w \in H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$. The uniqueness of ξ_j follows from coercivity of B_2 on $H^1_{\mathrm{per},0}(Y;\mathbb{R}^n)$. The existence and uniqueness of $\xi_{j,h} \in P_h$ follows from the Lax–Milgram theorem and the error bound follows from a standard Galerkin orthogonality argument. □

Let us give some comments regarding the homogenization of (4.2) in this weak regularity setting $(A,b) \in \mathcal{B}$, assuming that $\tilde{r} \in L^\infty_{\mathrm{per}}(Y;(0,\infty))$. Then, scaling equation (4.2) by the invariant measure r from (4.13) and applying the transformation argument from [4, 31] yields the problem

$$\begin{aligned} -\nabla\cdot(Q^\varepsilon\nabla u_\varepsilon) &= f \quad \text{in } \Omega,\\ u_\varepsilon &= g \quad \text{on } \partial\Omega, \end{aligned} \tag{4.15}$$

where $Q^\varepsilon := Q(\frac{\cdot}{\varepsilon})$ with $Q := rA + \Psi$, where $\Psi = (\psi_{ij})_{1\le i,j\le n}$ with $\psi_{ij} := \partial_i\phi_j - \partial_j\phi_i$ and

$$-\Delta\phi_j = \nabla\cdot(rAe_j) - rb_j \quad \text{in } Y, \qquad \phi_j \text{ is } Y\text{-periodic}, \qquad \int_Y \phi_j = 0.$$

If $\phi_j \in W^{1,\infty}_{\mathrm{per}}(Y)$, then for any sufficiently regular $f, g, \partial\Omega$, we have that $Q \in L^\infty_{\mathrm{per}}(Y;\mathbb{R}^{n\times n})$ is uniformly elliptic and there exists a unique solution $u_\varepsilon \in H^2(\Omega)$ to (4.15), which is equivalent to (4.2). As $\varepsilon \to 0$, the solution u_ε converges weakly in $H^1(\Omega)$ to the solution u of the homogenized problem (4.3) with $\overline{A}$ as in (4.4).

Corollary 4.2 (Approximation of $\overline{A}$ in setting $\mathcal{B}$)**.** *Suppose that the assumptions of Theorem 4.2 hold, and let ξ_h denote a finite element approximation of $D\chi$ obtained by Theorem 4.2(ii) with $\|D\chi - \xi_h\|_{H^1(Y;\mathbb{R}^{n\times n})} \to 0$, where $\chi = (\chi_1,\dots,\chi_n)$. Let $\tilde{r}$ be given by Lemma 2.2, and let $\tilde{r}_h$ denote its finite element approximation obtained by Theorem 2.4 with $\|\tilde{r} - \tilde{r}_h\|_{L^2(Y)} \to 0$. Suppose that $n \le 4$ and let $\overline{A}$ be given by (4.4) with r given by (4.13). Then, for $h > 0$ sufficiently small, by introducing*

$$\overline{A}_h := \int_Y r_h[I_n + \xi_h]A[I_n + \xi_h^{\mathrm{T}}], \qquad r_h := \frac{1}{\int_Y \gamma\tilde{r}_h}\gamma\tilde{r}_h,$$

we have the bound

$$\begin{aligned}\left|\overline{A}-\overline{A}_h\right| &\lesssim \|D\chi-\xi_h\|_{H^1(Y;\mathbb{R}^{n\times n})}\left(1+\|D\chi\|_{H^1(Y;\mathbb{R}^{n\times n})}\right)\\ &\quad+\|\tilde{r}-\tilde{r}_h\|_{L^2(Y)}\left(1+\|D\chi\|_{H^1(Y;\mathbb{R}^{n\times n})}\right)^2,\end{aligned}\tag{4.16}$$

where the constant absorbed in "$\lesssim$" only depends on (A,b) and n.

Proof. We begin by noting that $H^1(Y)\hookrightarrow L^4(Y)$ for $n\leq 4$ by Sobolev embedding, and that $\|r-r_h\|_{L^2(Y)}\lesssim\|\tilde{r}-\tilde{r}_h\|_{L^2(Y)}$. By the triangle inequality, we have that $\left|\overline{A}-\overline{A}_h\right|\leq T_1+T_2+T_3$, where

$$\begin{aligned}T_1&:=\left|\left\langle (I_n+D\chi)\,A\,(D\chi-\xi_h)^{\mathrm{T}}\right\rangle\right|\lesssim\|D\chi-\xi_h\|_{H^1(Y;\mathbb{R}^{n\times n})}\left(1+\|D\chi\|_{H^1(Y;\mathbb{R}^{n\times n})}\right),\\ T_2&:=\left|\left\langle (D\chi-\xi_h)\,A\left(I_n+\xi_h^{\mathrm{T}}\right)\right\rangle\right|\lesssim\|D\chi-\xi_h\|_{H^1(Y;\mathbb{R}^{n\times n})}\left(1+\|\xi_h\|_{H^1(Y;\mathbb{R}^{n\times n})}\right),\\ T_3&:=\left|\int_Y(r-r_h)\,(I_n+\xi_h)\,A\left(I_n+\xi_h^{\mathrm{T}}\right)\right|\lesssim\|\tilde{r}-\tilde{r}_h\|_{L^2(Y)}\left(1+\|\xi_h\|_{H^1(Y;\mathbb{R}^{n\times n})}\right)^2,\end{aligned}$$

and it follows that (4.16) holds for $h>0$ sufficiently small. $\square$

5. Numerical Experiments

We provide one numerical experiment for the setting $(A,b)\in\mathcal{A}$, and one numerical experiment for the setting $(A,b)\in\mathcal{B}$. Both experiments are for dimension $n=2$.

5.1. **Setting $\mathcal{A}$.** We choose $A:\mathbb{R}^2\to\mathbb{R}^{2\times 2}$ and $b:\mathbb{R}^2\to\mathbb{R}^2$ to be

$$\begin{aligned}A(y)&:=\begin{pmatrix}1+\arcsin(\sin^2(\pi y_1)) & \frac{1}{2}\sin(2\pi y_1)\\ \frac{1}{2}\sin(2\pi y_1) & 2+\cos^2(\pi y_1)\end{pmatrix},\\ b(y)&:=\mathrm{sign}(\sin(2\pi y_1))\begin{pmatrix}1\\1\end{pmatrix}\end{aligned}\tag{5.1}$$

for $y=(y_1,y_2)\in\mathbb{R}^2$, where the choice of A is as in [14]. We note that $(A,b)\in\mathcal{A}$ since $A\in W^{1,\infty}_{\mathrm{per}}(Y;\mathbb{R}^{2\times 2}_{\mathrm{sym}})$, $b\in L^{\infty}_{\mathrm{per}}(Y;\mathbb{R}^2)$, and A is uniformly elliptic.

First, we test the finite element scheme from Theorems 2.1 and 2.2 for the approximation of the invariant measure, i.e., the unique solution r to the FPK problem (2.6), where we choose R_h to be the space consisting of Y-periodic continuous piecewise affine functions with zero mean over Y on a periodic shape-regular triangulation $\mathcal{T}_h$ of the unit cell into triangles with vertices (ih,jh), $1\leq i,j\leq N=\frac{1}{h}\in\mathbb{N}$. All computations are performed in FreeFem++; see [28]. By introducing

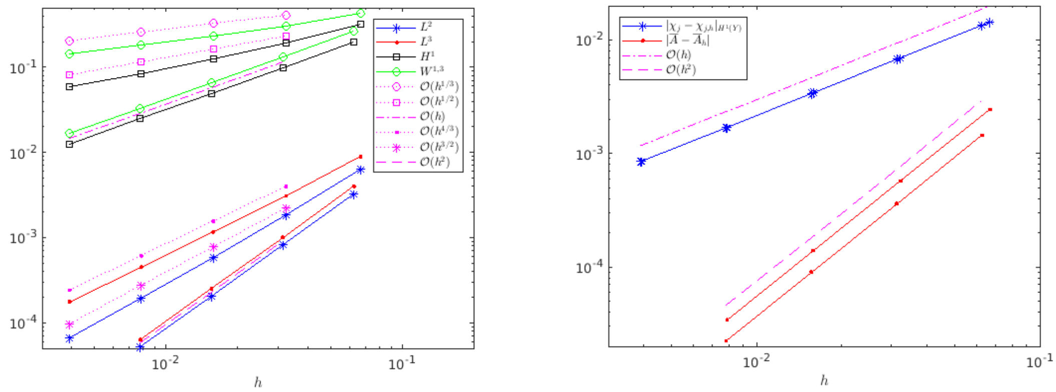
$$K:\mathbb{R}\to\mathbb{R},\qquad K(t):=\int_0^t\frac{\mathrm{sign}(\sin(2\pi x))}{1+\arcsin(\sin^2(\pi x))}\mathrm{d}x,$$

we compare our approximation with the true solution given by

$$r(y)=C_1^{-1}\frac{\mathrm{e}^{K(y_1)}}{1+\arcsin(\sin^2(\pi y_1))},\qquad C_1:=\int_0^1\frac{\mathrm{e}^{K(t)}}{1+\arcsin(\sin^2(\pi t))}\mathrm{d}t$$

for $y=(y_1,y_2)\in\mathbb{R}^2$; see [31].

The approximation errors are shown in Figure 1(A). For $p\in\{2,3\}$, we observe convergence of order $\mathcal{O}(h^{\frac{1}{p}})$ in the $W^{1,p}(Y)$-norm and convergence of order $\mathcal{O}(h^{1+\frac{1}{p}})$



(A) $\|r - r_h\|_{X(Y)}$ for $X \in \{L^2, L^3, H^1, W^{1,3}\}$. (B) $|\chi_j - \chi_{j,h}|_{H^1(Y)}$ and $|\overline{A} - \overline{A}_h|$.

FIGURE 1. Approximation errors for the approximation of the invariant measure r and the effective diffusion matrix $\overline{A}$ corresponding to $(A, b) \in \mathcal{A}$ defined in (5.1). We observe two curves, corresponding to whether or not there are elements of the triangulation whose interior intersects the line $\{y_1 = \frac{1}{2}\}$ along which $\partial_1 r$ exhibits a jump.

in the $L^p(Y)$-norm, which is consistent with the bounds from Theorems 2.1 and 2.2 since $r \in W^{1+s,p}(Y)$ for any $s \in [0, \frac{1}{p})$. Further, we observe superconvergence of order $\mathcal{O}(h)$ in the $W^{1,p}(Y)$-norm and of order $\mathcal{O}(h^2)$ in the $L^p(Y)$-norm when there are no elements of $\mathcal{T}_h$ whose interior intersects the line $\{y_1 = \frac{1}{2}\}$ along which $\partial_1 r$ jumps, which is expected since $r|_{Q\times(0,1)} \in H^2(Q \times (0,1))$ for $Q \in \{(0, \frac{1}{2}), (\frac{1}{2}, 1)\}$.

Since $K(1) = 0$, the centering condition (4.1) is satisfied; see [31]. We now test the finite element scheme from Theorem 4.1 for the approximation of (4.5). We compare with the true solution given by

$$\chi_j(y) = C_2^{-1} \int_0^{y_1} \mathrm{e}^{-K(t)} \mathrm{d}t - y_1 + c, \qquad C_2 := \int_0^1 \mathrm{e}^{-K(t)} \mathrm{d}t$$

for $y = (y_1, y_2) \in \mathbb{R}^2$ and $j \in \{1, 2\}$, where c is a constant such that $\int_Y \chi_j = 0$.

Finally, we test the approximation of the effective diffusion matrix from Corollary 4.1. We compare with the true effective diffusion matrix $\overline{A} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ given by (4.4). The approximation errors are shown in Figure 1(B). For the approximation of χ_j we observe convergence of order $\mathcal{O}(h)$ in the $H^1(Y)$-seminorm, and for the approximation of $\overline{A}$ we observe convergence of order $\mathcal{O}(h^2)$ in the Frobenius-norm. The results are consistent with, and better than the behavior expected from the bound in Corollary 4.1.

5.2. **Setting $\mathcal{B}$.** We choose $A : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ and $b : \mathbb{R}^2 \to \mathbb{R}^2$ to be

$$\begin{aligned} A(y) &:= \begin{pmatrix} 2 + \mathrm{sign}(\cos(\pi y_1)) \sin(\pi y_1) & \frac{1}{2}\sin(2\pi y_1) \\ \frac{1}{2}\sin(2\pi y_1) & 2 + \cos^2(\pi y_1) \end{pmatrix}, \\ b(y) &:= (\tfrac{1}{4} + \tfrac{3}{4}\mathrm{sign}(\sin(2\pi y_1))) \begin{pmatrix} 1 \\ 1 \end{pmatrix} \end{aligned} \tag{5.2}$$

for $y = (y_1, y_2) \in \mathbb{R}^2$. Note that $(A, b) \in \mathcal{B}$ since $A \in L^\infty_{\mathrm{per}}(Y; \mathbb{R}^{2\times 2}_{\mathrm{sym}})$, $b \in L^\infty_{\mathrm{per}}(Y; \mathbb{R}^2)$, A is uniformly elliptic, and the Cordes-type condition (2.5) is satisfied with, e.g., $\delta = \frac{1}{4} \in (\frac{2}{2+\pi^2}, 1]$.

First, we test the finite element scheme from Theorem 2.4 in conjunction with (4.13) for the approximation of the unique solution r to the FPK problem (2.32), where we choose $P_h \subset H^1_{\mathrm{per},0}(Y; \mathbb{R}^2)$ to be the space consisting of vector-valued functions whose components are Y-periodic continuous piecewise affine functions with zero mean over Y on a periodic shape-regular triangulation $\mathcal{T}_h$ of the unit cell into triangles with vertices (ih, jh), $1 \leq i, j \leq N = \frac{1}{h} \in \mathbb{N}$. By introducing

$$\tilde{K} : \mathbb{R} \to \mathbb{R}, \qquad \tilde{K}(t) := \int_0^t \frac{\frac{1}{4} + \frac{3}{4}\mathrm{sign}(\sin(2\pi x))}{2 + \mathrm{sign}(\cos(\pi x))\sin(\pi x)} \mathrm{d}x,$$

we compare our approximation with the true solution given by

$$r(y) = \tilde{C}_1^{-1} \frac{\mathrm{e}^{\tilde{K}(y_1)}}{2 + \mathrm{sign}(\cos(\pi y_1))\sin(\pi y_1)}, \qquad \tilde{C}_1 := \int_0^1 \frac{\mathrm{e}^{\tilde{K}(t)}}{2 + \mathrm{sign}(\cos(\pi t))\sin(\pi t)} \mathrm{d}t$$

for $y = (y_1, y_2) \in \mathbb{R}^2$; see [31]. The approximation error in the $L^2(Y)$-norm is shown in Figure 2(A). We observe convergence of order $\mathcal{O}(\sqrt{h})$, and superconvergence of order $\mathcal{O}(h)$ when there are no elements of $\mathcal{T}_h$ whose interior intersects the line $\{y_1 = \frac{1}{2}\}$ along which r jumps. This indicates that the function ρ from Theorem 2.4 belongs to $H^{1+s}(Y; \mathbb{R}^n)$ for any $s < \frac{1}{2}$, which is expected since $r \in H^s(Y)$ for any $s < \frac{1}{2}$. Note also that $r|_{Q\times(0,1)} \in H^1(Q \times (0, 1))$ for $Q \in \{(0, \frac{1}{2}), (\frac{1}{2}, 1)\}$.

Since $\tilde{K}(1) = 0$, the centering condition (4.14) is satisfied; see [31]. We now test the finite element scheme from Theorem 4.2 for the approximation of (4.5). We compare with the true solution given by

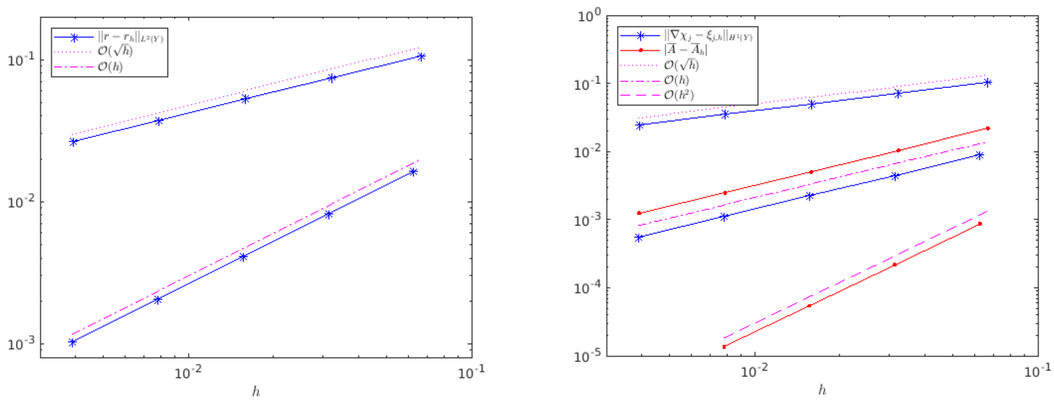
$$\chi_j(y) = \tilde{C}_2^{-1} \int_0^{y_1} \mathrm{e}^{-\tilde{K}(t)} \mathrm{d}t - y_1 + \tilde{c}, \qquad \tilde{C}_2 := \int_0^1 \mathrm{e}^{-\tilde{K}(t)} \mathrm{d}t$$

for $y = (y_1, y_2) \in \mathbb{R}^2$ and $j \in \{1, 2\}$, where $\tilde{c}$ is a constant such that $\int_Y \chi_j = 0$.

Finally, we test the approximation of the effective diffusion matrix from Corollary 4.2. We compare with the true effective diffusion matrix $\overline{A} \in \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ given by (4.4). The approximation errors are shown in Figure 2(B). For the approximation of $\nabla \chi_j$ in the $H^1(Y; \mathbb{R}^2)$-norm we observe convergence of order $\mathcal{O}(\sqrt{h})$, and superconvergence of order $\mathcal{O}(h)$ when there are no elements of $\mathcal{T}_h$ whose interior intersects the line $\{y_1 = \frac{1}{2}\}$. For the approximation of $\overline{A}$ in the Frobenius-norm, we observe convergence of order $\mathcal{O}(h)$, and superconvergence of order $\mathcal{O}(h^2)$ when there are no elements of $\mathcal{T}_h$ whose interior intersects the line $\{y_1 = \frac{1}{2}\}$. The results are consistent with, and actually better than the expected behavior from the bound in Corollary 4.2.

Acknowledgments

The work of ZZ is partially supported by the National Natural Science Foundation of China (Project 12171406), the Hong Kong RGC grant (Projects 17307921 and 17304324), and Seed Funding for Basic Research at HKU.



(A) $\|r - r_h\|_{L^2(Y)}$. (B) $\|\nabla\chi_j - \xi_{j,h}\|_{H^1(Y)}$ and $|\overline{A} - \overline{A}_h|$.

FIGURE 2. Approximation errors for the approximation of the invariant measure r and the effective diffusion matrix $\overline{A}$ corresponding to $(A, b) \in \mathcal{B}$ defined in (5.2). We observe two curves, corresponding to whether or not there are elements of the triangulation whose interior intersects the line $\{y_1 = \frac{1}{2}\}$ along which r exhibits a jump.

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